Deformation quantisation of derived Poisson structures

J.P.Pridham

Background

- 0-shifted quantisation of a k-CDGA (differential graded-commutative algebra) A is a differential graded associative $k[\![\hbar]\!]$-algebra $\tilde{A}$ deforming A.

- On $A = \tilde{A}/\hbar$, have Lie bracket

$$[a, b] := \hbar^{-1}(\tilde{a}\tilde{b} \mp \tilde{b}\tilde{a}) \mod \hbar.$$

$\Longrightarrow$ A a DG Poisson (a.k.a. P_1) algebra.

- When do Poisson structures quantise?

Underived quantisations, $n = 0$

- Kontsevich–Tamarkin: $\exists$ quantisations for Poisson structures on smooth k-algebras ($\mathbb{Q} \subset k$).
- Algebroid quantisations for smooth schemes (deforming the category of line bundles).

Digression: positively shifted quantisation

- E_k-algebra A has k homotopy compatible associative multiplications.
- P_k-algebras are CDGAs with biderivation Lie bracket of degree $1-k$; called $(k-1)$-shifted.
- Kontsevich formality: for $k \geq 2$,

$$E_k\text{-algebras} \simeq P_k\text{-algebras}$$

(after choosing Drinfeld associator).

$\implies$ Quantisation for P_k-algebras automatic ($k \geq 2$).

Sketch of K–T's quantisation I

- Associative deformations of an algebra A governed by Hochschild complex.
- $\mathrm{CC}^\bullet(A, A)$ an E_2-algebra (McClure–Smith).
- Formality gives a P_2-algebra (Gerstenhaber algebra) $\tilde{B}$ Lie quasi-isomorphic to $\mathrm{CC}^\bullet(A, A)$.
- $\tilde{B}$ effectively a deformation of $\mathrm{HH}^*(A, A) =: B$.
- For A smooth commutative, $B \cong \mathrm{Pol}(A, 1)$ (polyvectors, HKR).

Sketch of K–T's quantisation II

- When A smooth underived, $\mathrm{Pol}(A,1)$ doesn't deform, giving L_∞ quasi-iso
$$\mathrm{CC}^\bullet(A,A) \simeq \mathrm{Pol}(A,1).$$
- Thus equivalence between quantisations and Poisson structures in smooth underived setting:
  - Objects are $k[\![\hbar]\!]$-linear Poisson structures on $A[\![\hbar]\!]$;
  - Morphisms are Poisson isomorphisms $\equiv \mathrm{id} \mod \hbar$;
  - 2-automorphisms are elements $a \in \hbar^2 A[\![\hbar]\!]$ acting as conjugation by $\exp([a,-])$.
- All steps of K–T's quantisation extend to derived stacks except deformation argument.

K–T deformation argument recast I

- Good truncation filtration $\{\tau^{\leq n} V\}_n$ on any complex V:
  $\ldots \xrightarrow{\delta} V^0 \xrightarrow{\delta} \ldots \xrightarrow{\delta} V^{n-1} \xrightarrow{\delta} \mathrm{Z}^n V \to 0 \to \ldots$
- $\mathrm{Rees}(V, \tau) := \bigoplus_n (\tau^{\leq n} V)\hbar^n$ a $\mathbb{G}_m$-equivariant (i.e. weight decomposition) $k[\hbar]$-module.
- $\mathrm{Rees}(\tilde{B}, \tau)$ a $\mathbb{G}_m$-equivariant $P_2[\hbar]$-algebra, with $\mathbb{G}_m$-action on P_2 by degree.
- $\mathrm{Rees}(\tilde{B}, \tau)/\hbar = \mathrm{gr}^\tau \tilde{B} \simeq \mathrm{H}^* \tilde{B} = B$.
- $\mathrm{Rees}(\tilde{B}, \tau) = \varprojlim_n^{\mathbb{G}_m} \mathrm{Rees}(\tilde{B}, \tau)/\hbar^n$.

K–T deformation argument recast II

- $B[\hbar] \simeq \mathrm{Rees}(\tilde{B}, \tau)$ by induction:

$$\begin{tikzcd} & \mathrm{Rees}(\tilde{B}, \tau)/\hbar^{n+1} \arrow[d] \\ B[\hbar] \arrow[r] \arrow[ur, dashed] & \mathrm{Rees}(\tilde{B}, \tau)/\hbar^{n}. \end{tikzcd}$$

- Obstruction in $\mathrm{H}^1\mathrm{R}\underline{\mathrm{Der}}_{P_2}(B, \hbar^n B)^{\mathbb{G}_m}$:
- for $n \geq 2$, weights $\implies$ vanishing;
- for $n = 1$, affine symmetry argument.

Generalisation to CDGAs I

- $\mathrm{CC}^\bullet(A, A)$ total complex of a double complex

$$A \xrightarrow{b} \underline{\mathrm{Hom}}(A, A) \xrightarrow{b} \underline{\mathrm{Hom}}(A \otimes A, A) \xrightarrow{b} \dots$$

- τ^{HH}: good truncation in Hochschild direction.
- For $A^\#$ free, $\mathrm{gr}^{\tau^{\mathrm{HH}}}\mathrm{CC}^\bullet(A, A) \xrightarrow[\mathrm{HKR}]{\sim} \mathrm{Pol}(A, 1)$

$$A \xrightarrow{0} \underline{\mathrm{Hom}}_A(\Omega^1_A, A) \xrightarrow{0} \underline{\mathrm{Hom}}_A(\Omega^2_A, A) \xrightarrow{0} \dots .$$

- If A is (derived) locally of finite presentation ($\approx$ finitely generated as free algebra), then $B := \mathrm{Pol}(A, 1)$ generated in weights $0, 1$, so:

Generalisation to CDGAs II

- spectral sequence ($p \geq 1$)

$$\mathrm{Ext}^q_B(\mathrm{L}\Omega^p_B, \hbar^{p-1}M)^{\mathbb{G}_m} \implies \mathrm{H}^{p+q-1}\mathrm{R}\underline{\mathrm{Der}}_{P_2}(B, M)^{\mathbb{G}_m}$$

gives $\mathrm{R}\underline{\mathrm{Der}}_{P_2}(B, \hbar^n B)^{\mathbb{G}_m} \simeq 0 \quad \forall n \geq 2$, since $p < n + p - 1$.

- Obstructions in $\mathrm{H}^1\mathrm{R}\underline{\mathrm{Der}}_{P_2}(B, \hbar^n B)^{\mathbb{G}_m}$

$$\begin{tikzcd}
 & \mathrm{Rees}(\tilde{B}, \tau^{\mathrm{HH}})/\hbar^{n+1} \arrow[d] \\
B[\hbar] \arrow[r] \arrow[ur, dashed] & \mathrm{Rees}(\tilde{B}, \tau^{\mathrm{HH}})/\hbar^n
\end{tikzcd}$$

then vanish $\forall n \geq 2$, and unique lifts ($\mathrm{H}^{\leq 0} = 0$).

Generalisation to CDGAs III

- No affine symmetries, so still a potential obstruction for $n = 1$.

Eliminating the first-order obstruction I

- Cut the problem in half.
- $\mathrm{CC}^\bullet(A, A)$ has an involution

$$f^*(a_1, \ldots, a_n) := \pm f(a_n, \ldots, a_1),$$

  sending a deformation to its opposite algebra.
- Formality isomorphisms from *even* associators give $\tilde{B}$ an involution, with $(xy)^* = x^* y^*$ and $[x, y]^* = -[x^*, y^*]$.
- Thus $\tilde{B}$ deforms $B = \mathrm{Pol}(A, 1)$ as an *involutive* P_2-algebra.

Eliminating the first-order obstruction II

- Key property: For $f \in \mathsf{H}^*\mathrm{gr}_i^{\tau^{\mathsf{HH}}}\mathsf{CC}^\bullet(A, A)$, $f^* = (-1)^i f$.
- Involutive Rees construction

$$\mathrm{Rees}(V, F, *) = \bigoplus_n \{ v \in F_n V \ : \ v^* = (-1)^n v \} \hbar^n,$$

a $\mathbb{G}_m$-equivariant $k[\hbar^2]$-module.

- $\mathrm{Rees}(\tilde{B}, \tau^{\mathsf{HH}}, *)$ a $\mathbb{G}_m$-equivart $P_2[\hbar^2]$-algebra,
- $\mathrm{Rees}(\tilde{B}, \tau^{\mathsf{HH}}, *)/\hbar^2 \simeq B$.

Eliminating the first-order obstruction III

- Obstructions

$$
\begin{tikzcd}
 & \mathrm{Rees}(\tilde{B}, \tau^{\mathrm{HH}}, *)/\hbar^{2m+2} \arrow[d] \\
B[\hbar^2] \arrow[r] \arrow[ur, dashed] & \mathrm{Rees}(\tilde{B}, \tau^{\mathrm{HH}}, *)/\hbar^{2m}.
\end{tikzcd}
$$

- Already know

$$\mathrm{R}\underline{\mathrm{Der}}_{P_2}(B, \hbar^{2m} B)^{\mathbb{G}_m} \simeq 0 \quad \forall m \geq 1.$$

- Thus $\mathrm{Rees}(\tilde{B}, \tau^{\mathrm{HH}}, *) \simeq B[\hbar^2]$, canonically.

Deformation quantisation of CDGAs

- Hence filtered L_∞-quasi-isomorphism
  $(\mathrm{CC}^\bullet(A,A), \tau_p^{\mathrm{HH}}) \simeq (\mathrm{Pol}(A,1), \underline{\mathrm{Hom}}_A(\Omega_A^{\le p}, A))$
  $(*$ on left corresponds to $(-1)^p$ on $\Omega_A^p)$.
- Taking Maurer–Cartan gives:

Theorem (P)

For l.f.p. CDGAs A, each even associator gives

$$\mathcal{P}(A,0) \to Q\mathcal{P}^{sd}(A,0)$$

from the space of Poisson structures to the space of anti-involutive ($b \star_\hbar a = \pm a \star_{-\hbar} b$) algebroid quantisations.

Global deformation quantisations

Sheafifying gives:

Theorem (P)

For all l.f.p. derived schemes and DM stacks $\mathfrak{X}$,

$$\mathcal{P}(\mathfrak{X}, 0) \to Q\mathcal{P}^{sd}(\mathfrak{X}, 0)$$

(includes LCI schemes).

$\rightsquigarrow$ deformation of $\mathrm{Perf}_{\mathfrak{X}}$ as a dg category with duality.

- If $\mathrm{H}^2(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) = 0$, every quantisation is strict (i.e. by associative algebras, not algebroids); $\ker(\mathrm{CC}^\bullet(\mathscr{O}, \mathscr{O}) \to \mathscr{O})$ replaces $\mathrm{CC}^\bullet(\mathscr{O}, \mathscr{O})$.

The space of quantisations I

- Quantised polyvectors

$$\tilde{F}^i Q\widehat{\mathrm{Pol}}(A,0) := \prod_{p \geq i} \tau_p^{\mathrm{HH}} \mathrm{CC}^\bullet(A,A)\hbar^{p-1}.$$

- Maurer–Cartan gives space of algebroid quantisations

$$Q\mathcal{P}(A,0) := \underline{\mathrm{MC}}(\tilde{F}^2 Q\widehat{\mathrm{Pol}}(A,0)[1]).$$

- Imposing $f(\hbar)^* = f(-\hbar)$ gives anti-involutive quantisations $Q\mathcal{P}^{sd}(A,0) \subset Q\mathcal{P}(A,0)$.

The space of quantisations II

- τ^{HH} quasi-isomorphic to a filtration γ by shuffles.
- Corresponds to the BD_1 operad governing almost commutative algebras.
- Thus quantisation of A is $\tilde{A}$ associative with $\tilde{A}/\hbar$ a CDGA and $\tilde{A}/\hbar \simeq A$.
- $\mathrm{Im}\,(Q\widehat{\mathrm{Pol}}(A,0) \to A[\![\hbar]\!])$ gives curvature (algebroid) terms.

Generalisations

- Analytic and smooth settings: work with EFC and $\mathcal{C}^\infty$-rings; use polydifferential operators instead of Hochschild complex.
- Artin stacks/Lie groupoids: have to resolve using Lie algebroids, e.g. $[Y/G]$ as
$$[Y/\mathfrak{g}] \Leftarrow [Y \times G/\mathfrak{g}^{\oplus 2}] \Lleftarrow [Y \times G^2/\mathfrak{g}^{\oplus 3}] \ldots,$$
  and Chevalley–Eilenberg.
  - Similar argument works, but homological subtleties.
  - Gives curved deformation of $\mathrm{Perf}_{\mathfrak{X}}$ as a dg category with duality. [*curved* means $\delta^2 \neq 0$ (but close)]

Curved DGAs

Hochschild complex $\rightsquigarrow$ *curved* deformations:

- associative unital graded algebra B^*,
- curvature $\kappa \in B^2$,
- differential $\delta \colon B^* \to B^*[1]$,
- $\delta^2 b = [\kappa, b]$, $\delta\kappa = 0$, $\delta(1) = 0$.
- $\approx$ Morita deformations [LvdB].
- Curved dg categories defined similarly.

The space of quantisations — details

Complete classifications:

$$Q\mathcal{P}(A,0) \simeq \underline{\mathrm{MC}}(\prod_{p\geq 0} \hbar^{\max(0,2-p)}\underline{\mathrm{Hom}}_A(\Omega_A^p, A)[\![\hbar]\!][1-p])$$

Poisson algebras over $k[\![\hbar]\!]$, deforming A as a CDGA, with $\hbar^2$-curvature/2-automorphisms.

$$Q\mathcal{P}^{sd}(A,0) \simeq \underline{\mathrm{MC}}(\prod_{p\geq 0} \hbar^{2\max(0,1-\lfloor \frac{p}{2}\rfloor)}\underline{\mathrm{Hom}}_A(\Omega_A^p, A)[\![\hbar^2]\!][1-p])$$

Poisson algebras over $k[\![\hbar^2]\!]$, deforming A as a CDGA, with $\hbar^2$-curvature.

References

Maxim Kontsevich.

Deformation quantization of Poisson manifolds. *Lett. Math. Phys.*, 66(3):157–216, 2003.

Maxim Kontsevich.

Operads and motives in deformation quantization. *Lett. Math. Phys.*, 48(1):35–72, 1999. Moshé Flato (1937–1998).

Dmitry E. Tamarkin.

Operadic proof of M. Kontsevich's formality theorem. Thesis (Ph.D.)–The Pennsylvania State University.

Maxim Kontsevich.

Deformation quantization of algebraic varieties. *Lett. Math. Phys.*, 56(3):271–294, 2001. EuroConférence Moshé Flato 2000, Part III (Dijon).

J. P. Pridham.
Quantisation of derived Poisson structures.
arXiv: 1708.00496v4 [math.AG], 2019.

J. P. Pridham.
Deformation quantisation for unshifted symplectic structures on derived Artin stacks. arXiv: 1604.04458v2 [math.AG], 2016.

Wendy Lowen and Michel Van den Bergh.
The curvature problem for formal and infinitesimal deformations. arXiv:1505.03698, 2015.