Closed-string invariants (lite)

Note Title 11/6/2013

Defn: A Gerstenhaber algebra structure on a graded vector space V consists of

- a graded associative supercommutative product on V
- a graded super Lie bracket $[\,,\,]$ on $V[1]$

which are compatible in the sense that $[a,-]$ is a derivation of the product of degree $|a|-1$, for any pure $a \in V$. (Poisson identity).

The part of the closed-string A and B models we look at will be Gerstenhaber algebras.

The closed-string A-model (following Auroux' lectures on mirror symmetry).

Let (M, ω) be a symplectic manifold (means $\omega \in \Omega^2(M)$, $d\omega = 0$, $\omega^{\wedge n}$ = volume form).

An almost-complex structure on M is an endomorphism

$$J \in \mathrm{End}(TM), \quad J^2 = -\mathbb{1}$$

('almost' because J need not be integrable).

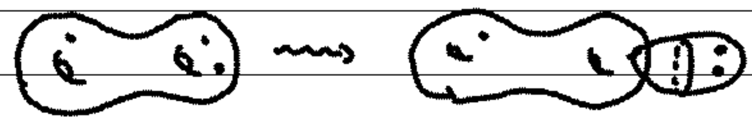
J is compatible with ω if $\omega(J\cdot, \cdot)$ is a Riem. metric.

The space of J's compatible with ω is contractible.

Let $(\Sigma_g, j, z_1, \dots, z_k)$ be a Riemann surface, genus g, complex structure j, with marked points $z_1, \dots, z_k \in \Sigma_g$.

$\mathcal{M}_{g,k}$ = moduli space of such, up to biholomorphism. It's a cpx orbifold of $\dim_{\mathbb{C}} = 3g - 3 + k$ (orbifold because some curves have symmetries)

$\overline{\mathcal{M}}_{g,k}$ = Deligne - Mumford compactification by 'stable' nodal curves:



when points come together, they 'bubble off' a sphere. This is a compact complex orbifold.

A map $u : \Sigma_g \longrightarrow M$ is called J-holomorphic if

$$Du \circ j = J \circ Du.$$

For any $\beta \in H_2(M)$, define

$$\mathcal{M}_{g,k}(M, J, \beta) := \left\{ u : \Sigma_g \to M \text{ J-holomorphic}, [u] = \beta \right\} / \sim$$

where '$\sim$' denotes holomorphic reparametrisation of Σ_g.

These moduli spaces generically come in finite-dimensional families.

Banach vector bundle

$$\begin{tikzcd} W^{\ell,p}(\Sigma, \Omega^{0,1}(\Sigma, u^* TM)) \arrow[d, shift right] \\ W^{\ell+1,p}(\Sigma, M;\beta) \times \mathcal{M}_{g,k} \arrow[u, shift right, "\bar\partial_J"'] \end{tikzcd}$$

depends on J and j

$$\mathcal{M}_{g,k}(M, J, \beta) := (\bar\partial_J)^{-1}(0), \quad \text{where}$$

$$\bar\partial_J u := \tfrac{1}{2}(Du + J \circ Du \circ j) \in \Omega^{0,1}(\Sigma, u^* TM).$$

The linearisation of $\bar\partial_J$ is Fredholm, with index

$$2d = 2c_1(TM)(\beta) + (n-3)(2-2g) + 2k \quad .$$

If $\bar\partial_J$ is transverse to the zero section, $\mathcal{M}_{g,k}(M, J, \beta)$ will be a smooth orbifold of dimension $2d$.

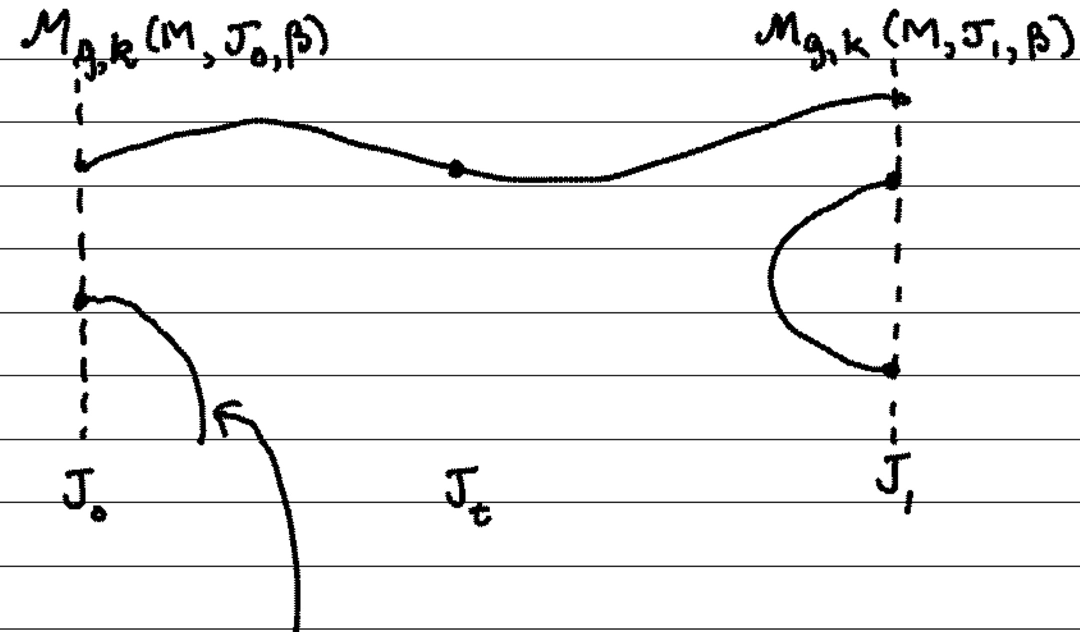
In some situations one can prove that $\bar\partial_J$ is transverse to 0 for 'generic' J, using the Sard-Smale theorem, but in general there are serious technical difficulties to be overcome.

We would like to use these moduli spaces $\mathcal{M}_{g,k}(M, J, \beta)$ to define invariants of (M, ω). E.g. if it is finite-dimensional we would like to count its points. For this to work we need the moduli space to be <u>compact</u>. This is necessary for two reasons:

1. So the count is finite;

2. So we can show the count is independent of J: given J_0 and J_1, there is a 1-parameter family J_t interpolating between them. We look at the 1-dim'l moduli space of curves which are J_t-holomorphic for <u>some</u> t:

$$\bigcup_{t \in [0,1]} \mathcal{M}_{g,k}(M, J_t, \beta)$$



bad: means signed count is different for J_0 and J_1.

Gromov compactness:

If $u : \Sigma \longrightarrow M$ is a J-holomorphic curve, define its energy

$$E(u) := \omega([u]) = \int_\Sigma u^* \omega.$$

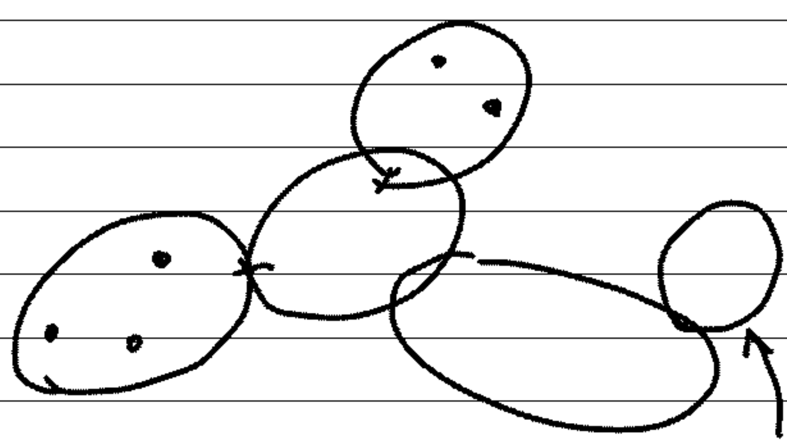
We have

$$u^*\omega(v, jv) = \omega(Du(v), J \circ Du(v)) \geqslant 0,$$

so $E(u) \geqslant 0$, with equality iff u is constant.

Thm (Gromov): If $u_i : \Sigma_i \to M$ is a sequence of J-hol. curves (genus 0, say), then there is a subsequence that converges to a stable map

$$u_\infty : \Sigma_\infty \longrightarrow M$$



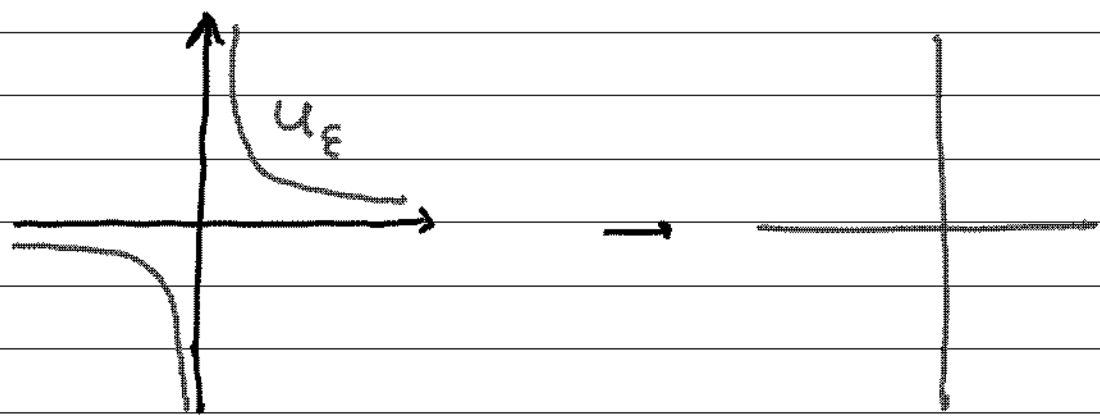
components not necessarily stable.

E.g. The holomorphic spheres

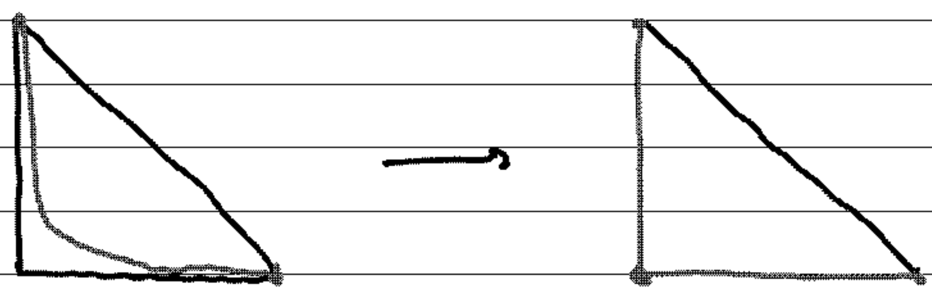
$$u_\varepsilon := \{xy = \varepsilon z^2\} \subset \mathbb{CP}^2$$

converge, as $\varepsilon \to 0$, to the nodal sphere

$$u_0 := \{x=0\} \cup \{y=0\}.$$



or, in a toric picture,



So we define

$$\overline{\mathcal{M}}_{g,k}(M, J, \beta) := \left\{ \begin{array}{l} \text{(possibly nodal) } J\text{-hol.} \\ \text{curves of genus } g \\ \text{in homology class } \beta \end{array} \right\} \Big/ \sim .$$

$\overline{\mathcal{M}}_{g,k}(-)$ is a compact complex orbifold (if you're lucky), of dimension

$$2d = 2c_1(TM)(\beta) + (n-3)(2-2g) + 2k.$$

It also has evaluation maps

$$ev_i : \overline{\mathcal{M}}_{g,k}(M, J, \beta) \longrightarrow M$$
$$u \longmapsto u(z_i)$$

$i = 1, \dots, k,$ $\quad d : \overline{\mathcal{M}}_{g,k}(M, J, \beta) \to \overline{\mathcal{M}}_{g,k}$

We define the <u>Gromov - Witten invariants:</u>

for $\alpha_1, \dots, \alpha_k \in H^*(M)$, $\psi \in H^*(\overline{\mathcal{M}}_{g,k})$

$$\langle \alpha_1, \dots, \alpha_k \rangle_{g,\beta}^{\psi} := \int_{\overline{\mathcal{M}}_{g,k}(M,J,\beta)} d^*\psi \wedge ev_1^*\alpha_1 \wedge \dots \wedge ev_k^*\alpha_k.$$

(it is 0 unless the sum of the degrees of the α_i and ψ is equal to $2d$).

Equivalently, if $\alpha_i = P.D.(c_i)$, and $\psi = P.D.(s)$

$\langle \alpha_1, \dots, \alpha_k \rangle_{g,\beta}^{\psi} :=$ # genus - g J- holomorphic curves in class β, meeting cycles $c_1, \dots, c_k$, whose domain lies in class $S \subset \overline{\mathcal{M}}_{g,k}$.

We then define

$$\langle \alpha_1, \dots, \alpha_k \rangle_g^{\psi} := \sum_{\beta \in H_2(M)} \langle \alpha_1, \dots, \alpha_k \rangle_{g,\beta}^{\psi} \, T^{\omega(\beta)} \in \Lambda_0$$

where Λ_0 is the <u>Novikov ring:</u>

$$\Lambda_0 := \left\{ \sum_{i=0}^{\infty} c_i T^{\lambda_i} \;\middle|\; c_i \in \mathbb{Q},\ \lambda_i \in \mathbb{R}_{\geq 0},\ \lambda_i \to +\infty \right\}$$

J-hol curves have $\omega(u) \geq 0$!

You should think of Λ_0 as an analogue of a formal power series ring, but with real powers. If ω is integral, then $\omega(\beta) \in \mathbb{Z}_{\geq 0}$, and we can replace Λ_0 by $\mathbb{Q}[[T]]$.

Note: if we tried to set $T=1$, i.e.

$$\langle \alpha_1, \dots, \alpha_k \rangle_g^{\mathbb{+}} := \sum_\beta \langle \alpha_1, \dots, \alpha_k \rangle_{g,\beta}^{\mathbb{+}},$$

we would run into trouble because the sum need not be finite. We need to use the Novikov ring because our moduli spaces are only compact if we have a bound on the energy.

Another note: you should think of T, the generator of Λ_0, as a coordinate on $\mathcal{M}_{symp}(M)$. We wish we could define

$$\langle \dots \rangle_g^{\mathbb{+}} := \sum_\beta \langle \dots \rangle_{g,\beta}^{\mathbb{+}} \, e^{-\omega(\beta)},$$

but we don't know if the RHS converges; so we put

$$T^{\omega(\beta)} = e^{-(-\log T\,\omega)(\beta)}$$

and think of this as the 'large-volume limit', i.e., in a neighbourhood of the singular point in $\mathcal{M}_{symp}(M)$ where ω is infinite.

When we actually prove examples of mirror symmetry, it is easier if we stay away from the singular point itself, where $T = 0$; so we often work over the <u>Novikov field</u>:

$$\Lambda := \Lambda_0[T^{-1}] = \{\textstyle\sum c_j T^{\lambda_j} : \lambda_j \in \mathbb{R},\ \lambda_j \to +\infty\}$$

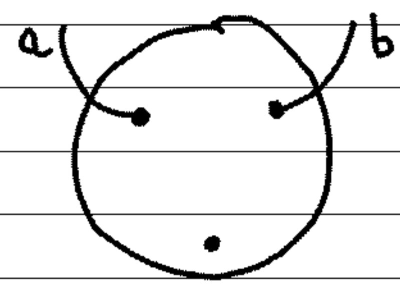
<u>Quantum cohomology:</u>

We use this to define the <u>quantum cup product</u> on $H^*(M; \Lambda_0)$:

$$\langle \alpha * \beta, \gamma \rangle := \langle \alpha, \beta, \gamma \rangle_{g=0}. \qquad (\psi = \text{trivial})$$

intersection pairing.

In other words: if α, β are Poincaré dual to cycles a, b, then we consider the moduli space of J-hol. spheres with 3 marked points, two constrained to lie on a, b:

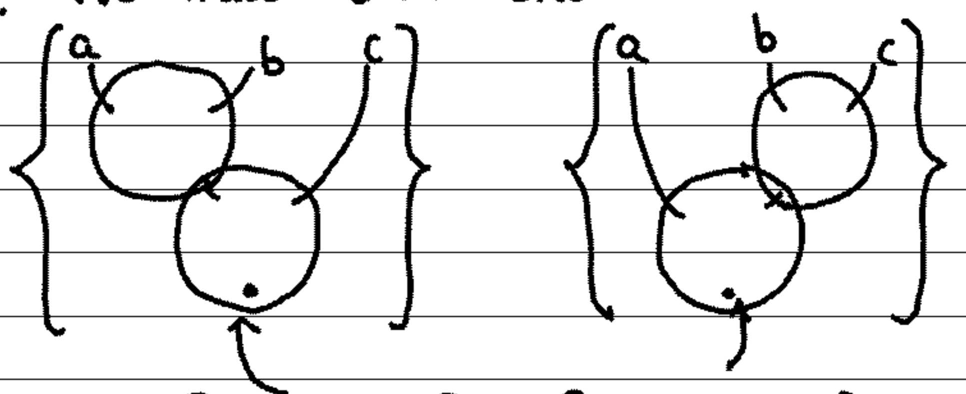


then the remaining marked point sweeps out a cycle c, which we weight by $T^{\omega(u)} \in \Lambda_0$. The Poincaré dual to this cycle c defines $\alpha * \beta$.

Note that the T^0 component of $\alpha * \beta$ counts J-hol spheres with energy $\omega(u) = 0$, which must be constant: so c is just the intersection of a and b. Intersection is P.D. to cup product, so quantum cup product specialises to ordinary cup product when $T = 0$. Hence the name: the intersection product gets 'smeared out'.

<u>Thm</u>: The quantum cup product is associative.

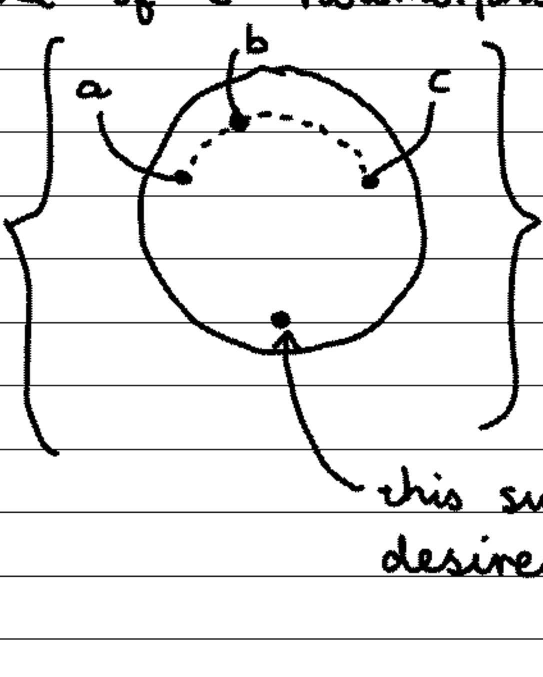
<u>Pf</u>: We must show that



[what this sweeps out] = [what this sweeps out].

To prove this, we find a chain whose boundary is [LHS] − [RHS].

This is given exactly by the moduli space of J-holomorphic spheres



the marked point labelled 'b' moves on a path from 'a' to 'c'.

this sweeps out the desired chain.

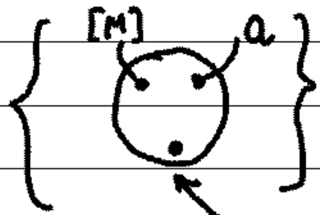
More formally, LHS $= \langle \alpha, \beta, \gamma, \delta \rangle_{0,4}^{\psi} =$ RHS where ψ is P.D. to the point class in $\overline{M}_{0,4}$: LHS corresponds to the point 

, RHS to 

.

<u>Thm</u>: The unit $e \in H^0(M; \Lambda_0)$ remains a unit in QH^*.

<u>Pf</u>: e is P.D. to $[M]$. $e * \alpha$ is (P.D. to):



what this sweeps out

this moduli space is zero-dimensional, by definition. But if the sphere is non-constant, we can move the marked point labelled by $[M]$ around on it; it doesn't matter where it is, as it's constrained to lie in $[M]$ = everything. But this makes a 2-dimensional moduli space, contradiction: the sphere must be constant, and we get the usual cup product

$$e * \alpha = e \cup \alpha = \alpha.$$

(by the same argument, $\alpha * \beta = \alpha \cup \beta$ for $\alpha \in H^1(M)$).

<u>Grading</u>: Recall

$$\dim \left(\mathcal{M}_{g,k}(M, J, \beta)\right) = 2c_1(TM)(\beta) + (n-3)(2g-2) + 2k$$

setting $g = 0$, $k = 3$, we get

$$\dim = 2c_1(TM)(\beta) + 2n.$$

It follows that the contribution to $\alpha * \gamma$ coming from curves in homology class β has degree

$$|\alpha * \gamma|_{\beta} = |\alpha| + |\gamma| + 2c_1(TM)(\beta).$$

In particular, if M is Calabi-Yau (i.e., $c_1(TM) = 0$), then $QH^*(M)$ remains $\mathbb{Z}$-graded; otherwise, it is $\mathbb{Z}/_{2N}$ graded, where

N = minimal Chern number of M.

$QH^*(M)$ becomes a graded, associative, supercommutative Λ_0-algebra, which recovers $H^*(M)$ when we set $T = 0$.

E.g. $M = T^2$. The only holomorphic spheres mapping into T^2 are constants, so the quantum cup product is undeformed:

$$QH^*(T^2) \cong \Lambda^*(\Lambda^2).$$

E.g. $M = \mathbb{CP}^1$. $QH^*(M; \Lambda) = \Lambda e \oplus \Lambda p$.

$$e * e = e \qquad e * p = p * e = p$$

$p * p = Te$ (from the obvious $\mathbb{CP}^1$).

So $QH^*(\mathbb{CP}^1) \cong \Lambda[p]/_{p^2 = T}$