

The structure of the pro- l -unipotent fundamental group of a smooth variety*

J.P.Pridham

May 5, 2005

Contents

Introduction	1
1 Functors on nilpotent Lie algebras	2
1.1 Differential graded algebras	7
2 SDCs over \mathcal{N}_k	10
2.1 Computing SDCs	12
2.2 The pro-unipotent fundamental group	13
2.3 Equivalence between SDCs and DGAs in characteristic 0	18
3 Structure of the fundamental group	20
3.1 The topological fundamental group	20
3.2 The algebraic fundamental group	21
Bibliography	26

Introduction

In [Pri04a], I used the theory of SDCs (developed in [Pri04b]) to prove results about the structure of the algebraic fundamental group of a smooth variety, similar to those proved in [GM88] for the structure of the fundamental group of a Kähler manifold. [GM88] was motivated by [DGMS75], which shows that the homotopy type of a compact Kähler manifold is a formal consequence of its cohomology, the idea being to replace statements about DGAs with those about DGLAs.

One consequence of [DGMS75] is that the de Rham fundamental group of a compact Kähler manifold is quadratically presented. In [Mor78], a weaker result is proved for smooth complex varieties. The primary aim of this paper is to prove an analogous result for smooth proper varieties in finite characteristic. The results in [DGMS75] and [Mor78] were inspired by the yoga of weights, motivated by the Weil Conjectures. With

*The author is supported by Trinity College, Cambridge and the Isle of Man Department of Education.

this in mind, it is reasonable to expect the characteristic zero results to have finite characteristic analogues.

In [Del75] is expressed the philosophy that, for algebraic varieties, there should be a weight decomposition not only on cohomology, but also on the minimal model (i.e. the homotopy theory), and that only non-zero weights are interesting. In Weil II ([Del80]), Deligne uses this idea to define the \mathbb{Q}_l -homotopy type of a variety and prove that it is formal for smooth proper algebraic varieties defined over finite fields.

The results of [DGMS75] and [Mor78] give explicit information about the structure of the fundamental group. Both papers make use of the Sullivan minimal model, developed in [Sul77] and shown to determine the de Rham fundamental group $\pi_1(X) \otimes \mathbb{R}$ (and, indeed, the entire Whitehead graded Lie algebra). However, since the proof involves the topological methods of [Sul77], no such information has been obtained from the formality result in Weil II. By developing a theory of deformations over nilpotent Lie algebras in Section 1, I obtain similar explicit information on the structure of the algebraic fundamental group. The Weil Conjectures imply that, for X smooth and proper, the pro-nilpotent fundamental group $\pi_1(X, \bar{x}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ is quadratically presented. If X is merely smooth, then the pro-nilpotent fundamental group is defined by equations of bracket length at most four. The Hard Lefschetz Theorem implies that, for X smooth and projective, the pro-nilpotent fundamental group cannot be free.

1 Functors on nilpotent Lie algebras

This section extends the ideas of [Sch68] to a slightly different context.

Fix a field k , and define \mathcal{N}_k to be the category of finite-dimensional nilpotent Lie algebras over k , and $\widehat{\mathcal{N}}_k$ to be the category of finitely generated (i.e. $\dim L/[L, L] < \infty$) pro-nilpotent Lie algebras over k .

Given $\mathcal{L} \in \widehat{\mathcal{N}}_k$, let $\mathcal{N}_{\mathcal{L},k}$ be the category of pairs $(N \in \mathcal{N}_k, \mathcal{L} \xrightarrow{\phi} N)$, and $\widehat{\mathcal{N}}_{\mathcal{L},k}$ the category of pairs $(N \in \widehat{\mathcal{N}}_k, \mathcal{L} \xrightarrow{\phi} N)$. We will almost always consider the case $\mathcal{L} = 0$ (note that $\mathcal{N}_{0,k} = \mathcal{N}_k$), although a few technical lemmas (Theorems 1.18 and 1.19 and Corollary 1.20) require the full generality of $\mathcal{N}_{\mathcal{L},k}$. In $\widehat{\mathcal{N}}_{\mathcal{L},k}$, \mathcal{L} is the initial object, and 0 the final object.

Definition 1.1. Given a finite dimensional vector space V/k , denote the free pro-nilpotent Lie algebra on generators V by $L(V)$. Explicitly, let $A(V)$ be the free complete associative algebra on generators V ,

$$A(V) := \varprojlim_n \left(\bigoplus_m V^{\otimes m} \right) / (V)^n,$$

(the universal enveloping algebra of $L(V)$), with coproduct $\Delta : A(V) \rightarrow A(V) \hat{\otimes} A(V)$ defined on generators by $\Delta(v) = v \otimes 1 + 1 \otimes v$. Then

$$L(V) := \{l \in A(V) : \Delta(l) = l \otimes 1 + 1 \otimes l\}$$

Definition 1.2. Given Lie algebras $N, M \in \widehat{\mathcal{N}}_k$, let $N * M$ be the completed free Lie algebra product, i.e. the completion with respect to the commutator filtration of the free product of the Lie algebras N and M . Note that $*$ is sum in $\widehat{\mathcal{N}}_k$ — the analogue in $\widehat{\mathcal{C}}_\Lambda$ is $\hat{\otimes}$.

Definition 1.3. Define the free Lie algebra $L_{\mathcal{L}}(V) := \mathcal{L} * L(V)$

We require that all functors on $\mathcal{N}_{\mathcal{L},k}$ satisfy

(H0) $F(0) = \bullet$, the one-point set.

We adapt the following definitions and results from [Sch68] (with identical proofs):

Definition 1.4. For $p : N \rightarrow M$ in $\mathcal{N}_{\mathcal{L},k}$ surjective, p is a small extension if $\ker p = (t)$, a principal ideal, such that $[N, (t)] = (0)$. To say that (t) is a principal ideal means that the map

$$\begin{array}{ccc} \mathcal{L}(k) & \rightarrow & (t) \\ 1 & \mapsto & t \end{array}$$

is surjective. Note that any surjection can be factorised as a composition of small extensions.

For $F : \mathcal{N}_{\mathcal{L},k} \rightarrow \text{Set}$, define $\hat{F} : \widehat{\mathcal{N}_{\mathcal{L},k}} \rightarrow \text{Set}$ by

$$\hat{F}(L) = \varprojlim_n F(L/\Gamma_n(L)),$$

where the $\Gamma_n(L)$ are defined inductively by

$$\Gamma_1(L) = L, \quad \Gamma_{n+1}(L) = [L, \Gamma_n(L)].$$

Note that $\hat{F}(L) \xrightarrow{\sim} \text{Hom}(h_L, F)$, where

$$\begin{array}{ccc} h_L : \mathcal{N}_{\mathcal{L},k} & \rightarrow & \text{Set}; \\ N & \mapsto & \text{Hom}(L, N). \end{array}$$

Definition 1.5. We will say a functor $F : \mathcal{N}_{\mathcal{L},k} \rightarrow \text{Set}$ is pro-representable if it is isomorphic to h_L , for some $L \in \widehat{\mathcal{N}_{\mathcal{L},k}}$. By the above remark, this isomorphism is determined by an element $\xi \in \hat{F}(L)$. We say the pro-couple (L, ξ) pro-represents F .

Definition 1.6. A natural transformation $\phi : F \rightarrow G$ in $[\mathcal{N}_{\mathcal{L},k}, \text{Set}]$ is called:

1. unramified if $\phi : t_F \rightarrow t_G$ is injective, where $t_F = F(L(\epsilon))$, where

$$L(\epsilon) := (L(\epsilon), \mathcal{L} \xrightarrow{0} L(\epsilon)).$$

2. smooth if for every $M \rightarrow N$ in $\mathcal{N}_{\mathcal{L},k}$, we have $F(M) \rightarrow G(M) \times_{G(N)} F(N)$
3. étale if it is smooth and unramified.

Definition 1.7. $F : \mathcal{N}_{\mathcal{L},k} \rightarrow \text{Set}$ is smooth if and only if $F \rightarrow \bullet$ is smooth.

Theorem 1.8. 1. Let $A \rightarrow B$ be a morphism in $\widehat{\mathcal{N}_{\mathcal{L},k}}$. Then $h_B \rightarrow h_A$ is smooth if and only if $B = A * L(X_1, \dots, X_n)$.

2. If $F \rightarrow G$ and $G \rightarrow H$ are smooth morphisms of functors, then the composition $F \rightarrow H$ is smooth.
3. If $u : F \rightarrow G$ and $v : G \rightarrow H$ are morphisms of functors such that u is surjective and vu is smooth, then v is smooth.

Definition 1.9. A pro-couple (L, ξ) is a hull for F if the induced map $h_L \rightarrow F$ is étale.

Theorem 1.10. Let $(L, \xi), (L', \xi')$ be hull of F . Then there exists an isomorphism $u : L \rightarrow L'$ such that $F(u)(\xi) = \xi'$.

Lemma 1.11. Suppose F is a functor such that

$$F(V \oplus W) \xrightarrow{\sim} F(V) \times F(W)$$

for vector spaces V and W over k , where $V \in \mathcal{N}_{\mathcal{L},k}$ is defined by $[V, V] = 0$, $\mathcal{L} \xrightarrow{0} V$. Then $F(V)$, and in particular t_F , has a canonical vector space structure, and $F(V) \cong t_F \otimes V$.

Theorem 1.12. Given $F : \mathcal{N}_{\mathcal{L},k} \rightarrow \text{Set}$, let $N' \rightarrow N$ and $N'' \rightarrow N$ be morphisms in $\mathcal{N}_{\mathcal{L},k}$, and consider the map:

$$(\dagger) F(N' \times_N N'') \rightarrow F(N') \times_{F(N)} F(N'').$$

Then

1. F has a hull if and only if F has properties (H1), (H2) and (H3) below:

- (H1) (\dagger) is a surjection whenever $N'' \rightarrow N$ is a small extension.
- (H2) (\dagger) is a bijection when $N = 0$, $N'' = L(\epsilon)$.
- (H3) $\dim_k(t_F) < \infty$.

2. F is pro-representable if and only if F has the additional property (H4):

$$(H4) F(N' \times_N N'') \xrightarrow{\sim} F(N') \times_{F(N)} F(N'').$$

for any small extension $N' \rightarrow N$.

Proof. [Sch68], Theorem 2.11. □

Definition 1.13. $F : \mathcal{N}_{\mathcal{L},k} \rightarrow \text{Set}$ is homogeneous if

$$\eta : F(N' \times_N N'') \rightarrow F(N') \times_{F(N)} F(N'')$$

is an isomorphism for every $N' \rightarrow N$.

Note that a homogeneous functor satisfies conditions (H1), (H2) and (H4).

Definition 1.14. $F : \mathcal{N}_{\mathcal{L},k} \rightarrow \text{Set}$ is a deformation functor if:

1. η is surjective whenever $N' \rightarrow N$.
2. η is an isomorphism whenever $N = 0$.

Note that a deformation functor satisfies conditions (H1) and (H2).

The following results are adapted from[Man99]:

Definition 1.15. Given $F : \mathcal{N}_{\mathcal{L},k} \rightarrow \text{Set}$, an obstruction theory (V, v_e) for F consists of: a k -vector space V , the obstruction space, together with obstruction maps $v_e : F(N) \rightarrow V \otimes_k I$ for each small extension

$$e : 0 \rightarrow I \rightarrow L \rightarrow N \rightarrow 0,$$

such that:

1. If $\xi \in F(N)$ can be lifted to $F(L)$ then $v_e(\xi) = 0$.
2. For every morphism $\alpha : e \rightarrow e'$ of small extensions, we have $v_{e'}(\alpha(\xi)) = (\text{id}_V \otimes \alpha)(v_e(\xi))$, for all $\xi \in F(N)$.

An obstruction theory (V, v_e) is called complete if $\xi \in F(N)$ can be lifted to $F(L)$ whenever $v_e(\xi) = 0$.

Theorem 1.16. (*Standard Smoothness Criterion*) Given $\phi : F \rightarrow G$, with $(V, v_e) \xrightarrow{\phi'} (W, w_e)$ a compatible morphism of obstruction theories, if (V, v_e) is complete, $V \xrightarrow{\phi'} W$ injective, and $t_F \rightarrow t_G$ surjective, then ϕ is smooth.

Proof. [Man99], Proposition 2.17. □

For functors $F : \mathcal{N}_{\mathcal{L},k} \rightarrow \text{Set}$ and $G : \mathcal{N}_{\mathcal{L},k} \rightarrow \text{Grp}$, we say that G acts on F if we have a functorial group action $G(N) \times F(N) \xrightarrow{*} F(N)$, for each N in \mathcal{N}_k . The quotient functor F/G is defined by $(F/G)(N) = F(N)/G(N)$.

Theorem 1.17. If $F : \mathcal{N}_{\mathcal{L},k} \rightarrow \text{Set}$, a deformation functor, and $G : \mathcal{N}_{\mathcal{L},k} \rightarrow \text{Grp}$ a smooth deformation functor, with G acting on F , then $D := F/G$ is a deformation functor, and if $\nu : t_G \rightarrow t_F$ denotes $h \mapsto h * 0$, then $t_D = \text{coker } \nu$, and the universal obstruction theories of D and F are isomorphic.

Proof. [Man99], Lemma 2.20. □

Theorem 1.18. For $F : \mathcal{N}_{\mathcal{L},k} \rightarrow \text{Set}$ homogeneous, and $G : \mathcal{N}_{\mathcal{L},k} \rightarrow \text{Grp}$ a deformation functor, given $a, b \in F(L)$, define $\text{Iso}(a, b) : \mathcal{N}_{L,k} \rightarrow \text{Set}$ by

$$\text{Iso}(a, b)(L \xrightarrow{f} N) = \{g \in G(N) \mid g * f(a) = f(b)\}.$$

Then $\text{Iso}(a, b)$ is a deformation functor, with tangent space $\ker \nu$ and, if G is moreover smooth, complete obstruction space $\text{coker } \nu = t_D$.

Proof. [Man99], Proposition 2.21. □

Theorem 1.19. If G, G' smooth deformation functors, acting on F, F' respectively, with F, F' homogeneous, $\ker \nu \rightarrow \ker \nu'$ surjective, and $\text{coker } \nu \rightarrow \text{coker } \nu'$ injective, then $F/G \rightarrow F'/G'$ is injective.

Proof. [Man99], Corollary 2.22. □

This final result does not appear in [Man99], but proves extremely useful:

Corollary 1.20. *If $F : \mathcal{N}_{\mathcal{L},k} \rightarrow \text{Set}$ and $G : \mathcal{N}_{\mathcal{L},k} \rightarrow \text{Grp}$ are deformation functors, with G acting on F , let $D := F/G$, then:*

1. *If G is smooth, then η_D is surjective for every $M \twoheadrightarrow N$ (i.e. D is a deformation functor).*
2. *If F is homogeneous and $\ker \nu = 0$, then η_D is injective for every $M \twoheadrightarrow N$.*

Thus, in particular, F/G will be homogeneous if F is homogeneous, G is a smooth deformation functor and $\ker \nu = 0$.

Proof. As for [Pri04b] Theorem 0.15. □

To summarise the results concerning the pro-representability of the quotient $D = F/G$, we have:

1. If F is a deformation functor and G a smooth deformation functor, with $\text{coker } \nu$ finite dimensional, then D has a hull.
2. If F is homogeneous and G a smooth deformation functor, with $\ker \nu = 0$ and $\text{coker } \nu$ finite dimensional, then D is pro-representable.

Example 1.21. There is one main class of examples of deformation functors on \mathcal{N}_k . Let k be of characteristic zero. Then it is well-known (see e.g. [ABC⁺96]) that there is an equivalence of categories

$$\mathcal{N}_k \xrightarrow{\text{exp}} \text{Uni}_k,$$

where Uni_k is the category of unipotent k -groups. Fix a finitely presented group G , and consider the functor

$$\begin{aligned} h_G : \text{Uni}_k &\rightarrow \text{Set} \\ U &\mapsto \text{Hom}(G, U). \end{aligned}$$

This is equivalent to the functor

$$\begin{aligned} F : \mathcal{N}_k &\rightarrow \text{Set} \\ N &\mapsto \text{Hom}(G, \text{exp}(N)). \end{aligned}$$

In fact, F is pro-representable by a pro-nilpotent Lie algebra L , and by definition

$$G \otimes_{\mathbb{Z}} k := \text{exp}(L).$$

1.1 Differential graded algebras

Definition 1.22. A DGA over a field k of characteristic 0 is a graded vector space $A = \bigoplus_{i \in \mathbb{Z}} A^i$ over k , equipped with operators $\cdot : A \times A \rightarrow A$ bilinear and $d : A \rightarrow A$ linear, satisfying:

1. $A^i \cdot L^j \subset A^{i+j}$.
2. $a \cdot b = (-1)^{\bar{a}\bar{b}} b \cdot a$.
3. $(a \cdot b) \cdot c = \cdot (b \cdot c)$.
4. $d(A^i) \subset A^{i+1}$.
5. $d \circ d = 0$.
6. $d(a \cdot b) = da \cdot b + (-1)^{\bar{a}} a \cdot db$

Here \bar{a} denotes the degree of a , mod 2, for a homogeneous. Note that our DGAs need not have a unit.

Fix a DGA A .

Definition 1.23. The Maurer-Cartan functor $\text{MC}_A : \mathcal{N}_k \rightarrow \text{Set}$ is defined by

$$\text{MC}_A(N) = \{x \in A^1 \otimes N \mid dx + \frac{1}{2}[x, x] = 0\}.$$

Observe that for $\omega \in A^1 \otimes N$,

$$d\omega + \frac{1}{2}[\omega, \omega] = 0 \Rightarrow (d + \text{ad}_\omega) \circ (d + \text{ad}_\omega) = 0,$$

so $(A \otimes N, [,], d + \text{ad}_\omega)$ is a DGLA.

Definition 1.24. Define the gauge functor $\text{G}_A : \mathcal{N}_k \rightarrow \text{Grp}$ by

$$\text{G}_A(N) = \exp(A^0 \otimes N).$$

We may now define the DGLA $(A \otimes N)_d$ as in [Man99]:

$$(A \otimes N)_d^i = \begin{cases} (A \otimes N)^1 \oplus kd & i = 1 \\ (A \otimes N)^i & i \neq 1, \end{cases}$$

with

$$d_d(d) = 0, \quad [d, d] = 0, \quad [d, a]_d = da, \quad \forall a \in (A \otimes N).$$

Lemma 1.25. $\exp(A^0 \otimes N)$ commutes with $[,]$ when acting on $(A \otimes N)_d$ via the adjoint action.

Corollary 1.26. Since $\exp(A^0 \otimes N)$ preserves $(A^1 \otimes N) + d \subset (A \otimes N)_d$ under the adjoint action, and

$$x \in \text{MC}_A(N) \Leftrightarrow [x + d, x + d] = 0,$$

the adjoint action of $\exp(A^0 \otimes N)$ on $A^1 \otimes N + d$ induces an action of $\text{G}_A(N)$ on $\text{MC}_A(N)$, which we will call the gauge action.

Definition 1.27. $\text{Def}_A = \text{MC}_A/\text{G}_A$, the quotient being given by the gauge action $\alpha(x) = \text{ad}_\alpha(x + d) - d$. Observe that G_A is homogeneous.

For $a \in \text{MC}_A(N)$, define $K_a : \mathcal{N}_{N,k} \rightarrow \text{Grp}$ by

$$K_a(M) = \exp((d + \text{ad}_a)A^{-1} \otimes M).$$

Note that this makes sense, since $(d + \text{ad}_a)^2 = 0$, so

$$(d + \text{ad}_a)A^{-1} \otimes M \leq A^0 \otimes M$$

is a Lie subalgebra. Note that K_a is then a subfunctor of $\text{Iso}(a, a) \leq \text{G}_A(A)$, so acts on $\text{Iso}(a, b)$ by right multiplication.

Define the deformation groupoid \mathfrak{Def}_A to have objects MC_A , and morphisms given by $\text{Iso}(a, b)/K_a$.

Now, $t_{\text{G}_A} = A^0\epsilon$, and $t_{\text{MC}_A} = \mathbb{Z}^1(A\epsilon)$, with action

$$\begin{aligned} t_{\text{G}_A} \times t_{\text{MC}_A} &\rightarrow t_{\text{MC}_A}; \\ (b, x) &\mapsto x + db, \text{ so} \end{aligned}$$

$$t_{\text{Def}_A} = \text{H}^1(A).$$

Lemma 1.28. $\text{H}^2(A)$ is a complete obstruction space for MC_A .

Proof. Given a small extension

$$e : 0 \rightarrow K \rightarrow N \rightarrow M \rightarrow 0,$$

and $x \in \text{MC}_A(M)$, lift x to $\tilde{x} \in A^1 \otimes N$, and let

$$h = d\tilde{x} + \frac{1}{2}[\tilde{x}, \tilde{x}] \in A^2 \otimes N.$$

In fact, $h \in A^2 \otimes K$, as $dx + \frac{1}{2}[x, x] = 0$.

Now,

$$dh = d^2\tilde{x} + [d\tilde{x}, \tilde{x}] = [h - \frac{1}{2}[\tilde{x}, \tilde{x}], \tilde{x}] = [h, \tilde{x}] = 0,$$

since $[[\tilde{x}, \tilde{x}], \tilde{x}] = 0$ and $[K, N] = 0$. Let

$$v_e(x) = [h] \in \text{H}^2(A \otimes K) = \text{H}^2(A) \otimes K.$$

This is well-defined: if $y = \tilde{x} + z$, for $z \in A^1 \otimes K$, then

$$dy + \frac{1}{2}[y, y] = d\tilde{x} + dz + \frac{1}{2}[\tilde{x}, \tilde{x}] + \frac{1}{2}[z, z] + [\tilde{x}, z] = h + dz,$$

as $[K, N] = 0$.

This construction is clearly functorial, so it follows that $(\text{H}^2(A), v_e)$ is a complete obstruction theory for MC_A . \square

Now Theorem 1.17 implies the following:

Theorem 1.29. Def_A is a deformation functor, $t_{\text{Def}_A} \cong \mathbb{H}^1(A)$, and $\mathbb{H}^2(A)$ is a complete obstruction theory for Def_A .

The other theorems can be used to prove:

Theorem 1.30. If $\phi : A \rightarrow B$ is a morphism of DGAs, and

$$\mathbb{H}^i(\phi) : \mathbb{H}^i(A) \rightarrow \mathbb{H}^i(B)$$

are the induced maps on cohomology, then:

1. If $\mathbb{H}^1(\phi)$ is bijective, and $\mathbb{H}^2(\phi)$ injective, then $\text{Def}_A \rightarrow \text{Def}_B$ is étale.
2. If also $\mathbb{H}^0(\phi)$ is surjective, then $\text{Def}_A \rightarrow \text{Def}_B$ is an isomorphism.
3. Provided condition 1 holds, $\mathfrak{D}\text{ef}_A \rightarrow \mathfrak{D}\text{ef}_B$ is an equivalence of functors of groupoids if and only if $\mathbb{H}^0(\phi)$ is an isomorphism.

Proof. [Man99], Theorem 3.1, mutatis mutandis. □

Theorem 1.31. If $\mathbb{H}^0(A) = 0$, then Def_A is homogeneous.

Proof. This is essentially Theorem 1.20, with some allowance made for K_a . □

Thus, in particular, a quasi-isomorphism of DGAs gives an isomorphism of deformation functors and of deformation groupoids.

Example 1.32. Fix a differentiable manifold X . The deformation functor

$$\begin{aligned} F : \mathcal{N}_{\mathbb{R}} &\rightarrow \text{Set} \\ \mathfrak{g} &\mapsto \text{Hom}(\pi_1(X, x), \exp(\mathfrak{g})) / \exp(\mathfrak{g}), \end{aligned}$$

where $\exp(\mathfrak{g})$ acts by conjugation, is governed by the DGA

$$A^\bullet := \Gamma(X, \mathscr{A}_{\mathbb{R}}^\bullet),$$

the correspondence being as follows. Given $\omega \in \text{MC}_A$, let

$$\mathbb{B}_\omega := D^{-1}(\omega),$$

where

$$\begin{aligned} D : \exp(\mathfrak{g} \otimes \mathscr{A}^0) &\rightarrow \mathfrak{g} \otimes \mathscr{A}^1 \\ \alpha &\mapsto d\alpha \cdot \alpha^{-1}. \end{aligned}$$

Then \mathbb{B}_ω is a principal $\exp(\mathfrak{g})$ -sheaf on X . There is a monodromy action of $\pi_1(X, x)$ on the stalk \mathbb{B}_x , and this gives us a representation $\pi_1(X, x) \rightarrow G$.

That this gives rise to an isomorphism of functors follows from the flabbiness of \mathscr{A}^\bullet (compare tangent and obstruction spaces). Of course, Def_A is not pro-representable, but its hull is $\pi_1(X, x) \otimes \mathbb{R}$.

2 SDCs over \mathcal{N}_k

The formal definitions, properties and constructions of SDCs over \mathcal{N}_k hold in exactly the same manner as those defined in [Pri04b] over \mathcal{C}_Λ . A summary follows.

Definition 2.1. A simplicial deformation complex S^\bullet consists of smooth homogeneous functors $S^n : \mathcal{N}_k \rightarrow \text{Set}$ for each $n \geq 0$, together with maps

$$\begin{aligned} \partial^i : S^n &\rightarrow S^{n+1} & 1 \leq i \leq n \\ \sigma^i : S^n &\rightarrow S^{n-1} & 0 \leq i < n \end{aligned}$$

an associative product $* : S^m \times S^n \rightarrow S^{m+n}$, with identity $1 : \bullet \rightarrow S^0$, where \bullet is the constant functor $\bullet(\mathfrak{g}) = \bullet$ on \mathcal{N}_k , such that:

1. $\partial^j \partial^i = \partial^i \partial^{j-1} \quad i < j$.
2. $\sigma^j \sigma^i = \sigma^i \sigma^{j+1} \quad i \leq j$.
3. $\sigma^j \partial^i = \begin{cases} \partial^i \sigma^{j-1} & i < j \\ \text{id} & i = j, i = j + 1. \\ \partial^{i-1} \sigma^j & i > j + 1 \end{cases}$.
4. $\partial^i(s) * t = \partial^i(s * t)$.
5. $s * \partial^i(t) = \partial^{i+m}(s * t)$, for $s \in S^m$.
6. $\sigma^i(s) * t = \sigma^i(s * t)$.
7. $s * \sigma^i(t) = \sigma^{i+m}(s * t)$, for $s \in S^m$.

Remark 2.2. If we set ω_0 to be the unique element of $S^1(0)$, then, since 0 is the initial object in \mathcal{N}_k , we may set $\partial^0(s) = \omega_0 * s$, and $\partial^{n+1}(s) = s * \omega_0$. S^\bullet then becomes a cosimplicial complex.

Definition 2.3. Let t_S^\bullet be the tangent space of S^\bullet , i.e. $t_S^n = S^n(L(\epsilon))$.

Definition 2.4. Define the Maurer-Cartan functor MC_S by

$$\text{MC}_S(\mathfrak{g}) = \{\omega \in S^1(\mathfrak{g}) : \omega * \omega = \partial^1(\omega)\}.$$

Lemma 2.5. S^0 is a group under multiplication.

Now, if $\omega \in \text{MC}_S(\mathfrak{g})$ and $g \in S^0(\mathfrak{g})$, then $g * \omega * g^{-1} \in \text{MC}_S(\mathfrak{g})$. We may therefore make the following definition:

Definition 2.6.

$$\text{Def}_S = \text{MC}_S / S^0,$$

the quotient being with respect to the adjoint action. The deformation groupoid

$$\mathfrak{Def}_S$$

has objects MC_S , and morphisms given by S^0 .

Lemma 2.7. *The action $S^0 \times S^n \rightarrow S^n$ is faithful (i.e. $s * h = t$ for some t only if $s = 1$).*

This implies that, for all $\omega \in \text{MC}_S(\mathfrak{g})$, $\sigma^0(\omega) = 1$.

Definition 2.8. Define the cohomology groups of S to be

$$H^i(S) := H^i(t_S^\bullet),$$

the cohomology groups of the cosimplicial complex t_S^\bullet .

Lemma 2.9. *The tangent space of MC_S is $Z^1(t_S^\bullet)$, with the action of S^0 giving $\nu(s) = \partial^1(s) - \partial^0(s)$.*

Lemma 2.10. $H^2(S)$ is a complete obstruction space for MC_S .

Theorems 1.16 to 1.20 now imply:

Theorem 2.11. Def_S is a deformation functor, with tangent space $H^1(S)$ and complete obstruction space $H^2(S)$. Moreover, if $H^0(S) = 0$, then Def_S is homogeneous.

Proof. Theorem 1.17 and Corollary 1.20. □

Theorem 2.12. *If $\phi : S \rightarrow T$ is a morphism of SDCs, and*

$$H^i(\phi) : H^i(S) \rightarrow H^i(T)$$

are the induced maps on cohomology, then:

1. *If $H^1(\phi)$ is bijective, and $H^2(\phi)$ injective, then $\text{Def}_S \rightarrow \text{Def}_T$ is étale.*
2. *If also $H^0(\phi)$ is surjective, then $\text{Def}_S \rightarrow \text{Def}_T$ is an isomorphism.*
3. *Provided condition 1 holds, $\mathfrak{D}\text{ef}_S \rightarrow \mathfrak{D}\text{ef}_T$ is an equivalence of functors of groupoids if and only if $H^0(\phi)$ is an isomorphism.*

Call a morphism $\phi : S \rightarrow T$ a quasi-isomorphism if the $H^i(\phi) : H^i(S) \rightarrow H^i(T)$ are all isomorphisms.

Definition 2.13. Given a morphism $\phi : S \rightarrow T$ of SDCs, define the groupoid

$$\mathfrak{D}\text{ef}_\phi$$

to be the fibre of the morphism

$$\mathfrak{D}\text{ef}_S \rightarrow \mathfrak{D}\text{ef}_T$$

over the unique point $x_0 \in \text{MC}_T(0)$.

Explicitly, $\mathfrak{D}\text{ef}_\phi(\mathfrak{g})$ has objects

$$\{(\omega, h) \in \text{MC}_S(\mathfrak{g}) \times T^0(\mathfrak{g}) \quad : \quad h\phi(\omega)h^{-1} = x_0\},$$

and morphisms

$$S^0(\mathfrak{g}), \quad \text{where} \quad g(\omega, h) = (g\omega g^{-1}, h\phi(g)^{-1}).$$

Theorem 2.14. *Let $S_\phi^n(A)$ be the fibre of $S^n(A) \xrightarrow{\phi} T^n(A)$ over x_0^n . Then S_x^\bullet is an SDC, and the canonical map*

$$\mathfrak{D}\text{ef}_{S_\phi^n} \rightarrow \mathfrak{D}\text{ef}_\phi$$

is an equivalence of groupoids.

2.1 Computing SDCs

Throughout this section, we will consider functors $\mathcal{D} : \mathcal{N}_k \rightarrow \text{Cat}$. We will *not* require that these functors satisfy (H0) (the condition that $F(0) = \bullet$).

Definition 2.15. Given a functor $\mathcal{D} : \mathcal{N}_k \rightarrow \text{Cat}$, and an object $D \in \text{Ob } \mathcal{D}(0)$, define $\mathfrak{Def}_{\mathcal{D}, D} : \mathcal{N}_k \rightarrow \text{Grpd}$ by setting $\mathfrak{Def}_{\mathcal{D}, D}(\mathfrak{g})$ to be the fibre of $\mathcal{D}(\mathfrak{g}) \rightarrow \mathcal{D}(0)$ over (D, id) .

Definition 2.16. We say a functor $\mathcal{E} : \mathcal{N}_k \rightarrow \text{Cat}$ has uniformly trivial deformation theory if the functor $\text{Mor } \mathcal{E}$ is smooth and homogeneous, and the functor $\text{Cmpts } \mathcal{E}$ is constant, i.e for $\mathfrak{g} \rightarrow \mathfrak{h}$ in \mathcal{N}_k , $\mathcal{E}(\mathfrak{g}) \rightarrow \mathcal{E}(\mathfrak{h})$ is full and essentially surjective.

Assume that we are given functors $\mathcal{A}, \mathcal{B}, \mathcal{D}, \mathcal{E} : \mathcal{N}_k \rightarrow \text{Cat}$, and a diagram

$$\begin{array}{ccc} \mathcal{D} & \begin{array}{c} \xrightarrow{U} \\ \dashv \! \dashv \\ \xleftarrow{F} \end{array} & \mathcal{A} \\ \begin{array}{c} \uparrow \! \uparrow \\ \dashv \! \dashv \\ \downarrow \end{array} G & & \begin{array}{c} \downarrow \! \downarrow \\ \dashv \! \dashv \\ \uparrow \end{array} G \\ \mathcal{B} & \begin{array}{c} \xrightarrow{U} \\ \dashv \! \dashv \\ \xleftarrow{F} \end{array} & \mathcal{E} \end{array}$$

where \mathcal{E} has uniformly trivial deformation theory, the horizontal adjunctions are monadic and the vertical adjunctions comonadic. Let

$$\begin{aligned} \top_{\mathfrak{h}} &= UF & \perp_{\mathfrak{h}} &= FU \\ \perp_{\mathfrak{v}} &= VG & \top_{\mathfrak{v}} &= GV, \end{aligned}$$

with

$$\eta : 1 \rightarrow \top_{\mathfrak{h}}, \quad \gamma : \perp_{\mathfrak{v}} \rightarrow 1, \quad \varepsilon : \perp_{\mathfrak{h}} \rightarrow 1 \text{ and } \alpha : 1 \rightarrow \top_{\mathfrak{v}}.$$

Assume moreover that the following identities hold:

$$\begin{aligned} GU &= UG \\ FV &= VF \\ UV &= VU \end{aligned}$$

$$\begin{aligned} V\varepsilon &= \varepsilon V & U\alpha &= \alpha U \\ V\eta &= \eta V & U\gamma &= \gamma U. \end{aligned}$$

Theorem 2.17. For $D \in \text{Ob } \mathcal{D}(0)$, let $E = UVD \in \text{Ob } \mathcal{E}(0)$, and write $E(\mathfrak{g})$ for the image of E under the map $\text{Ob } \mathcal{E}(0) \rightarrow \text{Ob } \mathcal{E}(\mathfrak{g})$. Set $S^n(\mathfrak{g})$ to be the fibre

$$S^n(\mathfrak{g}) = \text{Hom}_{\mathcal{E}(\mathfrak{g})}(\top_{\mathfrak{h}}^n E(\mathfrak{g}), \perp_{\mathfrak{v}}^n E(\mathfrak{g}))_{UV(\alpha_D^n \circ \varepsilon_D^n)}$$

of

$$\text{Hom}_{\mathcal{E}(\mathfrak{g})}(\top_{\mathfrak{h}}^n E(\mathfrak{g}), \perp_{\mathfrak{v}}^n E(\mathfrak{g})) \rightarrow \text{Hom}_{\mathcal{E}(0)}(\top_{\mathfrak{h}}^n E, \perp_{\mathfrak{v}}^n E)$$

over $UV(\alpha_D^n \circ \varepsilon_D^n)$.

We give S^n the structure of an SDC as in [Pri04b] Theorem 2.5.

Then

$$\mathfrak{Def}_{\mathcal{D}, D} \simeq \mathfrak{Def}_S.$$

Proof. As for [Pri04b] Theorem 2.5. □

2.2 The pro-unipotent fundamental group

Definition 2.18. Take a pro-finite group Γ . Consider the functor on $\mathcal{N}_{\mathbb{Q}_l}$ given by

$$\mathfrak{g} \mapsto \mathrm{Hom}(\Gamma, \exp(\mathfrak{g})).$$

This has tangent space $H^1(\Gamma, \mathbb{Q}_l)$, and universal obstruction space $H^2(\Gamma, \mathbb{Q}_l)$. If $\dim H^1(\Gamma, \mathbb{Q}_l) < \infty$, it is pro-representable, by $\mathcal{L} \in \widehat{\mathcal{N}_{\mathbb{Q}_l}}$ say. Define the pro- \mathbb{Q}_l -unipotent completion of Γ by

$$\Gamma \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_l := \exp(\mathcal{L}(\Gamma, \mathbb{Q}_l)),$$

and the \mathbb{Q}_l Malcev Lie algebra by

$$\mathcal{L}(\Gamma, \mathbb{Q}_l) := \mathcal{L}.$$

Remark 2.19. It follows that \mathcal{L} is a hull for the quotient functor

$$\mathfrak{g} \mapsto \mathrm{Hom}(\Gamma, \exp(\mathfrak{g})) / \exp(\mathfrak{g}).$$

Fix a connected scheme X and a geometric point $\bar{x} \rightarrow X$. We wish to study the structure of the group

$$\pi_1(X, \bar{x}) \otimes \mathbb{Q}_l.$$

Consider the functor

$$\mathrm{Hom}(\pi_1(X, \bar{x}), -) : \mathcal{N}_{\mathbb{Q}_l} \rightarrow \mathrm{Grpd},$$

which associates to $\mathfrak{g} \in \mathcal{N}_{\mathbb{Q}_l}$ the discrete groupoid

$$\mathrm{Hom}(\pi_1(X, \bar{x}), \exp(\mathfrak{g})).$$

Recall the following definition from [Pri04a] Section 2.2:

Definition 2.20. Given a pro- l group K , define a constructible principal K -sheaf to be a principal K -sheaf \mathbb{D} , such that

$$\mathbb{D} = \varprojlim_{K \rightarrow F \text{ finite}} F \times^K \mathbb{D}.$$

Given an l -adic Lie group G , a constructible principal G -sheaf is a G -sheaf \mathbb{B} for which there exists a constructible principal K -sheaf \mathbb{D} , for some $K \leq G$ compact, with $\mathbb{B} = G \times^K \mathbb{D}$ (observe that compact and totally disconnected is equivalent to pro-finite).

Definition 2.21. Define the functor

$$\mathfrak{PB}(X) : \mathcal{N}_{\mathbb{Q}_l} \rightarrow \mathrm{Grpd}$$

to associate to each $\mathfrak{g} \in \mathcal{N}_{\mathbb{Q}_l}$, the groupoid of principal constructible $\exp(\mathfrak{g})$ -sheaves on X .

Define the groupoid

$$\mathfrak{PB}_{\bar{x}}(X)(\mathfrak{g})$$

to be the fibre of

$$\mathfrak{PB}(X)(\mathfrak{g}) \rightarrow \mathfrak{PB}(\bar{x})(\mathfrak{g})$$

over $(\exp(\mathfrak{g}), \mathrm{id})$.

Alternatively, we may think of $\mathfrak{PB}_{\bar{x}}(X)(\mathfrak{g})$ as consisting of pairs (\mathbb{B}, θ) , for

$$\mathbb{B} \in \mathfrak{PB}(X)(\mathfrak{g}),$$

and θ an isomorphism

$$\theta : \mathbb{B}_{\bar{x}} \rightarrow \exp(\mathfrak{g}).$$

Thus the crucial difference between the groupoids $\mathfrak{PB}_{\bar{x}}(X)(\mathfrak{g})$ and $\mathfrak{PB}(X)(\mathfrak{g})$ is that only those morphisms which give the identity on $\mathbb{B}_{\bar{x}}$ are permitted in the former.

Theorem 2.22. *As in [Pri04b] Section 4.2, we have a functorial equivalence of groupoids*

$$\mathrm{Hom}(\pi_1(X, \bar{x}), \exp(\mathfrak{g})) / \exp(\mathfrak{g}) \rightarrow \mathfrak{PB}(X)(\mathfrak{g}).$$

In fact, the same construction gives us an equivalence

$$\mathrm{Hom}(\pi_1(X, \bar{x}), \exp(\mathfrak{g})) \rightarrow \mathfrak{PB}_{\bar{x}}(X)(\mathfrak{g}).$$

Remark 2.23. If we have an automorphism σ of the pair $\bar{x} \rightarrow X$, it yields automorphisms of both the groupoids above. The construction of the equivalence ensures that it is σ -equivariant.

Definition 2.24. Let

$$\mathfrak{FB}(X) : \mathcal{N}_{Q_t} \rightarrow \mathrm{Cat}$$

associate to \mathfrak{g} the category of faithful constructible $\exp(\mathfrak{g})$ -sheaves on X . Observe that $\mathfrak{PB}(X)$ is the fibre of $\mathfrak{FB}(X)$ over the trivial $\exp(0)$ -sheaf 1 (the constant sheaf of singleton sets).

As in [Pri04b], we have a functorial comonadic adjunction

$$\begin{array}{c} \mathfrak{FB}(X) \\ u^* \downarrow \dashv \uparrow u_* \\ \mathfrak{FB}(X'), \end{array}$$

where

$$X' = \coprod_{x \in X} \bar{x},$$

where for each $x \in X$, a geometric point $\bar{x} \rightarrow X$ has been chosen. We have maps $u_x : \bar{x} \rightarrow X$, giving a map $u : X' \rightarrow X$.

If X is a variety, we may instead take

$$X' = \coprod_{x \in |X|} \bar{x}.$$

Observe that the second category has uniformly trivial deformation theory, $\exp(\mathfrak{g})$ being smooth, so that deformations are described by the SDC

$$S_X^n(\mathfrak{g}) = \mathrm{Hom}_{\mathfrak{FB}(X')(\mathfrak{g})}(\mathbb{T}(\mathfrak{g}), (u^* u_*)^n \mathbb{T}(\mathfrak{g})),$$

where $\mathbb{T}(\mathfrak{g}) = \exp(\mathfrak{g})$, the trivial $\exp(\mathfrak{g})$ -sheaf. Note that $(u^*u_*)^n \mathbb{T}(0) = 1$, and $S_X^n(0)$ is one point, so we need not worry about taking the fibre over $UV(\alpha_D^n \circ \varepsilon_D^n)$ (in the notation of Theorem 2.17).

Now, write

$$C^n(X, \mathcal{F}) := \Gamma(X', (u^*u_*)^n u^* \mathcal{F}),$$

for sheaves \mathcal{F} on X , and

$$\mathcal{C}^n(\mathcal{F}) := (u_*u^*)^{n+1} \mathcal{F},$$

so that $C^n(X, \mathcal{F}) = \Gamma(X, \mathcal{C}^n(\mathcal{F}))$.

We then have isomorphisms

$$\begin{aligned} S^n(\mathfrak{g}) &\cong \Gamma(X', (u^*u_*)^n \mathbb{T}(\mathfrak{g})) \cong C^n(X, \exp(\mathfrak{g})), \\ (b \mapsto g \cdot b) &\leftarrow g, \end{aligned}$$

the latter being a cosimplicial complex of groups, becoming an SDC via the Alexander-Whitney cup product

$$g * h = (\partial^{m+n} \dots \partial^{m+2} \partial^{m+1} g) \cdot (\partial^0)^m h,$$

for $g \in C^m$, $h \in C^n$.

Lemma 2.25. *If $H^i(X, \mathbb{Z}_l)$ and $H^{i-1}(X, \mathbb{Z}_l)$ are finitely generated, then the cohomology groups of S_X^\bullet are isomorphic to*

$$H^*(X, \mathbb{Q}_l).$$

Proof. Observe that the tangent space of S^\bullet is

$$\varprojlim_n C^\bullet(X, \mathbb{Z}/l^n) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l,$$

where

$$C^\bullet(X, \mathbb{Z}/l^n)$$

is a complex computing

$$H_{\text{ét}}^*(X, \mathbb{Z}/l^n).$$

Now,

$$\begin{aligned} H^i(S_X) &= H^i(\varprojlim_n C^\bullet(X, \mathbb{Z}/l^n) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) \\ &= H^i(\varprojlim_n C^\bullet(X, \mathbb{Z}/l^n)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \\ &= H^i(\varprojlim_n C^\bullet(X, \mathbb{Z}_l) \otimes \mathbb{Z}/l^n) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l, \end{aligned}$$

where

$$C^\bullet(X, \mathbb{Z}_l) = \varprojlim_n C^\bullet(X, \mathbb{Z}/l^n),$$

noting that this is a flat l -adic system.

Now, the tower

$$\cdots \rightarrow C^\bullet(X, \mathbb{Z}_l) \otimes \mathbb{Z}/l^{n+1} \rightarrow C^\bullet(X, \mathbb{Z}_l) \otimes \mathbb{Z}/l^n \rightarrow \cdots$$

clearly satisfies the Mittag-Leffler condition, so, by [Wei94] Theorem 3.5.8, we have the exact sequence:

$$\begin{array}{ccccccc} 0 & \rightarrow & \varprojlim^1 H^{i-1}(C^\bullet(X, \mathbb{Z}_l)_n) & \rightarrow & H^i(\varprojlim_n C^\bullet(X, \mathbb{Z}_l)_n) & \rightarrow & \varprojlim_n H^i(C^\bullet(X, \mathbb{Z}_l)_n) \rightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ 0 & \rightarrow & \varprojlim^1 H^{i-1}(X, \mathbb{Z}/l^n) & \rightarrow & H^i(\varprojlim_n C^\bullet(X, \mathbb{Z}_l)_n) & \rightarrow & H^i(X, \mathbb{Z}_l) \rightarrow 0. \end{array}$$

From [Mil80] Lemma V 1.11, it follows that the $H^{i-1}(X, \mathbb{Z}/l^n)$ are finitely generated \mathbb{Z}/l^n -modules, so satisfy DCC on submodules, so this inverse system satisfies the Mittag-Leffler condition, making the \varprojlim^1 on the left vanish.

Thus

$$H^i(S) = H^i(X, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l = H^i(X, \mathbb{Q}_l).$$

□

Lemma 2.26. *Given a map $f : X \rightarrow Y$ of schemes, there is a canonical morphism*

$$f^* : S_X^\bullet \rightarrow S_Y^\bullet$$

of SDCs, inducing the usual map

$$f^* : H^i(Y, \mathbb{Q}_l) \rightarrow H^i(X, \mathbb{Q}_l)$$

on cohomology.

Proof. The map f induces a map $f' : X' \rightarrow Y'$, making the following diagram commute:

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ u \downarrow & & \downarrow v \\ X & \xrightarrow{f} & Y. \end{array}$$

We then have

$$S_Y^n(\mathfrak{g}) = \mathrm{Hom}_{Y'}(\mathfrak{g}, (v^*v_*)^n \mathfrak{g}) \xrightarrow{(f')^\sharp} \mathrm{Hom}_{X'}((f')^* \mathfrak{g}, (f')^*(v^*v_*)^n \mathfrak{g}).$$

Now, using the the canonical maps

$$\begin{aligned} (f')^*v_* &\cong u^*f^* \\ f^*v_* &\rightarrow u_*(f')^*, \end{aligned}$$

we get a map

$$\mathrm{Hom}_{X'}((f')^* \mathfrak{g}, (f')^*(v^*v_*)^n \mathfrak{g}) \rightarrow \mathrm{Hom}_{X'}((f')^* \mathfrak{g}, (u^*u_*)^n (f')^* \mathfrak{g}),$$

but \mathfrak{g} is just the constant sheaf, so there is a canonical isomorphism $(f')^* \mathfrak{g} \cong \mathfrak{g}$. Putting these maps together gives us the morphism

$$f^* : S_X^\bullet \rightarrow S_Y^\bullet$$

of SDCs.

□

Remark 2.27. As in [Pri04a] Remark 2.14, we may replace \mathbb{Q}_l by any finite extension E , or indeed by $\bar{\mathbb{Q}}_l$.

In fact, we can go further than this in describing $\pi_1(X, \bar{x}) \otimes \mathbb{Q}_l$:

Theorem 2.28. *If we consider the morphism*

$$S_X^\bullet \xrightarrow{u_x^*} S_{\bar{x}}^\bullet$$

of SDCs, define the SDC

$$S_{X, \bar{x}}^n \subset S_X^n$$

to be the fibre of u_x^ over the unique 0-valued point of $S_{\bar{x}}^n$. Then*

$$\mathfrak{D}\mathfrak{e}\mathfrak{f}_{S_{X, \bar{x}}}$$

is equivalent to the discrete groupoid pro-represented by $\mathcal{L}(\pi_1(X, \bar{x}), \mathbb{Q}_l)$.

The cohomology of $S_{X, \bar{x}}^\bullet$ is

$$H^i(X, \bar{x}; \mathbb{Q}_l) \cong \begin{cases} 0 & i = 0 \\ H^i(X, \mathbb{Q}_l) & i \neq 0, \end{cases}$$

provided that the $H^i(X, \mathbb{Z}_l)$ are finitely generated.

Proof. Recall, from Theorem 2.14, that $\mathfrak{D}\mathfrak{e}\mathfrak{f}_{S_{X, \bar{x}}}(\mathfrak{g})$ is equivalent to $\mathfrak{D}\mathfrak{e}\mathfrak{f}_{u_x^*}(\mathfrak{g})$, the fibre of

$$\mathfrak{D}\mathfrak{e}\mathfrak{f}_{S_X}(\mathfrak{g}) \xrightarrow{u_x^*} \mathfrak{D}\mathfrak{e}\mathfrak{f}_{S_{\bar{x}}}(\mathfrak{g})$$

over (a_0, id) , where $a_0 \in S_{\bar{x}}^1(0)$ is the unique 0-valued object. From the above discussion, it follows that this groupoid is equivalent to the fibre of

$$\mathfrak{P}\mathfrak{B}(X)(\mathfrak{g}) \rightarrow \mathfrak{P}\mathfrak{B}(\bar{x})(\mathfrak{g})$$

over $(\exp(\mathfrak{g}), \text{id})$, which is precisely

$$\mathfrak{P}\mathfrak{B}_{\bar{x}}(X)(\mathfrak{g}).$$

Hence, by Theorem 2.22, we have a canonical equivalence

$$\text{Hom}(\pi_1(X, \bar{x}), \mathfrak{g}) \rightarrow \mathfrak{D}\mathfrak{e}\mathfrak{f}_{S_{X, \bar{x}}}(\mathfrak{g}).$$

The statement about cohomology follows from Theorem 2.14. \square

Remark 2.29. Observe that, from Lemma 2.26, an automorphism σ of the pair $\bar{x} \xrightarrow{u_x} X$ will naturally give rise to an automorphism of

$$S_X^\bullet \xrightarrow{u_x^*} S_{\bar{x}}^\bullet,$$

and hence of $\mathfrak{D}\mathfrak{e}\mathfrak{f}_{u_x^*}(\mathfrak{g})$. It is clear that the equivalence in the theorem will be σ -equivariant.

Remark 2.30. Tangential base points.

If $X = Y - Z$, for Y smooth and Z a closed subscheme, there is a definition of the fundamental group $\pi_1(X, \bar{v})$, where v is a non-zero tangent vector at a point on the boundary of Z . This makes use of a morphism of sites:

$$\acute{\text{E}}\text{t}(X) \rightarrow \acute{\text{E}}\text{t}(\bar{v}).$$

In order to define an SDC governing this fundamental group, we want to replace X' by $X' \sqcup \bar{v}$. Instead, we make use of the morphism

$$\acute{\text{E}}\text{t}(X) \rightarrow \acute{\text{E}}\text{t}(X') \times \acute{\text{E}}\text{t}(\bar{v}) = \acute{\text{E}}\text{t}(\bar{v}) \times \prod_{x \in X} \acute{\text{E}}\text{t}(\bar{x}),$$

which is sufficient. In fact, working with the sites is implicitly what we were doing before, since infinite coproducts are not defined in the category of schemes.

We use $X' \sqcup \bar{v}$ to define $S_{X'}^\bullet$, and the morphism $\bar{v} \rightarrow X' \sqcup \bar{v}$ enables the construction of a map $S_{X'}^\bullet \rightarrow S_{\bar{v}}^\bullet$, which is what we require.

2.3 Equivalence between SDCs and DGAs in characteristic 0

The reasoning of [Pri04b] Section 5 carries over directly to this context. We define the category $\mathcal{N}_k^{\mathbb{N}_0}$ in the obvious manner. The only result we need to check is the analogue of [Pri04b] Lemma 5.6:

Lemma 2.31. *Given a smooth homogeneous functor $G : \mathcal{N}_k^{\mathbb{N}_0} \rightarrow \text{Grp}$, the tangent space A^* has the structure of a GA (graded-commutative associative algebra without unit). Moreover, there is a canonical isomorphism (functorial in $\mathfrak{g}_* \in \mathcal{N}_k^{\mathbb{N}_0}$):*

$$\exp\left(\bigoplus_n A^n \otimes \mathfrak{g}_n\right) \cong G(\mathfrak{g}_*).$$

Proof. Since G is smooth and homogeneous, it will be “pro-represented” by a “pro-Artinian” topological free graded Lie algebra \mathcal{L}_* . The product on G gives rise to a co-associative coproduct

$$\rho : \mathcal{L}_* \rightarrow \mathcal{L}_* * \mathcal{L}_*,$$

for which $\mathcal{L}_* \rightarrow 0$ is the co-identity (existence of a co-inverse is an automatic consequence of pro-nilpotence).

$$\begin{aligned} G(\mathfrak{g}_*) &= \text{Hom}_{\text{Lie}}(\mathcal{L}^*, \mathfrak{g}^*) \\ \bigoplus_n A^n \otimes \mathfrak{g}_n &= \text{Der}(\mathcal{L}^*, \mathfrak{g}^*). \end{aligned}$$

Hence both embed into $\text{Hom}_{\text{Vect}}(\mathcal{L}_*, \mathfrak{g}_*)$. Observe the isomorphism

$$\begin{aligned} \text{Hom}_{\text{Vect}}(\mathcal{L}_*, \mathfrak{g}_*) &\cong \text{End}_{\mathfrak{g}_* \text{-Mod}}^\rho(\mathcal{L}_* * \mathfrak{g}_*), \\ f &\mapsto \mathcal{L}_* * \mathfrak{g}_* \xrightarrow{\rho * 1_{\mathfrak{g}_*}} \mathcal{L}_* * \mathcal{L}_* * \mathfrak{g}_* \xrightarrow{1_{\mathcal{L}_*} * [f, 1_{\mathfrak{g}_*}]} \mathcal{L}_* * \mathfrak{g}_*, \\ \mathcal{L}_* &\hookrightarrow \mathcal{L}_* * \mathfrak{g}_* \xrightarrow{\alpha} \mathcal{L}_* * \mathfrak{g}_* \xrightarrow{0 * 1_{\mathfrak{g}_*}} \mathfrak{g}_* \hookleftarrow \alpha, \end{aligned}$$

where the latter set is defined to consist of those $\alpha \in \text{End}_{\mathfrak{g}_* \text{-Mod}}(\mathcal{L}_* * \mathfrak{g}_*)$ for which the following diagram commutes:

$$\begin{array}{ccc} \mathcal{L}_* * \mathfrak{g}_* & \xrightarrow{\alpha} & \mathcal{L}_* * \mathfrak{g}_* \\ \rho * \mathfrak{g}_* \downarrow & & \rho * \mathfrak{g}_* \downarrow \\ \mathcal{L}_* * \mathcal{L}_* * \mathfrak{g}_* & \xrightarrow{\mathcal{L}_* * \alpha} & \mathcal{L}_* * \mathcal{L}_* * \mathfrak{g}_*. \end{array}$$

This is an associative algebra, with product given by composition. Then $G(\mathfrak{g}_*)$ is the subset consisting of Lie algebra homomorphisms, composition corresponding to the original product. $\bigoplus_n A^n \otimes \mathfrak{g}_n$ is the subset of derivations, with Lie bracket given by the usual formula $[\alpha, \beta] = \alpha \circ \beta \mp \beta \circ \alpha$.

Consider the maps

$$\begin{aligned} \log(g) &= \sum_{n \geq 1} (-1)^{n-1} \frac{(g - \text{id})^{on}}{n} \\ \exp(\alpha) &= \sum_{n \geq 0} \frac{\alpha^{on}}{n!} \end{aligned}$$

on $\text{End}_{\mathfrak{g}_* \text{-Mod}}^\rho(\mathcal{L}_* * \mathfrak{g}_*)$. \exp takes derivations to homomorphisms, and \log is its inverse, taking homomorphisms to derivations.

Hence, for $\alpha \in \text{Der}(\mathcal{L}^*, \mathfrak{g}^*) = \bigoplus_n A^n \otimes \mathfrak{g}_n$,

$$\exp(\alpha) \in G(\mathfrak{g}_*).$$

Similarly, for $g \in \text{Hom}_{\text{Lie}}(\mathcal{L}^*, \mathfrak{g}^*) = G(\mathfrak{g}_*)$,

$$\log(g) \in \bigoplus_n A^n \otimes \mathfrak{g}_n.$$

It remains only to show that A^* is a graded-commutative algebra. This follows from the functorial (in \mathfrak{g}_*) isomorphism

$$\bigoplus_n A^n \otimes \mathfrak{g}_n = \text{Der}(\mathcal{L}^*, \mathfrak{g}^*),$$

the right hand object being a Lie algebra, since it is log of a group. \square

As in [Pri04b] Section 5, we then obtain functors

$$\text{DGA} \begin{array}{c} \xrightarrow{\mathcal{S}} \\ \xleftarrow{\mathcal{D}} \end{array} \text{SDC}$$

with functorial quasi-isomorphisms $1 \rightarrow \mathcal{SD}$, $1 \rightarrow \mathcal{DS}$ inducing an equivalence of the localised categories.

3 Structure of the fundamental group

3.1 The topological fundamental group

Fix a connected differentiable manifold X , and a point $x \in X$. Let \mathcal{A}^n be the sheaf of real-valued C^∞ n -forms on X .

As in Section 2.2, the functor

$$\mathfrak{R}(\pi_1(X, x)) : \mathcal{N}_{\mathbb{R}} \rightarrow \text{Grpd},$$

which associates to $\mathfrak{g} \in \mathcal{N}_{\mathbb{R}}$ the groupoid with objects

$$\text{Hom}(\pi_1(X, x), \exp(\mathfrak{g})),$$

and morphisms $\exp(\mathfrak{g})$ (acting by conjugation), is governed by the SDC

$$S^n(\mathfrak{g}) = C^n(X, \exp(\mathfrak{g})) = C^n(X, \mathcal{S}^n(\mathbb{R}))(\mathfrak{g}).$$

As in [Pri04a] Section 1, we have the following quasi-isomorphisms of SDCs (over $\mathcal{N}_{\mathbb{R}}$):

$$\begin{array}{c} C^n(X, \mathcal{S}^n(\mathbb{R})) \\ \downarrow \\ C^n(X, \mathcal{S}(\mathcal{A}^\bullet)^n) \\ \uparrow \\ \mathcal{S}(\Gamma(X, \mathcal{A}^\bullet))^n, \end{array}$$

since on cohomology we have:

$$\begin{array}{c} H^n(X, \mathbb{R}) \\ \downarrow \\ \mathbb{H}^n(X, \mathcal{A}^\bullet) \\ \uparrow \\ H^n(\Gamma(X, \mathcal{A}^\bullet)), \end{array}$$

the first quasi-isomorphism arising because $\mathbb{R} \rightarrow \mathcal{A}^\bullet$ is a resolution, and the second from the flabbiness of \mathcal{A}^\bullet .

Hence $\mathfrak{R}(\pi_1(X, x))$ is governed by the DGA

$$A^\bullet := \Gamma(X, \mathcal{A}^\bullet),$$

whose quasi-isomorphism class, also known as the real homotopy type of X , is an invariant which has been studied in great detail.

In [Sul77], the minimal model of a connected DGA is defined, from which the de Rham fundamental group $\pi_1(X) \otimes \mathbb{R}$ is recovered as the dual Lie algebra. We can, however, recover the de Rham fundamental group as the hull of the functor Def_A , so do not have to involve minimal models.

We can now adapt the results of [Pri04a] Section 1 to this context. In fact the proofs are slightly simpler, since we are working with the constant sheaf \mathbb{R} , rather than the twisted sheaf $\text{ad}\mathbb{B}_0$. [Pri04a] Section 1.1 then becomes the main result of [DGMS75], [Pri04a] Section 1.2 becomes the main result of [Mor78], and the only new result is from adapting [Pri04a] Section 1.3:

Theorem 3.1. *Let X be a complex variety. Then there is a mixed Hodge structure (hence a weight decomposition) on the rational minimal model for X . In particular, the Malcev Lie algebra*

$$\mathcal{L}(\pi_1(X) \otimes \mathbb{R}) \cong \mathbb{R}[[\mathbf{H}^1(X, \mathbb{R})^\vee]] / (f(\mathbf{H}^2(X, \mathbb{R})^\vee)),$$

where f preserves the mixed Hodge structures.

Proof. Identical to [Pri04a] Theorem 1.7. \square

This answers a question posed in [ABC⁺96] as to whether Hodge theory can provide any information about the homotopy type (and hence the structure of the fundamental group) for singular varieties. Since zero weights are permitted in the cohomology of arbitrary varieties, we can, in general, say nothing about the function f , vindicating the observation in [Del75] that zero weights are unremarkable. However, if in a particular case \mathbf{H}^1 is known to have no component of weight zero, then conclusions can be drawn.

3.2 The algebraic fundamental group

Let $k = \mathbb{F}_q$, take a connected variety X_k/k , and let $X = X_k \otimes_k \bar{k}$. Fix a closed point x of X , and denote the associated geometric point $x \rightarrow X$ by \bar{x} . Without loss of generality (increasing q if necessary), we assume that $k(x) \subset \mathbb{F}_q$.

From Section 2.2, it follows that the Malcev Lie algebra (see Definition 2.18)

$$\mathcal{L}(\pi_1(X, \bar{x}), \mathbb{Q}_l)$$

pro-represents the groupoid $\mathfrak{Def}_{S_{X, \bar{x}}}$, and hence the functor $\text{Def}_{S_{X, \bar{x}}}$.

Observe that, since $k(x) \subset \mathbb{F}_q$, there is a natural action of $\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ on $\pi_1(X, \bar{x})$. Let F be the geometric Frobenius element. It follows from the remarks in Section 2.2 that the action of F on $\mathcal{L}(\pi_1(X, \bar{x}), \mathbb{Q}_l)$ will correspond to the action of F^* on the SDC above.

For X either smooth or proper, [Del80] shows that all the eigenvalues of Frobenius acting on the cohomology group $\mathbf{H}^i(X, \mathbb{Q}_l)$ are algebraic numbers α , and for each α , there exists a weight n , such that all complex conjugates of α have norm $q^{n/2}$. This provides us with a weight decomposition

$$\mathbf{H}^i(X, \mathbb{Q}_l) = \bigoplus_n \mathcal{W}_n \mathbf{H}^i(X, \mathbb{Q}_l).$$

Theorem 3.2. *There is an isomorphism*

$$\mathcal{L}(\pi_1(X, \bar{x}), \mathbb{Q}_l) \cong L(\mathbf{H}^1(X, \mathbb{Q}_l)^\vee) / (f(\mathbf{H}^2(X, \mathbb{Q}_l)^\vee)),$$

where $L(V)$ is the free pro-nilpotent Lie algebra (as in 1.1), and

$$f : \mathbf{H}^2(X, \mathbb{Q}_l)^\vee \rightarrow \Gamma_2 L(\mathbf{H}^1(X, \mathbb{Q}_l)^\vee)$$

preserves the (Frobenius) weight decompositions of [Del80] 3.3.7.

Moreover, the quotient map

$$f : \mathbb{H}^2(X, \mathbb{Q}_l)^\vee \rightarrow \Gamma_2/\Gamma_3 \cong \bigwedge^2 (\mathbb{H}^1(X, \mathbb{Q}_l)^\vee)$$

is dual to half the cup product

$$\mathbb{H}^1(X, \mathbb{Q}_l) \times \mathbb{H}^1(X, \mathbb{Q}_l) \xrightarrow{\frac{1}{2}\cup} \mathbb{H}^2(X, \mathbb{Q}_l).$$

Proof. This will essentially be the same as [Pri04a] Theorem 2.11. From the remarks in Section 2.2, it follows that the isomorphism

$$\mathrm{Hom}(\mathcal{L}\pi_1(X, \bar{x}), \mathfrak{g}) \cong \mathrm{Def}_{S_{X, \bar{x}}}(\mathfrak{g})$$

is Frobenius equivariant. Write

$$\mathcal{L} := \mathcal{L}\pi_1(X, \bar{x}).$$

Since $\mathcal{L}/\Gamma_n(\mathcal{L})$ is a finite dimensional vector space over \mathbb{Q}_l for all n , we may use the Jordan decomposition (over $\bar{\mathbb{Q}}_l$) successively to lift the cotangent space $\mathcal{L}/\Gamma_2(\mathcal{L})$ to $\mathcal{L}/\Gamma_n(\mathcal{L})$, obtaining a space of generators $V \subset \mathcal{L}$, with the map $V \rightarrow \mathcal{L}/\Gamma_2(\mathcal{L})$ an isomorphism preserving the weight decompositions.

Now look at the map

$$L(V) \rightarrow \mathcal{L};$$

let its kernel be $J \subset [V, V]$. $(J/[V, J])^\vee$ is then a universal obstruction space for $h_{\mathcal{L}}$. Note that we may lift the Frobenius action on \mathcal{L} to an action on $L(V)$.

Since $\mathbb{H}^2(X, \mathbb{Q}_l)$ is an obstruction space for $\mathrm{Def}_{\bar{x}}$, there is a unique injective map of obstruction spaces ([Man99] Proposition 2.18)

$$(J/VJ)^\vee \hookrightarrow \mathbb{H}^2(X, \mathbb{Q}_l).$$

Observe that both of these obstruction spaces are Frobenius equivariant. Given a functor D on which F (Frobenius) acts, we say an obstruction theory (W, w_e) is Frobenius equivariant if there is a Frobenius action on the space W such that for every small extension

$$e : 0 \rightarrow I \rightarrow \mathfrak{h} \rightarrow \mathfrak{g} \rightarrow 0,$$

and every $d \in D(\mathfrak{g})$, we have $w_e(Fd) = Fw_e(d)$.

Since $h_{\mathcal{L}} \rightarrow \mathrm{Def}_{\bar{x}}$ commutes with Frobenius, the corresponding map of obstruction spaces does (by uniqueness).

As before, we may use the Jordan decomposition on the successive quotients to obtain a space of generators $W \subset J$, such that $W \rightarrow J/[V, J]$ is an isomorphism preserving the weight decompositions.

Hence we have maps

$$\mathbb{H}^2(X, \mathbb{Q}_l)^\vee \twoheadrightarrow J/[V, J] \cong W \hookrightarrow L(V),$$

and

$$V \cong \mathcal{L}/[\mathcal{L}, \mathcal{L}] \cong \mathbb{H}^1(X, \mathbb{Q}_l)^\vee,$$

preserving the Frobenius decompositions. Let

$$f : \mathbb{H}^2(X, \mathbb{Q}_l)^\vee \rightarrow L(\mathbb{H}^1(X, \mathbb{Q}_l)^\vee)$$

be the composition. Then

$$\mathcal{L} = L(V)/J \cong L(\mathbb{H}^1(X, \mathbb{Q}_l)^\vee)/(f(\mathbb{H}^2(X, \mathbb{Q}_l)^\vee)),$$

as required.

It only remains to prove the statement about the cup product:

Lemma 3.3. *Let $F : \mathcal{N}_K \rightarrow \text{Set}$ be a deformation functor. If V is an obstruction theory for F , then there is a canonically defined obstruction map $t_F \times t_F \rightarrow V$.*

Proof. (Of Lemma.) Since F is a deformation functor, $t_F \times t_F \cong F(L(\epsilon) \oplus L(\nu))$. Consider the small extension

$$e : 0 \rightarrow K[\epsilon, \nu] \rightarrow L(\epsilon, \nu)/([\epsilon, [\epsilon, \nu]], [\nu, [\nu, \epsilon]]) \rightarrow L(\epsilon) \oplus L(\nu) \rightarrow 0.$$

Sending $f \in F(L(\epsilon) \oplus L(\nu))$ to $v_e(f) \in V[\epsilon, \nu]$ defines the obstruction map $t_F \times t_F \rightarrow V$. Standard arguments show that this is bilinear and anti-symmetric. \square

Now, the obstruction map

$$V^\vee \otimes V^\vee \rightarrow W^\vee$$

is the dual map to $f : W \rightarrow \Gamma_2(V)/\Gamma_3(V)$, while the obstruction map

$$\mathbb{H}^1(X, \mathbb{Q}_l) \otimes \mathbb{H}^1(X, \mathbb{Q}_l) \rightarrow \mathbb{H}^2(X, \mathbb{Q}_l)$$

is the usual cup product. The result now follows from uniqueness of the obstruction map. \square

Corollary 3.4. *If X is smooth and proper, then*

$$\mathcal{L}(\pi_1(X, \bar{x}), \mathbb{Q}_l)$$

is quadratically presented.

In fact,

$$\mathcal{L}(\pi_1(X, \bar{x}), \mathbb{Q}_l) \cong L(\mathbb{H}^1(X, \mathbb{Q}_l)^\vee)/(\check{\cup} \mathbb{H}^2(X, \mathbb{Q}_l)^\vee),$$

where $\check{\cup}$ is dual to the cup product.

Proof. This follows since, under these hypotheses, [Del80] Corollaries 3.3.4–3.3.6 imply that $\mathbb{H}^1(X, \mathbb{Q}_l)$ is pure of weight 1, and $\mathbb{H}^2(X, \mathbb{Q}_l)$ is pure of weight 2. \square

Example 3.5. This implies that, for X smooth and proper, the pro- l quotient $\pi_1^l(X, \bar{x})$ of $\pi_1(X, \bar{x})$ cannot be the Heisenberg group

$$\mathcal{H}_3(\mathbb{Z}_l) = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \in \text{GL}_3(\mathbb{Z}_l) \right\},$$

since this is not of quadratic presentation (in particular, this can be inferred from the non-vanishing of the Massey triple product on $\mathbb{H}^1(\pi_1(X, \bar{x}), \mathbb{Q}_l)$ — see [ABC⁺96] Ch.3 §3 for criteria for a Lie algebra to be quadratically presented).

Corollary 3.6. *If X is smooth, then*

$$\mathcal{L}(\pi_1(X, \bar{x}), \mathbb{Q}_l)$$

is a quotient of the free Lie algebra $L(\mathbb{H}^1(X, \mathbb{Q}_l)^\vee)$ by an ideal which is finitely generated by elements of bracket length 2, 3, 4.

Proof. This follows since, under these hypotheses, [Del80] Corollaries 3.3.4–3.3.6 imply that $\mathbb{H}^1(X, \mathbb{Q}_l)$ is of weights 1 and 2, while $\mathbb{H}^2(X, \mathbb{Q}_l)$ is of weights 2, 3 and 4. Hence the image of $f : \mathbb{H}^2(X, \mathbb{Q}_l)^\vee \rightarrow L(\mathbb{H}^1(X, \mathbb{Q}_l)^\vee)$ gives equations:

$$\begin{aligned} (\mathcal{W}_{-1})^2 & \text{ weight } -2 \\ (\mathcal{W}_{-1})^3 + (\mathcal{W}_{-1})(\mathcal{W}_{-2}) & \text{ weight } -3 \\ (\mathcal{W}_{-1})^4 + (\mathcal{W}_{-1})^2(\mathcal{W}_{-2}) + (\mathcal{W}_{-2})^2 & \text{ weight } -4, \end{aligned}$$

hence equations of degree at most 4. □

Example 3.7. Thus $\pi_1^l(X, \bar{x})$ cannot be the group

$$\left\{ \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_5(\mathbb{Z}_l) \right\}.$$

Remark 3.8. If X is singular and proper, weights tell us nothing about the structure of the fundamental group, since zero weights are permitted, so any equations may arise.

Finally, a result which does not have an analogue in [Pri04a].

Theorem 3.9. *Let X be smooth and projective. The cup product*

$$\mathbb{H}^1(\pi_1(X, \bar{x}), \mathbb{Q}_l) \times \mathbb{H}^1(\pi_1(X, \bar{x}), \mathbb{Q}_l) \xrightarrow{\cup} \mathbb{H}^2(\pi_1(X, \bar{x}), \mathbb{Q}_l)$$

is non-degenerate.

Proof. Recall that $\mathbb{H}^1(\pi_1(X, \bar{x}), \mathbb{Q}_l)$ is a tangent space for our deformation functor, and $\mathbb{H}^2(\pi_1(X, \bar{x}), \mathbb{Q}_l)$ the *universal* obstruction space. This gives an isomorphism

$$\mathbb{H}^1(\pi_1(X, \bar{x}), \mathbb{Q}_l) \xrightarrow{\cong} \mathbb{H}^1(X, \mathbb{Q}_l),$$

and an injection

$$\mathbb{H}^2(\pi_1(X, \bar{x}), \mathbb{Q}_l) \hookrightarrow \mathbb{H}^2(X, \mathbb{Q}_l),$$

the maps being compatible with cup products. This yields:

$$\begin{array}{ccc} \mathbb{H}^1(\pi_1(X, \bar{x}), \mathbb{Q}_l) \otimes \mathbb{H}^1(\pi_1(X, \bar{x}), \mathbb{Q}_l) & \xrightarrow{\cup} & \mathbb{H}^2(\pi_1(X, \bar{x}), \mathbb{Q}_l) \\ \downarrow \cong & & \downarrow \\ \mathbb{H}^1(X, \mathbb{Q}_l) \otimes \mathbb{H}^1(X, \mathbb{Q}_l) & \xrightarrow{\cup} & \mathbb{H}^2(X, \mathbb{Q}_l) \\ \eta^{n-1} \cup \downarrow \cong & & \eta^{n-1} \cup \downarrow \\ \mathbb{H}^{2n-1}(X, \mathbb{Q}_l) \otimes \mathbb{H}^1(X, \mathbb{Q}_l) & \xrightarrow{\cup} & \mathbb{H}^{2n}(X, \mathbb{Q}_l) \cong \mathbb{Q}_l, \end{array}$$

the isomorphism on the bottom left being the Hard Lefschetz Theorem ([Del80] Theorem 4.1), η being the Chern class of an ample line bundle. The cup product along the bottom is non-degenerate by Poincaré duality, hence result. \square

Since the relations defining the hull arise from the cup product, this indicates that $\pi_1(X, \bar{x})$ must have some defining relations. More precisely:

Corollary 3.10. *If X is smooth and proper, then*

$$\mathcal{L}(\pi_1(X, \bar{x}))/\Gamma_3(\mathcal{L}(\pi_1(X, \bar{x}))) \cong L(V)/\Gamma_3(L(V)),$$

for any free Lie algebra $L(V)$.

Proof. As for [ABC⁺96] Proposition 3.25. \square

Remarks 3.11. Homotopy Theory

1. Classically, the DGA $\Gamma(X, \mathcal{A}^\bullet)$ (or, rather, its quasi-isomorphism class) is regarded as being the real homotopy type of a differentiable manifold X . It would therefore make sense to think of the canonical SDC

$$S^n(\mathfrak{g}) = C^n(X, \exp(\mathfrak{g})),$$

as capturing the \mathbb{Q}_l -homotopy theory of a scheme X , since it yields a DGA $\mathcal{D}(S)$. In fact, S is not merely an SDC, but a cosimplicial complex of group-valued functors. Under the duality of Lemma 2.31, this cosimplicial complex of functors is equivalent to the cosimplicial complex

$$C^*(X, \mathbb{Q}_l)$$

of algebras. It seems reasonable to regard the quasi-isomorphism class of this cosimplicial complex as being the \mathbb{Q}_l -homotopy type of X . The corresponding differential graded algebra will be quasi-isomorphic to the \mathbb{Q}_l -homotopy type defined in [Del80] §5, but the approach used there did not allow any practical consequences to be drawn from the formality of the homotopy type.

If we recall from [Sul77] that the correct notion of automorphisms of a homotopy type is that of *outer* automorphisms, i.e. automorphisms modulo homotopy, then the reasoning of [Del80] Corollary 5.3.7 shows that the homotopy type is formal, and that the quasi-isomorphism is Galois equivariant.

2. We may think of the complex $C^*(X, \bar{x}; \mathbb{Q}_l)$ as being the \mathbb{Q}_l -homotopy type of the pair $\bar{x} \rightarrow X$. This will be also be formal when X is smooth and proper.

References

- [ABC⁺96] J. Amorós, M. Burger, K. Corlette, D. Kotschick, and D. Toledo. *Fundamental groups of compact Kähler manifolds*, volume 44 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1996.
- [BK98] Sergey Barannikov and Maxim Kontsevich. Frobenius manifolds and formality of Lie algebras of polyvector fields. *Internat. Math. Res. Notices*, (4):201–215, 1998.
- [Del71] Pierre Deligne. Théorie de Hodge. II. *Inst. Hautes Études Sci. Publ. Math.*, (40):5–57, 1971.
- [Del75] Pierre Deligne. Poids dans la cohomologie des variétés algébriques. In *Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974)*, Vol. 1, pages 79–85. Canad. Math. Congress, Montreal, Que., 1975.
- [Del80] Pierre Deligne. La conjecture de Weil. II. *Inst. Hautes Études Sci. Publ. Math.*, (52):137–252, 1980.
- [DGMS75] Pierre Deligne, Phillip Griffiths, John Morgan, and Dennis Sullivan. Real homotopy theory of Kähler manifolds. *Invent. Math.*, 29(3):245–274, 1975.
- [GM88] William M. Goldman and John J. Millson. The deformation theory of representations of fundamental groups of compact Kähler manifolds. *Inst. Hautes Études Sci. Publ. Math.*, (67):43–96, 1988.
- [Man99] Marco Manetti. Deformation theory via differential graded Lie algebras. In *Algebraic Geometry Seminars, 1998–1999 (Italian) (Pisa)*, pages 21–48. Scuola Norm. Sup., Pisa, 1999.
- [Mil80] James S. Milne. *Étale cohomology*. Princeton University Press, Princeton, N.J., 1980.
- [Mok93] A. Mokrane. La suite spectrale des poids en cohomologie de Hyodo-Kato. *Duke Math. J.*, 72(2):301–337, 1993.
- [Mor78] John W. Morgan. The algebraic topology of smooth algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, (48):137–204, 1978.
- [Pri04a] J.P. Pridham. The deformation theory of representations of the fundamental group of a smooth variety. Preprint math.AG/0401344, 2004.
- [Pri04b] J.P. Pridham. Deformations via Simplicial Deformation Complexes. Preprint math.AG/0311168, 2004.
- [RZ82] M. Rapoport and Th. Zink. Über die lokale Zetafunktion von Shimuravarietäten. Monodromiefiltration und verschwindende Zyklen in ungleicher Charakteristik. *Invent. Math.*, 68(1):21–101, 1982.

- [Sch68] Michael Schlessinger. Functors of Artin rings. *Trans. Amer. Math. Soc.*, 130:208–222, 1968.
- [Ser92] Jean-Pierre Serre. *Lie algebras and Lie groups*. Springer-Verlag, Berlin, second edition, 1992. 1964 lectures given at Harvard University.
- [Sul77] Dennis Sullivan. Infinitesimal computations in topology. *Inst. Hautes Études Sci. Publ. Math.*, (47):269–331 (1978), 1977.
- [Tsu99] Takeshi Tsuji. p -adic étale cohomology and crystalline cohomology in the semi-stable reduction case. *Invent. Math.*, 137(2):233–411, 1999.
- [Wei94] Charles A. Weibel. *An introduction to homological algebra*. Cambridge University Press, Cambridge, 1994.