Closed-open maps

Note Title 3/8/2014

Recall: If X symplectic mfd, we defined $\mathcal{F}(M)$ = Fukaya category of M, under certain hypotheses. It's an A_∞ category. What is the correct notion of 'equivalence' of A_∞ categories?

Defn: If A and B are A_∞ algebras, an A_∞ homomorphism $F: A \dashrightarrow B$ is a collection of maps

$$F^s : A^{\otimes s} \to B \quad \text{for } s \geq 1,$$

of degree $1-s$, so that

$$\sum \pm F^*(a_1, \ldots a_i, \mu_A^*(a_{i+1}, \ldots, a_j), a_{j+1} \ldots a_s)$$
$$= \sum \pm \mu_B^k(F^*(a_1 \ldots a_{i_1}), F^*(a_{i_1+1}, \ldots, a_{i_2}), \ldots, F^*(a_{i_k+1}, \ldots, a_s))$$

for all $s \geq 1$.

$s=1$: $F^1 \circ \mu_A^1 = \mu_B^1 \circ F^1 \Rightarrow F^1$ a chain map, so it induces

$$[F^1] : H^*(A) \to H^*(B)$$

$s=2$:
$$F^2(\mu^1(a), b) \pm F^2(a, \mu^1(b)) \pm F^1(\mu^2(a,b))$$
$$= \pm \mu^1(F^2(a,b)) \pm \mu^2(F^1(a), F^1(b))$$

$\Rightarrow [F^1]$ is an algebra homomorphism.

There exists a notion of composing A_∞ morphisms.

If $[F']$ is an isomorphism, we call F an A_∞ <u>quasi-isomorphism</u>. This is the correct notion of equivalence between A_∞ algebras.

<u>Lem</u>: If $F: A \dashrightarrow B$ is an A_∞ quasi-isomorphism over a field of char $\neq 2$, then there exists an A_∞ quasi-iso

$$G : B \dashrightarrow A$$

so that $[F \circ G] = \mathrm{Id}$, $[G \circ F] = \mathrm{Id}$.

(so no formal inverting anything! ü).

We similarly define the notion of an A_∞ functor

$$F : \mathcal{C} \dashrightarrow \mathcal{D}$$

between A_∞ categories. An A_∞ functor is a quasi-equivalence if

$$[F'] : H^*(\mathcal{C}) \to H^*(\mathcal{D})$$

is an equivalence.

<u>Thm</u>: The Fukaya category is independent of choices made in its construction, up to A_∞ quasi-equivalence.

Now, recall an A_∞ category was defined in terms of its Hochschild cochain complex $CC^*(\mathcal{C})$: the structure maps $\mu \in CC^*(\mathcal{C})$ satisfy $\mu \circ \mu = 0$.

Given an A_∞ category $\mathcal{C}$, we define its Hochschild cohomology

$$HH^*(\mathcal{C}) := H^*(CC^*(\mathcal{C}), \overset{\delta_{cc}}{\overset{\shortparallel}{[\mu, -]}}).$$

As before, it's naturally a Gerstenhaber algebra.

<u>Lem</u>: If $\mathcal{C}$ and $\mathcal{D}$ are quasi-equivalent A_∞ categories, then

$$HH^*(\mathcal{C}) \cong HH^*(\mathcal{D}).$$

$$(\text{but } HH^*(\mathcal{C}) \ncong HH^*(H^*(\mathcal{C})) \text{ in general}).$$

There is a natural homomorphism of Gerstenhaber algebras, called the <u>closed-open string map</u>,

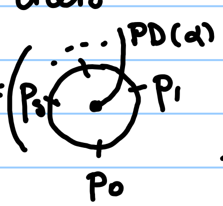
$$CO: QH^*(M) \longrightarrow HH^*(\mathcal{F}(M))$$

(or $\quad CO: SH^*(M) \longrightarrow HH^*(\mathcal{F}(M))$

if M is non-compact), defined as follows:

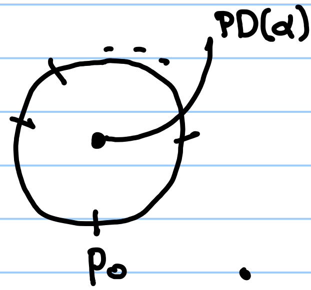
If $\alpha \in QH^*(M)$ is Poincaré dual to a smooth cycle P.D.(α), then

$$CO(\alpha)^s(p_1, \dots, p_s) := \sum_{p_0, \beta} \#\left(\quad \right) T^{\omega(\beta)} p_0.$$

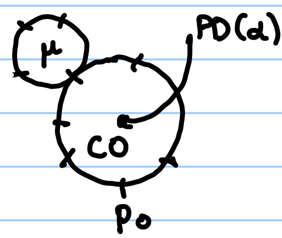


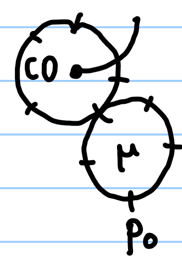
Claim: $\delta_{cc} CO(\alpha) = 0.$

Proof: Look at the boundary points of the Gromov compactification of the 1-dimensional component of the moduli space



They sum to 0, as usual; so





$$CO(\alpha)\circ\mu \quad \pm \quad \mu\circ CO(\alpha) \quad = \quad 0$$

$$[\mu, CO(\alpha)] = 0$$

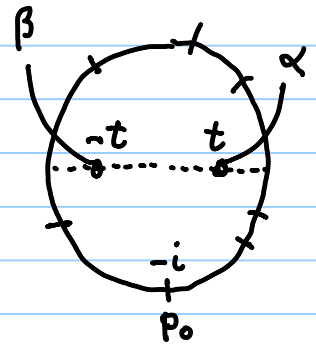
$$\delta_{cc}(CO(\alpha)) = 0.$$

Claim: CO is an algebra homomorphism.

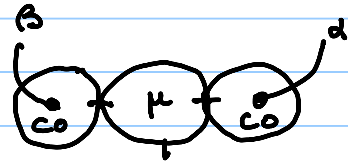
Proof: Recall the Yoneda product on $HH^*(\mathcal{C})$:

$$\alpha \cup \beta\,(a_1, \dots, a_s) := \sum \mu(\dots\alpha(\dots)\dots\beta(\dots)\dots).$$

Consider the 1-dimensional component of the moduli space



it has boundary points

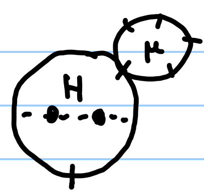


$t \to 1$

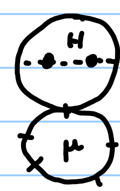
$\mu(\ldots co(\alpha)(\ldots) \ldots co(\beta)(\ldots) \ldots)$

$\shortparallel$

$co(\alpha) \cup co(\beta)$

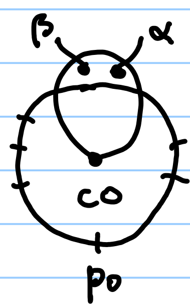


$H(\alpha,\beta) \circ \mu$



$\mu \circ H(\alpha,\beta)$

$t \in (0,1)$



$t \to 0.$

$CO(\alpha * \beta)$

quantum cup product

Hence $CO(\alpha) \cup CO(\beta) = CO(\alpha * \beta) + \delta H(\alpha,\beta).$

One similarly proves $[CO(\alpha), CO(\beta)] = CO([\alpha,\beta])$.
$+ [d, H(\alpha,\beta)].$

If we have some set $\mathcal{L}$ of Lagrangians in M, we can simply consider the full subcategory $\mathcal{F}(M)_{\mathcal{L}} \subset \mathcal{F}(M)$ with those objects.

Then there's a restriction map

$$HH^*(\mathcal{F}(M)) \to HH^*(\mathcal{F}(M)_{\mathcal{L}}),$$

so we can talk about

$$CO : QH^*(M) \to HH^*(\mathcal{F}(M)_{\mathcal{L}})$$

too.

<u>Rmk</u>: ① There exist results along the lines of "if $CO: QH^*(M) \to HH^*(\mathcal{F}(M)_{\mathcal{L}})$ is an isomorphism, then $\mathcal{L}$ 'generates' $\mathcal{F}(M)$ in a suitable sense.

② CO is expected to be an isomorphism for 'nice' M.

③ $\mathcal{F}(M)$ comes with some additional natural algebraic structures, beyond being an A_∞ category (e.g. 'Calabi-Yau' structure). These are expected to allow one to construct more refined operadic structure on $HH^*(\mathcal{F}(M))$, e.g. CohFT, and CO should also respect these. The exact statement depends on whether M is Calabi-Yau, Fano, a Landau-Ginzburg model, non-compact...

E.g.  $L \subset \mathbb{C}^*$

$$CF^*(L, L) \cong C^*(S^1) \cong \Lambda[\theta]/\theta^2$$

(and there's no 'higher' A_∞ structure).

Here, we can actually do without Λ, because all J-hol. disks in a fixed moduli space have same area, by Stokes' theorem.

This happens whenever M is exact ($\omega = d\alpha$) and $L \subset M$ is exact ($\alpha|_L = dh$).

So, $CF^*(L, L) \cong \mathbb{C}[\theta]/\theta^2$.

Thm (the HKR isomorphism): If θ_i anti-commute,

$$HH^*(\mathbb{C}[\theta_1, \dots, \theta_n]) \cong \mathbb{C}[[z_1, \dots, z_n]][\theta_1, \dots, \theta_n]$$

via the map

$$\varphi \xmapsto{HKR} \sum_{s \geq 0} \varphi(\underline{z}, \dots, \underline{z})$$

where $\underline{z} := \sum_{i=1}^{n} z_i \theta_i$.

Recall $SH^*(\mathbb{C}^*) \cong \mathbb{C}[z, z^{-1}][\theta]/\theta^2$.

The map

$$\begin{array}{ccc} CO: SH^*(\mathbb{C}^*) & \longrightarrow & HH^*(\mathcal{F}(\mathbb{C}^*)_L) \\ \wr\wr & & \wr\wr \\ \mathbb{C}[z^{\pm 1}][\theta]/\theta^2 & \longrightarrow & \mathbb{C}[\![z]\!][\theta]/\theta^2 \end{array}$$

sends

$$z^n \xmapsto{CO} e^{nz}$$

$$\theta \longmapsto \theta.$$

It is a homomorphism of Gerstenhaber algebras.

<u>Morse-Bott Floer cohomology</u>

Recall: $CF^*(L,L) \cong CM^*(L)$. When we deal with the A_∞ structure, it can get confusing to define, e.g.,

$$\mu^4: CF^*(L,L)^{\otimes 4} \to CF^*(L,L)$$

if we have to define

$$CF^*(L,L) := CF^*(L, \varphi(L)).$$

We are forced to define

$$\mu^4: CF^*(L,\varphi(L)) \otimes CF^*(\varphi(L), \varphi'(L)) \otimes \dots \to CF^*(L, \varphi'''(L))$$

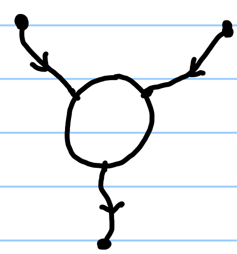
It turns out to be equivalent to define

$$CF^*(L,L) := CM^*(L)$$

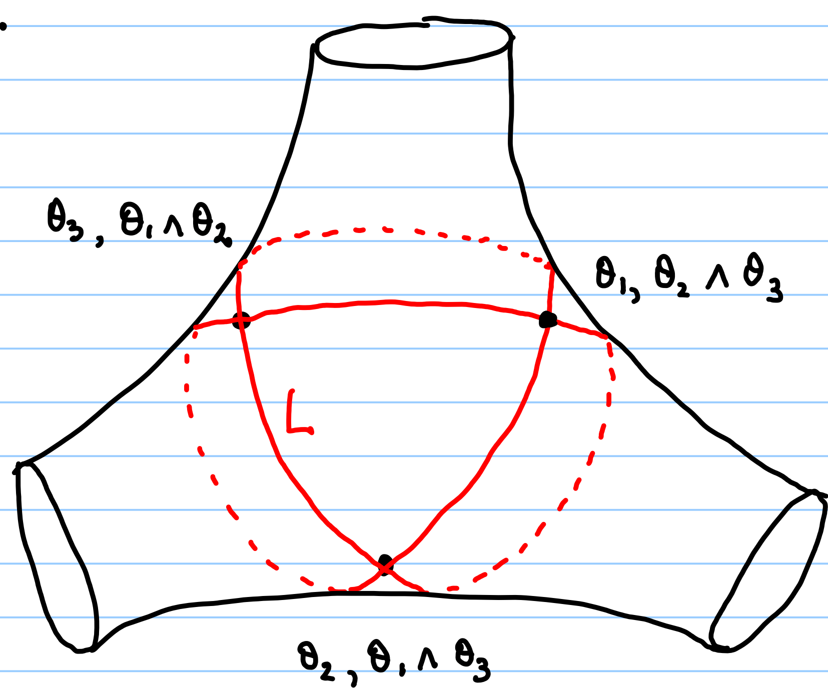
and, instead of counting J-hol. disk

with a boundary puncture, asymptotic at the puncture to an intersection point between L and $\varphi(L)$, count J-hol. disks with a boundary marked point, and a semi-infinite Morse flowline:

so the product would count



E.g.

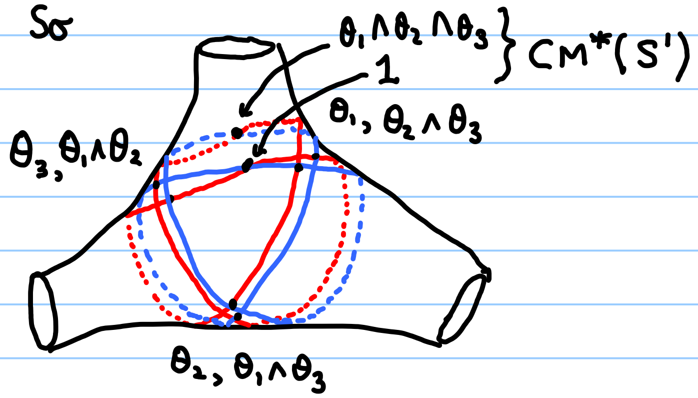


$CF^*(L,L)$ has generators from

$$CM^*(L) \cong \mathbb{C}\langle 1, \theta_1 \wedge \theta_2 \wedge \theta_3 \rangle$$

as well as 2 generators for each self-intersection point, as labelled above.

Recall we define $CF^*(L,L) := CF^*(L, \varphi(L))$.
So

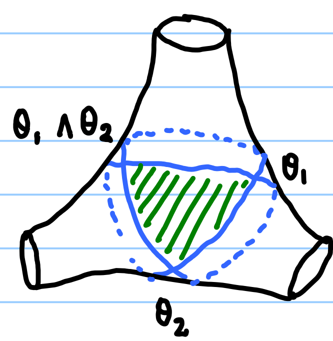


we get 2 generators for each self-intersection point, and 2 from Morse cohomology of S^1.

Hence, $CF^*(L,L) \cong \Lambda^* \mathbb{C}^3$.

You can check that $\mu^1 = 0$ (the only topological strips correspond to the Morse differential on $CM^*(S^1)$, which is 0).

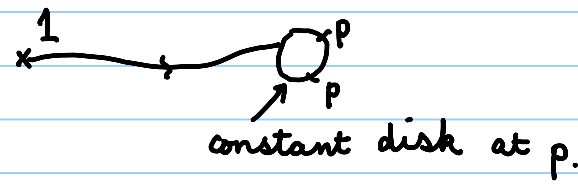
μ^2 is the exterior product:



green triangle: $\mu^2(\theta_1, \theta_2) = \theta_1 \wedge \theta_2$

other triangle (on back): $\mu^2(\theta_2, \theta_1) = -\theta_1 \wedge \theta_2$

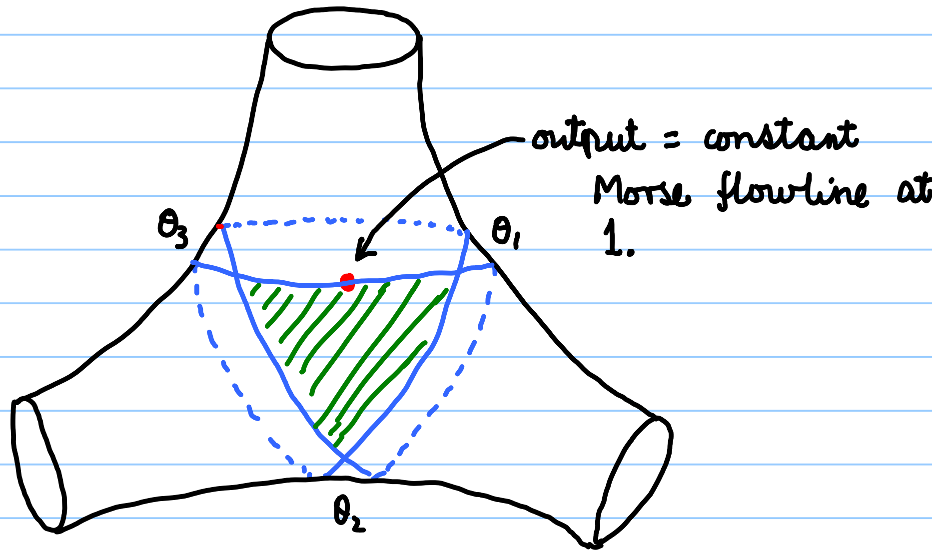
also $\mu^2(1, -) = \mu^2(-, 1) = $ identity:



One checks that $HF^*(L,L) \cong \Lambda^* \mathbb{C}^3$ <u>as an algebra</u>. Now what about the A_∞ structure?

There's one important higher product:

$\mu^3(\theta_1, \theta_2, \theta_3) = 1$:



This corresponds to the element

$$\mu^3(\underline{z}, \underline{z}, \underline{z}) = z_1 z_2 z_3 \in \mathbb{C}[[z_1, z_2, z_3]][\theta_1, \theta_2, \theta_3]$$

$$\cong HH^*(\mathbb{C}[\theta_1, \theta_2, \theta_3]).$$

We can compute $HH^*(CF^*(L,L))$ now: there's a spectral sequence induced by the <u>length filtration</u>:

$$CC^{s+t}(A)^s = \mathrm{Hom}^t(A^{\otimes s}, A)$$

$$F^s CC^*(A) := \prod_{s' \geqslant s} CC^*(A)^{s'}.$$

The E_2 page of the spectral sequence is Hochschild cohomology of the algebra $(A, \mu^2) = \mathbb{C}[\theta_1, \theta_2, \theta_3]$, which is (HKR)

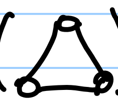
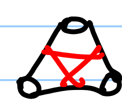
$$E_2 = \mathbb{C}[[z_1, z_2, z_3]][\theta_1, \theta_2, \theta_3].$$

The only subsequent non-trivial differential is

$$\delta_3 = [z_1 z_2 z_3, -]$$

Schouten-Nijenhuis bracket.

We obtain

CO: $SH^*($  $) \longrightarrow HH^*($  $)$

$$\cong$$

$$H^*(\mathbb{C}[z_1, z_2, z_3][\theta_1, \theta_2, \theta_3], [z_1 z_2 z_3, -])$$

$$\cong$$

$$HT^*(\mathbb{C}^3, z_1 z_2 z_3),$$

and this is an isomorphism of Gerstenhaber algebras

(NB. We have to play a game with gradings to get $\mathbb{C}[z_1, z_2, z_3]$ rather than $\mathbb{C}[[z_1, z_2, z_3]]$).