Mirror Symmetry 1

Note Title 11/3/2013

Introduction: what is mirror symmetry?

We'll consider invariants of complex manifolds:

(M, J)

smooth manifold $\nearrow$ $\nwarrow$ integrable complex structure

Invariants of a complex manifold will be called 'the B-model'.

E.g. 'the closed-string B-model' = Hodge structure, or alternatively, periods of differential forms.

E.g. 'the open-string B-model' = (bounded derived) category of coherent sheaves.

We'll also consider invariants of symplectic mflds:

(M, ω)

smooth manifold $\nearrow$ $\nwarrow$ symplectic form ω

(i.e., $\omega \in \Omega^2(M)$
$d\omega = 0$
$\omega^{\wedge \mathrm{top}}$ non-vanishing volume form).

Invariants of a symplectic manifold will be called 'the A-model'.

E.g. 'the closed-string A-model' = Gromov-Witten invariants = counts of holomorphic curves, weighted by $e^{-(\text{symplectic area})}$.
(like # complex lines through 2 generic points in $\mathbb{CP}^n$ = 1, or # degree-d curves on the quintic 3-fold).

Rmk. For an <u>open</u> symplectic manifold, the closed-string A-model is symplectic cohomology.

E.g. 'the open-string A-model' = (bounded derived) Fukaya category (subject of next lecture).

For us, complex and symplectic structures will appear as part of the structure on a <u>Kähler manifold</u>:

$$(M, \omega, J)$$

ω = symp. form, J = integrable complex structure

'compatible' in the sense that $\omega(\cdot, J\cdot)$ is a Riemannian metric.

<u>'Meta mirror symmetry'</u>:

> There exist pairs of Kähler 'spaces',
>
> $$(M, \omega, J) \quad \text{and} \quad (M^\vee, \omega^\vee, J^\vee),$$
>
> such that there are 'equivalences'
>
> $$\begin{tikzcd} A(M, \omega) \arrow[dr, leftrightarrow] & A(M^\vee, \omega^\vee) \\ B(M, J) \arrow[ur, leftrightarrow] & B(M^\vee, J^\vee) \end{tikzcd}$$
>
> (we call these 'mirror pairs').

Of course, symplectic and complex structures come in families. We denote

$$\mathcal{M}_{symp}(M) := \{\text{symp. forms on } M\} / \text{symplectom.}$$

$$\mathcal{M}_{cpx}(M) := \{\text{cpx structures on } M\}/\text{biholom.}$$

If M and $M^\vee$ are mirror, we expect there to be diffeomorphisms of moduli spaces:

$$\begin{tikzcd}
\mathcal{M}_{symp}(M) \arrow[drr, leftrightarrow] & & \mathcal{M}_{symp}(M^\vee) \arrow[dll, leftrightarrow] \\
\mathcal{M}_{cpx}(M) & & \mathcal{M}_{cpx}(M^\vee)
\end{tikzcd}$$

and the equivalences of A- and B-models hold in families.

These diffeomorphisms are called the <u>mirror maps</u>.

We have

$$T_\omega \mathcal{M}_{symp}(M) \cong H^2(M) \quad \text{(Moser's theorem)}$$

$$T_J \mathcal{M}_{cpx}(M) \cong H^1(M, TM). \quad \text{(BTT theorem)}$$

So the mirror map should identify

$$H^2(M) \cong H^1(M^\vee, TM^\vee).$$

In fact, there are 'extended moduli spaces' of symplectic and complex structures, with

$$T_\omega \mathcal{M}^{ext}_{symp}(M) \cong \bigoplus_{p,q} H^q(M, \Omega^p M) \cong H^*(M)$$

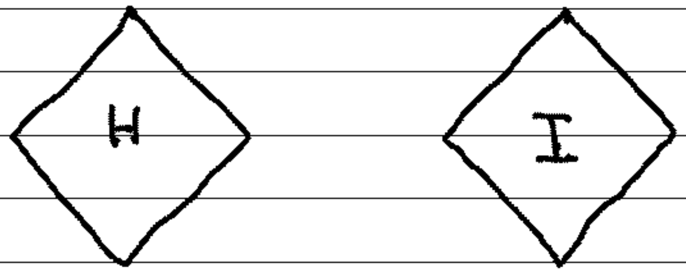
$$T_J \mathcal{M}^{ext}_{cpx}(M) \cong \bigoplus_{p,q} H^q(M, \Lambda^p TM),$$

and an 'extended mirror map' which identifies

$$H^q(M, \Omega^p M) \cong H^q(M^\vee, \Lambda^p TM^\vee).$$

$$\text{if } M^\vee \text{ Calabi-Yau} \quad \cong \quad H^q(M^\vee, \Omega^{n-p} M^\vee)$$

so the Hodge diamonds are 'mirror':



$M \qquad\qquad M^\vee$

<u>closed-string mirror symmetry:</u>

Predicts equality of numbers:

$$GW(M, \omega) \longleftrightarrow \text{Periods}(M^\vee, J^\vee)$$

$$\text{Periods}(M, J) \longleftrightarrow GW(M^\vee, \omega^\vee)$$

(up to some serious algebraic rearrangement).

this was used to spectacular effect by Candelas - de la Ossa - Green - Parkes '91 to correctly predict the number of degree-d curves on the quintic 3-fold; subsequently verified by Givental and Lian - Liu - Yau in many cases (all Calabi - Yau and Fano complete intersections in toric varieties).

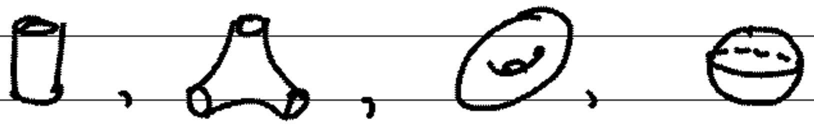
open-string/homological mirror symmetry (HMS):
(Kontsevich '94)

Predicts equivalences of categories:

$$\begin{tikzcd} D^b \mathcal{F}uk(M,\omega) \arrow[dr, leftrightarrow] & D^b \mathcal{F}uk(M^\vee, \omega^\vee) \arrow[dl, leftrightarrow] \\ D^b Coh(M,J) & D^b Coh(M^\vee, J^\vee). \end{tikzcd}$$

This is good because:

- it is much more general, and extends to some very simple and illuminating cases (e.g. we'll see



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which help build intuition... we don't have to start with Calabi-Yau 3-folds!

- Nevertheless, it is expected to imply closed-string mirror symmetry, including higher-genus GW invariants (not currently known beyond genus 1 for quintic 3-fold).

- Fuk(M) and Coh(M) are very rich:

Fuk(M): gauge theory, e.g. Heegaard-Floer theory, Seidel-Smith's version of Khovanov homology of knots; symplectic topology ("everything is a Lagrangian" - Weinstein)

Coh(M): representation theory, algebraic geometry...

being able to translate between the two worlds is very useful.

These lectures will focus on homological mirror symmetry. Topics we will cover:

- introduction to Fuk and Coh, and their variants (wrapped Fukaya category, Fukaya-Seidel category, matrix factorisations...)
- known examples of HMS
- the relationship between closed-string and open-string invariants, deformation theory, finding generators for the Fukaya category.
- The SYZ conjecture and tropical geometry (how to cook up examples of HMS, and how to prove it).