Spatial Soliton Evolution in Nematic Liquid Crystals in the Nonlinear Local Regime

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In the present work spatial solitons in nematic liquid crystals are considered in the regime of local response of the crystal, such solitons having been called nematicons in previous experimental studies. The general equations governing light propagation in a nematic liquid crystal form a coupled system consisting of a nonlinear Schrödinger (NLS) type equation for the light and a forced heat equation for the material response. In the limit of low light intensity and local material response, it is shown that these coupled equations reduce to a single higher order NLS equation. Modulation equations are derived for the evolution of a nematicon-like pulse, these modulation equations also including the dispersive radiation shed as the pulse evolves. The modulation equations show that a nematicon is stable, although this stability is weak, being due to a $|E|^6E$ term in the higher order NLS equation, where E is the electric field of the polarised light. Solutions of the modulation equations are compared with numerical solutions of the full equations governing the liquid crystal and good agreement is found when the light intensity is small. Finally, extensions of the present work to conform with experimental parameters are discussed.

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1. Introduction

In a remarkable set of experiments, Assanto and collaborators demonstrated the possibility of producing stable spatial solitons in nematic liquid crystals.[1,2] These spatial solitons show an initial oscillation at their waist near the point of injection of the light, but eventually settle onto stable, straight, spatially coherent structures, called nematicons.[1,2] In the experiments, the spatial solitons were produced for liquid crystals with a strong non-local response. Under this non-local material response regime, the injected light pulse is trapped, since the liquid crystal, under the influence of the light, forms a waveguide which is wider than the soliton. This self-trapping of light to form a spatial soliton was also studied in[2,3] both analytically and numerically.

In the present work, the production of nematicons in liquid crystals with a local response to light will be investigated both analytically and numerically. Under this local response regime, the waveguide width is the same as the width of the light beam. As the waveguide is narrow, in this local response limit it is possible that a nematicon could leak dispersive radiation and so become destabilised. Hence it is vital that the role of this dispersive radiation on the evolution of the nematicon pulse be fully understood, which is a major objective of the present work.

The basic equations governing the evolution of a nematicon in a liquid crystal form a coupled system consisting of a nonlinear Schrödinger (NLS)-like equation, which has two space dimensions transverse to the propagation direction and one space dimension along the propagation direction, governing the light propagation and a (steady) forced heat equation governing the response of the liquid crystal. The modulation equations for the evolution of a nematicon in a liquid crystal will then be derived from an averaged Lagrangian formulation of these basic governing equations, in a manner similar to that of[4] for the standard one space dimensional nonlinear Schrödinger (NLS) equation. These modulation equations will then be extended to include the dispersive radiation shed by a nematicon-type pulse as it evolves, that is for boundary conditions which do not produce an exact nematicon solution. This radiation is determined by the solution of an appropriate linearised form of the governing equations for the liquid crystal. The modulation equations show that the saturation of the liquid crystal director motion is responsible for the formation of stable nematicons. Furthermore, in the limit of local response for the liquid crystal, the spatial oscillation of the waist of the nematicon is shown to have a very long period, in contrast to the situation for a non-local response of the liquid crystal.[1,2] Lastly it is shown that the radiative losses are small and that the nematicon eventually stabilises into a straight, coherent structure.

It is found that the agreement between solutions of the modulation equations with radiative loss and full numerical solutions of the governing equations is very good in light of the drastic approximations made to reduce the highly nonlinear, coupled system of equations governing the liquid crystal to a system of just two ordinary differential equations.

2. Formulation

The basic physical situation for nematicon propagation in a nematic liquid crystal is as in.[1,2] The major difference from the present work is that here the external electric field required to force the nematic above the phase transition threshold is introduced explicitly. The basic geometry is then one of a slab of liquid crystal with the nematicon pre-tilted using an external electric field, as illustrated in Figure 1. The nematicon is then taken to propagate in the z direction, again as illustrated in Figure 1. Finally let us denote by ϕ the angle the optical axis makes with the propagation direction z. The equations for the electric field of the input polarised light $\mathbf{E}$ and the optical director angle ϕ are then given in[5] in the form

$$i\frac{\partial E}{\partial z} + \frac{1}{2}\nabla^2 E - \cos(2\phi)\, E = 0 \qquad (1)$$

$$\nu\nabla^2\phi + q\cos(2\phi + 2\psi) + 2|E|^2\sin(2\phi) = 0. \qquad (2)$$

Here E is the magnitude of the electric field, the parameter q is the relative strength of the static to the dynamic electric field and the angle ψ measures the tilt of the static electric field to the propagation direction z.[5] The non-locality parameter is ν, which is the ratio of the elastic energy of the nematicon to the energy of the electric field. In,[1,2] the limit of ν large was considered both experimentally and numerically. In the present work, the opposite limit $\nu \to 0$ will be considered. This local limit can be attained in a variety of physical circumstances. Strongly illuminated liquid crystals have a high q, so that the stiffness ν will be small. Also increasing temperature will typically result in decreasing elastic constants for the crystal,[6] decreasing ν.

The initial condition for E is one of a localised disturbance at $z = 0$ and the boundary conditions for ϕ are anchoring conditions at the walls of the liquid crystal cell.[5] These general liquid crystal equations can be simplified to the present physical situation of[1,2] as follows. The liquid crystal material is pre-tilted so that ψ can be chosen so that $\phi = \pi/4$ is an equilibrium state for the nematic liquid crystal when there is no light present ($E = 0$). In this case, $\psi = 0$. This equilibrium state clearly does not satisfy the anchoring conditions at the cell walls, but it is, however, a good approximation in the bulk of the crystal. For the dynamic state, the angle θ is taken as the deviation of the director angle from its equilibrium value, $\phi = \pi/4 + \theta$. Then for $\psi = 0$, the liquid crystal equations (1) and (2) reduce to

$$i\frac{\partial E}{\partial z} + \frac{1}{2}\nabla^2 E + \sin(2\theta)\, E \;=\; 0 \tag{3}$$

$$\nu\nabla^2\theta - q\sin(2\theta) + 2|E|^2\cos(2\theta) \;=\; 0. \tag{4}$$

The boundary condition in the director deviation θ is $\theta \to 0$ as $r^2 = x^2 + y^2 \to \infty$, so that the imposed pre-tilt is recovered away from the nematicon.

In the present limit of local material response, $\nu \to 0$, the director equation (4) can be approximated by the algebraic equation

$$-q\sin(2\theta) + 2|E|^2\cos(2\theta) = 0, \tag{5}$$

which has the solution

$$\tan(2\theta) = \frac{2|E|^2}{q}. \tag{6}$$

Then substituting this solution for θ into the electric field equation (3) finally results in a saturable NLS equation of the form

$$i\frac{\partial E}{\partial z} + \frac{1}{2}\nabla^2 E + \frac{2|E|^2 E}{\sqrt{q^2 + 4|E|^4}} = 0. \tag{7}$$

This saturable NLS equation is the basic equation describing the nematicon in the local response limit and is the equation which will be analysed in the present work. The initial condition at $z = 0$ is taken to be an NLS soliton-like pulse of the form

$$E(0, r) = A\,\mathrm{sech}\,\frac{r}{W}. \tag{8}$$

It should be noted that the algebraic equation (5) for θ may have multiple solutions. A stable branch has then been used, which gives one possible waveguide centred about the nematicon.

The nonlinearity in the saturable NLS equation (7) is difficult to handle analytically, so that in the derivation of the modulation equations describing the nematicon in the next section, the small amplitude (A in (8) small), or large q, limit is taken, resulting in the higher order NLS equation

$$i\frac{\partial E}{\partial z} + \frac{1}{2}\nabla^2 E + \frac{2}{q}|E|^2 E - \frac{4}{q^3}|E|^6 E = 0. \tag{9}$$

Approximate solutions of this NLS equation describing the evolution of a nematicon will now be obtained and compared with full numerical solutions of the full governing system (3) and (4) for small ν. This is the subject of the next two sections.

3. Approximate Equations for Nematicon

The modulation equations for the nematicon will be obtained by substituting a suitable trial function into the averaged Lagrangian for the higher order NLS equation (9). To this end, the Lagrangian for the higher order NLS equation (9), governing the nematicon in the limit as $\nu \to 0$, is

$$L = ir\left(E^* E_z - E E_z^*\right) - r\left|E_r\right|^2 + \frac{2r}{q}\left|E\right|^4 - \frac{2r}{q^3}\left|E\right|^8, \tag{10}$$

since the nematicon is taken to be radially symmetric for simplicity. The superscript * denotes the complex conjugate. The $|E|^8$ term in this Lagrangian is responsible for the stabilisation of the self-focusing soliton and preventing its collapse, as would be the case for the two dimensional NLS equation (with only a cubic nonlinearity).[7,8]

To obtain the approximate equations governing the nematicon from this Lagrangian, a suitable trial function must be taken for the functional form of the nematicon. Such a suitable trial function is

$$E = a \operatorname{sech}\frac{r}{w}\, e^{i\sigma} + ig e^{i\sigma}. \tag{11}$$

This trial function is just a radially symmetric generalisation of the trial function used by[4] for the one space dimensional NLS equation. Again the trial function consists of two parts: the first term is just a varying NLS soliton and the second term is a flat shelf of radiation which sits under the evolving pulse.[4] The group velocity for waves for the linearised NLS equation is $c_g = k$, where k is the wavenumber, so that long wavelength waves do not propagate away from the evolving pulse and so form a flat shelf of radiation under it. The form of the radiation away from the evolving pulse will be considered in the next section. As the perturbed NLS equation (9) is two dimensional, the shelf is now a disc confined to the region $r < \ell$.[4] The shelf radius ℓ will be determined later in this section. As deduced in[4] from the inverse scattering solution for the NLS equation, the shelf of radiation is $\pi/2$ out of phase with the evolving pulse, which accounts for the ig term in the trial function (11).

Substituting the trial function (11) into the Lagrangian (10) and averaging by integrating in r from 0 to ∞ yields the averaged Lagrangian

$$\mathcal{L} = -2\left(a^2w^2I_2 + \Lambda g^2\right)\sigma' + 2w^2gI_1a' + 4awgI_1w' - 2aw^2I_1g' - a^2I + \frac{2}{q}I_4a^4w^2 - \frac{2}{q^3}I_8a^8w^2, \quad (12)$$

where

$$\Lambda = \frac{1}{2}\ell^2. \quad (13)$$

The various integrals I, I_1, I_2, I_4 and I_8 occurring in this averaged Lagrangian are

$$I = \int_0^\infty x\,\mathrm{sech}^2\,x\tanh^2 x\,dx = \frac{1}{3}\log 2 + \frac{1}{6} \quad (14)$$

$$I_1 = \int_0^\infty x\,\mathrm{sech}\,x\,dx = 2C \quad (15)$$

$$I_2 = \int_0^\infty x\,\mathrm{sech}^2\,x\,dx = \log 2 \quad (16)$$

$$I_4 = \int_0^\infty x\,\mathrm{sech}^4\,x\,dx = \frac{2}{3}\log 2 - \frac{1}{6} \quad (17)$$

$$I_8 = \int_0^\infty x\,\mathrm{sech}^8\,x\,dx = \frac{16}{35}\log 2 - \frac{19}{105}, \quad (18)$$

where C is the Catalan constant $C = 0.915965594\ldots$.[9]

Taking variations of the averaged Lagrangian (12) with respect to the parameters a, w, g and σ then results in, after some algebra, the variational equations

$$\frac{d}{dz}\left(I_1aw^2\right) = \Lambda g\frac{d\sigma}{dz} \quad (19)$$

$$I_1\frac{dg}{dz} = \frac{Ia}{2w^2} - \frac{I_4}{q}a^3 + \frac{3I_8}{q^3}a^7 \quad (20)$$

$$I_2\frac{d\sigma}{dz} = -\frac{I}{w^2} + \frac{3I_4}{q}a^2 - \frac{7I_8}{q^3}a^6 \quad (21)$$

$$\frac{d}{dz}\left(a^2w^2I_2 + \Lambda g^2\right) = 0. \quad (22)$$

These modulation equations for the nematicon are similar to those for the one space dimensional NLS equation.[4] The modulation equation (22) is the equation for conservation of mass for the evolving pulse, as can be easily verified from the mass conservation equation

$$i\frac{\partial}{\partial z}\left(r|E|^2\right)+\frac{1}{2}\frac{\partial}{\partial r}\left[rE^*E_r-rEE_r^*\right]=0. \tag{23}$$

for the higher order NLS equation (9). In this context, mass conservation does not refer to mass in any sense for light, but mass in the sense of invariances of the Lagrangian for the higher order NLS equation.[10] The conserved quantity $|E|^2$ is actually the light power in the present situation.

The fixed point of these modulation equations is given by $g=0$, with the amplitude a and width w related by

$$w^2=\frac{qI}{2a^2}\left[I_4-\frac{3I_8}{q^2}\,a^4\right]^{-1}. \tag{24}$$

These relations in the trial function (11) give the nematicon solution of the governing equation (9).

Using Nöther's Theorem (invariances in z) on the Lagrangian (10), it can be shown that the higher order NLS equation has the energy conservation equation

$$i\frac{\partial}{\partial z}\left[r|E_r|^2-\frac{2r}{q}|E|^4+\frac{2r}{q^3}|E|^8\right]+\frac{\partial}{\partial r}\left[\frac{1}{2}r\left(E_r^*E_{rr}-E_rE_{rr}^*\right)+\frac{2}{q}r|E|^2\left(EE_r^*-E^*E_r\right)\right.$$
$$\left.-\frac{4}{q^3}r|E|^6\left(EE_r^*-E^*E_r\right)\right]=0. \tag{25}$$

Then substituting the trial function (11) into this energy conservation equation and averaging by integrating in r from 0 to ∞ yields the energy conservation equation for the evolving nematicon

$$\frac{dH}{dz}=\frac{d}{dz}\left[Ia^2-\frac{2}{q}I_4a^4w^2+\frac{2}{q^3}I_8a^8w^2\right]=0. \tag{26}$$

Given the initial amplitude A and width W, this energy conservation equation, together with the amplitude-width relation (24), determines the final steady nematicon amplitude $\hat{a}$ and width $\hat{w}$ as the solution of

$$I\hat{a}^2-\frac{2}{q}I_4\hat{a}^4\hat{w}^2+\frac{2}{q^3}I_8\hat{a}^8\hat{w}^2=IA^2-\frac{2}{q}I_4A^4W^2+\frac{2}{q^3}I_8A^8W^2=H. \tag{27}$$

For small amplitude this equation has solution

$$\hat{a}^6 = -\frac{q^2 I_4 H}{2 I I_8} \tag{28}$$

to first order in small amplitude, so that from (24)

$$\hat{w}^2 = \frac{qI}{2 I_4 \hat{a}^2} \tag{29}$$

to first order in small amplitude.

The final parameter to determine is the shelf radius ℓ. This length will be determined as in[4] for the one space dimensional NLS equation. Linearisation of the modulation equations (19)–(22) about the fixed point given by (24) and $g = 0$ shows that the nematicon fixed point is a centre, whose oscillation frequency ω depends on Λ. Matching this centre frequency to the nematicon frequency $\dot{\sigma}$ at the fixed point then yields the shelf radius as

$$\Lambda = \frac{1}{2}\ell^2 = \frac{q^3 I_1^2 I}{24 I_2 I_8 \hat{a}^6} \, . \tag{30}$$

This same expression for the shelf width ℓ will be assumed to hold away from the nematicon fixed point, as in.[4]

In summary, the nematicon fixed point of the modulation equations (19)–(22) and (26) has amplitude and width given by

$$\hat{a}^6 = -\frac{q^2 I_4 H}{2 I I_8} \quad \text{and} \quad \hat{w}^2 = \frac{qI}{2 I_4 \hat{a}^2}, \tag{31}$$

where H is the constant (and so initial) value of the energy, and phase $\hat{\sigma}$ given by the modulation equation (21) as

$$\frac{d\hat{\sigma}}{dz} = \frac{I_4}{q I_2}\, \hat{a}^2 - \frac{7 I_8}{q^3 I_2}\, \hat{a}^6. \tag{32}$$

As this nematicon fixed point is a centre, the solution of the modulation equations (19)–(22) oscillates around the nematicon fixed point and so the modulation equations predict the oscillatory behaviour of the "waist" of the spatial soliton, these oscillations being absent when the soliton has the fixed point power, as described by.[1,2] However the effect of the shed dispersive radiation has not, as yet, been included.[4] It will be shown in the following section how the oscillatory behaviour of the waist of the pulse is transient and how the pulse ultimately settles into a steady nematicon.

4. Radiation Loss

The radiation shed by the pulse as it evolves has small amplitude, which can be seen from the full numerical solution shown in Figure 2. Therefore, away from the pulse, the radiation is governed by the linearised NLS equation

$$i\frac{\partial E}{\partial z} + \frac{1}{2r}\frac{\partial}{\partial r}\left(r\frac{\partial E}{\partial r}\right) = 0. \tag{33}$$

To determine the shed dispersive radiation, this linear NLS equation must be solved together with a boundary condition that matches the shed radiation to the shelf under the pulse, so that $E = S(z)$ at $r = \ell$.

As for the one space dimensional NLS equation,[4] the major contribution of the shed radiation to the evolution of the pulse is via the mass shed from the pulse into this radiation. The mass conservation equation for the linearised NLS equation (33) is

$$i\frac{\partial}{\partial z}\left(r|E|^2\right) + \frac{1}{2}\frac{\partial}{\partial r}\left(rE^*E_r - rEE_r^*\right) = 0. \tag{34}$$

Integrating this mass equation from the edge of the shelf at $r = \ell$ to $r = \infty$ then gives the mass flux into dispersive radiation from the evolving pulse as

$$\frac{d}{dz}\int_{\ell}^{\infty} r|E|^2\, dr = \mathrm{Im}\left(rE^*E_r\right)|_{r=\ell} + O(\dot{\ell}(z)). \tag{35}$$

In order to determine this mass flux to radiation, the boundary condition $S(z)$ needs to be fixed in terms of the pulse parameters and, furthermore, a relation between E_r and E at the edge of the shelf $r = \ell$ needs to be obtained.

To determine the function $S(z)$, we note from numerical solutions, for example that shown in Figure 2, that the major contribution to the radiation loss in the vicinity of the pulse comes from the flat shelf.[4] Now in the vicinity of the disc $r \leq \ell$, the trial function (11) can be decomposed into the fixed point solution plus a component which must be eventually radiated away

$$E = E_{fixed} + E_1, \tag{36}$$

where E_1 is the radiated component. It will be assumed, based on numerical solutions, that $|E_1|$ is small. The equation for conservation of mass (23) then gives that the mass density decomposes as

$$\int_0^{\ell(z)} |E|^2 r\, dr = \int_0^{\ell(z)} \left[|E_{fixed}|^2 + 2\,\mathrm{Re}(E_{fixed}E_1) + |E_1|^2\right] r\, dr. \tag{37}$$

Now, on assuming small overlap between E_{fixed} and E_1, we have

$$I_2 a^2 w^2 + \Lambda g^2 \approx I_2 \hat{a}^2 \hat{w}^2 + \Lambda |E_1|^2|_{r=\ell}, \tag{38}$$

on noting the total mass given by the mass variational equation (22) and expanding about the fixed point amplitude $\hat{a}$ and width $\hat{w}$. Mass conservation therefore gives $S(z)$ as

$$|S(z)|^2 = \frac{1}{\Lambda}\left[I_2 a^2 w^2 - I_2 \hat{a}^2 \hat{w}^2 + \Lambda g^2\right]. \tag{39}$$

If a similar calculation for $S(z)$ were made for the one space dimensional NLS equation, then the result would be

$$|S(z)|^2 = \frac{1}{\ell}\left[2a^2 w - 2\hat{a}^2 \hat{w} + \ell g^2\right]. \tag{40}$$

This expression differs from the result of[4] by the factor multiplying the square brackets, which instead of $1/\ell$ is $3\hat{a}/8$ in.[4] Now for the 1D NLS equation, $\ell = 3\pi^2/(8\hat{a})$, independent of the expression for $S(z)$,[4] so that the ratio of the factor in (40) to that of[4] is $64/(9\pi^2) \approx 0.721$, which is close enough to 1 under the approximations made in the present work. This same ratio was noted in[4] from their derivation partly based on the inverse scattering solution of the NLS equation. In the subsequent approximations made in that work, it was also assumed that this ratio can be approximated by 1. So within the approximations made, the expression (40) is the same as that of,[4] and so by implication, the expression (39) has the same validity.

Expression (39) can also be derived in a manner similar to that of[4] for the 1D NLS equation. This method expands a near the fixed point as $a = \hat{a} + a_1$. Then the energy conservation equation (26) results in the expansion for the width

$$a^2 w^2 = \frac{q I_2}{2\left[1 - I_8 \hat{a}^4/(q^2 I_4)\right] I_4 \hat{a}^2}\left\{\hat{a}^2 I - H + \frac{6 I I_8 \hat{a}^4}{q^2 I_4 \left[1 - 3 I_8 \hat{a}^4/(q^2 I_4)\right]}\, a_1^2\right\} \tag{41}$$

on expanding about the nematicon fixed point. Now the total mass is

$$I_2 a^2 w^2 + \Lambda g^2 \tag{42}$$

from the mass conservation equation (22). Expanding this mass about the fixed point, using the width expansion (41) and subtracting the fixed point mass results in the same expression (39) for S on taking out a factor of Λ. A final point is that the first method for determining S is much simpler than the second method based on this expansion of energy about the fixed point.

The final quantity to be determined to calculate the mass flux (35) is $\mathrm{Im}(rE^* E_r)$. To calculate this quantity, we need to solve the linear Schrödinger equation (33) together with the boundary condition $E = S(z)$ on $r = \ell(z)$. Since the shelf width ℓ is assumed to be slowly varying, we can take it to be constant in the calculation of the shed radiation. Solving the linear Schrödinger equation (33) using Laplace transforms results in

$$E_r\,|_{r=\ell} = -\frac{1}{2\pi i}\int_C \sqrt{2s}\, e^{-i\pi/4}\frac{K_1\left(\sqrt{2s}\, e^{-i\pi/4}\ell\right)}{K_0\left(\sqrt{2s}\, e^{-i\pi/4}\ell\right)}\,\bar{E}_0 e^{sz}\, ds, \tag{43}$$

where E_0 is the boundary condition for E at $r = \ell$, s is the Laplace transform variable, K_0 and K_1 are modified Bessel functions of order 0 and 1 respectively and the contour C lies to the right of all singularities of the integrand.

Using this Laplace transform solution, we have that the required flux product is

$$E^* E_r = S^*(z)\int_0^z G(z-z')E_0(z')\, dz', \tag{44}$$

since $E = S$ at $r = \ell$, where the Green's function G is

$$G(\eta) = -\frac{1}{2\pi i}\int_C \sqrt{2s}\, e^{-i\pi/4}\frac{K_1\left(\sqrt{2s}\, e^{-i\pi/4}\ell\right)}{K_0\left(\sqrt{2s}\, e^{-i\pi/4}\ell\right)}\, e^{s\eta}\, ds. \tag{45}$$

Now expressing S in polar form as

$$S = R(z)e^{i\varphi(z)} \tag{46}$$

with $R(z) = |S(z)|$ and assuming that the phase φ is slowly varying, as in,[4] we obtain

$$E^* E_r = R(z)\int_0^z G(z-z')R(z')\, dz'. \tag{47}$$

Hence since $R = S$ is determined via (39), the radiation flux is now determined on calculating the Green's function G. For actual calculations, however, we shall now derive a useful approximation to this Green's function.

In order to calculate the Green's function G, the integrand in expression (45) for G can be re-arranged in the form

$$-\frac{2s}{\ell}\frac{d}{ds}\log\left(K_0(\sqrt{2s}\, e^{-i\pi/4}\ell)\right). \tag{48}$$

Since the long time behaviour of the radiation flux is required, this expression will now be expanded for $s \to 0$, using the asymptotic expansion $K_0(z) \sim -\log(z/2)$ as $z \to 0$.[9] Then using this asymptotic approximation as $s \to 0$, the integrand in definition (45) for G becomes

$$-\frac{1}{\ell}\,\frac{2}{\log s + \log\Lambda - i\pi/2}. \tag{49}$$

Hence the Green's function (45) becomes

$$G(\eta) = -\frac{1}{2\pi i\ell}\int_C \frac{2e^{s\eta}}{\log s + \log\Lambda - i\pi/2}\, ds. \qquad (50)$$

This integral for G can be evaluated by closing the contour C around the branch point of the log and changing variables to $-\operatorname{Re} s = \exp(\xi)$ to obtain

$$G(\eta) = \frac{1}{4\ell}\int_{-\infty}^{\infty} \frac{e^{-e^{\xi}\eta+\xi}}{\left(\xi/2 + \frac{1}{2}\log\Lambda\right)^2 - i(\pi/2)\left(\xi/2+\frac{1}{2}\log\Lambda\right) + 3\pi^2/16}\, d\xi. \qquad (51)$$

The integral (51) is now approximated for large z using the standard method of stationary phase to finally obtain

$$E_r = -\frac{\sqrt{2\pi}}{4e\ell}\int_0^z \frac{R(z')}{z-z'}\frac{1}{\left[(1/2)\log((z-z')/\Lambda) - i\pi/4\right]^2 + \pi^2/4}\, dz'. \qquad (52)$$

This integral is clearly convergent as $z \to z'$ due to the square of the log in the denominator of the integrand. On identifying R with S (39), the calculation of the mass flux from the pulse is now complete on using (35).

It now remains to add the mass loss (35) due to shed dispersive radiation to the variational equations of the previous section. The mass flux given by (35) can be added as a loss term to the mass conservation equation (22), resulting in a complete set of equations describing the evolution of the nematicon, including loss to dispersive radiation. However, as in[4] for the 1D NLS equation, it is more convenient to replace the mass equation by the energy conservation equation (26), since the energy of the pulse is conserved to first order, while the mass of the pulse is not, due to the calculated dispersive loss. As in,[4] on eliminating the mass equation, it can be found, on using the energy conservation equation (26), that the mass loss (35) is then added to equation (20) for g, resulting in the modified equation for g

$$I_1\frac{dg}{dz} = \frac{Ia}{2w^2} - \frac{I_4}{q}a^3 + \frac{3I_8}{q^3}a^7 - 2\alpha g, \qquad (53)$$

where the loss coefficient α is

$$\alpha = -\frac{\sqrt{2\pi}I_1}{32eR\Lambda}\int_0^z \frac{\pi R(z')\log((z-z')/\Lambda)}{\left\{\left[\frac{1}{4}\log((z-z')/\Lambda)\right]^2 + 3\pi^2/16\right\}^2 + \pi^2\left[\log((z-z')/\Lambda)\right]^2/16}\frac{dz'}{(z-z')} \qquad (54)$$

and

$$R^2 = \frac{1}{\Lambda}\left[I_2a^2w^2 - I_2\hat{a}^2\hat{w}^2 + \Lambda g^2\right]. \qquad (55)$$

It should be noted that in obtaining the loss coefficient (54) from the mass flux (35), $r \operatorname{Im} E^* E_r$ has been calculated from expression (52) for E_r. A full set of modulation equations describing the evolution of the nematicon is then the variational equations (19), (21) of the previous section, the energy conservation equation (26) and the modified g equation (53).

Before comparing solutions of the modulation equations with full numerical solutions of the liquid crystal equations (3) and (4), it should be noted that if the higher order term $|E|^6 E$ were absent from the NLS equation (9), then the modulation equations (19)–(22) would become linearly degenerate at the fixed point. This degeneracy explains in simple terms the appearance of a threshold, this being a steady soliton, between collapse to zero for initial conditions below the threshold and blow-up for initial conditions above the threshold. The solutions of the next section will be found to have long oscillation periods. This degeneracy is again responsible for the long period, since the period is due to the $|E|^6 E$ term and the amplitude is small.

5. Comparison with Numerical Solutions

In this section, full numerical solutions of the liquid crystal equations (3) and (4) will be compared with solutions of the modulation equations (19), (21), (26) and (53). The NLS-type equation (3) was solved using the pseudo-spectral method of[11] and the Poisson equation (4) was solved using an FFT-based boundary value numerical scheme.[12] In this manner, the liquid crystal equations were solved using Fourier methods. The modulation equations (19), (21), (26) and (53) were solved using the standard 4th order Runge-Kutta method.

Figure 3 shows a comparison between the pulse amplitude as given by the numerical and modulation solutions for the initial conditions $A = 0.3$, $W = 4$ with $q = 2$. It can be seen that there is good agreement in the amplitude of the pulse, with the amplitude decay predicted by the loss term (54) being slightly weaker than the decay given by the full numerical solution. The period of the modulation amplitude oscillation is longer than the numerical period. As noted at the end of the previous section, the period of these amplitude oscillations is large due to their being determined by the $|E|^6 E$ term in the higher order NLS equation (9) as the nematicon is only marginally stable. It should also be noted that the numerical solution is the solution of the full liquid crystal equations, while the modulation equations are for the approximate higher order NLS equation (9) derived from these general equations. Hence exact agreement between the numerical and modulation solutions cannot be expected.

A further amplitude comparison is shown in Figure 4 for the initial conditions $A = 0.45$ and $W = 2$ with $q = 1$. The agreement shown in this figure is not as good as that shown in Figure 3 with the decay of the modulation amplitude being much slower than the numerical decay. As the period of the amplitude oscillations is linked to the amplitude, the oscillation period as predicted by the modulation equations is much longer than the numerical period. The reason for the worse agreement shown in this figure is the effect of the approximations made in deriving the higher order NLS equation (9), these being that the amplitude a of the nematicon is small, or equivalently, that q is large. It can be seen that the amplitude in Figure 4 is approaching 0.7, so that the small amplitude assumption is not always valid. Furthermore, q has been reduced from 2 to 1 from Figure 3. While the amplitude in Figure 3 approaches 0.8, the higher value of q in this figure results in better agreement.

This tendency for the agreement to get worse as the amplitude increases and q decreases is further illustrated in Figure 5, for which the initial conditions are $A = 0.4$ and $W = 2.5$ with $q = 1$. It can be seen that there is a marked difference between the numerical and modulation amplitudes and periods, as expected since q is small and the amplitude is now approaching 0.85, so that good agreement is not expected.

A general conclusion from these amplitude comparisons is that the modulation equations for the reduced higher order NLS equation (9) are in good agreement with solutions of the full liquid crystal equations (3) and (4) for low light intensity and large, or largish, values of q. The radiation damping, represented by the loss coefficient (54), under-estimates the amplitude damping. However it can be seen from Figures 3–5 that the steady nematicon amplitude, given by the mean of the amplitude oscillations, is well predicted by the modulation equations. Therefore the oscillations in the waist of the nematicon, seen in the experimental results,[1,2] stabilise over relatively long distances, in agreement with the experimental results. A further point is that the full numerical solutions indicate that there is a non-unique nematicon solution when the initial amplitude A is around 1 or higher. This possible non-uniqueness is the subject of present investigation.

6. Conclusions

It has been shown both asymptotically and numerically that spatial solitons, so-called nematicons, can exist in a nematic liquid crystal which has a local material response, which may be due to temperature or strong illumination of the liquid crystal. In this situation, the distortion of the liquid crystal is localised around the nematicon. However in the experimental studies of,[1,2] liquid crystals with highly nonlocal response were used. This experimental work demonstrated the possibility of the existence of spatial solitons due to the nonlocality of the nematic. This non-locality then balances the cubic nonlinearity of the NLS equation (9) without the $|E|^6E$ term, resulting in spatial solitons. The present work suggests the possibility of the production of spatial solitons in the limit of local response of the nematic, which are stabilised by the saturating nonlinearity. The dynamics of solitons in the limit of nonlocal material response, as in the experimental work of,[1,2] is under investigation at present.

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List of Figure Captions

Figure 1. Schematic of liquid crystal geometry. The volume of liquid crystal V has a nematicon of radius r propagating in the z direction. The electric field $\mathbf{E}$ is polarised in the transverse, upwards direction.

Figure 2. Full numerical solution of liquid crystal equations (3) and (4) at $t = 400$ for $q = 2$ and $\nu = 0.01$ for the initial conditions $A = 0.3$ and $W = 4.0$.

((a)) $|E|$.

((b)) $\mathrm{Im}\, E$.

Figure 3. Amplitude a of the nematicon as a function of z for the initial conditions $A = 0.3$, $W = 4$ with $q = 2$. Numerical solution of governing equations (3) and (4) with $\nu = 0.01$: — ; solution of modulation equations (19), (21), (26) and (53): − − − .

Figure 4. Amplitude a of the nematicon as a function of z for the initial conditions $A = 0.45$, $W = 2$ with $q = 1$. Numerical solution of governing equations (3) and (4) with $\nu = 0.01$: — ; solution of modulation equations (19), (21), (26) and (53): − − − .

Figure 5. Amplitude a of the nematicon as a function of z for the initial conditions $A = 0.4$, $W = 2.5$ with $q = 1$. Numerical solution of governing equations (3) and (4) with $\nu = 0.01$: — ; solution of modulation equations (19), (21), (26) and (53): − − − .

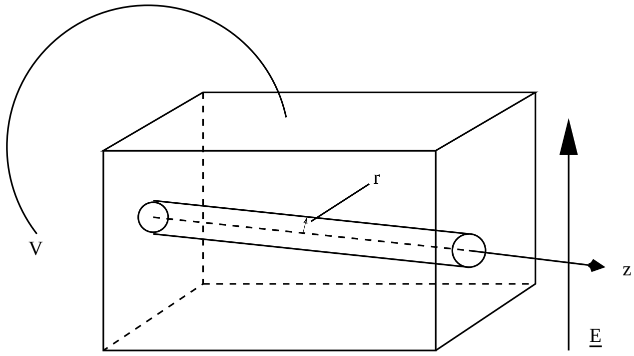
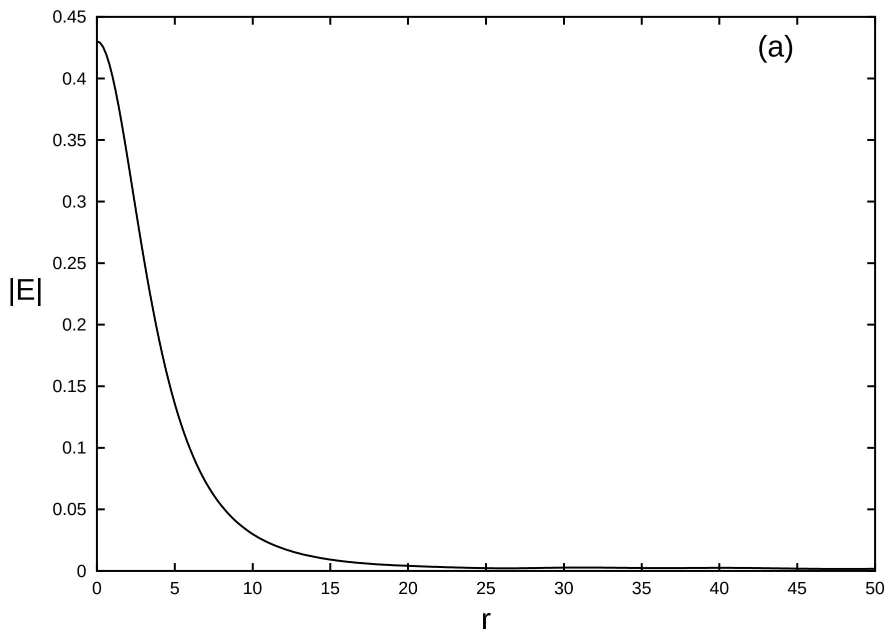


Figure 1.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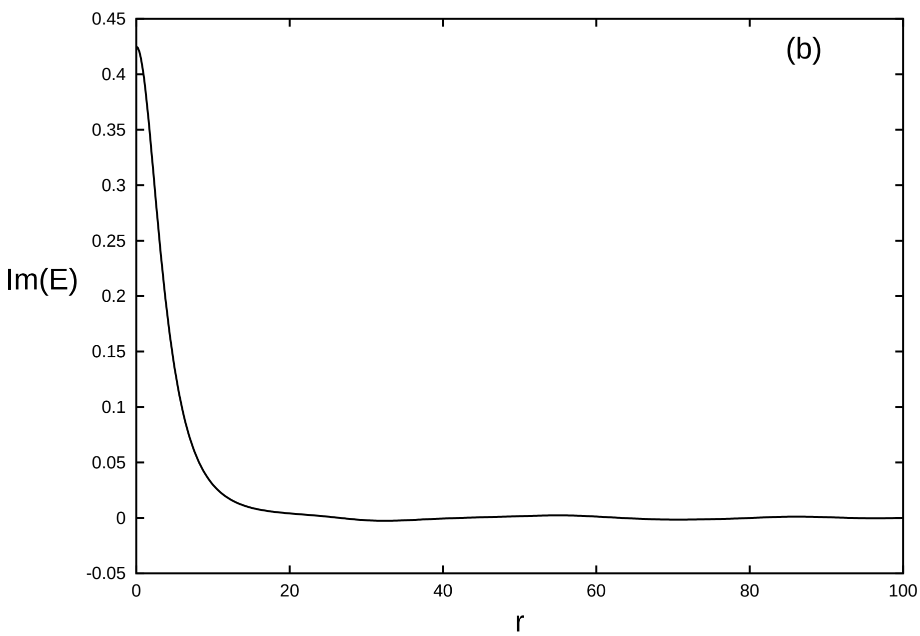
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Figure 2.

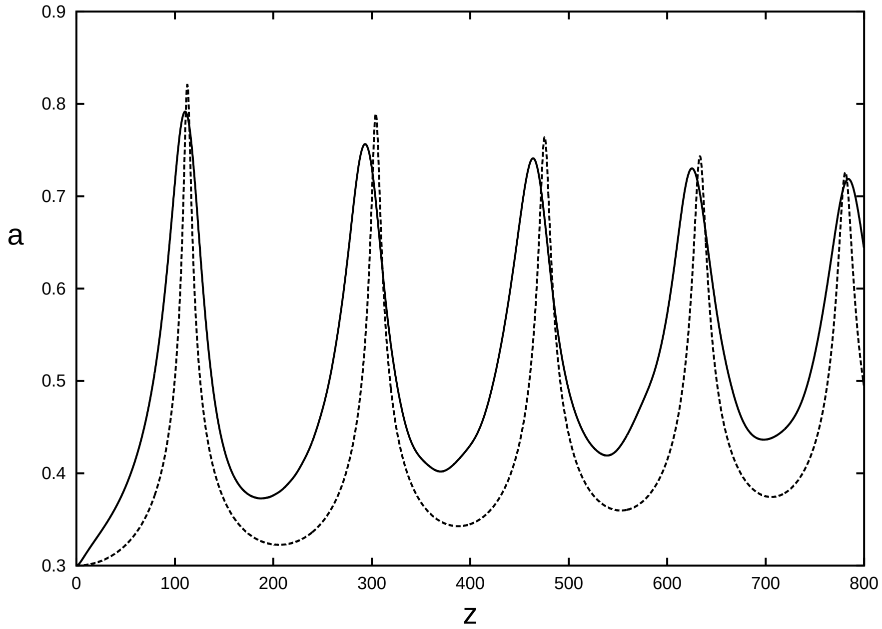

Figure 3.

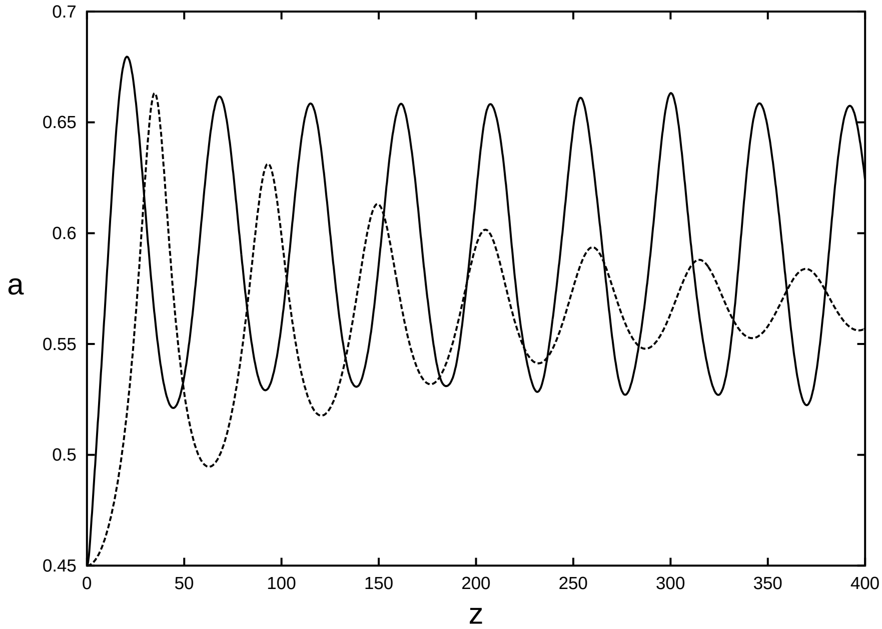

Figure 4.

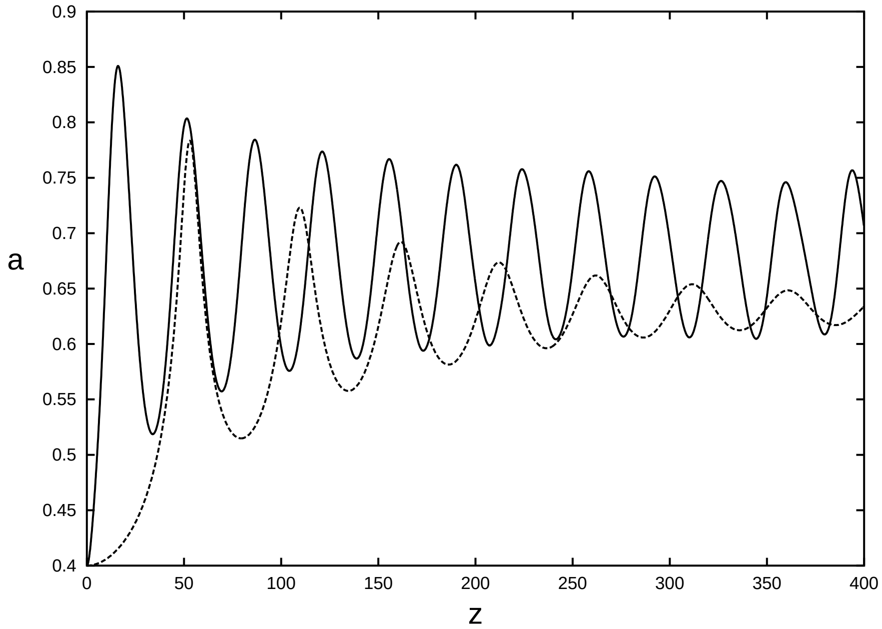
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Figure 5.