Closed-string invariants and operads

Note Title 11/9/2013

<u>Recall</u>: Given (M,ω) symplectic, and a choice of J, we defined

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,k}(M,J,\beta) \arrow[r, "ev"] \arrow[d, "s"'] & M^k \\ \overline{\mathcal{M}}_{g,k} & \end{tikzcd}$$

$$\dim_{\mathbb{R}} = 2c_1(TM)(\beta) + (n-3)(2-2g) + 2k =: 2d.$$

We used this to define GW invariants:

$$\langle \alpha_1,\dots,\alpha_k \rangle^{\psi}_{g,\beta} := \sum_{\beta\in H_2(M)} \int_{\overline{\mathcal{M}}_{g,k}} s^*\psi \wedge ev^*(\alpha_1\otimes\dots\otimes\alpha_k)\, T^{\omega(\beta)}$$
$$\in \Lambda_0,$$

which we also thought of informally as

'# J-hol. curves of genus g, meeting $c_1,\dots,c_k$ at their marked points, with domain in $S \subset \overline{\mathcal{M}}_{g,k}$'

where c_i are P.D. to α_i, S is P.D. to ψ.

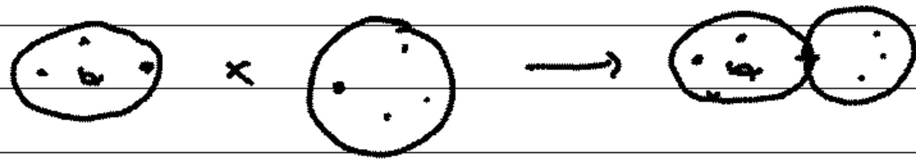
This gives maps

$$I_{g,k} : H^*(\overline{\mathcal{M}}_{g,k}) \otimes H^*(M)^{\otimes k} \to \Lambda_0,$$

where the part coming from class β has degree $2d$.
(if $c_1(M)=0$, i.e., M is Calabi-Yau, we can say $I_{g,k}$ has degree $(n-3)(2-2g)+2k$ w.r.t. an absolute $\mathbb{Z}$-grading; otherwise we can only say this w.r.t. a $\mathbb{Z}/2N$ grading. N = minimal Chern number of M).

These maps have a lot of interesting structure: e.g.,

$$\overline{\mathcal{M}}_{g_1,k_1+1} \times \overline{\mathcal{M}}_{g_2,k_2+1} \longrightarrow \overline{\mathcal{M}}_{g_1+g_2,k_1+k_2}$$

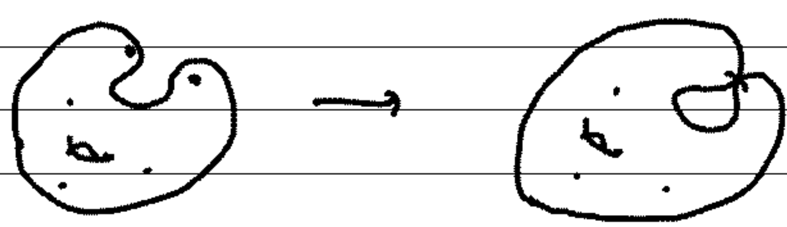


These induce maps

$$H^*(\overline{\mathcal{M}}_{g_1+g_2,k_1+k_2}) \longrightarrow H^*(\overline{\mathcal{M}}_{g_1,k_1+1}) \otimes H^*(\overline{\mathcal{M}}_{g_2,k_2+1})$$

Similarly there are maps

$$\overline{\mathcal{M}}_{g,k+2} \longrightarrow \overline{\mathcal{M}}_{g+1,k}$$



The maps $I_{g,k}$ should be compatible with these maps (think of the Poincaré dual interpretation in terms of cycles). A system of maps

$$I_{g,k} : H^*(\overline{\mathcal{M}}_{g,k}) \otimes V^{\otimes k} \longrightarrow \Lambda_0$$

satisfying these axioms (+ symmetry in inputs) is called a Cohomological Field Theory (Kontsevich-Manin).

From now on, we'll restrict to the $g=0$ case.

A nice way to algebraically package a genus-0 CohFT is as follows:

regard one of the k marked points as an output, via Poincaré duality, and leave the rest as inputs: then we get Λ_0-linear maps

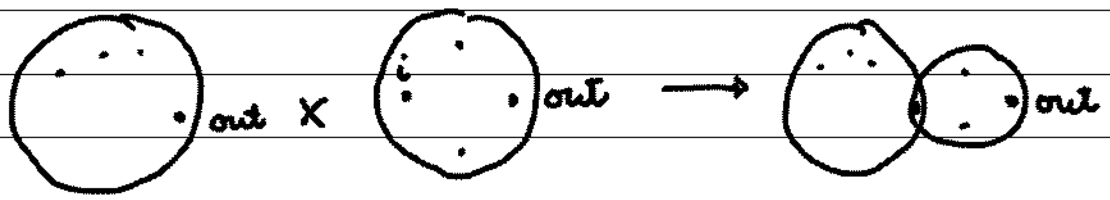
$$\tilde{I}_k : H_*(\overline{\mathcal{M}}_{0,k+1}) \longrightarrow \mathrm{Hom}\left(H^*(M,\Lambda_0)^{\otimes k}, H^*(M,\Lambda_0)\right).$$

$$\langle \tilde{I}_k(S)(\alpha_1,\dots,\alpha_k), \alpha_{k+1} \rangle := \langle \alpha_1,\dots,\alpha_{k+1} \rangle^{P.D.(S)}_{g=0}.$$

Then the maps $\tilde{I}_k$ must satisfy

$$\tilde{I}_{k_1}(S_1)(\alpha_1,\dots \tilde{I}_{k_2}(S_2)(\alpha_{i+1}\dots),\alpha_{i+k_2+1}\dots \alpha_{k_1+k_2-1})$$
$$= \tilde{I}_{k_1+k_2-1}(S_1 \circ_i S_2)(\alpha_1,\dots,\alpha_{k_1+k_2-1}).$$

where the operation '$\circ_i$' is induced by



$$H_*(\overline{\mathcal{M}}_{0,k_1+1}) \otimes H_*(\overline{\mathcal{M}}_{0,k_2+1}) \longrightarrow H_*(\overline{\mathcal{M}}_{0,k_1+k_2+1}).$$

This fits into the formalism of <u>operads</u>.

<u>Defn</u>: A (non-unital, non-symmetric) <u>operad</u> of vector spaces consists of a collection of vector spaces

$$\mathcal{P}(m) \quad \text{for all } m \geq 0,$$

together with 'composition' maps

$$\circ_i : \mathcal{P}(m) \otimes \mathcal{P}(n) \longrightarrow \mathcal{P}(m+n-1) \qquad i = 1, \dots, m,$$

satisfying certain relations. Think of $\mathcal{P}(m)$ as a factory with m inputs and one output, and $\circ_i$ as taking the output of $\mathcal{P}(n)$ and inserting it into the ith input of $\mathcal{P}(m)$; the relations satisfied by $\circ_i$ say that this 'insertion' operation is associative.

<u>Defn</u>: An <u>algebra</u> over an operad $\mathcal{P}$ is a vector space V together with maps

$$\Phi_m : \mathcal{P}(m) \longrightarrow \mathrm{Hom}(V^{\otimes m}, V)$$

compatible with composition in the sense that, for any $\varphi \in \mathcal{P}(m)$, $\psi \in \mathcal{P}(n)$,

$$\Phi_{m+n-1}(\varphi \circ_i \psi)(v_1, \dots, v_{m+n-1})$$

$$= \Phi_m(\varphi)(v_1, \dots v_{i-1}, \Phi_n(\psi)(v_i, \dots v_{i+n-1}) \dots v_{m+n-1}).$$

Think: Operad ⇝ Algebra ⇝ Module.

E.g. $\mathcal{P} = \mathrm{Ass}$: $\mathcal{P}(0) = \mathcal{P}(1) = 0$

$\mathcal{P}(m) = k \quad \forall\, m \geq 2$

(k = ground field)

$\circ_i$ = identity $\forall\, m, n, i$.

An algebra over $\mathcal{P}$ is an associative k-algebra, with multiplication
$$\Phi_2(1) \in \mathrm{Hom}(V^{\otimes 2}, V).$$

Proof of associativity:

$$\Phi_2(1)\big(\Phi_2(1)(a,b), c\big) = \Phi_3(1)(a,b,c)$$
$$= \Phi_2(1)\big(a, \Phi_2(1)(b,c)\big).$$

<u>Defn</u>: A <u>symmetric</u> operad comes with an action of Σ_m on $\mathcal{P}(m)$, and all composition maps $\circ_i$ are equivariant w.r.t. this action, in appropriate sense. Algebras over symmetric operads are required to respect the Σ_m action too.

E.g. $\mathcal{P} = \mathrm{Lie}$: $\mathcal{P}(0) = \mathcal{P}(1) = 0$

$\mathcal{P}(2) = k$ with Σ_2 acting <u>nontrivially</u>.

$\mathcal{P}(3) = k^3/k$

$\ldots$

$\Phi_2(1) \in \mathrm{Hom}(V^{\otimes 2}, V)$ gives the Lie bracket; Σ_2-equivariance says it is antisymmetric.

$\mathcal{P}(3) = k^3/k$, generated by

$$[a,[b,c]],\quad [b,[c,a]],\quad [c,[a,b]],$$

modulo their sum being zero (Jacobi).

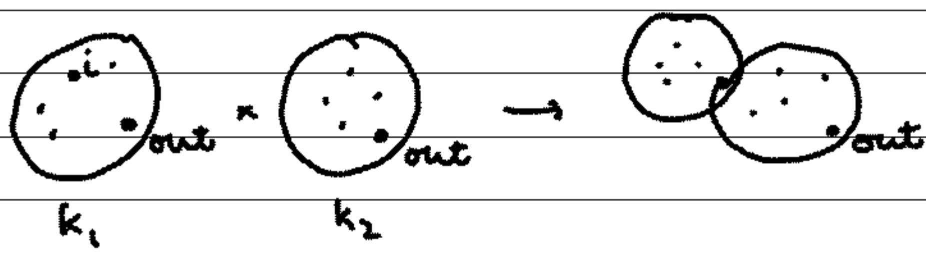
$\mathcal{P}(m) =$ space of operations obtained by all possible Lie brackets on m elements, modulo antisymmetry and Jacobi relation.

For the genus-0 GW invariants, we define the operad

$$\mathcal{P}(k) := H_*(\overline{\mathcal{M}}_{0,k+1}),$$

with composition operations given by gluing:

$$\circ_i : \mathcal{P}(k_1) \otimes \mathcal{P}(k_2) \longrightarrow \mathcal{P}(k_1 + k_2 - 1)$$



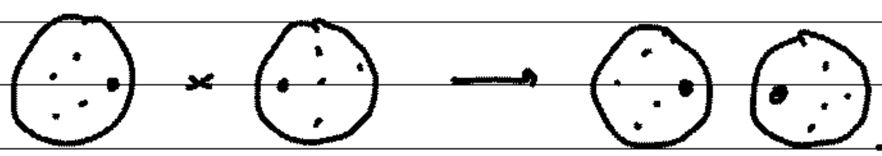
glue the output onto the ith input.

This defines a symmetric operad, called the <u>modular operad</u>, and the genus zero GW invariants make $H^*(M;\Lambda_0)$ into a Λ_0-linear algebra over this operad.

There's an obvious map of operads

$$\mathrm{Ass} \longrightarrow \mathcal{P} = H_*(\overline{\mathcal{M}}_{0,k+1})$$

sending each k into $H_0(\overline{\mathcal{M}}_{0,k+1})$. In particular this is compatible with the splitting maps

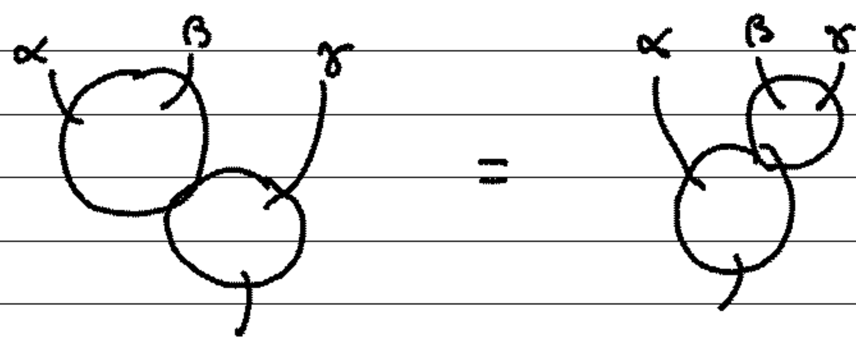


Therefore, any $H_*(\overline{\mathcal{M}}_{0,k+1})$ algebra becomes an associative algebra (like if $R \to S$ is a map of algebras, any S-module becomes an R-module).

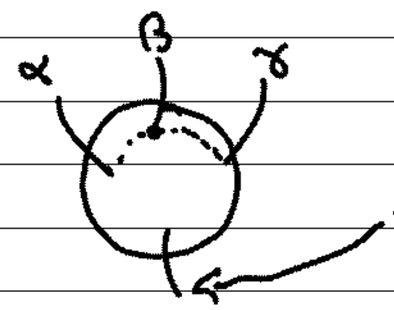
For the GW invariants, the corresponding product is the quantum cup product introduced last time.

How does the proof of associativity look, in practice?

$$(\alpha * \beta) * \gamma \quad = \quad \alpha * (\beta * \gamma)$$



these correspond to two points in $\overline{\mathcal{M}}_{0,4} \cong \mathbb{CP}^1$. They are homologous; so if we look at the moduli space of 4-pointed J-hol. Spheres with domain



this sweeps out a homology between LHS and RHS.

More generally: restricting to the fundamental class $[\overline{M}_{0,k+1}]$ gives us (antisymmetric) maps

$$(\dots) : H^*(M)^{\otimes k} \longrightarrow H^*(M).$$

From these we define a new product: for any $v \in H^*(M; \Lambda_0)$, let

$$\alpha *_v \beta := \sum_{k=0}^{\infty} \frac{1}{k!} (\alpha, \beta, \underbrace{v, \dots, v}_{k}).$$

To ensure convergence we need to work with some completion in the v variables. So we define a new coefficient ring

$$\Lambda_0^{ext} := \Lambda_0 [\![H^*(M)]\!]$$

The new product $*_v$ defines a product on $H^*(M; \Lambda_0^{ext})$. This is called the 'big' quantum cup product, and $H^*(M; \Lambda_0^{ext}, *_v)$ the 'big' quantum cohomology.

<u>Thm</u>: This product is associative. In fact, giving such a product and requiring it to be associative is equivalent to giving $H^*(M; \Lambda_0)$ the structure of an operad over $H_*(\overline{\mathcal{M}}_{0,k+1})$.

(Kontsevich - Manin).

<u>Rmk</u>: You should think of Λ_0^{ext} as giving parameters on the <u>extended</u> symplectic moduli space, $\mathcal{M}^{ext}_{symp}$, which morally has tangent space $= H^*(M)$.

(recall that this contains the 'classical' tangent space, which is $H^2(M)$ by Moser's theorem).

Remark on grading: if M is Calabi-Yau, then $*_v$ respects $\mathbb{Z}$-gradings, as long as we grade the coefficient ring:

$$\Lambda_0^{ext} = \Lambda_0 [\![H^*(M)[2]]\!].$$

If M is not Calabi-Yau, $*_v$ only respects a $\mathbb{Z}/_{2N}$ grading, where N = minimal Chern number.