The Fukaya category

Note Title 2/23/2014

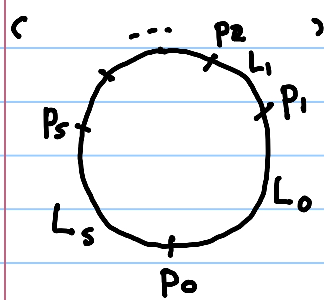
Given $L_0, L_1 \subset M$ Lagrangian submanifolds, define

$$CF^*(L_0, L_1) := \Lambda \langle L_0 \cap L_1 \rangle$$

Given $L_0, L_1, \ldots, L_s$ Lagrangians, and $p_i \in L_{i-1} \cap L_i$ for all i (mod $s+1$), we

define a moduli space

$$\mathbb{D} := \{ |z| \leq 1 \} \subset \mathbb{C}$$



$:= \{ (\zeta_0, \zeta_1, \ldots, \zeta_s)$ ordered points on $\partial\mathbb{D}$,

$u : \mathbb{D} \setminus \{\zeta_0, \ldots, \zeta_s\} \to M$

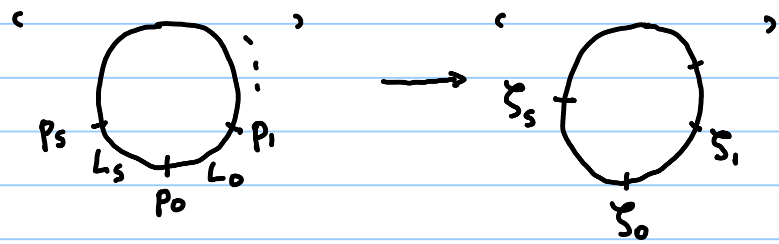
J-holomorphic,

u maps component of $\partial\mathbb{D} \setminus \{\zeta_0, \ldots, \zeta_s\}$ between ζ_i and ζ_{i+1} to L_i,

u asymptotic to p_i at $\zeta_i \} / \sim$

where $\sim$ denotes the action of $PGL(2, \mathbb{R}) = \{$holomorphic automorphisms of $\mathbb{D}\}$ by reparametrisation.

There is a forgetful map



$$\zeta_0 \;\; \| \;\; \mathcal{R}^s := \{(\zeta_0, \dots, \zeta_s) \text{ ordered points on } \partial\mathbb{D}\} / PGL(2,\mathbb{R})$$

which just remembers the domain of u.

$\dim(\mathcal{R}^s) = s-2$; e.g. $\mathcal{R}^2 = \{\text{point}\}$, because $PGL(2,\mathbb{R})$ acts transitively on triples of boundary points.

The fibres of the forgetful map have dimension $\operatorname{ind}(\beta)$ (β = homotopy class of map u).

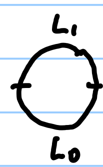
So total dimension is $\operatorname{ind}(\beta) + s - 2$.

Last time, we defined

$$\partial : CF^*(L_0, L_1) \longrightarrow CF^*(L_0, L_1)$$

by

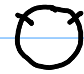
$$\partial p = \sum_{q,\beta} \# \Big(q \Big) \quad T^{\omega(\beta)} q$$



where, by assumption,

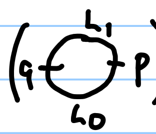
- β = homotopy class of map u
- '#' means 'signed count of the

0-dimensional component of the moduli space'.

This coincides with the definition given last time: any  is biholomorphic

to a strip; then the subgroup of $PGL(2,\mathbb{R})$ fixing (ζ_0,ζ_1) is $\mathbb{R}$, acting by 'translation'. We have $s=1$, so the 0-dimensional component of the moduli space has

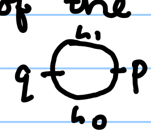
$$\operatorname{ind}(\beta)+s-2=0$$
$$\Rightarrow \operatorname{ind}(\beta)=1.$$

So $\#\Big($ 

$\Big) = \#\big($ J-holomorphic strips of index $\operatorname{ind}(\beta)=1$, modulo translation$\big)$.

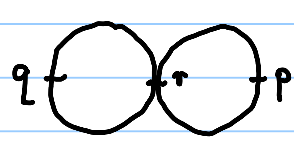
Under suitable assumptions (e.g., $[\omega]\cdot\pi_2(M,L_i)=0$), we showed

$$\partial^2=0$$

because the boundary points of the Gromov compactification of the 1-dimensional component of 

are in 1:1

correspondence with



= terms of $\partial^2(p)$.

We define

$$HF^*(L_0, L_1) := H^*\big(CF^*(L_0, L_1), \partial\big).$$

If $[\omega]\cdot\pi_2(M, L) = 0$, then

$$HF^*(L, L) \cong H^*(L; \Lambda).$$

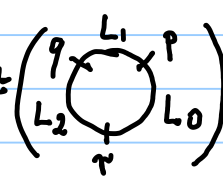
(in more general circumstances where Floer cohomology is defined, there is a spectral sequence
$$H^*(L) \Rightarrow HF^*(L, L)).$$

Next, we define

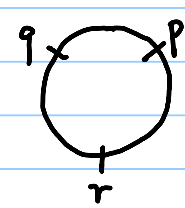
$$\mu^2 : CF^*(L_0, L_1) \otimes CF^*(L_1, L_2) \to CF^*(L_0, L_2)$$

by

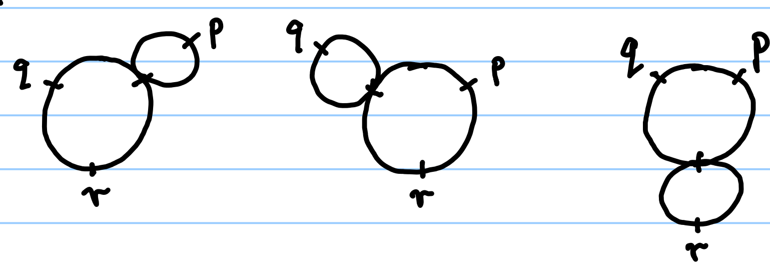
$$\mu^2(p, q) := \sum_{\beta, r} \#\left(\text{disk}\right) T^{\omega(\beta)}\, r.$$



We saw the boundary points of the Gromov compactification of the 1-dimensional component of



are



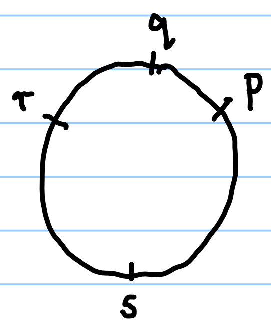
which correspond to terms in the expression

$$\partial \mu^2(p,q) \pm \mu^2(\partial p, q) \pm \mu^2(p, \partial q) = 0.$$

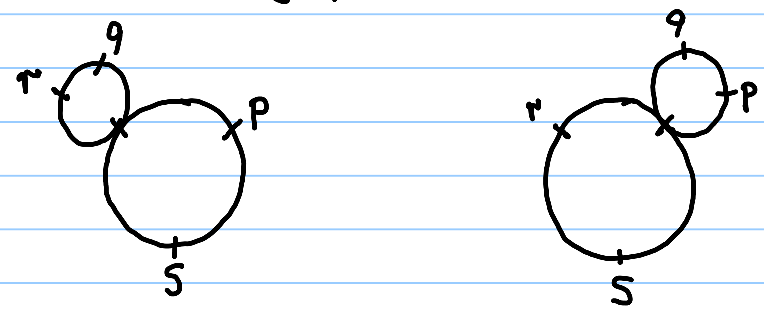
Hence, μ^2 descends to

$$[\mu^2] : HF^*(L_0, L_1) \otimes HF^*(L_1, L_2) \to HF^*(L_0, L_2).$$

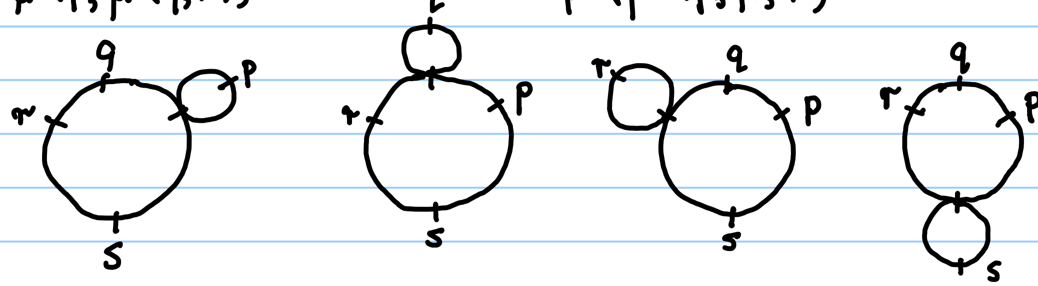
Is $[\mu^2]$ associative? The Gromov compactification of the 1-dimensional component of the moduli space



has boundary points

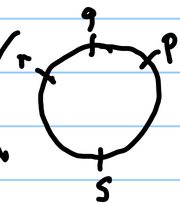


$\mu^2(p, \mu^2(q,r))$ $\qquad\qquad$ $\mu^2(\mu^2(p,q),r)$



so, if we define

$$\mu^3(p,q,r) := \sum_{s,\beta} \# \Bigg($$



$$\Bigg) \, T^{\omega(\beta)} \, s$$

then

$$\mu^2(p,\mu^2(q,r)) \pm \mu^2(\mu^2(p,q),r) =$$

$$\pm\, \mu^3(\partial p, q, r) \pm \mu^3(p, \partial q, r) \pm \mu^3(p,q,\partial r) \pm \partial\mu^3(p,q,r),$$

$\Rightarrow$ $[\mu^2]$ is associative on the level of cohomology.

Lem: If $[\omega]\cdot\pi_2(M,L) = 0$ then $HF^*(L,L) \cong H^*(L)$, and

$[\mu^2] : HF^*(L,L)^{\otimes 2} \longrightarrow HF^*(L,L)$

corresponds to the cup product. Furthermore, we have $e_L \in HF^0(L,L)$ corresponding to the identity; and for any $p \in HF^*(L_0,L_1)$,

$$[\mu^2](e_{L_0}, p) = p = [\mu^2](p, e_{L_1}).$$

<u>Defn</u>: The <u>Donaldson-Fukaya category</u> of M, $DF(M)$, is a Λ-linear category with:

- Objects are Lagrangians $L \subset M$ (with $[\omega]\cdot\pi_2(M, L) = 0$, and a Pin structure, and a grading) ($\mathbb{Z}$- or $\mathbb{Z}/2N$-graded)
- Morphism spaces are Λ-vector spaces

  $$\mathrm{Hom}^*(L_0, L_1) := HF^*(L_0, L_1)$$

- Composition maps are degree-0 maps

  $$[\mu^2] : HF^*(L_0, L_1) \otimes_{\Lambda} HF^*(L_1, L_2) \to HF^*(L_0, L_2)$$

  (N.B. Using opposite from usual convention for composition of functions in categories).

- Identity morphisms are $e_L \in HF^0(L, L)$.

However, the Donaldson-Fukaya category does <u>not</u> contain all the information we need to study mirror symmetry. We need to work with the <u>Fukaya category</u> of M, $\mathcal{F}(M)$. This is an <u>A_∞ category</u>:

Associative $\mathbb{k}$-algebra : $\mathbb{k}$-linear category ::

$\mathbb{k}$-linear A_∞ algebra : $\mathbb{k}$-linear A_∞ category.

A $\Bbbk$-linear pre-category $\mathcal{C}$ consists of:

- a set of objects $Ob(\mathcal{C})$
- for any pair of objects (X, Y), a ($\mathbb{Z}$- or $\mathbb{Z}/2N$-graded) $\Bbbk$-vector space

$$hom^*(X, Y).$$

We introduce convenient notation, for $X_0, \dots, X_s \in Ob(\mathcal{C})$,

$$\mathcal{C}(X_0, \dots, X_s) := hom^*(X_0, X_1) \otimes hom^*(X_1, X_2) \otimes \dots hom^*(X_{s-1}, X_s),$$

then

$$CC^{s+t}(\mathcal{C})^s := \prod_{X_0, \dots, X_s \in Ob(\mathcal{C})} Hom^t(\mathcal{C}(X_0, \dots, X_s), \mathcal{C}(X_0, X_s)).$$

and

$$CC^*(\mathcal{C}) := \prod_{s \geq 0} CC^*(\mathcal{C})^s.$$

If $\mathcal{C}$ has one object, there's only one hom space

$$hom^*(X, X) := A$$

and $CC^*(\mathcal{C}) \cong CC^*(A)$ agrees with our previous definition.

We define the Gerstenhaber product

$$\circ : CC^*(\mathcal{C}) \otimes CC^*(\mathcal{C}) \longrightarrow CC^*(\mathcal{C})$$

by the same formula as previously:

$$\varphi \circ \psi(a_1, \dots, a_s) := \sum \pm \varphi(\dots \psi(\dots)\dots),$$

and the Gerstenhaber bracket

$$[\varphi, \psi] := \varphi \circ \psi - (-1)^{|\varphi|'|\psi|'} \psi \circ \varphi.$$

A <u>(non-unital) $\Bbbk$-linear category</u> is given by a choice of

$$\mu^2 \in CC^2(\mathcal{C})^2$$

satisfying $\mu^2 \circ \mu^2 = 0$. I.e., we have composition maps

$$\mu^2 : \mathcal{C}(X_0, X_1, X_2) \to \mathcal{C}(X_0, X_2)$$

of degree 0, for all $X_0, X_1, X_2 \in Ob(\mathcal{C})$, and they're associative.

A (unital) $\Bbbk$-linear category is this, plus identity endomorphisms $e_X \in hom^*(X, X)$ satisfying the obvious.

A <u>$\Bbbk$-linear A_∞ category</u> is given by a choice of

$$\mu \in CC^2(\mathcal{C})^{s \geq 1}$$

satisfying $\mu \circ \mu = 0$. I.e., we have

$$\mu^s : \mathcal{C}(X_0, \dots, X_s) \longrightarrow \mathcal{C}(X_0, X_s) \quad \text{for } s \geq 1,$$

satisfying the <u>A_∞ relations</u>

$$\mu^1(\mu^1(a)) = 0.$$

$$\mu^1(\mu^2(a,b)) \pm \mu^2(\mu^1(a), b) \pm \mu^2(a, \mu^1(b)) = 0.$$

$$\mu^1(\mu^3(a,b,c)) \pm \mu^3(\mu^1(a), b, c) \pm \mu^3(a, \mu^1(b), c)$$

$$\pm \mu^3(a, b, \mu^1(c)) \pm \mu^2(\mu^2(a,b), c) \pm \mu^2(a, \mu^2(b,c)) = 0$$

$\dots$

Given a $\Bbbk$-linear A_∞ category $\mathcal{C}$, we can define a non-unital $\Bbbk$-linear $H^*(\mathcal{C})$, with

- the same set of objects
- $\mathrm{Hom}^*_{H^*(\mathcal{C})}(X, Y) := H^*(\hom^*_{\mathcal{C}}(X, Y), \mu^1)$
- composition maps given by $[\mu^2]$.

If $H^*(\mathcal{C})$ is unital, we say $\mathcal{C}$ is <u>cohomologically unital</u>.

Defn: The Fukaya category of M, $\mathcal{F}(M)$, is a Λ-linear A_∞ category with

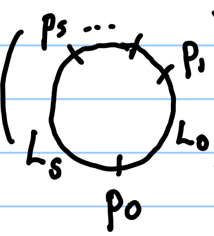
- Objects = Lagrangians $L \subset M$ (with $[\omega]\cdot\pi_2(M,L) = 0$, and Pin structure, and grading)
- hom-spaces

$$\hom^*_{\mathcal{F}(M)}(L_0,L_1) := CF^*(L_0,L_1)$$

- $\mu^s : \mathcal{F}(M)(L_0,\dots,L_s) \to \mathcal{F}(M)(L_0,L_s)$

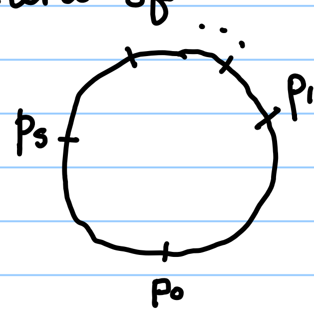
defined by

$$\mu^s(p_1,\dots,p_s) := \sum_{p_0,\beta} \#\Big($$

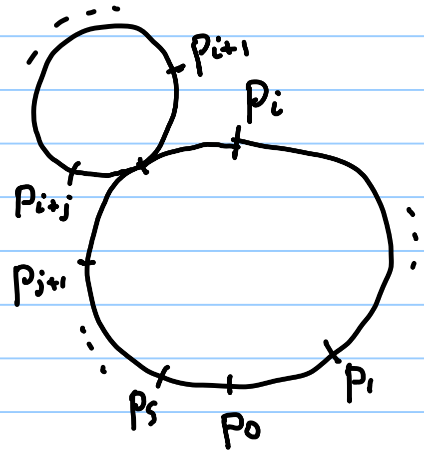


$$\Big)\, T^{\omega(\beta)}\, p_0$$

The fact that $\mu\circ\mu = 0$ follows from the fact that the boundary points of the Gromov compactification of the 1-dimensional component of



are in 1-1 correspondence with



$$\mu^{s+1-j}(p_1,\dots,p_i,\mu^j(p_{i+1},\dots,p_{i+j}),p_{i+j+1},\dots,p_s)$$

We have already seen $\mu^1 = \partial$, μ^2, and μ^3, and the first 3 A_∞ equations.

Note that

$$D\mathcal{F}(M) := H^*(\mathcal{F}(M)).$$

$D\mathcal{F}(M)$ is unital, so $\mathcal{F}(M)$ is cohomologically unital.

Why A_∞ categories appear:

$\bar{\mathcal{R}}^k$ = Deligne-Mumford compactification of $\mathcal{R}^k$ by stable nodal disks

has the structure of a polytope, with boundary facets covered by the maps

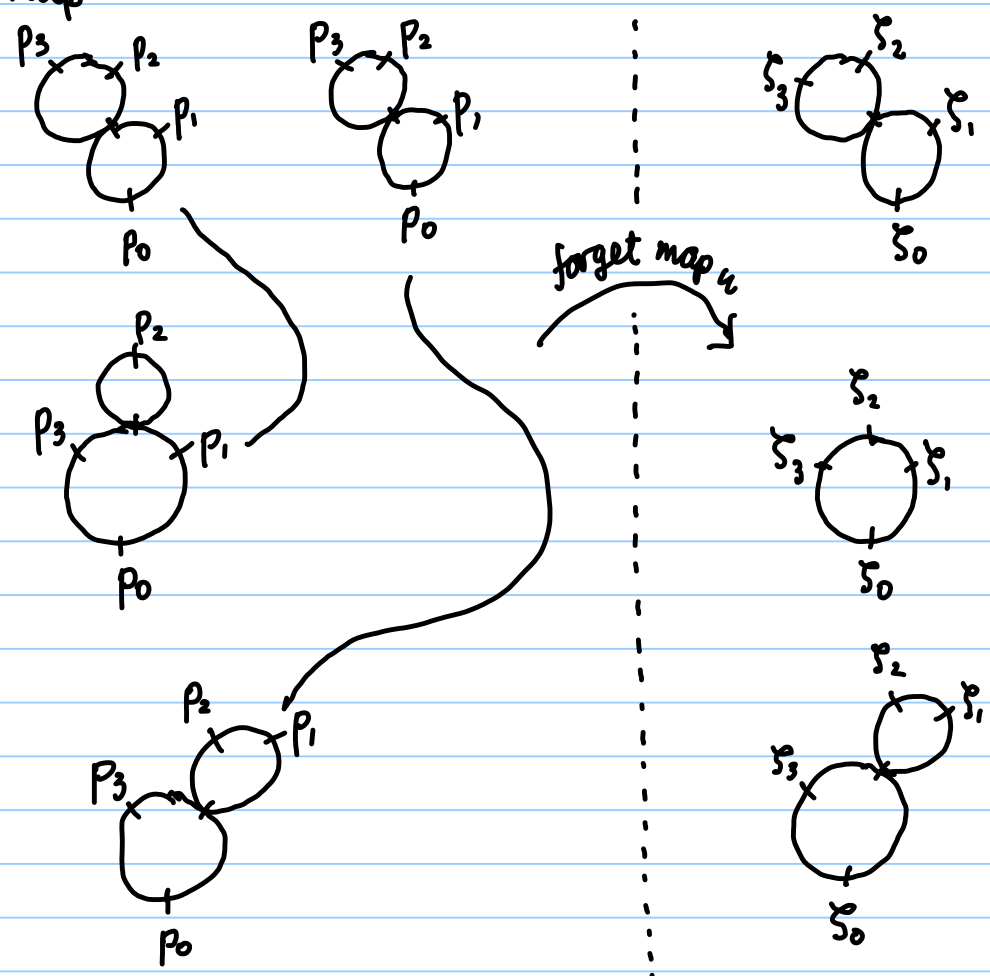
$$\circ_i : \bar{\mathcal{R}}^{k_1} \times \bar{\mathcal{R}}^{k_2} \to \bar{\mathcal{R}}^k \qquad \text{for } k = k_1 + k_2 - 1$$

So we can define a DG operad

$$P(k) := C_*^{\text{cellular}}(\bar{\mathcal{R}}^k)$$

and DG algebras over this DG operad are equivalent to A_∞ algebras; the operations μ^s correspond to the top-dimensional cell of $\bar{R}^k$.

It's necessarily DG, because the forgetful map



can degenerate in the middle, by energy concentrating, without the domain degenerating.