Closed strings and operads, cont'd

Last time:

We saw (non-unital, non-symmetric) operads, e.g. Ass, $H_*(\overline{M}_{0,n+1})$. In fact $H_*(\overline{M}_{0,n+1})$ is symmetric: it comes with an action of Sym_n, permuting the n marked points on $\overline{M}_{0,n+1}$.

We saw that the genus zero GW invariants of M equip $H^*(M;\Lambda_0)$ with the structure of an algebra over $H_*(\overline{M}_{0,n+1})$ (this also means it respects the Sym_n actions).

There's an obvious morphism of operads

$$\mathrm{Ass} \cong H_0(\overline{M}_{0,n+1}) \hookrightarrow H_*(\overline{M}_{0,n+1})$$

so $H^*(M)$ becomes a Λ_0-linear associative algebra: the 'small' quantum cohomology algebra $QH^*(M)$.

Let's denote by

$$(\ ,\dots,\ )_k : H^*(M)^{\otimes k} \to H^*(M)$$

the operation coming from the fundamental class in $H_{2k-4}(\overline{M}_{0,k+1})$. Now for any

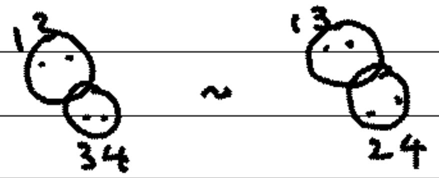
$$v \in H^*(M) \quad \text{and} \quad \alpha, \beta \in H^*(M),$$

define

$$\alpha *_v \beta := \sum_{k=0}^{\infty} \frac{(\alpha, \beta, v, \dots, v)_{k+2}}{k!}.$$

This may not converge, but leaving that aside for the moment, ‘Thm’: (Modulo convergence). $*_v$ is an associative product for all $v \in H^*(M)$.

Pf: We saw the relation



in $H_*(\overline{M}_{0,4})$

gives associativity of $QH^*(M)$.
We pull this relation back to $H_*(\overline{M}_{0,k+1})$ via the map forgetting $k-3$ of the marked points; the proof of associativity follows as before.

To deal with non-convergence, we work in a formal neighbourhood of the origin in $H^*(M)$. So we get:

Thm: $*_v$ defines an associative product on
$$H^*(M) \otimes \Lambda_0 [\![H^*(M)]\!].$$
This is called the ‘big’ quantum cohomology algebra.

Thm (Keel): The operations $(\ldots)_k$ generate the operad $H_*(\overline{M}_{0,k+1})$; and
(Kontsevich-Manin): associativity of $*_v$ is the only relation between these (i.e. it generates all relations).

Rmk: You should think of Λ_0^{ext} as giving parameters on the extended symplectic moduli space, M^{ext}_{symp}, which morally has tangent space $= H^*(M)$.

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(recall that this contains the 'classical' tangent space, which is $H^2(M)$ by Moser's theorem).

GW invariants satisfy some extra axioms... see Kontsevich-Manin. E.g. Frobenius algebra.

E.g. $\mathbb{CP}^2$. The moduli space of genus $= 0$, degree-d curves in $\mathbb{CP}^2$ has dimension

$$2n + 2c_1(\beta) - 6 = 6d - 2.$$

So the most interesting GW invariants to count are degree-d curves passing through $3d-1$ generic points. Let the number of these be N_d.

Let $\langle 1, u, p \rangle$ be a basis for $H^*(\mathbb{CP}^2)$.
We have ($p =$ P.D. to point class).

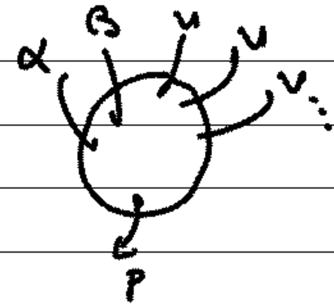
$$N_d := \langle p, p, \dots, p \rangle_{\underset{\text{genus}}{0},\ \underset{\#\text{marked points}}{3d-1},\ \underset{\text{homology class}}{d}}$$

The coefficient ring of big quantum cohomology is

$$\Lambda_0[[\underset{\text{dual to } p}{P}, \underset{u}{Q}, \underset{1}{R}]]$$

We have

$$u * u = p + Au + B$$
$$u * p = Cu + D$$
$$p * p = Eu + F$$

(why do p's look like that? For p to be an output, some moduli space of holomorphic disks has to sweep out a point class: 

but we can move p about on the sphere $\Rightarrow$ unless the sphere is <u>constant</u> it won't sweep out a point class.
Constant spheres just give the usual intersection/cup products, so the only contribution comes from $u \smile u = p$).

Associativity says

$$(u * u) * p = u * (u * p)$$

$$(p + Au + B) * p = u * (Cu + D)$$

$$\Rightarrow Eu + F + A(Cu + D) + Bp = C(p + Au + B) + Du$$

$$\Rightarrow \quad F = BC - AD.$$

Let's extract the $T^d P^{3(d-1)-1}$ coefficients out of this. The coefficient on the LHS is

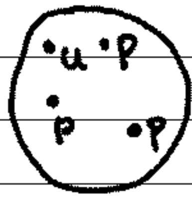
$$\frac{\langle \overbrace{\bar{p}, \bar{p}, \dots, }^{\text{inputs}} \overbrace{\bar{p}}^{\text{output}} \rangle_{0,3d-1,d}}{(3d-4)!}$$

while on the RHS it is

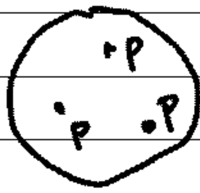
$$\sum_{d_1+d_2=d} \frac{\langle \bar{u}, \bar{u}, p, \dots, \bar{p} \rangle_{0,3d_1+1,d_1}}{(3d_1-2)!} \frac{\langle \bar{u}, \bar{p}, p \dots, \bar{u} \rangle_{0,3d_2+1,d_2}}{(3d_2-2)!}$$

$$-\sum_{d_1+d_2=d} \frac{\langle \bar{u}, \bar{u}, p \dots p, \bar{u} \rangle_{0,3d_1+2,d_1}}{(3d_1-1)!} \frac{\langle \bar{u}, \bar{p}, p \dots, \bar{p} \rangle_{0,3d_2,d_2}}{(3d_2-3)!}$$

Now note that counting spheres of degree d like this:



is the same as counting spheres like this



multiplied by d: the sphere intersects u in d places, so there are d places you can put the marked point (divisor axiom). So the above equation becomes

$$N_d = \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0}} \binom{3d-4}{3d_1-2} d_1^2 d_2^2 N_{d_1} N_{d_2} - \binom{3d-4}{3d_1-1} d_1^3 d_2 N_{d_1} N_{d_2}.$$

(Kontsevich recursion relation).

We know $N_1 = 1$, so this can be solved recursively to give $1, 1, 12, 620 \ldots$

$$N_2 = \binom{2}{1} - \binom{2}{2} = 1$$

$$N_3 = \binom{5}{1} \cdot 4 - \binom{5}{2} \cdot 2 + \binom{5}{4} \cdot 4 - \binom{5}{5} \cdot 8$$

$$= 20 - 20 + 20 - 8$$

$$= 12$$

$N_4 = 620$ etc.

Remark on grading: if M is Calabi-Yau, then $*_v$ respects $\mathbb{Z}$-gradings, as long as we grade the coefficient ring:

$$\Lambda_0^{ext} = \Lambda_0 [\![H^*(M)[2]]\!].$$

If M is not Calabi-Yau, $*_v$ only respects a $\mathbb{Z}/_{2N}$ grading, where N = minimal Chern number.