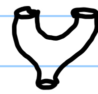
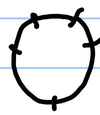
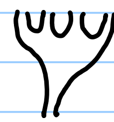
Coherent sheaves

Recall, we saw

	A-model	B-model
closed string (Gerst., BV alg.)	  $QH^*(M)$, $SH^*(M)$  CO 	$H^*(\Lambda^* TM)$, $H^*(\Lambda^* TM, [W,-])$ HKR, HKR
relation	$HH^*(\mathcal{F}(M))$	$HH^*(D^b Coh(M))$, $HH^*(D^b Sing(M,W))$
open string (A_∞ category)	$\mathcal{F}(M)$  or 	$\tilde{D}^b Coh(M)$, $\tilde{D}^b Sing(M,W)$ today's focus

$\mathcal{F}(M)$ is an $\underline{A_\infty \text{ category}}$: if its hom-spaces are $\mathcal{F}(L_0, L_1)$, there are maps

$$\mu^s : \mathcal{F}(L_0, L_1) \otimes \dots \otimes \mathcal{F}(L_{s-1}, L_s) \to \mathcal{F}(L_0, L_s)$$

for all $s \geq 1$, so that A_∞ relations are satisfied.

As a consequence, we can define the <u>cohomological category</u> $H^*\mathcal{F}(M)$, with Hom-spaces

$$H^*(\mathcal{F}(L_0, L_1), \mu^1)$$

and composition maps induced by m^2.

The open-string B-model is the bounded derived category of coherent sheaves on M, $D^b Coh(M)$.

Objects = (bounded complexes of) coherent sheaves on M
(think: holomorphic vector bundles on complex submanifolds).

Morphisms = $Ext^*(\mathcal{E}, \mathcal{F})$

Composition = composition of Ext's,

$$\circ : Ext^*(\mathcal{E}, \mathcal{F}) \otimes Ext^*(\mathcal{F}, \mathcal{G}) \to Ext^*(\mathcal{E}, \mathcal{G}).$$

This is the analogue of $H^*\mathcal{F}(M)$: it's the cohomological category of an underlying A_∞ category.

Namely, when we compute Ext^*, it's always as the cohomology of an underlying complex (e.g. a Čech complex, or the complex $Hom(\mathcal{I}^\bullet_{\mathcal{E}}, \mathcal{I}^\bullet_{\mathcal{F}})$ where $\mathcal{I}^\bullet_{\mathcal{E}}, \mathcal{I}^\bullet_{\mathcal{F}}$ are injective resolutions of $\mathcal{E}, \mathcal{F}$. We define an A_∞ category $\tilde{D}^b Coh(M)$ with

Objects = same as for $D^b Coh(M)$

Morphisms = cochain complex underlying Ext^*, in whichever model you prefer

A_∞ structure maps = μ^1 is the differential on the cochain complex

μ^2 is composition map
(defined on cochain level)

$\mu^{\geq 3} = 0.$

Such an A_∞ category is called a <u>differential graded</u> or <u>DG</u> category.

The <u>homological mirror symmetry</u> conjecture predicts that, if M and $M^\vee$ are mirror Calabi-Yau varieties, then

$$\begin{tikzcd}
\tilde{D}^b \mathcal{F}(M) \arrow[drr, leftrightarrow, "\simeq"] & & \tilde{D}^b \mathcal{F}(M^\vee) \arrow[dll, leftrightarrow, "\simeq"'] \\
\tilde{D}^b \mathrm{Coh}(M) & & \tilde{D}^b \mathrm{Coh}(M^\vee)
\end{tikzcd}$$

quasi-equivalence of A_∞ categories

What does $\tilde{D}^b \mathcal{F}$ mean?

There is a procedure for taking the 'derived' category of an A_∞ category $\mathcal{C}$, whose objects are 'complexes' of objects in $\mathcal{C}$. The result is a 'triangulated' A_∞ category $\tilde{D}^b(\mathcal{C})$. (this is possible for A_∞ or DG categories but not triangulated categories, due to non-functoriality of cones).

<u>Rmk</u>: In many situations we need to further enlarge $\tilde{D}^b \mathcal{F}(M)$ by formally adding new objects, corresponding to 'direct summands' of objects; i.e., for each idempotent endomorphism of an object X, put a new object representing its image. The result is called $\tilde{D}^\pi \mathcal{C}$.

Rmk: It seems to be necessary to use $\tilde{D}^{\pi \text{ or } b}\mathcal{F}(M)$ for HMS to have a chance at being true; $\mathcal{F}(M)$ 'wants' to have more general objects (singular Lagrangians, coisotropics) but they're too hard to do Floer theory for.

Rmk: If M is a closed manifold, then $QH^*(M)$ and $\mathcal{F}(M)$ are defined over Λ. Therefore, for the HMS equivalence to hold, $M^\vee$ should really be a variety over Λ, so that $\tilde{D}^b Coh(M^\vee)$ is a Λ-linear A_∞ category like $\tilde{D}^b\mathcal{F}(M)$ is.

Typically, we have some degenerating family of varieties parametrized by r, i.e., over Spec $\mathbb{C}[\![r]\!]$, with a singular fibre at $r=0$, and we obtain a smooth family by doing a base change via

$$\begin{aligned} \mathbb{C}[\![r]\!] &\to \Lambda \\ r &\mapsto r. \end{aligned}$$

What is the relationship between the open- and closed-string B-model?

In the A-model, we had

$$QH^*(X) \cong HH^*(\mathcal{F}(X)).$$

conjectured iso of Gerst. alg.'s.

In the B-model, we have

$$H^*(X; \Lambda^* TX) \cong HH^*(\tilde{D}^b Coh(X))$$

Hochschild cohomology as an A_∞ category

Why? There's a functor

$$\tilde{D}^b\mathrm{Coh}(X\times Y) \longrightarrow \mathrm{Fun}(\tilde{D}^b\mathrm{Coh}(X), \tilde{D}^b\mathrm{Coh}(Y))$$

DG-category of 'DG functors'

by regarding $\xi \in \tilde{D}^b\mathrm{Coh}(X\times Y)$ as a <u>Fourier-Mukai kernel</u>: it gives a functor

$$\begin{tikzcd} & X\times Y \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ X & & Y \end{tikzcd} \qquad pr_{2*}(pr_1^*\xi \otimes -)$$

all appropriately derived.

E.g. $\mathcal{O}_\Delta := \Delta_*\mathcal{O}_X \in \tilde{D}^b\mathrm{Coh}(X\times X)$ corresponds to the identity functor

$$\mathrm{Id}: \tilde{D}^b\mathrm{Coh}(X) \longrightarrow \tilde{D}^b\mathrm{Coh}(X).$$

(where $\Delta: X \hookrightarrow X\times X$ is the diagonal morphism).

It's a theorem of Toën that, under mild hypotheses on X and Y, this functor is a quasi-equivalence of DG categories.

In particular,

$$HH^*(\tilde{D}^b\mathrm{Coh}(X)) \cong \mathrm{Hom}^*_{\mathrm{Fun}(\tilde{D}^b\mathrm{Coh}(X),\tilde{D}^b\mathrm{Coh}(X))}(\mathrm{Id},\mathrm{Id})$$

the definition we gave is equivalent to this, properly interpreted

$$\cong \mathrm{Hom}^*_{\tilde{D}^b\mathrm{Coh}(X\times X)}(\mathcal{O}_\Delta,\mathcal{O}_\Delta)$$

Toën's theorem

$$\cong \mathrm{Ext}^*_{X\times X}(\Delta_* \mathcal{O}_X, \Delta_* \mathcal{O}_X)$$

$$\cong \mathrm{Ext}^*_X(\Delta^* \mathcal{O}_\Delta, \mathcal{O}_X)$$

zero differential

Now $\Delta^* \mathcal{O}_\Delta \cong \bigoplus_i \Omega^i(X)[i]$

(this is called the Hochschild-Kostant-Rosenberg isomorphism; it was proven in the affine case by HKR, and in the sheafified version by Yekutieli).

$$\cong H^*(\mathcal{H}om(\bigoplus_i \Omega^i(X)[i], \mathcal{O}_X))$$

$$\cong H^*(\bigoplus_i \Lambda^i TX[-i])$$

$$=: HT^*(X).$$

In fact, this can be made into an isomorphism of Gerst. alg.'s.

Therefore, if $\tilde{D}^b\mathcal{F}(M) \cong \tilde{D}^b Coh(M^\vee)$, we have

$$\begin{tikzcd}
HH^*(\tilde{D}^b\mathcal{F}(M)) \arrow[r, phantom, "\cong"] & HH^*(\tilde{D}^b Coh(M^\vee)) \arrow[d, equal, "\cong\ \mathrm{HKR}"] \\
QH^*(M) \arrow[u, "CO"', "?\cong"] \arrow[r, dashed, leftrightarrow, "\cong"'] & HT^*(M^\vee)
\end{tikzcd}$$

So (open-string) homological mirror symmetry

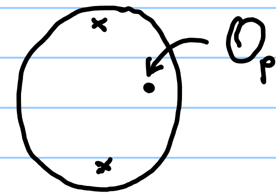
$\overset{?}{\Rightarrow}$ closed-string mirror symmetry.

E.g. $M = \mathbb{C}^*$

$CF^*(L,L) \cong \mathbb{C}[\theta]/\theta^2$

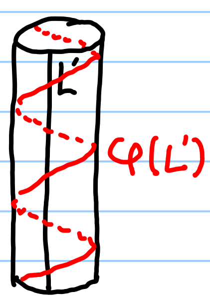
$M^\vee = \mathbb{C}^* = \operatorname{Spec} S$

$S = \mathbb{C}[z, z^{-1}]$



$$
\begin{aligned}
&\operatorname{Ext}^*_{\mathbb{C}^*}(\mathcal{O}_p, \mathcal{O}_p) \\
&\cong \operatorname{Ext}^*_{S\text{-mod}}(S/(z-p), S/(z-p)) \\
&\cong \operatorname{Hom}_S(S \xrightarrow[z-p]{} S, S/(z-p)) \\
&\cong S/(z-p) \oplus S/z-p \\
&\cong \underset{\deg 0}{\mathbb{C}} \oplus \underset{\deg 1}{\mathbb{C}} \\
&\cong \mathbb{C}[\theta]/\theta^2
\end{aligned}
$$

There is no 'higher' A_∞ structure in this case.



$CF^*(L', L') \cong \mathbb{C}[z, z^{-1}]$

'wrapped' Floer cohomology

$$
\begin{aligned}
&\operatorname{Ext}^*_{\mathbb{C}^*}(\mathcal{O}, \mathcal{O}) \\
&\cong \operatorname{Ext}^*_{S\text{-mod}}(S, S) \\
&\cong \operatorname{Hom}_S(S, S) \\
&\cong S = \mathbb{C}[z, z^{-1}].
\end{aligned}
$$

again no 'higher' A_∞ structure.

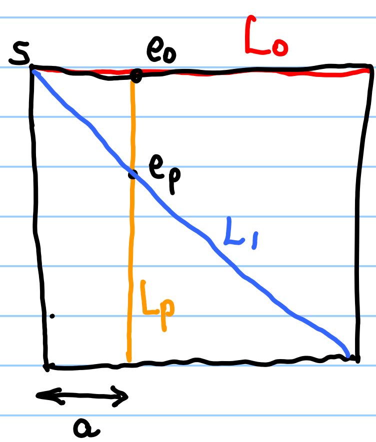
$\mathbb{C}^*$ is mirror to $\mathbb{C}^*$; the quasi-isomorphism

$$\tilde{D}^b \mathcal{F}(\mathbb{C}^*) \cong \tilde{D}^b Coh(\mathbb{C}^*) \quad \text{sends}$$
$$L \longmapsto \mathcal{O}_p$$
$$L' \longmapsto \mathcal{O}.$$

E.g. Elliptic curve (Polishchuk - Zaslow, following exposition of Auroux). We'll pretend $\tau \in \mathbb{R}^+$, and we have convergence everywhere.

$M = \mathbb{R}^2/\mathbb{Z}^2$

$\omega = dx \wedge dy$



$\mu^2(s, e_p) = \underbrace{\;?\;}\, e_0$

$p \in \mathbb{C}^*/\mathbb{Z}$, $p = e^{a+ib}$

equip L_p with a $U(1)$ local system ξ_p with monodromy e^{ib}.

$M^\vee_\tau = \mathbb{C}^*/_{\tau}\mathbb{Z}$

family of elliptic curves parametrised by τ.

$\mathcal{O}$ $\quad$ $\mathcal{L}$ $\quad$ $\mathcal{O}_p$

structure sheaf

line bundle with $c_1(\mathcal{L}) = 1$

skyscraper sheaf at $q = \tau^a e^{ib} \in M^\vee_\tau$

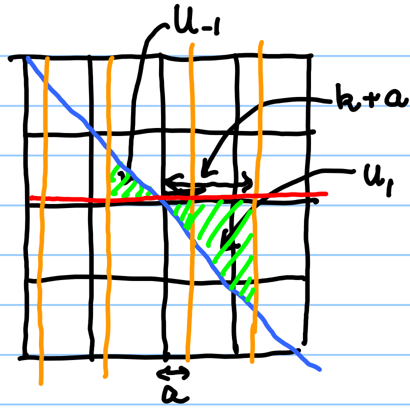
(families of coherent sheaves)

$$\mathcal{O} \xrightarrow{s} \mathcal{L} \xrightarrow{ev_q} \mathcal{O}_p$$

holomorphic section of $\mathcal{L}$

evaluation at q (in trivialisation of $\mathcal{L}|_q$)

composition = value $s(q)$ of s at q.



$$\mu^2(s, e_p) = \sum_{k=-\infty}^{\infty} r^{\omega(u_k)} \operatorname{mon}(\xi_p)\, e_0$$

$$= \sum_{k=-\infty}^{\infty} r^{\frac{1}{2}(k+a)^2} e^{i(k+a)b} e_0$$

$$= r^{\frac{1}{2}a^2 + iab} \sum_{k=-\infty}^{\infty} r^{\frac{1}{2}k^2 + ak} e^{ikb} e_0$$

$$= r^{\frac{1}{2}a^2 + iab} f(q)\, e_0$$

where $q = r^a e^{ib}$,

$$f(q) := \sum_{k=-\infty}^{\infty} r^{\frac{1}{2}k^2} q^k$$

Note: $f(q)$ satisfies

$$f(r \cdot q) = \sum r^{\frac{1}{2}k^2 + k} q^k$$
$$= r^{-1} q^{-1} \sum r^{\frac{1}{2}(k+1)^2} q^{k+1}$$
$$= r^{-1} q^{-1} f(q).$$

On the B-model, we put $\mathcal{L}$ as the line bundle whose total space is

$$\operatorname{Tot}(\mathcal{L}) := (\mathbb{C}^* \times \mathbb{C})/\sim \qquad (z, v) \sim (r \cdot z, r^{-1} z^{-1} v).$$

Then a section of $\mathcal{L}$ (written in a local trivialization) has the form

$$s(z) = \sum_{k=-\infty}^{\infty} a_k z^k, \quad \text{so that}$$

$$s(\tau \cdot z) = \tau^{-1} z^{-1} s(z). \quad (s \text{ is a 'theta function'}).$$

This functional equation characterises s, up to scaling by an element of Λ (it gives a recursion for a_k); so we can choose a basis vector $f(z)$, where f is as above. Then, if we define the functor to send

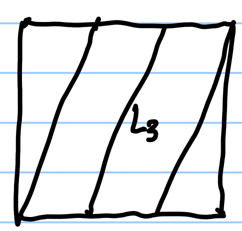
$$CF^*(L_0, L_1) \longrightarrow \mathrm{Hom}(\mathcal{O}, \mathcal{L})$$

$$s \longmapsto f(z)$$

then it respects the composition maps.

More generally, if L_k is a Lagrangian of slope k, it corresponds to $\mathcal{L}_k$, where

$$\mathrm{Tot}(\mathcal{L}_k) = (\mathbb{C}^* \times \mathbb{C})/\sim, \quad (z,v) \sim (\tau \cdot z, \tau^{-k} z^{-k} v).$$



$CF^*(L_0, L_k)$ is rank k, $\mathrm{Hom}^*(\mathcal{O}, \mathcal{L}_k)$ is rank k, and a similar thing happens: you get theta functions which are sections of $\mathcal{L}_k$.

There is more to proving HMS (the higher A_∞ products) but we won't do it (Polishchuk)

Recall that although the mirror to a Calabi-Yau variety $M^\vee$ is another Calabi-Yau variety M, the mirror to a Fano variety $M^\vee$ is a Landau-Ginzburg model (M, W).

The open-string B-model associated to (M, W) is the DG category of matrix factorizations of W, $MF(M, W)$.

If $M = \operatorname{Spec} S$ is affine, $W \in S$, then

$Ob(MF(M,W))$ = finitely-generated projective $\mathbb{Z}/2$-graded S-modules, with a differential d of degree 1 which squares to $W \cdot id$:

$$\bar{P} := \{ P_0 \underset{d_1}{\overset{d_0}{\rightleftarrows}} P_1 \}$$

$$d_0 d_1 = W \cdot id$$

$$d_1 d_0 = W \cdot id$$

$\hom(\bar{P}, \bar{Q}) := \operatorname{Hom}_S(P_0 \oplus P_1, Q_0 \oplus Q_1)$ with natural $\mathbb{Z}/2$-grading

$$\mu^1(F) = F \circ d_P - d_Q \circ F$$

$$\mu^2(F, G) = G \circ F.$$

Everything works analogously to $\tilde{D}^b Coh$:

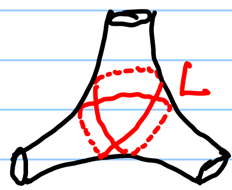
$$HH^*(MF(M,W)) \cong HJ^*(M, W)$$

as Gerstenhaber algebras.

Alternatively, one has

$$MF(M,W) \cong \tilde{D}^b Coh(W^{-1}(0)) / Perf(W^{-1}(0)).$$

E.g.



$CF^*(L,L) \cong \Lambda^* \mathbb{C}^3$, with standard algebra structure, and one important higher A_∞ product:

$$\mu^3(\theta_1, \theta_2, \theta_3) = \mathbb{1},$$

corresponding to

$W = xyz$

$$\in HH^*(\Lambda^* \mathbb{C}^3) \cong \mathbb{C}[[x,y,z]] \otimes \Lambda^* \mathbb{C}^3$$

$(\mathbb{C}^3, W = xyz)$

$\mathcal{O}_0$

$$Hom_{MF}(\mathcal{O}_0, \mathcal{O}_0)$$

$$\cong Ext^*_{\mathbb{C}^3}(\mathcal{O}_0, \mathcal{O}_0)$$

$$\cong \Lambda^* \mathbb{C}^3$$ as an algebra;

the higher A_∞ products correspond to $W \in HH^*(\Lambda^* \mathbb{C}^3)$

So

$$CF^*(L,L) \cong Hom_{MF(W)}(\mathcal{O}_0, \mathcal{O}_0)$$

quasi-equivalence of A_∞ algebras.

As a consequence, one extracts

$$HH^*(CF^*(L,L)) \cong HH^*(Hom_{MF}(\mathcal{O}_0, \mathcal{O}_0))$$

$$\wr\wr \qquad\qquad\qquad\qquad \wr\wr$$

$SH^*($  $)$ $\qquad$ $HT^*(\mathbb{C}^3, W)$

$\Rightarrow$ we get closed-string mirror symmetry out.