Multigraded regularity and the Koszul property

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Abstract

We give a criterion for the section ring of an ample line bundle to be Koszul in terms of multigraded regularity. We discuss applications to adjoint bundles on toric varieties as well as to polytopal semigroup rings.

Key words: multigraded regularity, Koszul property, polytopal semigroup rings

1. Introduction

Let A be an ample and globally generated line bundle on a projective variety X over a field k, and let $R(A) = \bigoplus_{\ell \geq 0} H^0\left(X, A^\ell\right)$ be the section ring associated to A. Recall that a graded k algebra $R = k \oplus R_1 \oplus R_2 \oplus \cdots$ is called *Koszul* (or *wonderful*) if k admits a linear free resolution over R. It is well known that a Koszul algebra is generated in degree 1, and that the ideal of relations between its generators is generated by quadrics.

The purpose of this note is to give criteria for the section ring of an ample line bundle to be Koszul in terms of the regularity of the line bundle. The following theorem illustrates the flavour of our main result, Theorem 3.

Theorem 1. *Let A be an ample and globally generated line bundle on a projective variety X over an infinite field k. Assume that $H^i(X, A^{m-i}) = 0$ for $i > 0$. Then the section ring $R(A^m) = \bigoplus_{\ell \geq 0} H^0(X, A^{\ell m})$ is Koszul.*

It is well known that section rings of high enough powers of ample line bundles are Koszul (see [1]). Moreover, Eisenbud, Reeves and Totaro [9] give criteria for Veronese subalgebras of graded k-algebras to be Koszul in terms of the algebraic Castelnuovo-Mumford regularity. We illustrate the relationship between our theorem and these criteria in the end of Section 2.

Sufficient criteria for powers of ample line bundles to have Koszul section rings are known for curves [35, 4, 30, 25, 6], homogeneous spaces [17, 2, 31], elliptic ruled surfaces [13], abelian varieties [18, 32], and toric varieties [3]. The Koszul property has also been studied for points in projective spaces, see [7,

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19, 29], and toric varieties admitting additional combinatorial structure, see [33, 28, 16, 23, 26, 27].

We will prove a more general version of Theorem 1 in terms of multigraded regularity (compare [21] and [15]), see Theorem 3. It generalizes a result for line bundles on surfaces by Gallego and Purnaprajna [12, Theorem 5.4] that they use to give criteria for line bundles on elliptic ruled surfaces to have a Koszul section ring. The proof is based on a vanishing theorem due to Lazarsfeld and uses methods very similar to those in [12] and [15]. Generalizing the results in [9], Conca, Herzog and Trung [5] give criteria for diagonal subalgebras of standard bigraded algebras to be Koszul in terms of the bigraded Betti numbers, see also Remark 12.

In vein of Fujita's conjectures [11], it is a natural question to ask under what conditions adjoint line bundles of the form $A \otimes K_X$ for an ample line bundle A on a Gorenstein projective variety X have a Koszul section ring. This question was studied by Pareschi [24] for very ample line bundles on smooth varieties. As an application of our main theorem, we give a criterion for adjoint line bundles on Gorenstein toric varieties to have Koszul section ring.

Moreover, we show how the criteria for polytopal semigroup rings to be Koszul due to Bruns, Gubeladze and Trung [3, Theorem 1.3.3.] can be improved if multiples of the polytope do not contain interior lattice points, see Section 4. Theorem 3 is part of my thesis [14].

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2. Multigraded regularity and proof of theorem

Let X be a projective variety over a field k. We will assume for the remainder of the paper that k is infinite. Let $\mathrm{B}_1, \ldots, \mathrm{B}_r$ be globally generated line bundles on X. For $\mathbf{u} \in \mathbb{Z}^r$, we let $\mathrm{B}^{\mathbf{u}} := \mathrm{B}_1^{u_1} \otimes \cdots \otimes \mathrm{B}_r^{u_r}$ and $|\mathbf{u}| = u_1 + \cdots + u_r$. Let $\mathcal{B} = \{\mathrm{B}^{\mathbf{u}} \mid \mathbf{u} \in \mathbb{N}^r\} \subset \operatorname{Pic}(X)$ be the submonoid of $\operatorname{Pic}(X)$ generated by $\mathrm{B}_1, \ldots, \mathrm{B}_r$.

Definition 2. Let L be a line bundle on X. A sheaf $\mathcal{F}$ is called L-regular with respect to $\mathrm{B}_1, \ldots, \mathrm{B}_r$ if

$$H^i\left(X, \mathcal{F} \otimes L \otimes \mathrm{B}^{-\mathbf{u}}\right) = 0$$

for all $i > 0$ and for all $\mathbf{u} \in \mathbb{N}^r$ with $|\mathbf{u}| = i$.

Obeserve that for $r = 1$, this is the usual definition for Castelnuovo-Mumford regularity, compare [20, 1.8.4.].

Now we are ready to state the main theorem.

Theorem 3. *Let* $\mathrm{B}_1, \ldots, \mathrm{B}_r$ *be a set of globally generated line bundles on* X *generating a semigroup* $\mathcal{B}$*. Let* $A \in \mathcal{B}$ *be an ample line bundle and assume that* $A \otimes \mathrm{B}_i^{-1} \in \mathcal{B}$ *for all* $i = 1, \ldots, r$*. If* A *is* $\mathcal{O}_X$*-regular with respect to* $\mathrm{B}_1, \ldots, \mathrm{B}_r$*, then the section ring* $R(A) = \bigoplus_{\ell \geq 0} H^0(X, A^\ell)$ *is Koszul.*

Theorem 1 is the special case when $r = 1$. The proof is based on a vanishing theorem of Lazarsfeld (Lemma 4 below) and a multigraded version of Mumford's theorem, see [15, Theorem 2.1.]. Note that the proof of Mumford's theorem only requires k to be infinite.

To a globally generated vector bundle E is associated a vector bundle M_E, the kernel of the evaluation map

$$0 \to M_E \to H^0\left(X, E\right) \otimes \mathcal{O}_X \to E \to 0. \tag{1}$$

For $h \in \mathbb{N}$ and A globally generated, we define vector bundles $M^{(h)}$ inductively, by letting $M^{(0)} = A$ and $M^{(h)} = M_{M^{(h-1)}} \otimes A$, provided $M^{(h-1)}$ is globally generated.

Lemma 4 (Lazarsfeld, see [24, Lemma 1])**.** *Let* X *be a projective variety, let* A *be an ample line bundle on* X*, and let* $R(A) = \bigoplus_{\ell \geq 0} H^0\left(X, A^\ell\right)$ *be the section ring associated to* A*. Assume that the vector bundles* $M^{(h)}$ *are globally generated for all* $h \geq 0$*. If* $H^1\left(X, M^{(h)} \otimes A^\ell\right) = 0$ *for all* $\ell \geq 0$ *and all* $h \geq 0$ *then* $R(A)$ *is Koszul. Moreover, if* $H^1\left(X, A^\ell\right) = 0$ *for all* $\ell \geq 1$*, the converse also holds.*

Observe that the proof of this lemma is valid for projective varieties over any field.

Proof of Theorem 3. We say that a sheaf $\mathcal{F}$ is L-regular if it is L-regular with respect to $B_1, \ldots, B_r$. We will use induction on h to show that $M^{(h)}$ is $\mathcal{O}_X$-regular. It then follows from Mumfords theorem [15, Theorem 2.1., (3)] that $M^{(h)}$ is globally generated, and $M^{(h+1)}$ is defined.

Note that $M^{(0)} = A$ is $\mathcal{O}_X$-regular by assumption. For the induction step we apply [15, Lemma 2.2.] to the short exact sequence (1) with $E = M^{(h)}$ twisted by A. Since by assumption $A \otimes B^{-e_j} \in \mathcal{B}$, it follows that $A \otimes B^{-e_j} \cong B^{u'}$ for $u' \in \mathbb{N}^r$. By the induction hypothesis and Mumfords theorem [15, Theorem 2.1.,(1)], $M^{(h-1)}$ is $A \otimes B^{-e_j}$-regular for all j, and so $M^{(h-1)} \otimes A$ is B^{-e_j}-regular for all j. Similarly, by Mumfords theorem [15, Theorem 2.1., (2)], the natural map $H^0\left(X, M^{(h-1)}\right) \otimes H^0\left(X, A \otimes B^{-e_j}\right) \to H^0\left(X, M^{(h-1)} \otimes A \otimes B^{-e_j}\right)$ is surjective for all $1 \leq j \leq r$.

Mumfords theorem [15, Theorem 2.1., (1)] implies $M^{(h)}$ is also B^u-regular for all $u \in \mathbb{N}^r$. Hence $H^1\left(X, M^{(h)} \otimes A^\ell\right) = 0$ for all $h \geq 0$ and $\ell \geq 0$, and Lemma 4 implies that $R\left(A\right)$ is Koszul. □

Example 5. The fact that the section ring of high enough powers of ample line bundles is Koszul follows easily from this result: By Serre vanishing, L^d is $\mathcal{O}_X$-regular with respect to L for d large enough, hence the associated section ring is Koszul.

Remark 6. Let $R \cong k[x_0, \dots, x_N]/I$, where $I \subset k[x_0, \dots, x_N]$ is a homogeneous ideal. If I admits a quadratic Gröbner basis with respect to some monomial ordering, then R is Koszul. However, a Koszul algebra need not admit a presentation whose ideal admits a quadratic Gröbner basis, see [9].

Remark 7. It is well known that if the section ring of a line bundle L is Koszul, then L satisfies Green's property N_1 (see [20, 1.8.C] for an introduction to property N_p). On the other hand, Sturmfels [34, Theorem 3.1] exhibited an example of a smooth projectively normal curve whose coordinate ring is presented by quadrics but is not Koszul. However, in many cases, criteria for line bundles to satisfy N_p imply that their section ring is Koszul and even that its ideal admits a quadratic Gröbner basis when $p \geq 1$. For example, the conditions for Theorem 3 agree with those of [15] for a line bundle to satisfy N_1.

Eisenbud, Reeves and Totaro [9] give criteria for Veronese subrings of finitely generated graded k-algebras to be Koszul in terms of algebraic regularity. Translating their result into the language of ample line bundles, we obtain a better bound than Theorem 1 for normally generated line bundles.

Definition 8. An ample line bundle is called *normally generated*, if the natural map

$$\underbrace{H^0(X,L) \otimes \cdots \otimes H^0(X,L)}_{m} \to H^0(X,L^m)$$

is surjective for all m.

Corollary 9. *Let A be a normally generated line bundle. Suppose A^m is $\mathcal{O}_X$-regular with respect to A. Then if $d \geq \frac{m}{2}$, the ideal of the section ring associated to A^d admits a quadratic Gröbner basis; in particular, the section ring of A^d is Koszul.*

To see how this follows from the criteria in [9], we first review the notion of algebraic regularity.

Definition 10. Let $S = k[x_0, \dots, x_N]$ be a polynomial ring over k. A finitely generated graded S-module M is m-regular if $\operatorname{Tor}_i^S(M,k)_j = 0$ for $j > i + m$ and $i \geq 0$.

Let $I \subset S$ be a homogeneous ideal, and let $R = S/I$. We denote with $R^{(d)} = \bigoplus_{m \in \mathbb{N}} R_{md}$ the d'th Veronese subalgebra of R. Keeping in mind that S/I is m-regular if and only if I is $(m+1)$-regular, the following theorem is an easy consequence of the results proved in Eisenbud, Reeves and Totaro [9].

Theorem 11 ([9])**.** *If R is $(m-1)$-regular and $d \geq \frac{m}{2}$, then the ideal of $R^{(d)}$ admits a quadratic Gröbner basis.*

Proof of Corollary 9. Since A is normally generated, it is very ample. Moreover, the section ring R associated to A is generated in degree 1, and it agrees with the homogeneous coordinate ring of the embedding $\iota \colon X \hookrightarrow \mathbb{P} := \mathbb{P}\left(H^0\left(X, A\right)\right)$ induced by A. In particular, R is of the form S/I for $S = \operatorname{Sym}^\bullet H^0(X,A)$

and I a homogeneous ideal in S. Now A^m is $\mathcal{O}_X$-regular with respect to itself if and only if $\iota_* A$ is $\mathcal{O}_{\mathbb{P}}(m-1)$-regular with respect to $\mathcal{O}_{\mathbb{P}}(1)$ if and only if $R(\mathbb{P}, \iota_* A) = R(X, A)$ is $(m-1)$-regular as a S-module, (see for example [8, Exercise 20.20.] or [20, 1.8.26.]). Since the section ring associated to A^d agrees with $R(X, A)^{(d)}$, the corollary follows from Theorem 11. □

Remark 12. Conca, Herzog and Trung generalize the results of [9] to the bigraded case, and their methods generalize to the multigraded case. In [5, Theorem 6.2] they give criteria for diagonal subalgebras of a standard bigraded algebra R to be Koszul in terms of the bigraded Betti numbers of the resolution of R. Given two ample line bundles L and M, we associate a bigraded algebra $R = \oplus_{(a,b)\in\mathbb{Z}^2_{\geq 0}} H^0(X, L^a \otimes M^b)$ over $\operatorname{Sym}^\bullet H^0(X, L) \otimes \operatorname{Sym}^\bullet H^0(X, M)$. Note that R is standard if and only if the natural map $\operatorname{Sym}^a H^0(X, L) \otimes \operatorname{Sym}^b H^0(X, M) \to H^0(X, L^a \otimes M^b)$ is surjective for all $(a, b) \in \mathbb{Z}^2_{\geq 0}$. The section ring of $L^a \otimes M^b$ agrees with the diagonal subalgebra $\oplus_{s\in\mathbb{Z}_{\geq 0}} R_{(as,bs)}$. However, the relationship between the multigraded regularity of the sheaves L, M and the Betti numbers of the bigraded resolution of R is subtle and and it is not obvious how to apply [5, Theorem 6.2] in this case.

3. Applications to adjoint line bundles on toric varieties

In [24], Pareschi proved that if A is a very ample line bundle on a smooth projective variety X of dimension n, then for $d \geq n+1$, the section ring of $A^d \otimes K_X$ is Koszul, unless $X \cong \mathbb{P}^n$ and $A \cong \mathcal{O}_{\mathbb{P}^n}(1)$.

On a smooth toric variety every ample line bundle is very ample, so Pareschi's theorem applies to ample line bundles on smooth toric varieties. Using Theorem 3, we obtain a similar criterion for Gorenstein toric varieties. Note that the semigroup of integral nef divisors on a toric variety is finitely generated, and that every nef divisor on a toric variety is globally generated.

Theorem 13. *Let X be a Gorenstein toric variety of dimension n, and let $\{B_1, \ldots, B_r\}$ be a set of nef divisors generating the nef cone of X. Let A be an ample line bundle such that $A \otimes B_i^{-1}$ is nef for all $1 \leq i \leq r$. Then $R(X, A^{n+1} \otimes K_X)$ is Koszul unless $X \cong \mathbb{P}^n$ and $A = \mathcal{O}_{\mathbb{P}^n}(1)$.*

Proof. (Compare proof of Corollary 1.6 in [15].) It follows from [22, Theorem 3.4] that $A^{n+1} \otimes K_X$ is $\mathcal{O}_X$-regular with respect to $\{B_1, \ldots, B_r\}$. Moreover, by [10, Corollary 0.2], $A^n \otimes K_X$ is nef, hence $A^{n+1} \otimes K_X \otimes B_i^{-1}$ is nef for all i. The claim now follows from Theorem 3. □

4. Applications to polytopal semigroup rings

The question which powers of ample line bundles on toric varieties have Koszul section ring was studied by Bruns, Gubeladze and Trung in [3]. They show under a mild assumption that for an ample line bundle A on a toric variety of dimension n that the section ring $R(A^k)$ is Koszul for $k \geq n$. Observe that the Koszul property also follows easily from Theorem 1. In fact, since the higher cohomology of an ample line bundle on a toric variety vanishes, A^n is $\mathcal{O}_X$-regular.

A more careful study of the regularity of a line bundle on a toric variety shows that if r is the number of integer roots of the Hilbert polynomial of A, then A^{n-r} is $\mathcal{O}_X$-regular (see [15, Lemma 4.1]), and we obtain the following Corollary.

Corollary 14. *Let A be an ample line bundle on a toric variety X, and let r be the number of integer roots of the Hilbert polynomial of A. Then $R(A^k)$ is Koszul for $k \geq n - r$.*

In terms of lattice polytopes and polytopal semigroup rings this can be rephrased as follows. Let $M \cong \mathbb{Z}^n$ be a lattice, and $P \subset M \otimes \mathbb{R} := M_{\mathbb{R}}$ be a lattice polytope. P determines a semigroup $S_P \subset M \times \mathbb{Z}$, the semigroup generated by $\{(p, 1) \in M \times \mathbb{Z} \mid p \in P \cap M\}$. Let $k[S_P]$ be the semigroup algebra associated to S_P.

Corollary 15. *Let P be a lattice polytope of dimension n, and let r be the largest positive integer such that rP does not contain any interior lattice points. Then the polytopal semigroup ring $k[S_{kP}]$ is Koszul for $k \geq n - r$.*

This follows from the fact that the largest positive integer r such that rP does not contain any interior lattice points is equal to the number of integer roots of the Hilbert polynomial of the ample line bundle associated to P (see for example [15, Section 4]) and that $(n - r)P$ is normal, see [15, Corollary 1.3].

Remark 16. In fact, in [3] the authors prove that for a lattice polytope P of dimension n such that the group generated by S_P is the full lattice, the semigroup algebra $k[S_{kP}]$ admits a presentation such that the corresponding ideal admits a quadratic Gröbner basis for $k \geq n$. It follows similarly from [3, Theorem 1.4.1.] that in the situation of Corollary 15 the semigroup algebra $k[S_{kP}]$ admits a presentation such that the corresponding ideal admits a quadratic Gröbner basis for $k \geq n - r$.

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