

SHIFTED SYMPLECTIC STRUCTURES ON DERIVED ANALYTIC MODULI OF ℓ -ADIC LOCAL SYSTEMS AND GALOIS REPRESENTATIONS

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ABSTRACT. We develop a characterisation of non-Archimedean derived analytic geometry based on dg enhancements of dagger algebras. This allows us to formulate derived analytic moduli functors for many types of pro-étale sheaves, and to construct shifted symplectic structures on them by transgression using arithmetic duality theorems. In order to handle duality functors involving Tate twists, we introduce weighted shifted symplectic structures on formal weighted moduli stacks, with the usual moduli stacks given by taking \mathbb{G}_m -invariants.

In particular, this establishes the existence of shifted symplectic and Lagrangian structures on derived moduli stacks of ℓ -adic constructible complexes on smooth varieties via Poincaré duality, and on derived moduli stacks of ℓ -adic Galois representations via Tate and Poitou–Tate duality; the latter proves a conjecture of Minhyong Kim.

INTRODUCTION

In [Kim, §10], Kim outlined an approach to interpreting Selmer groups as Lagrangian intersections of suitable moduli spaces, and proposed various generalisations. The main purpose of this paper is to provide the necessary foundations to make constructions of this sort precise in a derived geometric setting. This involves two substantial new pieces of theory: a characterisation of non-Archimedean derived analytic geometry well-suited to the study of ℓ -adic sheaves, and a theory of shifted symplectic structures which can handle the Tate twists featuring in arithmetic duality theorems.

Section 1 develops a theory of derived non-Archimedean analytic geometry built from differential graded enhancements of the dagger algebras of [GK], i.e. rings of overconvergent functions. These give rise to the same theory as that proposed in [Pri15] using rings with entire functional calculus. By Corollary 1.32, this theory is equivalent to the dagger-analytic analogue of the non-Archimedean derived analytic spaces from [PY1, Lur2], and agrees with that theory on spaces without boundary, despite involving much less data; it satisfies similar representability theorems. In §1.4, pro-étale sheaves are associated to derived dagger algebras using their natural topology, a construction which is hard to mimic in other models of analytic geometry. Section 2 then translates the theory of derived analytic symplectic geometry from [Pri15] to this dagger analytic context, where it takes a very natural form.

The principle underpinning Section 3 is that given a derived algebraic moduli functor F over \mathbb{Q}_ℓ or a similar field, and any scheme X , we can naturally form a derived analytic moduli functor of F -valued (hyper)sheaves on the pro-étale site of X . For instance, when $F = BGL_r$ this gives us a derived analytic moduli stack of locally free \mathbb{Q}_ℓ -sheaves of rank r . Moreover, if F is n -shifted symplectic¹ and X is smooth of dimension m over an algebraically closed field, then Poincaré duality endows the derived moduli stack of

¹for instance BGL_r is 2-shifted symplectic

suitable F -valued sheaves on X with an $(n - 2m)$ -shifted symplectic structure when X is proper (Examples 3.13), or an $(n + 1 - 2m)$ -shifted Lagrangian structure in general (Examples 3.21).

The algebraic closure hypothesis is necessary for the examples in §3 because of the Tate twists featuring in duality theorems, so the purpose of Section 4 is to develop a generalisation of the theory of shifted symplectic structures to address cases where the dualising bundle is non-trivial. The considerations introduced here work equally well in algebraic and analytic settings, applying to moduli of maps from spaces which are not Calabi–Yau but have a dualising line bundle, and this weighted theory can be thought of as a special case of the theory of \mathcal{P} -shifted symplectic structures from the seminal manuscript [BG]. The idea is to characterise the moduli space as the \mathbb{G}_m -invariant locus of a natural \mathbb{G}_m -equivariant formal thickening, with that thickening carrying a shifted symplectic structure of non-zero weight with respect to the \mathbb{G}_m -action. The resulting constructions have a similar flavour to Iwasawa theory, but with \mathbb{G}_m -actions rather than $\hat{\mathbb{Z}}^*$ -actions. Remark 5.14 describes local forms for weighted shifted symplectic spaces in terms of twisted shifted cotangent bundles, and §4.4 establishes weighted representability results.

Section 5 contains the main applications of the paper, constructing weighted shifted symplectic (Examples 5.13) and Lagrangian (Examples 5.18) structures on a wide range of derived analytic moduli stacks of pro-étale sheaves. Poincaré duality for smooth proper schemes and local Tate duality for local fields tends to give rise to symplectic structures for moduli of sheaves, while Poincaré duality for smooth schemes and Poitou–Tate duality for number fields give rise to Lagrangian structures. There are various ways to combine these, including Lagrangian intersections yielding the Selmer-type constructions envisaged by Kim (Example 5.18.(7)).

In Section 6, we set up the theory of shifted Poisson structures in the weighted setting. The main result is Theorem 6.8 and its generalisation in §6.6.2, extending the equivalence between shifted symplectic and non-degenerate shifted Poisson structures to incorporate weights (and even to handle \mathcal{P} -shifted structures in Remark 6.9). Quantisation results in the weighted setting are summarised in §6.5; the only modification needed to enable them is to give the formal parameter \hbar the same weight as the Poisson structure. As a consequence, the examples of Section 5 all give rise to weighted shifted Poisson structures, many of which (depending on shift and non-degeneracy) admit quantisations of various flavours.

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Notation and terminology. Throughout the paper, we will usually denote chain differentials by δ . The graded vector space underlying a chain (resp. cochain) complex V is denoted by $V_{\#}$ (resp. $V^{\#}$). When we have to work with chain and cochain structures separately, we denote shifts as subscripts and superscripts, respectively, so $(V_{[i]})_n := V_{i+n}$ and $(V^{[i]})^n := V^{i+n}$.

When we need to compare chain and cochain complexes, we silently make use of the equivalence u from chain complexes to cochain complexes given by $(uV)^i := V_{-i}$. On suspensions, this has the effect that $u(V_{[n]}) = (uV)^{[-n]}$; accordingly, we often write $V[n] := V^{[n]}$ for cochain complexes and $V[n] := V_{[-n]}$ for chain complexes when no confusion is likely.

Given an associative algebra A in chain complexes, and A -modules M, N in chain complexes, we write $\underline{\mathrm{Hom}}_A(M, N)$ for the cochain complex given by

$$\underline{\mathrm{Hom}}_A(M, N)^i = \mathrm{Hom}_{A_{\#}}(M_{\#[i]}, N_{\#}),$$

with differential $f \mapsto \delta_N \circ f \pm f \circ \delta_M$.

We refer to commutative algebras in chain complexes as CDGAs (i.e. commutative differential graded algebras); these are assumed unital unless stated otherwise.

Here and elsewhere, we use the symbol \pm to denote the sign in the total complex of a double complex, or in induced constructions such as tensor powers of, and monomial operations on, chain complexes, noting that internal tensor products are total complexes of external tensor products. The sign is determined by the property that \pm takes the value $+$ when all inputs have degree 0. The symbol \mp then denotes the opposite sign.

1. AFFINOID DAGGER DG SPACES

Let K be a field of characteristic 0, complete with respect to a non-Archimedean valuation. The non-Archimedean hypothesis is not strictly necessary, and in particular our statements all have complex analogues, which however mostly rely on different references.

We will be working with affinoid dagger spaces in the sense of [GK].

Definition 1.1. Given strictly positive real numbers r_1, \dots, r_n , recall that the Tate algebra $K\langle \frac{x_1}{r_1}, \dots, \frac{x_n}{r_n} \rangle$ is the ring of power series $\sum_{\nu \in \mathbb{N}_0^n} a_\nu x^\nu$ for $a_\nu \in K$ such that the set

$$\{|a_\nu| r_1^{\nu_1} \dots r_n^{\nu_n}\}_\nu$$

is bounded above.

This is a Banach algebra, with norm $\|\sum_\nu a_\nu x^\nu\| = \sup_\nu |a_\nu| r_1^{\nu_1} \dots r_n^{\nu_n}$.

The Tate algebra is thought of as analytic functions on the closed polydisc of radii (r_1, \dots, r_n) .

An affinoid algebra is then a quotient algebra $A = K\langle \frac{x_1}{r_1}, \dots, \frac{x_n}{r_n} \rangle / I$ for some (r_1, r_2, \dots, r_n) and some ideal I . The associated affinoid space $\mathrm{Sp}(A)$ is then the set of maximal ideals of A , equipped with the obvious structure sheaf on open affinoid subdomains.

As in [GK], this idea is adapted as follows to study overconvergent analytic functions on the closed polydisc.

Definition 1.2. Given strictly positive real numbers r_1, \dots, r_n , recall that the Washnitzer algebra $K\langle \frac{x_1}{r_1}, \dots, \frac{x_n}{r_n} \rangle^\dagger$ is the nested union

$$\bigcup_{\rho_i > r_i} K\langle \frac{x_1}{\rho_1}, \dots, \frac{x_n}{\rho_n} \rangle = \bigcup_{\rho_i > r_i} T(\rho_1, \dots, \rho_n)$$

of Tate algebras.

Explicitly, elements are power series $\sum_\nu a_\nu x^\nu$ for $a_\nu \in K$, such that

$$|a_\nu| \rho_1^{\nu_1} \dots \rho_n^{\nu_n} \xrightarrow{|\nu| \rightarrow \infty} 0$$

for some $\rho_i > r_i$.

A dagger algebra is then defined in [GK] to be a quotient algebra $A = K\langle \frac{x_1}{r_1}, \dots, \frac{x_n}{r_n} \rangle^\dagger / I$ for some $r_i > 0$ and some ideal I . The associated affinoid dagger space $\mathrm{Sp}(A)$ is then the set of maximal ideals of A , equipped with the obvious structure sheaf on open affinoid subdomains.

1.1. Definitions.

1.1.1. *Dagger dg affinoids.* The following is the natural generalisation to the dagger affinoid setting of the dg schemes of [Kon, CFK]:

Definition 1.3. Define an affinoid dagger dg space X over K to consist of an affinoid dagger space X^0 over K together with an \mathcal{O}_{X^0} -CDGA $\mathcal{O}_{X, \geq 0}$ in coherent sheaves on X^0 , with $\mathcal{O}_{X, 0} = \mathcal{O}_{X^0}$.

Explicitly, this means that we have coherent sheaves $\{\mathcal{O}_{X, i}\}_{i \geq 0}$, an associative graded-commutative product $\mathcal{O}_{X, i} \otimes_{\mathcal{O}_{X^0}} \mathcal{O}_{X, j} \xrightarrow{\sim} \mathcal{O}_{X, i+j}$, and a derivation $\delta : \mathcal{O}_{X, i} \rightarrow \mathcal{O}_{X, i-1}$ with $\delta \circ \delta = 0$.

A morphism $f: X \rightarrow Y$ of affinoid dagger dg spaces consists of a morphism $f^0: X^0 \rightarrow Y^0$ of affinoid dagger spaces, together with a morphism $f^\sharp: (f^0)^* \mathcal{O}_Y \rightarrow \mathcal{O}_X$ of CDGAs in coherent sheaves on X^0 .

We define a dagger dg algebra A to be a K -algebra of the form $\Gamma(X^0, \mathcal{O}_X)$ for X an affinoid dagger dg space. Equivalently, this is a K -CDGA $(A_{\geq 0}, \delta)$ such that A_0 is a dagger algebra and the A_0 -modules A_m are all finite.

Note that the functor $X \mapsto \Gamma(X^0, \mathcal{O}_X)$ gives a contravariant equivalence between affinoid dagger dg spaces and dagger dg algebras.

Definition 1.4. Given an affinoid dagger dg space X , we define the underived truncation $\pi^0 X \subset X^0$ to be the closed affinoid subspace defined by the ideal sheaf $\delta \mathcal{O}_{X,1} \subset \mathcal{O}_{X,0}$.

Remark 1.5. Although the corresponding construction for dg schemes is denoted by $\pi_0 X$ in [CFK], we use the notation $\pi^0 X$ to avoid confusion with path components of the simplicial constructions we will see later, and because superscripts are more appropriate than subscripts to denote equalisers.

Definition 1.6. We say that a morphism $f: X \rightarrow Y$ of affinoid dagger dg spaces is a quasi-isomorphism if it induces an isomorphism $\pi^0 f: \pi^0 X \rightarrow \pi^0 Y$ on underived truncations together with isomorphisms $f^{-1} \mathcal{H}_n(\mathcal{O}_Y) \cong \mathcal{H}_n(\mathcal{O}_X)$ of sheaves on X^0 .

The category of coherent sheaves on a dagger affinoid $\mathrm{Sp}(B)$ is equivalent to the category of finite B -modules via the global sections functor, by [GK, Theorem 2.18 and Proposition 3.1]. We may therefore rephrase Definition 1.3 as saying that an affinoid dagger dg space consists of a K -CDGA $A_{\geq 0}$ with A_0 a dagger algebra (i.e. a quotient of some Washnitzer algebra $K\langle x_1, \dots, x_n \rangle^\dagger$ of overconvergent functions on a closed polydisc). Morphisms are then just CDGA morphisms, and quasi-isomorphisms of affinoid dagger dg spaces just correspond to quasi-isomorphisms of these K -CDGAs.

Definition 1.7. Say that a dagger dg algebra A is quasi-free if A_0 is isomorphic to a Washnitzer algebra $K\langle x_1, \dots, x_n \rangle^\dagger$ for some n , and A is freely generated as a graded-commutative A_0 -algebra.

Say that a morphism $f: R \rightarrow A$ of dagger dg algebras is quasi-free if A_0 is isomorphic over R_0 to a relative Washnitzer algebra $R_0\langle x_1, \dots, x_n \rangle^\dagger$, and if A is freely generated as a graded-commutative algebra over $A_0 \otimes_{R_0} R$.

Here, $R_0\langle x_1, \dots, x_n \rangle^\dagger$ is the ring of overconvergent functions on the relative polydisc $\mathrm{Sp}(R_0) \times \mathrm{Sp}(K\langle x_1, \dots, x_n \rangle^\dagger)$, given by $K\langle y_1, \dots, y_m, x_1, \dots, x_n \rangle^\dagger / (I)$ when $R_0 = K\langle y_1, \dots, y_m \rangle^\dagger / I$.

Lemma 1.8. *Every morphism $f: A \rightarrow B$ of dagger dg algebras admits a factorisation $A \xrightarrow{p} \check{B} \xrightarrow{r} B$ with p quasi-free and r a surjective quasi-isomorphism, and a factorisation $A \xrightarrow{q} \check{B} \xrightarrow{s} B$ with q a quasi-free quasi-isomorphism and s surjective in strictly positive degrees.*

Proof. Since B_0 is a dagger algebra, it admits a surjection $K\langle x_1, \dots, x_n \rangle^\dagger \twoheadrightarrow B_0$ for some n . Combined with f_0 , this gives a morphism $r_0: A_0\langle x_1, \dots, x_n \rangle^\dagger \twoheadrightarrow B_0$, which is automatically also surjective, so we set $\check{B}_0 := A_0\langle x_1, \dots, x_n \rangle^\dagger$, with the obvious map $p_0: A_0 \rightarrow \check{B}_0$.

We thus have a morphism $A \otimes_{A_0} \check{B}_0 \rightarrow B$ of CDGAs in coherent \check{B}_0 -modules, which is surjective in degree 0. The standard construction of a cofibration-trivial fibration

factorisation of CDGAs then allows us to factorise this as

$$A \otimes_{A_0} \check{B}_0 \xrightarrow{p'} \check{B} \xrightarrow{r} B$$

for some CDGA \check{B} extending \check{B}_0 , such that p' is quasi-free and r a surjective quasi-isomorphism.

Noetherianity of \check{B}_0 ensures that we can (non-functorially) choose finitely many generators in each degree for \check{B} over $A \otimes_{A_0} \check{B}_0$, since every module arising in the inductive construction of a resolution is finite over \check{B}_0 . This choice ensures that the \check{B}_0 -modules \check{B}_m are finite, so correspond to coherent sheaves, meaning that \check{B} is indeed a dagger dg algebra. The construction has ensured that p and r are of the desired form.

For the other factorisation, just take the standard construction of a trivial cofibration-fibration factorisation of CDGAs $A \rightarrow \check{B} \rightarrow B$, but with free dagger algebras rather than polynomials in degree 0. \square

1.1.2. *Localised dg dagger affinoids.* Now, in common with dg schemes, a problem with affinoid dagger dg spaces (X^0, \mathcal{O}_X) is that the space X^0 has no geometric significance and tends to get in the way. For instance, for any open affinoid subspace $U \subset X^0$ containing $\pi^0 X$, the map $(U, \mathcal{O}_X|_U) \rightarrow (X^0, \mathcal{O}_X)$ is a quasi-isomorphism. As a consequence, the category of affinoid dagger dg spaces has too few morphisms, even after localising at quasi-isomorphisms. Accordingly, we now introduce a localised version.

Definition 1.9. Given non-negative real numbers r_1, \dots, r_n , define the quasi-Washnitzer algebra $K\langle \frac{x_1}{r_1}, \dots, \frac{x_n}{r_n} \rangle^\dagger$ to be the nested union

$$\bigcup_{\rho_i > r_i} K\langle \frac{x_1}{\rho_1}, \dots, \frac{x_n}{\rho_n} \rangle = \bigcup_{\rho_i > r_i} T(\rho_1, \dots, \rho_n)$$

of Tate algebras.

Explicitly, elements are power series $\sum_\nu a_\nu x^\nu$ for $a_\nu \in K$, such that

$$|a_\nu| \rho_1^{\nu_1} \dots \rho_n^{\nu_n} \xrightarrow{|\nu| \rightarrow \infty} 0$$

for some $\rho_i > r_i$.

Note that we are allowing the numbers r_i to be 0. In particular, we allow the algebra $K\langle x_1, \dots, x_m, \frac{x_{m+1}}{0}, \dots, \frac{x_n}{0} \rangle^\dagger$, which can be thought of as $\Gamma(\mathbb{D}^m, i^{-1}\mathcal{O}_{\mathbb{D}^n})$ for the inclusion $i: \mathbb{D}^m \hookrightarrow \mathbb{D}^n$ of polydiscs, so we are looking at germs of overconvergent functions on the dagger space \mathbb{D}^n restricted to \mathbb{D}^m .

Although only stated for the Washnitzer algebra (the case $r_i = 1$ for all i), the proofs of [Gün, §2] (which adapt Rückert's Basis Theorem and reduce to Tate algebras) apply for any non-negative set of radii, so $K\langle \frac{x_1}{r_1}, \dots, \frac{x_n}{r_n} \rangle^\dagger$ is a Noetherian factorial Jacobson ring, even when some of the r_i are 0.

Definition 1.10. Define a quasi-dagger algebra to be a quotient algebra $A = K\langle \frac{x_1}{r_1}, \dots, \frac{x_n}{r_n} \rangle^\dagger / I$ for some $r_i \geq 0$. Define the associated affinoid quasi-dagger space $\mathrm{Sp}(A)$ to be the set of maximal ideals of A , equipped with the obvious structure sheaf on open localised affinoid subdomains.

The category of quasi-dagger algebras is then defined by letting morphisms be all K -algebra homomorphisms between quasi-dagger algebras, and the category of affinoid quasi-dagger spaces is it opposite.

In other words, affinoid quasi-dagger spaces (X, \mathcal{O}_X) are ringed spaces of the form $(\bar{X}, i^{-1}\mathcal{O}_Y)$ for closed immersions $i: \bar{X} \rightarrow Y$ of affinoid dagger spaces.

Definition 1.11. Given a quasi-dagger algebra $A = K\langle \frac{x_1}{r_1}, \dots, \frac{x_n}{r_n} \rangle^\dagger / I$, we define the affinoid algebra \bar{A} to be the completion of A with respect to the quotient semi-norm on A induced by the norm

$$\| \sum a_{j_1, \dots, j_n} x_1^{j_1} \dots x_n^{j_n} \|_{\underline{r}} := \sup_{j_1, \dots, j_n} |a_{j_1, \dots, j_n}| r_1^{j_1} \dots r_n^{j_n}$$

on $K\langle \frac{x_1}{r_1}, \dots, \frac{x_n}{r_n} \rangle$.

Note that in contrast to the situation for dagger algebras in [GK, §1], the completion map $A \rightarrow \bar{A}$ will not be injective if any of the radii r_i are 0.

Proposition 1.12. *There is a natural fully faithful functor from the category of quasi-dagger algebras to the ind-category of affinoid algebras.*

Proof. Since the algebras $K\langle \frac{x_1}{r_1}, \dots, \frac{x_n}{r_n} \rangle^\dagger$ are all Noetherian, any quasi-dagger algebra A can be expressed as a quotient of the form $K\langle \frac{x_1}{r_1}, \dots, \frac{x_n}{r_n} \rangle^\dagger / (f_1, \dots, f_m)$. The elements f_i must all lie in $K\langle \frac{x_1}{\rho_1}, \dots, \frac{x_n}{\rho_n} \rangle$ for some $\underline{\rho} > \underline{r}$, giving rise to a direct system

$$\{ K\langle \frac{x_1}{\rho'_1}, \dots, \frac{x_n}{\rho'_n} \rangle / (f_1, \dots, f_m) \}_{\underline{\rho} \geq \underline{\rho}' > \underline{r};}$$

this is the ind-affinoid algebra to which our functor G sends A .

We now need to establish natural isomorphisms $\mathrm{Hom}_{\mathrm{Alg}_K}(A, B) \cong \mathrm{Hom}_{\mathrm{ind}(\mathrm{Alg}_K)}(GA, GB)$; this will make G a fully faithful functor, and ensure that the choices in the previous paragraph did not matter up to isomorphism. Writing $A(\underline{\rho}') := K\langle \frac{x_1}{\rho'_1}, \dots, \frac{x_n}{\rho'_n} \rangle / (f_1, \dots, f_m)$ and similarly for B , this means that we need a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Alg}_K}(A, B) \cong \varprojlim_{\underline{\rho} \geq \underline{\rho}' > \underline{r}} \varinjlim_{\underline{\sigma} \geq \underline{\sigma}' > \underline{s}} \mathrm{Hom}_{\mathrm{Alg}_K}(A(\underline{\rho}'), B(\underline{\sigma}')).$$

Since $A \cong \varinjlim_{\underline{\rho} \geq \underline{\rho}' > \underline{r}} A(\underline{\rho}')$, we already have

$$\mathrm{Hom}_{\mathrm{Alg}_K}(A, B) \cong \varprojlim_{\underline{\rho} \geq \underline{\rho}' > \underline{r}} \mathrm{Hom}_{\mathrm{Alg}_K}(A(\underline{\rho}'), B),$$

so it suffices to prove that for any affinoid algebra C , the natural map

$$\varinjlim_{\underline{\sigma} \geq \underline{\sigma}' > \underline{s}} \mathrm{Hom}_{\mathrm{Alg}_K}(C, B(\underline{\sigma}')) \rightarrow \mathrm{Hom}_{\mathrm{Alg}_K}(C, B)$$

is an isomorphism.

We now adapt the proof of [GK, Lemma 1.8]. As in the proof of [BGR, 6.1.5/4], C is topologically generated by a finite set T of topological nilpotents. Given a K -algebra homomorphism $\phi: C \rightarrow B$, we can look at the induced homomorphism $\bar{\phi}: C \rightarrow \bar{B}$, for the completion \bar{B} of Definition 1.11; this is necessarily continuous, since it is a homomorphism of affinoid algebras. The elements $\bar{\phi}(t)$ for $t \in T$ are topologically nilpotent and hence power-bounded (in the sense that $\{t^n : n \in \mathbb{N}\}$ is a bounded set); since \bar{B} is affinoid, [BGR, 6.2.3/1] then implies that $\|\bar{\phi}(t)\|_{\underline{s}} \leq 1$ for all $t \in T$, with topological nilpotence implying that for some n we have $\|\bar{\phi}(t)^n\|_{\underline{s}} < 1$ for all $t \in T$.

Since $B = \varinjlim_{\underline{\sigma}' > \underline{s}} B(\underline{\sigma}')$, there exists some $\underline{\sigma}'$ for which the elements $\phi(t) \in B$ all lift to elements in $\widetilde{\phi(t)} \in B(\underline{\sigma}')$. Now, since $\|b\|_{\underline{s}} = \lim_{\underline{\sigma} \rightarrow \underline{s}} \|b\|_{\underline{\sigma}}$ for all $b \in B(\underline{\sigma}')$, we deduce

that for some $\underline{\sigma}' \geq \underline{\sigma}'' > \underline{s}$, we have $\|\widetilde{\phi}(t)\|_{\sigma''}^i < 1$ for all $t \in T$ and $n \leq i < 2n$. The images in $B(\underline{\sigma}'')$ of the elements $\widetilde{\phi}(t)$ are thus topologically nilpotent for all $t \in T$. In particular, they are power-bounded so must have $\|\widetilde{\phi}(t)\|_{\sigma''} \leq 1$, by [BGR, 6.2.3/1]. We have thus constructed a commutative diagram

$$\begin{array}{ccc} K\langle T \rangle & \xrightarrow{\tilde{\phi}} & B(\underline{\sigma}'') \\ \downarrow & & \downarrow \\ C & \xrightarrow{\phi} & B \end{array}$$

Since $K\langle T \rangle$ is Noetherian, the kernel of $K\langle T \rangle \rightarrow C$ is finitely generated, by a set U , say. The image of $\tilde{\phi}(U)$ in B is 0; since U is finite, this means that the image must be 0 in $B(\underline{\sigma}''')$ for some $\underline{\sigma}'' \geq \underline{\sigma}''' > \underline{s}$. That gives us a homomorphism $C \rightarrow B(\underline{\sigma}''')$ lifting ϕ , thus establishing surjectivity.

To see that the map is injective, consider another choice $\widehat{\phi}(t) \in B(\underline{\sigma}')$ of lift of $\phi(t)$ for each t . Since these have the same image in B , there must be some $\underline{\sigma}' \geq \underline{\sigma}^* > \underline{s}$ for which $\widehat{\phi}(t)$ and $\widetilde{\phi}(t)$ have the same image in $B(\underline{\sigma}^*)$; the set T being finite, we can choose the same $\underline{\sigma}^*$ for all T . Our element of $\varinjlim_{\underline{\sigma} \geq \underline{\sigma}^* > \underline{s}} \text{Hom}_{\text{Alg}_K}(C, B(\underline{\sigma}'))$ is thus unchanged by this choice, establishing injectivity. \square

Definition 1.13. Define an affinoid quasi-dagger dg space X over K to consist of an affinoid quasi-dagger space X^0 over K together with an \mathcal{O}_{X^0} -CDGA $\mathcal{O}_{X, \geq 0}$ in coherent sheaves on X^0 , with $\mathcal{O}_{X,0} = \mathcal{O}_{X^0}$. We then define an affinoid quasi-dagger dg algebra A over K to be a dg K -algebra of the form $\Gamma(X^0, \mathcal{O}_X)$ for an affinoid quasi-dagger dg space X ; this is equivalent to saying that A_0 is a quasi-dagger algebra and the A_0 -modules A_m are all finite.

We say that an affinoid quasi-dagger dg space X is a localised affinoid dagger dg space if the vanishing locus $\pi^0 X$ of δ is dagger affinoid and the closed immersion $i: \pi^0 X \rightarrow X^0$ gives an isomorphism on the underlying sets of points. We then define a localised dagger dg algebra A over K to be a dg K -algebra of the form $\Gamma(X^0, \mathcal{O}_X)$ for a localised affinoid dagger dg space X .

A morphism $f: X \rightarrow Y$ of affinoid quasi-dagger dg spaces consists of a morphism $f^0: X^0 \rightarrow Y^0$ of affinoid quasi-dagger spaces, together with a morphism $f^\#: (f^0)^* \mathcal{O}_Y \rightarrow \mathcal{O}_X$ of CDGAs in coherent sheaves on X^0 . Equivalently a morphism $A \rightarrow B$ of quasi-dagger dg algebras is just a homomorphism of dg K -algebras.

We denote the category of localised dagger dg algebras by $dg_+ \text{AffdAlg}_K^{\text{loc}, \dagger}$.

Definition 1.14. Given an affinoid dagger dg space X , we define the associated localised affinoid dagger dg space X_{loc} as follows. The affinoid quasi-dagger space X_{loc}^0 consists of the topological space underlying $\pi^0 X$ equipped with the sheaf $i^{-1} \mathcal{O}_{X^0}$, for the closed immersion $i: \pi^0 X \rightarrow X^0$. The sheaf $\mathcal{O}_{X_{\text{loc}}}$ of CDGAs on X_{loc}^0 is then given by $i^{-1} \mathcal{O}_X$. If $A = \Gamma(X^0, \mathcal{O}_X)$, we then write A^{loc} for the localised dagger dg algebra $\Gamma(\pi^0 X, i^{-1} \mathcal{O}_X)$.

Note that Noetherianity ensures that every localised affinoid dagger dg space is isomorphic to X_{loc} for some affinoid dagger dg space X , since the ideal defining $\pi^0 X \subset X^0$ must have finitely many generators, which converge on some larger polydisc.

Definition 1.15. We say that a morphism $f: X \rightarrow Y$ of affinoid quasi-dagger dg spaces is a quasi-isomorphism if it induces an isomorphism $\pi^0 f: \pi^0 X \rightarrow \pi^0 Y$ on underived truncations together with isomorphisms $f^{-1} \mathcal{H}_n(\mathcal{O}_Y) \cong \mathcal{H}_n(\mathcal{O}_X)$ of sheaves on X^0 .

The following is an immediate consequence of the sheaves $\mathcal{H}_n(\mathcal{O}_X)$ being supported on $\pi^0 X$

Lemma 1.16. *For any affinoid dagger dg space X , the canonical map $X_{\text{loc}} \rightarrow X$ is a quasi-isomorphism.*

The definitions of quasi-free morphisms and strictly closed immersions now adapt to affinoid quasi-dagger dg spaces in the obvious way, and the analogue of Lemma 1.8 also holds, since quasi-dagger algebras are Noetherian.

1.2. Derived mapping spaces and comparison with dg EFC algebras. We now recall some definitions from [Pri15], based on [CR].

Definition 1.17. Define a K -algebra A with entire functional calculus (or EFC K -algebra for short) to be a product-preserving set-valued functor $\mathbb{A}_K^n \mapsto A^n$ on the full subcategory of rigid analytic varieties with objects the affine spaces $\{\mathbb{A}_K^n\}_{n \geq 0}$.

Thus an EFC K -algebra A is a commutative K -algebra equipped with a systematic and consistent way of evaluating expressions of the form

$$\sum_{m_1, \dots, m_n=0}^{\infty} \lambda_{m_1, \dots, m_n} a_1^{m_1} \cdots a_n^{m_n}$$

in A whenever the coefficients $\lambda_{m_1, \dots, m_n} \in K$ satisfy

$$\lim_{\sum m_i \rightarrow \infty} |\lambda_{m_1, \dots, m_n}|^{1/\sum m_i} = 0.$$

Examples of EFC K -algebras include rings of functions on rigid K -analytic spaces, and any K -algebra colimits of such rings (in particular, rings of functions on dagger K -analytic spaces arise in this way). By [Pri15, Proposition 1.24], taking rings of functions gives a contravariant equivalence of categories between globally finitely presented Stein spaces over K and finitely presented EFC K -algebras. Since the EFC monad preserves filtered colimits, this also gives a contravariant equivalence between EFC K -algebras and the pro-category of globally finitely presented Stein spaces.

Definition 1.18. Define a non-negatively graded EFC-differential graded K -algebra (EFC-DGA for short) to be a chain complex $A_{\bullet} = (A_{\geq 0}, \delta)$ of K -vector spaces equipped with:

- an associative graded multiplication, graded-commutative in the sense that $ab = (-1)^{\bar{a}\bar{b}}ba$ for all $a, b \in A$, where \bar{a} is the parity of a (i.e. the degree modulo 2), and
- an enhancement of the K -algebra structure on A_0 to an EFC K -algebra structure,

such that δ is a graded derivation in the sense that $\delta(ab) = \delta(a)b + (-1)^{\bar{a}}a\delta(b)$ for all $a, b \in A$.

Proposition 1.19. *There is a natural fully faithful functor from quasi-dagger algebras A to EFC algebras.*

Proof. By Proposition 1.12, A naturally has the structure of an ind-affinoid algebra, and hence an ind-Banach algebra. Since every Banach algebra carries an entire functional calculus, this gives A_0 an EFC-algebra structure. Morphisms automatically preserve the EFC algebra structure, since morphisms of affinoid algebras are automatically continuous.

In other words, Proposition 1.12 gives us a fully faithful functor from quasi-dagger algebras A to ind-affinoid algebras $\{A(\underline{\rho})\}_{\underline{\rho}>\underline{r}}$, and there is a natural fully faithful functor from affinoid algebras to EFC algebras, so we have a fully faithful composite functor from quasi-dagger algebras to ind-EFC algebras, and hence a functor to EFC algebras by taking colimits. We therefore need to show that for such algebras A, B , the natural map

$$\mathrm{Hom}_{\mathrm{ind}(EFC)}(\{A(\underline{\rho})\}_{\underline{\rho}>\underline{r}}, \{B(\underline{\sigma})\}_{\underline{\sigma}>\underline{s}}) \rightarrow \mathrm{Hom}_{EFC}(\varinjlim_{\underline{\rho}>\underline{r}} A(\underline{\rho}), \varinjlim_{\underline{\sigma}>\underline{s}} B(\underline{\sigma})),$$

is an isomorphism, where the left-hand side expands out as

$$\varinjlim_{\underline{\rho}>\underline{r}} \varinjlim_{\underline{\sigma}>\underline{s}} \mathrm{Hom}_{EFC}(A(\underline{\rho}), B(\underline{\sigma})).$$

Since affinoid algebras are not obviously finitely presented as EFC algebras, we proceed by reducing to Stein algebras. Instead of regarding the Washnitzer algebra $K\langle\frac{x_1}{r_1}, \dots, \frac{x_n}{r_n}\rangle^\dagger$ as the nested union $\bigcup_{\rho_i>r_i} K\langle\frac{x_1}{\rho_1}, \dots, \frac{x_n}{\rho_n}\rangle$ of Tate algebras, we can look at it as the nested union $\bigcup_{\rho_i>r_i} O(\Delta(\rho_1, \dots, \rho_n))$ of Stein algebras, where $O(\Delta(\rho_1, \dots, \rho_n)) := \varprojlim_{\varrho_i<\rho_i} K\langle\frac{x_1}{\varrho_1}, \dots, \frac{x_n}{\varrho_n}\rangle$ is the ring of analytic functions on the open polydisc $\Delta(\rho_1, \dots, \rho_n)$ with radii ρ_i . Note that for $\underline{\varrho} < \underline{\rho}$ we have natural maps

$$K\langle\frac{x_1}{\rho_1}, \dots, \frac{x_n}{\rho_n}\rangle \rightarrow O(\Delta(\rho_1, \dots, \rho_n)) \rightarrow K\langle\frac{x_1}{\varrho_1}, \dots, \frac{x_n}{\varrho_n}\rangle,$$

so the direct systems $\{K\langle\frac{x_1}{\rho_1}, \dots, \frac{x_n}{\rho_n}\rangle\}_{\underline{\rho}>\underline{r}}$ and $\{O(\Delta(\rho_1, \dots, \rho_n))\}_{\underline{\rho}>\underline{r}}$ are isomorphic as ind-algebras.

Setting $A(\underline{\rho}') := A(\underline{\rho}) \otimes_{K\langle\frac{x_1}{\rho_1}, \dots, \frac{x_n}{\rho_n}\rangle} O(\Delta(\rho_1, \dots, \rho_n))$, we then have an ind-EFC algebra isomorphism $\{A(\underline{\rho}')\}_{\underline{\rho}>\underline{r}} \cong \{A(\underline{\rho})\}_{\underline{\rho}>\underline{r}}$. Since $A(\underline{\rho}')$ is a globally finitely presented Stein algebra, it is of finite presentation as an EFC-algebra by [Pri15, Proposition 1.24], so

$$\mathrm{Hom}_{EFC}(A(\underline{\rho}'), \varinjlim_{\underline{\sigma}>\underline{s}} B(\underline{\sigma})) \cong \varinjlim_{\underline{\sigma}>\underline{s}} \mathrm{Hom}_{EFC}(A(\underline{\rho}'), B(\underline{\sigma})),$$

and passing to limits completes the proof. \square

Applying Proposition 1.19 in degree 0 gives:

Corollary 1.20. *There is a natural fully faithful functor from quasi-dagger dg algebras to EFC-DGAs.*

We now set about establishing similar statements for the corresponding ∞ -categories given by inverting quasi-isomorphisms. The first subtlety we encounter is that for the standard model structure of [Pri15, Proposition 2.13], none of the objects with which we work is cofibrant. However, there is a Quillen-equivalent model structure which resolves this, constructed as follows.

Definition 1.21. Given a morphism $f: A \rightarrow B$ of EFC-algebras, as a special case of [Pri15, Definition 3.5] define the localisation $(A/B)^{\mathrm{loc}}$ of A along B as follows. If

A and B are finitely presented, then f takes the form $O(V) \rightarrow O(U)$ for a morphism $g: U \rightarrow V$ of globally finitely presented Stein spaces, and we set

$$(A/B)^{\text{loc}} := \Gamma(U, g^{-1}\mathcal{O}_V).$$

For the general case, we write the morphism as a filtered colimit of morphisms $f(i): A(i) \rightarrow B(i)$ of finitely presented EFC-algebras, indexed by some poset I , and then set $(A/B)^{\text{loc}} := \varinjlim_{i \in I} (A(i)/B(i))^{\text{loc}}$.

Remark 1.22. For a localised dagger dg algebra A , observe that by definition we have $A_0 \cong (A_0/H_0A)^{\text{loc}}$, identifying A with the underlying EFC-DGA given by Corollary 1.20.

The following is [Pri15, Proposition 3.12] specialised to our setting; the final statements follow because localised dagger affinoid spaces are constructed as inverse limits of open Stein subspaces of affine space, regarding the closed dagger polydisc \mathbb{D}^n as a limit of open discs.

Proposition 1.23. *There is a cofibrantly generated model structure (the local model structure) on the category of those non-negatively graded EFC-DGAs A_\bullet with $A_0 \cong (A_0/H_0A)^{\text{loc}}$, in which weak equivalence are quasi-isomorphisms and fibrations are surjective in strictly positive chain degrees. The inclusion functor to the category of all non-negatively graded EFC-DGAs is then a right Quillen equivalence.*

For any open immersion $U \rightarrow V$ of Stein spaces, the corresponding morphism of Stein algebras is a cofibration in this model structure, as are transfinite compositions of such, and compositions of these with quasi-free morphisms of EFC-DGAs. In particular, any localisation of a quasi-free dagger dg algebra is cofibrant in this model structure.

With respect to the model structure of Proposition 1.23, Lemma 1.8 ensures that we have cofibrant replacement within the subcategory of EFC-DGAs associated to localised dagger dg algebras. Moreover, all objects in that model structure are fibrant.

Lemma 1.24. *Given a small category \mathcal{C} and a subcategory \mathcal{W} , take an object $A \in \mathcal{C}$ and assume that we have an augmented cosimplicial diagram $\tilde{A}^\bullet \rightarrow A$ in \mathcal{C} such that*

- (1) *the morphisms $\tilde{A}^m \rightarrow A$ all lie in \mathcal{W} , and*
- (2) *for any morphism $B \rightarrow C$ in \mathcal{W} , the map of simplicial sets given in level m by*

$$\text{Hom}_{\mathcal{C}}(\tilde{A}^m, B) \rightarrow \text{Hom}_{\mathcal{C}}(\tilde{A}^m, C)$$

is a weak equivalence.

Then in the simplicial localisation $L_{\mathcal{W}}(\mathcal{C})$ of \mathcal{C} at \mathcal{W} , the simplicial set-valued functor

$$\underline{\text{Hom}}_{L_{\mathcal{W}}(\mathcal{C})}(A, -)$$

is weakly equivalent to $m \mapsto \text{Hom}_{\mathcal{C}}(\tilde{A}^m, B)$.

Proof. We can consider the model category of restricted diagrams from [TV1, §2.3.2], applied to our category, so objects are functors from \mathcal{C} to simplicial sets, and fibrant objects are objectwise fibrant functors which send morphisms in \mathcal{W} to weak equivalences. Writing $h_A := \text{Hom}_{\mathcal{C}}(A, -)$, in that model structure the morphisms $h_A \rightarrow h_{\tilde{A}^m}$ are all weak equivalences, since the maps $\tilde{A}^m \rightarrow A$ are in \mathcal{W} . The map $h_A \rightarrow \text{holim}_{m \in \Delta} h_{\tilde{A}^m}$ is thus a weak equivalence, but the latter is just the functor $H_{\tilde{A}}$ sending B to the simplicial set $m \mapsto \text{Hom}_{\mathcal{C}}(\tilde{A}^m, B)$.

By hypothesis, $H_{\tilde{A}}$ sends morphisms in \mathcal{W} to weak equivalences, so taking objectwise fibrant replacement gives us a weakly equivalent functor $H'_{\tilde{A}}$ which is fibrant in the model category of restricted diagrams, and $h_A \rightarrow H'_{\tilde{A}}$ is fibrant replacement. By [DK1], as interpreted in [TV1, Theorem 2.3.5], this means that $H_{\tilde{A}}(B)$ is weakly equivalent to the space of maps from A to B in $L_{\mathcal{W}}(\mathcal{C})$. \square

Proposition 1.25. *The functor from localised dagger dg algebras to EFC-DGAs given by Corollary 1.20 induces a fully faithful functor on simplicial categories after simplicial localisation at quasi-isomorphisms, as does its restriction to quasi-free localised dagger dg algebras.*

Proof. This effectively follows with the same reasoning as [DK2], since Lemma 1.8 means that localised dagger dg algebras come close to being a model subcategory of the model category in Proposition 1.23. However, since it is not closed under finite limits, we now give the details.

Given a dagger dg algebra A , repeated application of Lemma 1.8 gives us a quasi-free cosimplicial resolution \tilde{A}^\bullet in the following sense, using Reedy category techniques as in [Hov, §5]. Firstly, each \tilde{A}^n is a quasi-free dagger dg algebra, and moreover the latching maps $(\partial^0, \partial^1): \tilde{A}^0 \hat{\otimes}^\pi \tilde{A}^0 \rightarrow \tilde{A}^1$ etc. are quasi-free morphisms of dagger dg algebras, where $\hat{\otimes}^\pi$ is the completed projective tensor product (corresponding to product of affinoid dagger spaces). Secondly, the degeneracy maps are all quasi-isomorphisms and we have a quasi-isomorphism $\tilde{A}^0 \rightarrow A$.

Now, any localised dagger dg algebra A' is of the form A'^{loc} for some dagger dg algebra A , in the notation of Definition 1.14, with underlying EFC algebra $A' \cong A \otimes_{A_0} (A_0/H_0A)^{\text{loc}}$. The construction above then gives us a quasi-free cosimplicial resolution \tilde{A}'^\bullet of A' by setting $\tilde{A}'^m := (\tilde{A}^m)^{\text{loc}}$. On the underlying EFC algebras, this is a Reedy cofibrant cosimplicial resolution with respect to the model structure of Proposition 1.23. In particular, for any EFC-DGA B which is fibrant in that model structure, the space of maps $\mathbf{Rmap}_{EFC,DG}(A, B)$ is weakly equivalent to the simplicial set $m \mapsto \text{Hom}_{EFC,DG}(\tilde{A}'^m, B)$.

Since Corollary 1.20 gives a fully faithful functor from quasi-dagger dg algebras to EFC-DGAs, we have $\text{Hom}_{\text{Alg}_K}(\tilde{A}'^m, C) \cong \text{Hom}_{DG,EFC}(\tilde{A}'^m, C)$ for all quasi-dagger dg algebras C . If C is a localised dagger dg algebra, its underlying EFC-DGA is fibrant in the model structure of Proposition 1.23. Since A' is a cofibrant cosimplicial resolution, we deduce that the functor $H_{\tilde{A}'}(-)$ sending C to the simplicial set $m \mapsto \text{Hom}_{DGA\text{Alg}_K}(\tilde{A}'^m, C)$ is a model for $\mathbf{Rmap}_{EFC,DG}(A, -)$, and in particular sends quasi-isomorphisms in $dg_+\text{AffdAlg}_K^{\text{loc},\dagger}$ to weak equivalences. We may therefore appeal to Lemma 1.24, from which it follows that $H_{\tilde{A}'}(-)$ is also a model for the mapping space $\mathbf{Rmap}_{dg_+\text{AffdAlg}_K^{\text{loc},\dagger}}(A, -)$.

Finally, since the objects of \tilde{A} are all quasi-free, exactly the same reasoning applies to the category of restricted diagrams from quasi-free localised dagger dg algebras to simplicial sets. This means that for A and B quasi-free, $H_{\tilde{A}}(B)$ is also weakly equivalent to the space of maps from A to B in simplicial localisation at quasi-isomorphisms of the category of quasi-free localised dagger dg algebras. \square

Proposition 1.26. *Under the fully faithful functor from localised dagger dg algebras to EFC-DGAs in Corollary 1.20 and Proposition 1.25, the essential image consists of*

those EFC-DGAs A_\bullet for which H_0A is a dagger algebra and the H_0A -modules H_mA are all finite.

Proof. For any localised dagger dg algebra C , we have that H_0C is a dagger algebra, and the modules H_mB are finite, since coherent.

Given an EFC-DGA A satisfying the conditions above, it thus suffices to construct a dagger dg algebra C quasi-isomorphic to A , since we can then localise C at H_0A to give a quasi-isomorphic localised dagger dg algebra. We begin by replacing A with the quasi-isomorphic EFC-DGA $A \otimes_{A_0} (A_0/H_0A)^{\text{loc}}$.

Now, since H_0A is a dagger algebra, there exists a quasi-free dagger algebra $C(0)$ mapping surjectively to H_0A , and a surjection $C(0) \rightarrow H_0A$ of EFC algebras. Since we have localised A and $C(0)$ is quasi-free, this lifts to a morphism $f(0): C(0) \rightarrow A_0$ of EFC algebras, and hence to a map $C(0) \rightarrow A$ of EFC-DGAs which is surjective on H_0 .

Now assume inductively that we have constructed a sequence $C(0) \rightarrow C(1) \rightarrow \dots \rightarrow C(n)$ of quasi-free morphisms dagger dg algebras, with $C(i)$ generated over $C(i-1)$ by generators in degree i . Also assume that we have a morphism $f(n): C(n) \rightarrow A$ which gives isomorphisms on $H_{<n}$ and is surjective on H_n ; this hypotheses amounts to saying that $H_i \text{cone}(f(n)) = 0$ for all $i \leq n$. Since $H_{n+1}A$ is a finite H_0A -module and $C(n)$ is a complex of finite modules over the Noetherian ring $C(0) = C(n)_0$, it follows that $H_{n+1} \text{cone}(f(n))$ is a finite $C(0)$ -module. We can therefore pick a finite set S of generators and lift them to $Z_{n+1} \text{cone}(f(n))$, giving us a map

$$(\delta, f(n+1)): S \rightarrow \{(a, b) \in C(n)_n \times A_{n+1} : \delta a = 0, f(n)(a) = \delta b\},$$

and hence, placing S in degree $n+1$ a map $C(n+1) := (C(n)[S], \delta) \rightarrow A$ satisfying the hypotheses, which completes the inductive step.

The dagger dg algebra $C := \bigcup_n C(n)$ then has finitely many generators in each level, so is levelwise finitely generated over $C_0 = C(0)$, making it a quasi-free localised dagger dg algebra, and it is equipped with a quasi-isomorphism $C \rightarrow A$ of EFC-DGAs. \square

The significance of Propositions 1.25 and 1.26 is that we can use localised affinoid dagger dg spaces as the building blocks for derived dagger stacks satisfying a coherence condition, and hence for partially proper derived K -analytic stacks (e.g. derived K -analytic stacks without boundary) satisfying coherence conditions, as in [Pri15, §4.2].

1.3. Dagger dg spaces and stacks.

1.3.1. Definitions.

Definition 1.27. Define a K -dagger dg space X to be a pair $(\pi^0 X, \mathcal{O}_X)$ where $\pi^0 X$ is a K -dagger space in the sense of [GK, 2.12] and \mathcal{O}_X is a presheaf of quasi-dagger dg K -algebras (Definition 1.13) on the site of open affinoid subdomains of $\pi^0 X$, such that the homology presheaf $H_0 \mathcal{O}_X$ is just $\mathcal{O}_{\pi^0 X}$, and the homology presheaves $H_i \mathcal{O}_X$ are all coherent $\mathcal{O}_{\pi^0 X}$ -modules.

Example 1.28. Given an affinoid dagger dg space $X = (X^0, \mathcal{O}_X)$, there is an associated dagger dg space given by $(\pi^0 X, \iota^{-1} \mathcal{O}_X)$, for the closed immersion $\iota: \pi^0 X \rightarrow X^0$.

Definition 1.29. A morphism $f: X \rightarrow Y$ of K -dagger dg spaces is said to be a quasi-isomorphism if it induces an isomorphism $\pi^0 f: \pi^0 X \rightarrow \pi^0 Y$ of dagger spaces and isomorphisms $H_i(f^{-1} \mathcal{O}_Y) \rightarrow H_i \mathcal{O}_X$ for all i .

Similarly:

Definition 1.30. Define a K -dagger dg analytic Deligne–Mumford stack X to be a pair $(\pi^0 X, \mathcal{O}_X)$ where $\pi^0 X$ is a K -dagger analytic Deligne–Mumford stack and \mathcal{O}_X is a presheaf of quasi-dagger dg K -algebras on the site of dagger affinoid spaces étale over $\pi^0 X$, such that the homology presheaf $H_0 \mathcal{O}_X$ is just $\mathcal{O}_{\pi^0 X}$, and the homology presheaves $H_i \mathcal{O}_X$ are all coherent $\mathcal{O}_{\pi^0 X}$ -modules.

Remark 1.31. Similar definitions exist for N -stacks, in which case the étale site has higher categorical structure.

For an alternative characterisation of dg dagger spaces and stacks, and a generalisation to derived Artin stacks, we can use the approach via Čech nerve-type constructions as in [Pri5] and [EP, §6]. Instead of defining a presheaf \mathcal{O}_X on a site associated to $\pi^0 X$, we can just take a hypercover Z_\bullet of $\pi^0 X$ with each Z_n a disjoint union of dagger affinoid spaces U , and then give a dg dagger algebra $\mathcal{O}_{X,\bullet}(U)$ for each U , such that $H_0 \mathcal{O}_{X,\bullet}(U) \cong O(U)$ and $H_i \mathcal{O}_{X,\bullet}(V) \cong H_i \mathcal{O}_{X,\bullet}(U) \otimes_{O(U)} O(V)$ for each morphism $V \rightarrow U$. Equivalences are then generated by certain hypercovers. If we restrict to compact stacks with compact (higher) diagonals, we can take each Z_n to be dagger affinoid rather than a disjoint union of such.

Combining Corollary 1.20, Proposition 1.26 and [Pri15, Remark 4.6], we get:

Corollary 1.32. *The dagger-analytic analogue of the ∞ -category of derived K -analytic spaces from [PY1] is equivalent to the simplicial localisation at quasi-isomorphisms of the category of K -dagger dg spaces.*

Moreover, the ∞ -category of partially proper derived K -analytic spaces from [PY1] is equivalent to the simplicial localisation at quasi-isomorphisms of the category of partially proper K -dagger dg spaces.

The analogous statements for derived K -dagger analytic Deligne–Mumford and Artin (N -)stacks also hold. In particular, they can be regarded as functors from localised dagger dg K -algebras to simplicial sets.

Here, we are saying that X is partially proper if and only its underived truncation $\pi^0 X$ is so; essentially this means that the space does not have a boundary.

1.3.2. Representability.

Definition 1.33. We denote the category of localised dagger dg K -algebras by $dg_+ \text{AffdAlg}_K^{\dagger, \text{loc}}$. We then denote its full subcategory of objects which are bounded as chain complexes by $dg_+ \text{AffdAlg}_K^{\dagger, \text{loc}, \text{b}}$.

Definition 1.34. Say that a simplicial set-valued functor $F: dg_+ \text{AffdAlg}_K^{\dagger, \text{loc}} \rightarrow \text{sSet}$ is homotopy-preserving if it maps quasi-isomorphisms to weak equivalences.

Definition 1.35. We say that a map $A \rightarrow B$ in $dg_+ \text{AffdAlg}_K^{\dagger, \text{loc}}$ is a square-zero extension if it is surjective and the kernel I is square-zero, i.e. satisfies $I^2 = 0$.

Lemma 1.36. *If $A \rightarrow B$ and $C \rightarrow B$ are surjective maps in $dg_+ \text{AffdAlg}_K^{\dagger, \text{loc}}$, with $A \rightarrow B$ a square-zero extension, then the fibre product exists $A \times_B C \rightarrow C$ in $dg_+ \text{AffdAlg}_K^{\dagger, \text{loc}}$. Similar statements hold for dagger and quasi-dagger dg algebras.*

Proof. First observe that if $D \rightarrow C$ is a square-zero extension of K -CDGAs with kernel J , and each C_0 -module J_r is finitely generated, then $D \in dg_+ \text{AffdAlg}_K^{\dagger, \text{loc}}$, with generators in degree 0 given by combining those for J_0 with lifts of those for C_0 ;

the dg quasi-dagger algebra D is localised because the maps $\mathrm{Sp}(C_0) \rightarrow \mathrm{Sp}(D_0)$ and $\mathrm{Sp}(\mathrm{H}_0 C) \rightarrow \mathrm{Sp}(\mathrm{H}_0 D)$ give isomorphisms on the underlying sets of points.

Now observe that our hypotheses imply that $A \times_B C \rightarrow C$ is a surjection with kernel I . Moreover, I is levelwise finitely generated as a C_0 -module (since it is as a B_0 -module, and C_0 surjects onto B_0), so $A \times_B C \in dg_+ \mathrm{AffdAlg}_K^{\dagger, \mathrm{loc}}$. Beware that this would not be true if $C \rightarrow B$ were not surjective. \square

Definition 1.37. We say that a functor

$$F: dg_+ \mathrm{AffdAlg}_K^{\dagger, \mathrm{loc}} \rightarrow s\mathrm{Set}$$

is homogeneous if for all square-zero extensions $A \rightarrow B$ and all surjections $C \rightarrow B$ in $dg_+ \mathrm{AffdAlg}_K^{\dagger, \mathrm{loc}}$, the natural map

$$F(A \times_B C) \rightarrow F(A) \times_{F(B)}^h F(C)$$

to the homotopy fibre product is a weak equivalence.

Remark 1.38. This terminology is based on that from [Pri4], which was inspired by earlier usage in derived deformation theory, such as [Man], and is a natural generalisation of Schlessinger’s conditions for set-valued deformation functors from [Sch, Theorem 2.11] and [Art, 2.2, S1]. Note however that in algebraic setting of [Pri4], the morphism $C \rightarrow B$ was not required to be surjective; here, we have imposed surjectivity to ensure that the fibre product exists in our category. Homotopy-homogeneity differs from the notion of infinitesimal cohesion in [Lur1] in that we only require one of the morphisms to be nilpotent; our notion does not appear in [Lur1], but its influence is such that nowadays homogeneity is frequently referred to as “infinitesimal cohesion on one factor”.

Definition 1.39. Given a dg algebra A , we say that an A -module M is levelwise finitely generated if as a graded A -module it has a generating set with finitely many elements in each degree. We then let $dg_+ \mathrm{Coh}_A \subset dg_+ \mathrm{Mod}_A$ be the category of levelwise finitely generated modules.

The significance of this condition is that if A is an quasi-dagger dg algebra and M a levelwise f.g. A -module, then $A \oplus M$ is also an quasi-dagger dg algebra, where the multiplication is defined so that M is square-zero.

Definition 1.40. Given a homotopy-preserving homogeneous functor $F: dg_+ \mathrm{AffdAlg}_K^{\dagger, \mathrm{loc}} \rightarrow s\mathrm{Set}$, an object $A \in dg_+ \mathrm{AffdAlg}_K^{\dagger, \mathrm{loc}}$ and a point $x \in F(A)$, define the tangent functor $T_x F$

$$T_x F: dg_+ \mathrm{Coh}_A \rightarrow s\mathrm{Set},$$

by

$$T_x F(M) := F(A \oplus M) \times_{F(A)}^h \{x\},$$

where $A \oplus M$ is given the multiplication $(a_1, m_1)(a_2, m_2) := (a_1 a_2, a_1 m_2 + m_1 a_2)$.

As for instance in [Pri4, Lemma 1.12], the space $T_x F(M_{[-1]})$ deloops $T_x F(M)$, so we may define tangent cohomology groups by $D_x^{n-i}(F, M) := \pi_i(F(A \oplus M[n]) \times_{F(A)}^h \{x\})$.

Definition 1.41. In the setting of Definition 1.40, we say that F has a coherent cotangent complex $\mathbb{L}^{F,x}$ at x if there is a levelwise finitely generated A -module $\mathbb{L}^{F,x}$ in chain complexes, bounded below in chain degrees, representing $T_x(F)$ homotopically in the sense that the simplicial mapping space

$$\mathbf{Rmap}_{dg\mathrm{Mod}_A}(\mathbb{L}^{F,x}, -)$$

is weakly equivalent to $T_x(F)$ when restricted to dg_+Coh_A .

In particular, this means that

$$\pi_i T_x(F)(M) \cong \mathbb{E}xt_A^{-i}(\mathbb{L}^{F,x}, M)$$

for all $M \in dg_+Coh_A$.

Lemma 1.42. *In the setting of Definition 1.40, if $f: A \rightarrow B$ is a morphism in $dg_+AffdAlg_K^{\dagger,loc}$ and $x \in F(A)$ a point at which F has a coherent cotangent complex, then there is a natural quasi-isomorphism $\mathbb{L}^{F,f_*x} \simeq \mathbb{L}^{F,x} \otimes_A^{\mathbf{L}} B$ of B -modules.*

Proof. For any morphism $f: A \rightarrow B$ in $dg_+AffdAlg_K^{\dagger,loc}$ and any $M \in dg_+Coh_A$, flat over A as a graded module, the B -module $M \otimes_A B$ is levelwise finitely generated over B , so the morphism $A \oplus M \rightarrow B \oplus (M \otimes_A B)$ in $dg_+AffdAlg_K^{\dagger,loc}$ gives us a map $T_x(F)(M) \rightarrow T_{f_*x}(F)(M \otimes_A B)$. Since F is homotopy-preserving, this gives us a map $T_x(F)(M) \rightarrow T_{f_*x}(F)(M \otimes_A^{\mathbf{L}} B)$ in the derived category for all $M \in dg_+Coh_A$. By universality, this induces a morphism $\mathbb{L}^{F,f_*x} \rightarrow \mathbb{L}^{F,x} \otimes_A^{\mathbf{L}} B$.

Now, when f is surjective, for any $M \in dg_+Coh_B$, we have $A \oplus M \in dg_+AffdAlg_K^{\dagger,loc}$, and then homotopy-homogeneity applied to the fibre product $(A \oplus M) \cong (B \oplus M) \times_B A$ gives us a weak equivalence $T_x(F, M) \rightarrow T_{f_*x}(F)(M)$. By adjunction, this implies that the map $T_x(F)(M) \rightarrow T_{f_*x}(F)(M \otimes_A^{\mathbf{L}} B)$ is a quasi-isomorphism. Since F is homotopy-preserving, the same conclusion holds whenever $H_0f: H_0A \rightarrow H_0B$ is surjective.

In general, we can factorise f as a composite $A \rightarrow A\langle \frac{z_1}{r_1}, \dots, \frac{z_m}{r_m} \rangle^{\dagger} \rightarrow B$, so by transitivity it remains to consider the case when f is of the form $A \rightarrow A\langle \frac{z}{r} \rangle^{\dagger}$. We have a morphism $\eta: \mathbb{L}^{F,f_*x} \rightarrow \mathbb{L}^{F,x} \otimes_A A\langle \frac{z}{r} \rangle^{\dagger}$ in $dg_+Coh_{A\langle \frac{z}{r} \rangle^{\dagger}}$, and know that this becomes a quasi-isomorphism on base change along $A\langle \frac{z}{r} \rangle^{\dagger} \rightarrow C$ whenever the induced map $H_0A \rightarrow H_0C$ is surjective. We can now argue as for instance in the proof of [Lur1, Theorem 7.4.1], using Washnitzer algebras instead of polynomial rings. If we look at $\text{cone}(\eta)$, then its homology groups are finite $H_0A\langle \frac{z}{r} \rangle^{\dagger}$ -modules, but the base change result above implies that the support of the lowest non-zero homology group contains no closed points, which is a contradiction. \square

Definition 1.43. We say that $F: dg_+AffdAlg_K^{\dagger,loc} \rightarrow s\text{Set}$ is nilcomplete if for all $B \in dg_+AffdAlg_K^{\dagger,loc}$, applying F to the Postnikov tower of B gives an equivalence

$$F(B) \rightarrow \varprojlim_n F(B/\tau_{>n}B),$$

where $\tau_{>n}B \subset B$ is the dg ideal given by good truncation in degrees above n .

Lemma 1.44. *If $F: dg_+AffdAlg_K^{\dagger,loc} \rightarrow s\text{Set}$ is a homotopy-preserving, homogeneous, nilcomplete functor such that for all dagger algebras A and all points $x \in F(A)$, the groups $D_x^i(F, A)$ are all finitely generated A -modules and vanish for $i \ll 0$, then F has coherent cotangent complexes $\mathbb{L}^{F,y}$ at all points $y \in F(B)$ for all $B \in dg_+AffdAlg_K^{\dagger,loc}$.*

Proof. By [GK, Proposition 1.5], Washnitzer algebras are regular. Any $B \in dg_+AffdAlg_K^{\dagger,loc}$ is almost of finite presentation over a Washnitzer algebra by Lemma 1.8, so has a dualising module by [Lur1, Theorem 3.6.8]. Although our tangent functor is only defined on coherent complexes, the relevant sections of the proof of [Lur1, Theorem 3.6.9] establish the existence of $\mathbb{L}^{F,y}$ provided that $D_y^i(F, M)$ is a finitely generated H_0B -module for all finitely generated H_0B -modules M and that there exists some n

with $D_y^i(F, M) = 0$ for all $i < -n$ and all such M . We may take a projective resolution of M , and then homogeneity combines with nilcompleteness to imply that these conditions hold provided $D_x^i(F, H_0B)$ is finitely presented, and vanishes for $i < -n$.

Since $B \rightarrow H_0B$ is surjective, homogeneity gives $D_{\bar{x}}^i(F, M) \cong D_x^i(F, M)$, where \bar{x} is the image of x in $F(H_0B)$. Thus the conditions are satisfied by hypothesis. \square

We are now in a position to state a weak derived representability result.

Corollary 1.45. *A homotopy-preserving functor $F: dg_+ \text{AffdAlg}_K^{\dagger, \text{loc}} \rightarrow s\text{Set}$ is a dagger-analytic derived Artin n -stack if and only if the following conditions hold*

- (1) *The restriction $\pi^0 F: \text{AffdAlg}_K^{\dagger} \rightarrow s\text{Set}$ to underived dagger algebras is represented by a dagger-analytic Artin n -stack.*
- (2) *F is homogeneous.*
- (3) *F is nilcomplete.*
- (4) *for all dagger algebras $A \in \text{AffdAlg}_K^{\dagger}$, all $x \in F(A)_0$ and all étale morphisms $f: A \rightarrow A'$, the maps*

$$D_x^*(F, A) \otimes_A A' \rightarrow D_{f_x}^*(F, A')$$

are isomorphisms.

- (5) *for all dagger algebras A and all $x \in F(A)$, the groups $D_x^i(F, A)$ are all finitely generated A -modules.*

Proof. By Lemma 1.44, the conditions imply that F has coherent cotangent complexes $\mathbb{L}^{F, x}$ at all points $y \in F(A)$ for all $A \in dg_+ \text{AffdAlg}_K^{\dagger, \text{loc}}$, and by Lemma 1.42 we have quasi-isomorphisms $\mathbb{L}^{F, f_*x} \simeq \mathbb{L}^{F, x} \otimes_A^L B$ for all morphisms $f: A \rightarrow B$ in $dg_+ \text{AffdAlg}_K^{\dagger, \text{loc}}$. Arguing as in the proof of [Pri4, Corollary 1.36], it follows from [Pri4, Proposition 1.32] that F is an étale hypersheaf. The final stage of the proof of [PY2, Theorem 7.1] (itself based on the algebraic setting of [TV2, Theorem C0.9], based on [Lur1]) then adapts directly to give the desired result. \square

Remarks 1.46. This result is significantly weaker than the representability results of [Lur1, Pri4] in that it assumes that $\pi^0 F$ is representable. Direct analogues of stronger forms of the representability theorem cannot exist in our setting, because $dg_+ \text{AffdAlg}_K^{\dagger, \text{loc}}$ does not have filtered colimits or contain complete local rings. However, an analogue of the combined formal effectiveness and finite presentation conditions would be to require that for every complete local K -algebra A with residue field finite over K , the morphism

$$\lim_{\substack{\rightarrow \\ A' \subset A \\ \text{dagger} \\ \text{affinoid}}} F(A') \rightarrow \text{holim}_{\leftarrow m} F(A/\mathfrak{m}_A^m)$$

is a weak equivalence; a stronger representability result incorporating such a condition is plausible, but may be of limited use.

There is a simpler version of Corollary 1.45 in which we just take $F: dg_+ \text{AffdAlg}_K^{\dagger, \text{loc}, b} \rightarrow s\text{Set}$ to be a homotopy-preserving functor on bounded objects, and consequently drop the nilcompleteness condition. Such functors correspond to nilcomplete functors on $dg_+ \text{AffdAlg}_K^{\dagger, \text{loc}}$, the correspondence given by setting $F(B) := \text{holim}_{\leftarrow n} F(B/\tau_{>n} B)$.

The following is an immediate consequence of Corollary 1.45

Corollary 1.47. *Assume that $F: dg_+ \text{AffdAlg}_K^{\dagger, \text{loc}} \rightarrow s\text{Set}$ is homotopy-preserving, nil-complete and homogeneous. Take a K -dagger analytic Artin n -stack \mathfrak{X} , together with a natural transformation $\mathfrak{X} \rightarrow \pi^0 F$ of functors $\text{AffdAlg}_K^{\dagger, \text{loc}} \rightarrow s\text{Set}$ which is formally étale in the sense that for all square-zero extensions $A \rightarrow B$, the map*

$$\mathfrak{X}(A) \rightarrow F(A) \times_{F(B)}^h \mathfrak{X}(B)$$

is a weak equivalence.

Assume that for all points x in image of $\mathfrak{X}(A) \rightarrow \pi^0 F(A)$, the groups $D_x^i(F, A)$ are all finitely generated A -modules, and that for all étale morphisms $f: A \rightarrow A'$, the maps

$$D_x^*(F, A) \otimes_A A' \rightarrow D_{f_x}^*(F, A')$$

are isomorphisms.

Then the functor

$$A \mapsto F(A) \times_{F(H_0 A)}^h \mathfrak{X}(H_0 A)$$

on $dg_+ \text{AffdAlg}_K^{\dagger, \text{loc}}$ is representable by an n -geometric (resp. ∞ -geometric) K -dagger dg analytic Artin stack \mathfrak{X}

1.4. Pro-étale sheaves associated to affinoid dagger dg spaces. For our purposes, the great advantage of dagger algebras over EFC algebras is that the former are equipped with canonical topologies, which we now exploit to produce condensed algebras. From now on, we assume that the valuation on our base field K is discretely valued, so the ring $\mathcal{O}_K := \{\lambda \in K : |\lambda| \leq 1\}$ is a DVR with maximal ideal $\mathfrak{m}_K := \{\lambda \in K : |\lambda| < 1\}$, and the topology on K induced by the norm is the ϖ -adic topology, where ϖ is an element of \mathfrak{m}_K of maximum norm (and hence a generator of that ideal).

Now, every affinoid K -algebra A is a Banach K -algebra, and every finite A -module M then inherits the structure of a Banach space (up to Banach space isomorphism), since it admits a surjection $A^n \rightarrow M$ for some n , with kernel necessarily closed by [BGR, Proposition 5.2.7.1].

Lemma 1.48. *Every surjection $f: M \rightarrow N$ of finite modules over an affinoid K -algebra A admits a continuous K -linear section.*

Proof. Since the kernel L of f is (necessarily) closed, we have a topological A -linear isomorphism $M/L \cong N$. The norm on N is thus equivalent to the quotient norm induced from M . With respect to these norms, we can then let $M^\circ := \{m \in M : |m| \leq 1\}$ and similarly for $N^\circ \subset N$. These are \mathcal{O}_K -modules, with $M = \bigcup_n \varpi^{-n} M^\circ \cong M^\circ \otimes_{\mathcal{O}_K} K$ and similarly for N , where ϖ generates \mathfrak{m}_K .

Now, since the norm on N is ultrametric and discrete (taking the same values as the norm on K), the topology on the \mathcal{O}_K -module N° is the ϖ -adic topology. Pick a basis for the $\mathcal{O}_K/\mathfrak{m}_K$ -vector space $N^\circ/\varpi N^\circ = N^\circ/\mathfrak{m}_K N^\circ$, and lift to a set S of elements of N° . Then we have a topological \mathcal{O}_K -linear isomorphism $\varprojlim_n ((\mathcal{O}_K/\varpi^n).S) \rightarrow N^\circ$, since N° is ϖ -adically complete.

Picking pre-images of S in M° then gives us a continuous \mathcal{O}_K -linear section of $f: M^\circ \rightarrow N^\circ$, and hence a continuous K -linear section of $f: M \rightarrow N$ on tensoring with K . \square

Proposition 1.49. *Given a pro-finite set S and an affinoid K -algebra A , the functor*

$$\text{Hom}_{\text{cts}}(S, -)$$

on the abelian category of finite A -modules is exact.

In consequence, for all finite A -modules M the natural map $\mathrm{Hom}_{\mathrm{cts}}(S, A) \otimes_A M \rightarrow \mathrm{Hom}_{\mathrm{cts}}(S, M)$ is an isomorphism.

Proof. The functor is obviously left exact, preserving finite limits. Lemma 1.48 implies that for any short exact sequence of finite A -modules, the underlying sequence of topological K -modules splits. Since $\mathrm{Hom}_{\mathrm{cts}}(S, -)$ depends only on the topological abelian group structure, left exactness thus guarantees exactness.

The final statement then follows by applying the functor to a finite presentation $\mathrm{coker}(A^m \rightarrow A^n)$ of M , noting that finite A -modules are all finitely presented because A is Noetherian. \square

Definition 1.50. Given a topological K -vector space V , we let \underline{V} be the functor from pro-finite sets to K -vector spaces given by

$$S \mapsto \mathrm{Hom}_{\mathrm{cts}}(S, V),$$

the space of continuous functions from S to V .

We extend this definition to quasi-dagger dg algebras A and their modules via Proposition 1.12, writing A as a filtered colimit $A = \varinjlim_{i \in I} A(i)$ of dg affinoid algebras, and setting

$$\underline{A}(S) := \varinjlim_{i \in I} \underline{A}(i)(S),$$

and similarly for modules.

Corollary 1.51. *For any pro-finite set S , the functor $A \mapsto \underline{A}(S)$ from quasi-dagger dg algebras to CDGAs over $\underline{K}(S)$ preserves quasi-isomorphisms. The functor is moreover naturally isomorphic to the functor $A \mapsto \underline{A}_0(S) \otimes_{A_0} A$.*

Proof. Because filtered colimits are exact, Proposition 1.49 also applies to finite modules over Noetherian filtered colimits of affinoid algebras, hence over quasi-dagger algebras; Noetherianity means every finite A_0 -module is finitely presented, so induced by a module over some affinoid subalgebra. Since the A_0 -modules A_n are all finite, it follows immediately that $\underline{A}(S) \cong \underline{A}_0(S) \otimes_{A_0} A$, giving the second statement.

By applying Lemma 1.8, we see that any quasi-isomorphism $A \rightarrow B$ of quasi-dagger dg algebras admits a factorisation $A \rightarrow C \rightarrow B$ into quasi-isomorphisms for which the first map admits a retraction and the second is surjective. It thus suffices to prove that the functor $A \mapsto \underline{A}(S)$ sends surjective quasi-isomorphisms to quasi-isomorphisms.

Now, given a surjective quasi-isomorphism $A \rightarrow B$ of quasi-dagger dg algebras, the A_0 -modules A_n and B_n are all finite. Since the functor $M \mapsto \underline{M}(S)$ is exact on finite A_0 -modules, it preserves quasi-isomorphisms, giving the first statement. \square

Definition 1.52. Given a scheme X and a quasi-dagger dg algebra A , we define the sheaf \underline{A}_X of CDGAs on the affine pro-étale site $X_{\mathrm{pro\acute{e}t}}^{\mathrm{aff}}$ of X (see [BS, Definition 4.2.1]) by

$$U \mapsto \underline{A}(\pi_0 U),$$

where $\pi_0 U$ is the pro-finite set of components of the quasi-compact quasi-separated scheme U , constructed as in [BS, §2]. It follows from [BS, Lemma 4.2.12] that the presheaves \underline{A}_X are indeed sheaves.

We can use this to construct moduli functors of various flavours of sheaf on the pro-étale site:

Definition 1.53. Given a scheme X and a functor $F: dg_+ \text{CAlg}_K \rightarrow s\text{Set}$ from differential graded-commutative K -algebras in non-negative chain degrees to simplicial sets, define the functor

$$F(X_{\text{proét}}, -): dg_+ \text{AffdAlg}_K^{\text{loc}, \dagger} \rightarrow s\text{Set}$$

from localised dagger dg algebras to simplicial sets by

$$A \mapsto \mathbf{R}\Gamma(X_{\text{proét}}, F(\underline{A}_X))$$

where $\mathbf{R}\Gamma$ is the right-derived functor of the global sections functor Γ in simplicial sets.

Example 1.54. If G is an algebraic group over K , then we can let F be the derived stack BG , parametrising G -torsors. The functor $BG(X_{\text{proét}}, -)$ then parametrises G -torsors on $X_{\text{proét}}$. In particular, when A is a dagger algebra (concentrated in degree 0), $BG(X_{\text{proét}}, A)$ is the nerve of the groupoid of $G(\underline{A}_X)$ -torsors on $X_{\text{proét}}$.

If X is locally topologically Noetherian and connected, with a geometric point x , then for $A \in \text{AffdAlg}_K^\dagger$, this means that we have the nerve

$$BG(X_{\text{proét}}, A) \simeq B[\text{Hom}(\pi_1^{\text{proét}}(X, x), G(A))/G(A)],$$

of the groupoid of continuous group homomorphisms, where $G(A)$ is topologised using the topology on A (take the coarsest topology for which the maps $G(A) \rightarrow A$ given by elements of \mathcal{O}_G are all continuous), $\pi_1^{\text{proét}}$ is the pro-étale fundamental group of [BS, §7], and $G(A)$ acts on the set $\text{Hom}(\pi_1^{\text{proét}}(X, x), G(A))$ by conjugation.

Lemma 1.55. (1) *If F is homotopy-preserving in the sense that it maps quasi-isomorphisms to weak equivalences, then so is $F(X_{\text{proét}}, -)$.*
 (2) *Assume that the natural map $F(A' \times_{B'} C') \rightarrow F(A') \times_{F(B')}^h F(C')$ is a weak equivalence for all surjections $A' \rightarrow B' \leftarrow C'$ of CDGAs with $\ker(A'_0 \rightarrow B'_0)$ nilpotent; in particular this holds if F is homotopy-homogeneous in the sense of [Pri4]. Then for any surjective morphisms $A \rightarrow B \leftarrow C$ of localised dagger dg algebras, with $\ker(A_0 \rightarrow B_0)$ nilpotent, the object $A \times_B C$ is also a localised dagger dg algebra and the natural map*

$$F(X_{\text{proét}}, A \times_B C) \rightarrow F(X_{\text{proét}}, A) \times_{F(X_{\text{proét}}, B)}^h F(X_{\text{proét}}, C)$$

is a weak equivalence.

Proof. The first statement is an immediate consequence of Corollary 1.51.

For the second statement, begin by noting that since both $A \rightarrow B$ and $C \rightarrow B$ are surjective, the modules $(A_n \times_{B_n} C_n)$ must all be finite over $A_0 \times_{C_0} B_0$, so $A \times_B C$ is a quasi-dagger dg algebra. Since $A_0 \rightarrow B_0$ is a nilpotent surjection, the map $A_0 \times_{B_0} C_0 \rightarrow C_0$ is also a nilpotent surjection. Any element x in $\ker(A_0 \times_{B_0} C_0) \rightarrow \text{H}_0(A \times_B C)$ maps to 0 in $\text{H}_0 C$, so $1 + x$ maps to a unit in C_0 , since C is a localised dagger algebra. Nilpotence of $A_0 \times_{B_0} C_0 \rightarrow C_0$ then implies that $1 + x$ is a unit, so $A \times_B C$ is also localised.

The maps $\underline{A}_X \rightarrow \underline{B}_X \leftarrow \underline{C}_X$ are now objectwise surjections of CDGAs by Proposition 1.49, and the first is objectwise nilpotent. Since the functor $A \mapsto \underline{A}_X$ preserves finite limits, we have $\underline{A \times_B C}_X \cong \underline{A}_X \times_{\underline{B}_X} \underline{C}_X$. The hypothesis on F then gives

$$F(\underline{A \times_B C}_X) \simeq F(\underline{A}_X) \times_{F(\underline{B}_X)}^h F(\underline{C}_X),$$

and the desired statement then follows because derived global sections preserve homotopy limits. \square

2. SHIFTED SYMPLECTIC STRUCTURES

2.1. Structures on EFC algebras. We now recall how standard algebraic definitions adapt to the analytic setting, as sketched in [Pri15, §4.4].

Definition 2.1. Given an EFC-DGA A , define the complex Ω_A^1 to be the A -module in chain complexes representing the functor

$$M \mapsto \mathrm{Hom}_{\mathrm{EFC-DGA}}(A, A \oplus M) \times_{\mathrm{Hom}_{\mathrm{EFC-DGA}}(A, A)} \{\mathrm{id}\}$$

of closed EFC derivations from A to M of degree 0. Here, the EFC structure on $A \oplus M$ is determined by requiring it to be a group object over A ; explicitly, for a holomorphic function $f(z_1, \dots, z_n)$ in n variables, we set

$$f(a_1 + m_1, \dots, a_n + m_n) := f(a_1, \dots, a_n) + \sum_i \frac{\partial f}{\partial z_i}(a_1, \dots, a_n) m_i,$$

which in particular means that the multiplication on M is 0.

Given a morphism $R \rightarrow A$ of EFC-DGAs, define $\Omega_{A/R}^1$ to be the cokernel of $\Omega_R^1 \otimes_R A \rightarrow \Omega_A^1$.

As in [Pri15, §3.2], we can then follow the approach of [Qui] to give EFC cotangent complexes.

Definition 2.2. Denote by $A \mapsto (A, \mathbf{L}\Omega_A^1)$ the left-derived functor of the functor $A \mapsto (A, \Omega_A^1)$ from EFC-DGAs to the category of pairs (A, M) of EFC-DGAs and modules. We refer to $\mathbf{L}\Omega_A^1$ as the cotangent complex of A . Given a morphism $R \rightarrow A$ of EFC-DGAs, write $\mathbf{L}\Omega_{A/R}^1$ for the cone of the natural map $\mathbf{L}\Omega_R^1 \otimes_R^{\mathbf{L}} A \rightarrow \mathbf{L}\Omega_A^1$.

Definition 2.3. Given an EFC-DGA A , write $\Omega_A^p := \Lambda_A^p \Omega_A^1$, and define the de Rham complex $\mathrm{DR}(A)$ to be the product total cochain complex of the double complex

$$A \xrightarrow{d} \Omega_A^1 \xrightarrow{d} \Omega_A^2 \xrightarrow{d} \dots,$$

so the total differential is $d \pm \delta$.

We define the Hodge filtration F on $\mathrm{DR}(A)$ by setting $F^p \mathrm{DR}(A) \subset \mathrm{DR}(A)$ to consist of terms Ω_A^i with $i \geq p$.

Define $\mathbf{LDR}(A)$ to be $\mathrm{DR}(\tilde{A})$ for any cofibrant replacement of A .

Properties of the product total complex ensure that a map $f: A \rightarrow B$ induces a filtered quasi-isomorphism $\mathrm{DR}(A) \rightarrow \mathrm{DR}(B)$ whenever the maps $\Omega_A^p \rightarrow \Omega_B^p$ are quasi-isomorphisms, which will happen whenever f is a weak equivalence between cofibrant EFC-DGAs.

Note that if \tilde{A} is a cofibrant replacement for A , there is a natural \tilde{A} -linear quasi-isomorphism $\Omega_{\tilde{A}}^1 \rightarrow \mathbf{L}\Omega_A^1$.

Definition 2.4. Define an n -shifted pre-symplectic structure ω on an EFC-DGA A to be an element

$$\omega \in Z^{n+2} F^2 \mathbf{LDR}(A).$$

Explicitly, this means that ω is given by an infinite sum $\omega = \sum_{i \geq 2} \omega_i$, with $\omega_i \in (\Omega_{\tilde{A}}^i)_{i-n-2}$ and $d\omega_i = \pm \delta \omega_{i+1}$.

Definition 2.5. Define an n -shifted symplectic structure ω on A to be an n -shifted pre-symplectic structure ω for which the component $\omega_2 \in Z^n \Omega_{\tilde{A}}^2$ induces a quasi-isomorphism

$$\omega_2^\sharp: \underline{\mathrm{Hom}}_{\tilde{A}}(\Omega_{\tilde{A}}^1, \tilde{A}) \rightarrow (\Omega_{\tilde{A}}^1)_{[-n]},$$

with $\mathbf{L}\Omega_{\tilde{A}}^1$ perfect as an A -module.

Definition 2.6. Define the space of n -shifted pre-symplectic structures on an EFC-DGA A to be the simplicial set $\mathrm{PreSp}(A, n)$ given by Dold–Kan denormalisation of the chain complex

$$\tau_{\geq 0}(\mathbf{L}F^2\mathrm{DR}(A)^{[n+2]}),$$

where $\tau_{\geq 0}$ denotes good truncation in non-negative chain degrees.

Set $\mathrm{Sp}(A, n) \subset \mathrm{PreSp}(A, n)$ to consist of the symplectic structures — this is a union of path-components.

Observe that for any morphism $f: A \rightarrow B$ of EFC-DGAs, we have a natural map $\mathrm{PreSp}(A, n) \rightarrow \mathrm{PreSp}(B, n)$, but that this does not restrict to the subspaces of symplectic structures unless f gives a quasi-isomorphism $B \otimes_A^{\mathbf{L}} \mathbf{L}\Omega_A^1 \rightarrow \mathbf{L}\Omega_B^1$ on cotangent complexes, i.e. unless f is homotopy ind-étale.

Definition 2.7. Take a morphism $f: A \rightarrow B$ of EFC-DGAs, with an n -shifted pre-symplectic structure ω on A . We then define an isotropic structure on B relative to ω to be an element (ω, λ) of

$$Z^{n+1}\mathrm{cone}(\mathbf{L}F^2\mathrm{DR}(A) \rightarrow \mathbf{L}F^2\mathrm{DR}(B))$$

lifting ω .

This structure is called Lagrangian if $\mathbf{L}\Omega_A^1$ and $\mathbf{L}\Omega_B^1$ are perfect as an A -module and a B -module respectively, if ω is symplectic, and if contraction with the image (ω_2, λ_2) of (ω, λ) in $Z^{n-1}\mathrm{cone}(\Omega_A^2 \rightarrow \Omega_B^2)$ induces a quasi-isomorphism

$$(f \circ \omega_2^\sharp, \lambda_2^\sharp): \mathrm{cone}(\underline{\mathrm{Hom}}_{\tilde{B}}(\Omega_{\tilde{B}}^1, \tilde{B}) \rightarrow \underline{\mathrm{Hom}}_{\tilde{A}}(\Omega_{\tilde{A}}^1, \tilde{B})) \rightarrow (\Omega_{\tilde{B}}^1)_{[-n]}.$$

Definition 2.8. Given a morphism $A \rightarrow B$ of EFC-DGAs, define the space $\mathrm{Iso}(A, B; n)$ of n -shifted isotropic structures on the pair (A, B) to be the simplicial set given by Dold–Kan denormalisation of the chain complex

$$\tau_{\geq 0}(\mathrm{cone}(\mathbf{L}F^2\mathrm{DR}(A) \rightarrow \mathbf{L}F^2\mathrm{DR}(B))^{[n+1]}).$$

Set $\mathrm{Lag}(A, B; n) \subset \mathrm{Iso}(A, B; n)$ to consist of the Lagrangians on symplectic structures — this is a union of path-components.

2.2. Structures on quasi-dagger algebras.

Lemma 2.9. *If A is the EFC-algebra underlying the quasi-dagger algebra $K\langle \frac{x_1}{r_1}, \dots, \frac{x_n}{r_n} \rangle^\dagger$, for $r_1, \dots, r_n \geq 0$, then*

$$\Omega_A^1 \cong \bigoplus_{i=1}^n A.d x_i,$$

and the natural map $\mathbf{L}\Omega_A^1 \rightarrow \Omega_A^1$ is a quasi-isomorphism.

Proof. The free EFC-algebra on n -generators is given by the ring

$$\mathcal{O}(\mathbb{A}_K^n) \cong \varprojlim_{s_1, \dots, s_n \rightarrow \infty} K\langle \frac{x_1}{s_1}, \dots, \frac{x_n}{s_n} \rangle^\dagger$$

of analytic functions on affine n -space, so

$$\Omega_{O(\mathbb{A}_K^n)}^1 \cong \bigoplus_{i=1}^n O(\mathbb{A}_K^n).dx_i.$$

Since the inclusions $\Delta(\rho_1, \dots, \rho_n) \subset \mathbb{A}_K^n$ of open polydiscs are open immersions, [Pri15, Lemma 3.17] then implies that

$$\Omega_{O(\Delta(\rho_1, \dots, \rho_n))}^1 \cong \bigoplus_{i=1}^n O(\Delta(\rho_1, \dots, \rho_n)).dx_i,$$

while [Pri15, Lemma 3.21] implies that

$$\mathbf{L}\Omega_{O(\Delta(\rho_1, \dots, \rho_n))}^1 \simeq \Omega_{O(\Delta(\rho_1, \dots, \rho_n))}^1.$$

The result then follows by passing to the filtered colimit over $\rho_i > r_i$. \square

Definition 2.10. Given a quasi-dagger DGA A , we define the A -module Ω_A^1 such that $d: A \rightarrow \Omega_A^1$ is the universal K -linear derivation from A to A -modules M in complexes which are *coherent* in the sense of Definition 1.39, generalising the definition for dagger algebras in [GK, 4.1]; this exists with similar reasoning.

Beware that this is not usually the same as the algebraic module $\Omega_{A^{\text{alg}}}^1$ of Kähler differentials, which is universal with respect to K -linear derivations from A to *all* A -modules in complexes; there is a canonical map from $\Omega_{A^{\text{alg}}}^1 \rightarrow \Omega_A^1$, by universality.

Lemma 2.11. *If A is a quasi-free quasi-dagger DGA, then the A -module Ω_A^1 is a model for the cotangent complex of the EFC-DGA B underlying A .*

Proof. Since A_0 is a free quasi-dagger algebra, it is smooth, so [GK, 4.1] implies that $\Omega_{A_0}^1$ is the free A_0 -module of Lemma 2.9, which by that lemma is a model for the cotangent complex $\mathbf{L}\Omega_{B_0}^1$ of the EFC-algebra underlying A_0 .

Now A is freely generated as an A_0 -algebra, with generators in strictly positive degrees, so we have a short exact sequence

$$0 \rightarrow \Omega_{A_0}^1 \otimes_{A_0} A \rightarrow \Omega_A^1 \rightarrow \Omega_{A/A_0}^1 \rightarrow 0.$$

Moreover, [Pri15, Lemma 3.20] implies that $\mathbf{L}\Omega_{B/B_0}^1$ is given by the cotangent complex for commutative algebras, which is just Ω_{A/A_0}^1 by quasi-freeness. Thus the short exact sequence above must be quasi-isomorphic to the exact triangle

$$\mathbf{L}\Omega_{B_0}^1 \otimes_{B_0} B \rightarrow \mathbf{L}\Omega_B^1 \rightarrow \mathbf{L}\Omega_{B/B_0}^1 \xrightarrow{[1]} \dots,$$

and hence $\mathbf{L}\Omega_B^1 \simeq \Omega_A^1$. \square

Again, beware that the EFC cotangent complex will not be the same as the cotangent complex for commutative algebras, as the CDGAs underlying quasi-free quasi-dagger DGAs are not sufficiently close to being cofibrant, unlike the underlying EFC-DGAs.

The following is now immediate, and allows us to interpret shifted symplectic structures in terms of dagger affinoid constructions:

Corollary 2.12. *Given a quasi-free quasi-dagger DGA A with underlying EFC-DGA B , the de Rham complex $\mathbf{LDR}(B)$ of Definition 2.3 is quasi-isomorphic to the product total cochain complex of the double complex $\mathbf{DR}(A)$ given by*

$$A \xrightarrow{d} \Omega_A^1 \xrightarrow{d} \Omega_A^2 \xrightarrow{d} \dots,$$

where $\Omega_A^p := \Lambda_A^p \Omega_A^1$

Moreover the complex $F^p \mathbf{LDR}(B)$ is quasi-isomorphic to the subcomplex of $\mathrm{DR}(A)$ to consisting of terms Ω_A^i with $i \geq p$.

Corollary 2.13. *Given a quasi-dagger DGA A with underlying EFC-DGA B , and a quasi-isomorphism $\tilde{A} \rightarrow A$ from a quasi-free quasi-dagger DGA, the space $\mathrm{PreSp}(B, n)$ of n -shifted pre-symplectic structures on B is equivalent to the Dold–Kan denormalisation of the chain complex*

$$\tau_{\geq 0}(F^2 \mathrm{DR}(A)^{[n+2]}).$$

The space $\mathrm{Sp}(B, n) \subset \mathrm{PreSp}(B, n)$ then corresponds to the subspace of points ω for which the component $\omega_2 \in \mathbb{Z}^n \Omega_{\tilde{A}}^2$ induces a quasi-isomorphism

$$\omega_2^\sharp: \underline{\mathrm{Hom}}_{\tilde{A}}(\Omega_{\tilde{A}}^1, \tilde{A}) \rightarrow (\Omega_{\tilde{A}}^1)_{[-n]},$$

with $\Omega_{\tilde{A}}^1$ perfect as an \tilde{A} -module.

From now on, we will simply refer to these as (pre-)symplectic structures on A , denoted $\mathrm{PreSp}(A, n)$ and $\mathrm{Sp}(A, n)$.

2.3. Structures on dagger dg spaces and stacks.

Definition 2.14. Given a K -dagger dg space or K -dagger dg DM stack $X = (\pi^0 X, \mathcal{O}_X)$ as in §1.3, we define the spaces $\mathrm{PreSp}(X, n)$ and $\mathrm{Sp}(X, n)$ of n -shifted pre-symplectic and symplectic structures on X as follows. First take an objectwise quasi-free replacement $\tilde{\mathcal{O}}_X$ of X (so each $\tilde{\mathcal{O}}_X(U)$ is a quasi-free quasi-dagger dg algebra, and $\tilde{\mathcal{O}}_X \rightarrow \mathcal{O}_X$ is a quasi-isomorphism). We then set

$$\mathrm{PreSp}(X, n) := \mathbf{R}\Gamma(\pi^0 X, \mathrm{PreSp}(\tilde{\mathcal{O}}_X)) \quad \mathrm{Sp}(X, n) := \mathbf{R}\Gamma(\pi^0 X, \mathrm{Sp}(\tilde{\mathcal{O}}_X)).$$

Note that this is well-defined (i.e. independent of the choice $\tilde{\mathcal{O}}_X$) as a consequence of Proposition 1.19.

Definition 2.15. Similarly, given a morphism $f: X \rightarrow Y$ of K -dagger dg spaces or K -dagger dg DM stacks, we generalise Definition 2.8 to define the space $\mathrm{Iso}(X, Y; n)$ of n -shifted isotropic structures on X over Y to be the homotopy fibre product

$$\mathrm{Iso}(X, Y; n) := \mathrm{PreSp}(Y, n) \times_{\mathrm{PreSp}(X, n)}^h \{0\},$$

so that isotropic structures on X over a fixed n -shifted pre-symplectic structure ω on Y are given by the homotopy fibre $\{f^* \omega\} \times_{\mathrm{PreSp}(X, n)}^h \{0\}$ of $\mathrm{Iso}(X, Y; n) \rightarrow \mathrm{PreSp}(Y, n)$ over ω .

We then define the space $\mathrm{Lag}(X, Y; n) \subset \mathrm{Iso}(X, Y; n)$ of n -shifted Lagrangian structures on X over Y to consist of the path components of objects (ω, λ) which are non-degenerate in the sense that ω is symplectic and contraction with the image (ω_2, λ_2) of (ω, λ) in $\mathbb{Z}^{n-1} \mathrm{cone}(\Omega_{\tilde{\mathcal{O}}_Y}^2 \rightarrow \Omega_{\tilde{\mathcal{O}}_X}^2)$ induces a quasi-isomorphism

$$(f^\sharp \circ \omega_2^\sharp, \lambda_2^\sharp): \mathrm{cone}(\underline{\mathrm{Hom}}_{\tilde{\mathcal{O}}_X}(\Omega_{\tilde{\mathcal{O}}_X}^1, \tilde{\mathcal{O}}_X) \rightarrow \underline{\mathrm{Hom}}_{f^{-1}\tilde{\mathcal{O}}_Y}(f^{-1}\Omega_{\tilde{\mathcal{O}}_Y}^1, \tilde{\mathcal{O}}_X)) \rightarrow (\Omega_{\tilde{\mathcal{O}}_X}^1)_{[-n]}.$$

Remark 2.16. As a consequence of Corollary 2.13, these correspond to shifted symplectic and Lagrangian structures in the sense of [Pri15, §4.4] on the derived analytic spaces and stacks associated to K -dagger dg spaces and K -dagger dg DM stacks by Corollary 1.32.

For K -dagger dg Artin stacks, we now have to take a little more care, as non-degeneracy is not preserved by smooth morphisms. Given such a stack $X = (\pi^0 X, \mathcal{O}_X)$, with \mathcal{O}_X a sheaf on the site of affinoids which are smooth over $\pi^0 X$, the cotangent complex \mathbb{L}^X is a Cartesian \mathcal{O}_X -module in chain complexes representing the functor of derived derivations. It is determined by the property that for all Cartesian \mathcal{O}_X -modules M with $M(U) \in dg_+ \mathcal{C}oh_{\mathcal{O}_X(U)}$, the space of maps $(\pi^0 X, \mathcal{O}_X \oplus M) \rightarrow (\pi^0 X, \mathcal{O}_X)$ extending the identity map is the Dold–Kan denormalisation of the good truncation of $\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{L}^X, M)$.

Definition 2.17. Given a K -dagger dg Artin stack X we define the space $\mathrm{PreSp}(X, n)$ of n -shifted pre-symplectic structures on X by $\mathrm{PreSp}(X, n) := \mathbf{R}\Gamma(\pi^0 X, \mathrm{PreSp}(\tilde{\mathcal{O}}_X))$.

We then define $\mathrm{Sp}(X, n) \subset \mathrm{PreSp}(X, n)$ to consist of the path components of objects ω which are non-degenerate in the sense that the essentially unique morphism $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathbb{L}^X, \mathcal{O}_X) \rightarrow \mathbb{L}^X$ making the diagrams

$$\begin{array}{ccc} \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathbb{L}^X, \mathcal{O}_X)(U) & \longrightarrow & \mathbb{L}^X(U) \\ \downarrow & & \uparrow \\ \mathbf{R}\mathcal{H}om_{\tilde{\mathcal{O}}_X(U)}(\Omega_{\tilde{\mathcal{O}}_X(U)}^1, \mathcal{O}_X(U)) & \xrightarrow{\omega_2^\sharp} & \Omega_{\tilde{\mathcal{O}}_X(U)}^1 \end{array}$$

commute is a quasi-isomorphism, where \mathbb{L} denotes the cotangent complex.

We can then define shifted isotropic and Lagrangian structures exactly as in Definition 2.15, with the non-degeneracy conditions being imposed on cotangent complexes.

Remark 2.18. As in Remark 1.31, similar definitions exist for N -stacks, the only difference being that derived global sections are taken over sites with higher categorical structure.

Alternatively, we could proceed as in [Pri7, §2] and define these structures via Čech nerve-type constructions as in [Pri5] and [EP, §6]. Given a suitable simplicial hypercover Z_\bullet of $\pi^0 X$, the definitions then reduce to taking cosimplicial homotopy limits, so that

$$\mathrm{PreSp}(X, n) := \mathrm{holim}_{i \in \Delta} \mathrm{PreSp}(\Gamma(Z_i, \mathcal{O}_X(Z_i)), n),$$

with the symplectic condition having to be imposed globally on path components in the Artin case.

For more general homotopy-preserving functors $F: dg_+ \mathrm{AffdAlg}_K^{\dagger, \mathrm{loc}} \rightarrow s\mathrm{Set}$, we can generalise this definition to look at the space $\mathbf{R}\mathrm{map}(F, \mathrm{PreSp}(-, n))$ of maps of homotopy-preserving presheaves from F to $\mathrm{PreSp}(-, n)$, thus functorially associating an n -shifted pre-symplectic structure on A to each point in $F(A)$. In order to define a subspace of shifted symplectic structures, we need F to moreover be homogeneous with a cotangent complex in order to formulate the non-degeneracy condition. See §6.6 for a slightly subtler, but more effective, definition of a space of symplectic structures on a homogeneous homotopy-preserving functor; it admits a canonical map from the space considered here, with the two agreeing for derived DM stacks.

3. SHIFTED SYMPLECTIC STRUCTURES ASSOCIATED TO PRO-ÉTALE SHEAVES

3.1. Pre-symplectic structures.

Lemma 3.1. *Given a quasi-free quasi-dagger dg algebra A , we have natural maps*

$$F^p\mathbf{LDR}(\underline{A}(S)^{\text{alg}}) \rightarrow F^p\mathbf{DR}(A)(S),$$

functorial in pro-finite sets S , where the domain is the algebraic derived de Rham cohomology of the CDGA $\underline{A}(S)$, as in [Pri7].

Proof. We automatically have a natural map $F^p\mathbf{LDR}(\underline{A}(S)^{\text{alg}}) \rightarrow F^p\mathbf{DR}(\underline{A}(S)^{\text{alg}})$. Since the derivation $d: A \rightarrow \Omega_A^1$ is continuous, it induces a derivation $\underline{A}(S) \rightarrow \underline{\Omega}_A^1(S)$, and hence an $\underline{A}(S)$ -linear map $\Omega_{\underline{A}(S)^{\text{alg}}}^1 \rightarrow \underline{\Omega}_A^1(S)$, by universality. On passing to alternating powers, we then get compatible maps $\Omega_{\underline{A}(S)^{\text{alg}}}^j \rightarrow \underline{\Omega}_A^j(S)$, and the desired morphisms arise by passing to product total complexes. \square

Proposition 3.2. *Let ℓ be the unique integral prime in \mathfrak{m}_K , and assume that X is a topologically Noetherian scheme with a constructible \mathbb{Z}_ℓ -complex \mathbb{D} such that the étale cohomology groups $H^i(X_{\text{ét}}, \mathbb{D}/\ell^n)$ vanish for $i > d$ and are equipped with compatible maps*

$$\text{tr}: H^d(X_{\text{ét}}, \mathbb{D}/\ell^n) \rightarrow \mathbb{Z}/\ell^n.$$

Then given a quasi-dagger dg K -algebra A and a finite A -module M in complexes, we have a natural map

$$\mathbf{R}\Gamma(X_{\text{proét}}, \underline{M}_X \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{D}) \rightarrow M[-d]$$

in the ∞ -category of cochain complexes, depending only on the structure of M as a complex of topological abelian groups.

Proof. The hypotheses give us \mathbb{Z}/ℓ^n -linear zigzags

$$\mathbf{R}\Gamma(X_{\text{ét}}, \mathbb{D}/\ell^n) \xleftarrow{\sim} \tau^{\leq d} \mathbf{R}\Gamma(X_{\text{ét}}, \mathbb{D}/\ell^n) \rightarrow H^d(X_{\text{ét}}, \mathbb{D}/\ell^n)[-d] \rightarrow \mathbb{Z}/\ell^n[-d].$$

Since the rings \mathbb{Z}/ℓ^n are discrete, [BS, Corollary 5.1.6] then gives $\mathbf{R}\Gamma(X_{\text{ét}}, \mathbb{D}/\ell^n) \simeq \mathbf{R}\Gamma(X_{\text{proét}}, \mathbb{D}/\ell^n)$. We thus have a map

$$\mathbf{R}\Gamma(X_{\text{proét}}, \underline{N}_X \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{D}) \rightarrow N[-d]$$

for all finite ℓ -torsion abelian groups N . Passing to derived inverse limits and completing the tensor product, this map extends to pro- ℓ abelian groups N .

Given a finite module N over an affinoid K -algebra B , as in Lemma 1.48 we can take elements of norm ≤ 1 to give a pro- ℓ algebra B° and a finite B° -module (in particular pro- ℓ) N° with $N \cong N^\circ \otimes \mathbb{Q}$. By [BS, Lemma 6.8.12], the functor $\mathbf{R}\Gamma(X_{\text{proét}}, -)$ commutes with filtered colimits, giving us

$$\mathbf{R}\Gamma(X_{\text{proét}}, \underline{N}_X \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{D}) \rightarrow N[-d]$$

for all such modules N . Passing to filtered colimits again, this extends to the case where B is a dg affinoid K -algebra, and N a B -module in complexes.

Now, our quasi-dagger dg K -algebra A is a filtered colimit of affinoid dg K -algebras B , and M is of the form $A \otimes_B N$ for some such B and some finite B -module N . Passing to filtered colimits yet again gives us the desired map

$$\mathbf{R}\Gamma(X_{\text{proét}}, \underline{M}_X \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{D}) \rightarrow M[-d]$$

of complexes. \square

Examples 3.3. If X is an ℓ -coprime proper scheme, then trace maps tr of the form required Proposition 3.2 arise from the six functors formalism (and specifically Poincaré duality) whenever we have a form of duality on the base. Examples include the cases:

- (1) X is a proper separated scheme over a separably closed field k prime to ℓ . In particular, when X is moreover smooth of dimension m over k , we just have $\mathbb{D} \simeq \underline{\mathbb{Z}}_\ell(m)$, with trace

$$\mathrm{H}^{2m}(X_{\mathrm{pro\acute{e}t}}, \underline{\mathbb{Z}}_\ell(m)) \rightarrow \mathbb{Z}_\ell.$$

- (2) X is a proper separated scheme over a non-Archimedean local field k , with trace given by combining Poincaré duality with local Tate duality. When X is moreover smooth of dimension m over k , we have $\mathbb{D} \simeq \underline{\mathbb{Z}}_\ell(m+1)$, with trace

$$\mathrm{H}^{2m+2}(X_{\mathrm{pro\acute{e}t}}, \underline{\mathbb{Z}}_\ell(m+1)) \rightarrow \mathbb{Z}_\ell.$$

- (3) X is a proper separated scheme over a finite field k prime to ℓ . This follows similarly, by combining Poincaré duality with duality for $\mathrm{Gal}(k) \cong \hat{\mathbb{Z}}$ -representations. In particular, when X is moreover smooth of dimension m over k , we have $\mathbb{D} \simeq \underline{\mathbb{Z}}_\ell(m)$, with trace

$$\mathrm{H}^{2m+1}(X_{\mathrm{pro\acute{e}t}}, \underline{\mathbb{Z}}_\ell(m)) \rightarrow \mathbb{Z}_\ell.$$

- (4) X is a proper separated scheme over the extension $k_\infty := k(\mu_{\ell^\infty})$ of a non-Archimedean local field k , with trace given by combining Poincaré duality with a form of local Cassels–Tate pairing similar to [Nek, §10].

In detail, replacing k with a finite extension if necessary, we may assume that $\Gamma := \mathrm{Gal}(k_\infty/k) \cong \mathbb{Z}_\ell$, so $\Lambda := \mathbb{Z}_\ell[\Gamma] \cong \mathbb{Z}_\ell[[t]]$ and for $F_\Gamma(N) := \mathrm{Hom}_{\mathbb{Z}_\ell, \mathrm{cts}}(\Lambda, N)$ we have $\mathbf{R}\Gamma((k_\infty)_{\mathrm{pro\acute{e}t}}, \underline{N}) \simeq \mathbf{R}\Gamma(k_{\mathrm{pro\acute{e}t}}, \underline{F_\Gamma(N)})$. The kernel I of the augmentation map $\Lambda \rightarrow \mathbb{Z}_\ell$ is then a Λ -module of rank 1, and we have a short exact sequence

$$0 \rightarrow \Gamma \otimes_{\mathbb{Z}_\ell} \Lambda \rightarrow \Gamma \otimes_{\mathbb{Z}_\ell} \mathrm{Hom}_\Lambda(I, \Lambda) \rightarrow \mathbb{Z}_\ell \rightarrow 0$$

coming from the isomorphism $I \otimes_\Lambda \mathbb{Z}_\ell \cong \Gamma$. The resulting class in $\mathrm{Ext}_\Lambda^1(\mathbb{Z}_\ell, \Gamma \otimes_{\mathbb{Z}_\ell} \Lambda)$ gives a Λ -module morphism $F_\Gamma(N) \rightarrow (\underline{N} \otimes_{\mathbb{Z}_\ell} \Gamma)[1]$ for all finite ℓ -torsion groups N . This combines with local Tate duality to give a trace

$$\mathrm{H}^1((k_\infty)_{\mathrm{pro\acute{e}t}}, \underline{\Gamma^*(1)}) \rightarrow \mathbb{Z}_\ell,$$

where we write $\Gamma^* := \mathrm{Hom}_{\mathbb{Z}_\ell}(\Gamma, \mathbb{Z}_\ell)$.

More generally for X smooth and proper of dimension m over k_∞ , incorporating Poincaré duality gives a trace

$$\mathrm{H}^{2m+1}(X_{\mathrm{pro\acute{e}t}}, \underline{\Gamma^*(m+1)}) \rightarrow \mathbb{Z}_\ell.$$

- (5) An example with a similar flavour, but not strictly of the form in Proposition 3.2, is given by starting from a variety U over one of the fields k above, and taking an open immersion $j: U \hookrightarrow \bar{U}$ into a complete variety, with complement $i: Z \rightarrow \bar{U}$. If we have a trace map $\mathbf{R}\Gamma_c(U_{\mathrm{pro\acute{e}t}}, \mathbb{D})[r] \rightarrow \mathbb{Z}_\ell$ from cohomology with compact supports, then there is a composite trace map

$$\mathbf{R}\Gamma(Z_{\mathrm{pro\acute{e}t}}, i^* \mathbf{R}j_* \mathbb{D}) \rightarrow \mathbf{R}\Gamma_c(U_{\mathrm{pro\acute{e}t}}, \mathbb{D})[1] \rightarrow \mathbb{Z}_\ell[1-r],$$

via the equivalence

$$\mathbf{R}\Gamma(Z_{\mathrm{pro\acute{e}t}}, i^* \mathbf{R}j_* \mathbb{D}) \simeq \mathrm{cone}(\mathbf{R}\Gamma_c(U_{\mathrm{pro\acute{e}t}}, \mathbb{D}) \rightarrow \mathbf{R}\Gamma(U_{\mathrm{pro\acute{e}t}}, \mathbb{D})),$$

which follows from [Mil2, Prop III.1.2.9].

Unlike cohomology with compact supports, the cohomology theory $\mathbf{R}\Gamma(Z_{\mathrm{pro\acute{e}t}}, i^* \mathbf{R}j_* -)$ carries a cup product; the resulting pairing induced by the

trace then corresponds to the Poincaré duality pairing between $\mathbf{R}\Gamma(U, -)$ and $\mathbf{R}\Gamma_c(U, -)$. This leads to traces acting on cohomology one degree lower than those above. Explicitly, when U is smooth of dimension m over a separably closed base field, we have $\mathbb{D} \simeq \underline{\mathbb{Z}}_\ell(m)$ with trace given by the composite

$$\mathrm{H}^{2m-1}(Z_{\mathrm{proét}}, i^* \mathbf{R}j_* \underline{\mathbb{Z}}_\ell(m)) \rightarrow \mathrm{H}_c^{2m}(U_{\mathrm{proét}}, \underline{\mathbb{Z}}_\ell(m)) \rightarrow \mathbb{Z}_\ell;$$

there are similar statements for local and finite fields, but with different twists and shifts.

Now, for Deligne's oriented fibre product $Z_{\mathrm{proét}} \overset{\leftarrow}{\times}_{\bar{U}_{\mathrm{proét}}} U_{\mathrm{proét}}$, known as the deleted tubular neighbourhood of Z in X , we have

$$\mathbf{R}\Gamma(Z_{\mathrm{proét}} \overset{\leftarrow}{\times}_{\bar{U}_{\mathrm{proét}}} U_{\mathrm{proét}}, p^* L) \simeq \mathbf{R}\Gamma(Z_{\mathrm{proét}}, i^* \mathbf{R}j_* j^* L);$$

see for instance [Ill2, 6.4.4]. This example thus fits into the framework of Proposition 3.2 if we generalise from locally Noetherian schemes to topoi equipped with a cosheaf π_0 of pro-finite sets. The deleted tubular neighbourhood is a pro-étale analogue of the boundary ∂U at infinity of U featuring for instance in [PT, Definition 4.2].

Corollary 3.4. *If X is a topologically Noetherian scheme satisfying the conditions of Proposition 3.2 with $\mathbb{D} = \mathbb{Z}_\ell$, then for any n -shifted pre-symplectic (in the terminology of [Pri7]) derived ∞ -geometric Artin stack $F: dg_+ \mathrm{CAlg}_K \rightarrow s\mathrm{Set}$, the functor*

$$F(X_{\mathrm{proét}}, -): dg_+ \mathrm{AffdAlg}_K^{\mathrm{loc}, \dagger} \rightarrow s\mathrm{Set}$$

of Definition 1.53 carries a functorial $(n - d)$ -shifted pre-symplectic structure at all points; in particular, this implies that any formally étale map $Y \rightarrow F(X_{\mathrm{proét}}, -)$ from a dg dagger-analytic Artin ∞ -stack Y induces an $(n - d)$ -shifted pre-symplectic structure on Y .

Proof. The space $\mathrm{PreSp}(F, n)$ of n -shifted pre-symplectic structures on F is the space of maps from F to the functor $\mathrm{PreSp}^{\mathrm{alg}}(-, n): dg_+ \mathrm{CAlg}_K \rightarrow s\mathrm{Set}$ defined similarly to $\mathrm{PreSp}(-, n)$ in Definition 2.6, but based on algebraic cotangent complexes $\mathbf{L}\Omega_{A^{\mathrm{alg}}}$.

For any quasi-free quasi-dagger dg algebra A , Lemma 3.1 combines with Proposition 3.2 to give us maps

$$\mathbf{R}\Gamma(X_{\mathrm{proét}}, F^2 \mathbf{LDR}(\underline{A}_X^{\mathrm{alg}})) \rightarrow \mathbf{R}\Gamma(X_{\mathrm{proét}}, \underline{F^2 \mathrm{DR}(A)}_X) \rightarrow F^2 \mathrm{DR}(A)[-d],$$

and hence

$$\mathbf{R}\Gamma(X_{\mathrm{proét}}, \mathrm{PreSp}^{\mathrm{alg}}(\underline{A}_X^{\mathrm{alg}}, n)) \rightarrow \mathrm{PreSp}(A, n - d).$$

Since F maps quasi-isomorphisms to weak equivalences, Proposition 1.25 applies, and in particular it suffices to work with the restriction of $F(X_{\mathrm{proét}}, -)$ to objects which are quasi-free. Because $\mathrm{PreSp}^{\mathrm{alg}}(F, n) \simeq \mathbf{R}\mathrm{map}(F, \mathrm{PreSp}^{\mathrm{alg}}(-, n))$, the maps above combine to give a composite natural transformation

$$\mathrm{PreSp}^{\mathrm{alg}}(F, n) \times F(X_{\mathrm{proét}}, -) \rightarrow \mathrm{PreSp}^{\mathrm{alg}}(X_{\mathrm{proét}}, -, n) \rightarrow \mathrm{PreSp}(-, n - d),$$

of functors on quasi-free localised dagger dg algebras.

By adjunction, we can rephrase this as a morphism

$$\mathrm{PreSp}^{\mathrm{alg}}(F, n) \rightarrow \mathbf{R}\mathrm{map}(F(X_{\mathrm{proét}}, -), \mathrm{PreSp}(-, n - d))$$

in the ∞ -category of simplicial sets, where the space of maps is taken in the ∞ -category of homotopy-preserving simplicial set-valued functors on $dg_+ \mathrm{AffdAlg}_K^{\mathrm{loc}, \dagger}$. \square

3.2. Symplectic structures. If the shifted pre-symplectic structure on F is in fact symplectic and the trace from Proposition 3.2 leads to a duality theory, then one might expect that the pre-symplectic structure from Corollary 3.4 is in fact symplectic, as happens for analogous constructions for topological spaces as in [PTVV, §2.1]. In order to establish similar results, we will be cutting down to an open subfunctor on which duality does behave well.

Fix a topologically Noetherian scheme X and assume that we have a trace map

$$\mathrm{tr}: \mathbf{R}\Gamma(X_{\mathrm{pro\acute{e}t}}, \underline{M}_X \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{D}) \rightarrow M[-d]$$

as in Proposition 3.2; assume moreover that \mathbb{D} is constructible, with dual \mathbb{D}^* .

Definition 3.5. Given a quasi-dagger dg algebra A and a presheaf N of \underline{A}_X -modules in chain complexes on $X_{\mathrm{pro\acute{e}t}}$, we say that N satisfies weak duality with respect to the trace tr if for all morphisms $A \rightarrow C$ of quasi-dagger dg algebras, the map

$$\mathbf{R}\underline{\mathrm{Hom}}_{\underline{A}_X}(N, \underline{C}_X) \rightarrow \mathbf{R}\underline{\mathrm{Hom}}_A(\mathbf{R}\Gamma(X, N \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{D})[d], C)$$

induced by the pairing

$$\mathbf{R}\Gamma(X, N \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{D}) \otimes_A^{\mathbf{L}} \mathbf{R}\underline{\mathrm{Hom}}_{\underline{A}_X}(N, \underline{C}_X) \rightarrow \mathbf{R}\Gamma(X, \underline{C}_X \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{D}) \rightarrow C[-d]$$

is a quasi-isomorphism, and $\mathbf{R}\Gamma(X, N \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{D})$ is a perfect A -module.

Note that by taking $C = A \oplus M$, we can deduce a similar quasi-isomorphism for all $M \in dg_+ \mathrm{Coh}_A$ in place of C .

Lemma 3.6. *Given a surjection $g: A \rightarrow B$ of quasi-dagger dg algebras with square-zero kernel I , a presheaf N of \underline{A}_X -modules in chain complexes satisfies weak duality if and only if the presheaf $\bar{N} := N \otimes_{\underline{A}_X}^{\mathbf{L}} \underline{B}_X$ of \underline{B}_X -modules does so.*

Proof. We have an exact triangle

$$\bar{N} \otimes_{\underline{B}_X}^{\mathbf{L}} \underline{I}_X \rightarrow N \rightarrow \bar{N} \rightarrow,$$

and the result then follows immediately by substitution into Definition 3.5. \square

Lemma 3.7. *Given a quasi-dagger dg algebra A , a module $N \in dg_+ \mathrm{Mod}_{\underline{A}_X}$ satisfies weak duality if and only if the presheaf $\bar{N} := N \otimes_{\underline{A}_X}^{\mathbf{L}} \underline{H}_0 \underline{A}_X$ of $\underline{H}_0 \underline{A}_X$ -modules does so.*

Proof. If N satisfies weak duality, then by base change \bar{N} does so. Conversely, if \bar{N} satisfies weak duality then we may apply Lemma 3.6 to the Postnikov tower $\{A/\tau_{>k}A\}$ of A (a sequence of homotopy square-zero extensions), since Proposition 1.49 gives quasi-isomorphism-invariance, and thus deduce that the $\underline{A}/\tau_{>k}\underline{A}_X$ -modules $N \otimes_{\underline{A}_X}^{\mathbf{L}} \underline{A}/\tau_{>k}\underline{A}_X$ satisfy weak duality for all k . The result now follows by writing each quasi-dagger dg C -algebra as $\varprojlim_k (C/\tau_{>k}C)$ and taking homotopy limits in Definition 3.5. \square

Lemma 3.8. *Let X be a quasi-compact and quasi-separated scheme, and assume that we have an ℓ -adically complete commutative Noetherian ring R , together with a constructible \hat{R}_X -complex L in the sense of [BS, §6.5]. Then for any quasi-dagger algebra C equipped with a \mathbb{Z}_ℓ -algebra homomorphism $R \rightarrow C$, we have $\mathbf{R}\Gamma(X_{\mathrm{pro\acute{e}t}}, L \otimes_{\hat{R}_X}^{\mathbf{L}} \underline{C}_X) \simeq \mathbf{R}\Gamma(X_{\mathrm{pro\acute{e}t}}, L) \otimes_R^{\mathbf{L}} C$ and $\mathbf{R}\underline{\mathrm{Hom}}_{\hat{R}_X}(L, \underline{C}_X) \simeq \mathbf{R}\underline{\mathrm{Hom}}_{\hat{R}_X}(L, \hat{R}_X) \otimes_R^{\mathbf{L}} C$.*

Proof. As in §1.4, all quasi-dagger algebras C over R can be written as filtered colimits of ℓ -adically complete R -modules $C(\rho)$. Since L is constructible, the compactness property of [BS, Lemmas 6.3.14 and 6.8.12] gives the second statement. Similarly, cohomology of X preserves filtered colimits giving the first statement. \square

Examples 3.9. Roughly speaking, a sufficient condition for an \underline{A}_X -module N on an ℓ -coprime proper scheme X to satisfy weak duality is that its sheafification is constructible in an appropriate sense; by Lemma 3.7, we can reduce to looking at the $\underline{H}_0\underline{A}_X$ -module $N \otimes_{\underline{A}_X}^{\mathbf{L}} \underline{H}_0\underline{A}_X$. Also note that the category of modules satisfying weak duality is triangulated and idempotent-complete. Although [BS, §6] is developed for fields over \mathbb{Q}_ℓ rather than arbitrary dagger algebras, most of the arguments do generalise.

If there exists an ℓ -adically complete Noetherian ring R , a constructible \hat{R}_X -complex L in the sense of [BS, §6.5] and a K -algebra homomorphism $R \rightarrow H_0A$, then an $\underline{H}_0\underline{A}_X$ -module N with sheafification $L \otimes_{\hat{R}_X} \underline{H}_0\underline{A}_X$ satisfies weak duality whenever X is one of the schemes from Examples 3.3. This follows by combining Lemma 3.8 with the results of [BS, §6.5] and the six functors formalism, which leads to equivalences

$$\mathbf{R}\Gamma(X_{\text{proét}}, L^*) \simeq \mathbf{R}\underline{\text{Hom}}_{\mathbb{Z}_\ell}(\mathbf{R}\Gamma(X_{\text{proét}}, L \otimes_{\mathbb{Z}_\ell} \mathbb{D})[d], \mathbb{Z}_\ell)$$

in each case.

Every derived ∞ -geometric Artin stack F has a cotangent complex \mathbb{L}^F , consisting of suitably functorial A -modules $\mathbb{L}^{F,A,x}$ in chain complexes for each $x \in F(A)$.

Lemma 3.10. *Let X be a topologically Noetherian scheme satisfying the conditions of Proposition 3.2, and $F: dg_+\text{CAlg}_K \rightarrow s\text{Set}$ a derived ∞ -geometric Artin stack. At any point $\phi \in F(X_{\text{proét}}, A)$ at which the presheaf $\mathbb{L}^{F,\underline{A}_X,\phi}$ of \underline{A}_X -modules satisfies weak duality in the sense of Definition 3.5, the functor*

$$T_\phi(F(X_{\text{proét}}, -), -): M \mapsto F(X_{\text{proét}}, A \oplus M) \times_{F(X_{\text{proét}}, A)}^h \{\phi\}$$

on levelwise f.g. A -modules M is represented by the perfect complex

$$\mathbf{R}\Gamma(X, \mathbb{L}^{F,\underline{A}_X,\phi} \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{D})[d].$$

Proof. Since homotopy limits commute, we have

$$\begin{aligned} F(X_{\text{proét}}, A \oplus M) \times_{F(X_{\text{proét}}, A)}^h \{\phi\} &\simeq \mathbf{R}\Gamma(X_{\text{proét}}, F(\underline{A}_X \oplus \underline{M}_X) \times_{F(\underline{A}_X)}^h \{\phi\}) \\ &\simeq \mathbf{R}\Gamma(X_{\text{proét}}, N^{-1}\tau_{\geq 0}\mathbf{R}\mathcal{H}om_{\underline{A}_X}(\mathbb{L}^{F,\underline{A}_X,\phi}, \underline{M}_X)) \\ &\simeq N^{-1}\tau_{\geq 0}\mathbf{R}\underline{\text{Hom}}_{\underline{A}_X}(\mathbb{L}^{F,\underline{A}_X,\phi}, \underline{M}_X), \end{aligned}$$

where N^{-1} is Dold–Kan denormalisation and $\tau_{\geq 0}$ denotes good truncation of a chain complex.

By weak duality, this in turn is equivalent to

$$N^{-1}\tau_{\geq 0}\mathbf{R}\underline{\text{Hom}}_A(\mathbf{R}\Gamma(X, \mathbb{L}^{F,\underline{A}_X,\phi} \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{D})[d], M)$$

so is represented by $\mathbf{R}\Gamma(X, \mathbb{L}^{F,\underline{A}_X,\phi} \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{D})[d]$. \square

Corollary 3.11. *Let X be a topologically Noetherian scheme satisfying the conditions of Proposition 3.2 with $\mathbb{D} = \mathbb{Z}_\ell$, and let $F: dg_+\text{CAlg}_K \rightarrow s\text{Set}$ be an n -shifted symplectic derived Artin ∞ -stack. Then there is a natural $(n-d)$ -shifted symplectic structure (in the sense of Remark 2.18) on the full subfunctor $F(X_{\text{proét}}, -)^{wd} \subset F(X_{\text{proét}}, -)$ consisting of points ϕ at which the presheaf $\mathbb{L}^{F,\underline{A}_X,\phi}$ satisfies weak duality.*

Proof. The $(n-d)$ -shifted pre-symplectic structure ω of Corollary 3.4 pulls back along the inclusion map $F(X_{\text{proét}}, -)^{wd} \subset F(X_{\text{proét}}, -)$ to give an $(n-d)$ -shifted pre-symplectic structure, and we need to check that it is non-degenerate. By Lemma 3.6, the inclusion map is formally étale, so $\mathbb{L}^{F(X_{\text{proét}}, -)^{wd}, \phi} \simeq \mathbb{L}^{F(X_{\text{proét}}, -), \phi}$ at all points ϕ .

By Lemma 3.10 and its proof, we have

$$\begin{aligned} \mathbb{L}^{F(X_{\text{proét}}, -), \phi} &\simeq \mathbf{R}\Gamma(X, \mathbb{L}^{F, \underline{A}_X, \phi})[d], \\ \mathbf{R}\underline{\text{Hom}}_A(\mathbb{L}^{F(X_{\text{proét}}, -), \phi}, A) &\simeq \mathbf{R}\Gamma(X_{\text{proét}}, \mathbf{R}\mathcal{H}om_{\underline{A}_X}(\mathbb{L}^{F, \underline{A}_X, \phi}, \underline{A}_X)) \end{aligned}$$

since ω is induced by an n -shifted symplectic structure on F , the map

$$\omega_2^\sharp: \mathbf{R}\underline{\text{Hom}}_A(\mathbb{L}^{F(X_{\text{proét}}, -), \phi}, A) \rightarrow (\mathbb{L}^{F(X_{\text{proét}}, -), \phi})_{[d-n]}$$

then comes from derived global sections of the quasi-isomorphism

$$\mathbf{R}\mathcal{H}om_{\underline{A}_X}(\mathbb{L}^{F, \underline{A}_X, \phi}, \underline{A}_X) \rightarrow \mathbb{L}^{F, \underline{A}_X, \phi}[n],$$

so is itself a quasi-isomorphism. \square

Corollary 3.12. *Under the conditions of Corollary 3.11, take an (underived) dagger Artin analytic ∞ -stack Y equipped with a formally étale morphism*

$$\eta: Y \rightarrow \pi^0 F(X_{\text{proét}}, -)$$

of functors $\text{AffdAlg}_K^{\text{loc}, \dagger} \rightarrow \text{sSet}$, such that at all points ϕ in the image of η , the presheaf $\mathbb{L}^{F, \underline{A}_X, \phi}$ satisfies weak duality.

Then the functor $\tilde{Y}: A \mapsto Y(\mathbf{H}_0 A) \times_{F(X_{\text{proét}}, \mathbf{H}_0 A)}^h F(X_{\text{proét}}, A)$ on $\text{dg}_+ \text{AffdAlg}_K^{\text{loc}, \dagger}$ is a dg dagger Artin analytic ∞ -stack carrying a natural $(n-d)$ -shifted symplectic structure.

Proof. Since F is homotopy-preserving, nilcomplete and homogeneous, Corollary 1.51 implies that the same is true of $A \mapsto F(\underline{A}_X(U))$ for all U ; it is thus also true for $F(X_{\text{proét}}, -)$ by passing to homotopy limits. Moreover, Lemmas 3.6 and 3.7 imply that the same is true of the full subfunctor $F(X_{\text{proét}}, -)^{wd}$; they moreover imply that $F(X_{\text{proét}}, A)^{wd} \simeq F(X_{\text{proét}}, \mathbf{H}_0 A)^{wd} \times_{F(X_{\text{proét}}, \mathbf{H}_0 A)}^h F(X_{\text{proét}}, A)$, which in particular gives us a map $\tilde{Y} \rightarrow F(X_{\text{proét}}, -)^{wd}$.

Existence of a perfect cotangent complex from Lemma 3.10 then ensures that for any dagger algebra A and any $\phi \in F(X_{\text{proét}}, A)$ at which $\mathbb{L}^{F, \underline{A}_X, \phi}$ satisfies weak duality, the A -module $\text{Ext}_A^i(\mathbb{L}^{F(X_{\text{proét}}, -), \phi}, A)$ is finitely generated. Moreover, the universal nature of Definition 3.5 ensures that $\mathbb{L}^{F(X_{\text{proét}}, -), \phi'} \simeq \mathbb{L}^{F(X_{\text{proét}}, -), \phi} \otimes_A^{\mathbf{L}} A'$ for ϕ' the image of ϕ under $F(A) \rightarrow F(A')$, so flat base change gives

$$\text{Ext}_{A'}^i(\mathbb{L}^{F(X_{\text{proét}}, -), \phi}, A') \cong \text{Ext}_A^i(\mathbb{L}^{F(X_{\text{proét}}, -), \phi}, A) \otimes_A A'$$

whenever $A \rightarrow A'$ is étale.

Thus all the conditions of Corollary 1.47 are satisfied, making \tilde{Y} a dg dagger Artin analytic ∞ -stack, with cotangent complex $\mathbb{L}^{\tilde{Y}, \phi} \simeq \mathbb{L}^{F(X_{\text{proét}}, -), \phi}$. It has a shifted symplectic structure induced by pulling back along the formally étale map $\tilde{Y} \rightarrow F(X_{\text{proét}}, -)$. \square

Examples 3.13. Examples 3.3 and 3.9 now lead to some instances of $(n-d)$ -shifted symplectic moduli stacks when substituted into Corollary 3.12. If we take the derived stack F to be $B\text{GL}_n$, or BG for any other affine algebraic group equipped with a G -equivariant inner product on its Lie algebra, or to be the moduli stack of perfect complexes, then $n = 2$, so the corollary produces $(2-d)$ -shifted symplectic structures on moduli of G -torsors or of complexes of pro-étale sheaves on X , provided we impose some constructibility constraints.

In particular:

- (1) If X is a smooth proper scheme of dimension m over a separably closed field prime to ℓ , we have $(n - 2m)$ -shifted symplectic structures on suitable open substacks of the derived moduli stack of F -valued sheaves on $X_{\text{proét}}$, depending on a choice of isomorphism $\mathbb{Z}_\ell(m) \cong \mathbb{Z}_\ell$.
- (2) If X is a smooth proper scheme of dimension m over a cyclotomic extension $k_\infty := k(\mu_{\ell^\infty})$ of a non-Archimedean local field k , we have $(n - 2m - 1)$ -shifted symplectic structures on suitable open substacks of the derived moduli stack of F -valued sheaves on $X_{\text{proét}}$, depending on a choice of isomorphism $\mathbb{Z}_\ell(m + 1) \cong \text{Gal}(k_\infty/k)$.
- (3) If U is a smooth scheme of dimension m over a separably closed field prime to ℓ , a choice of trivialisation of $\mathbb{Z}_\ell(m)$ similarly leads to an $(n + 1 - 2m)$ -shifted symplectic structure on suitable open substacks of the derived moduli stack

$$A \mapsto \mathbf{R}\Gamma(Z_{\text{proét}}, i^* \mathbf{R}j_* F(\underline{A}_X))$$

of F -valued sheaves on the deleted tubular neighbourhood $Z \overset{\leftarrow}{\times}_{\bar{U}} U$ (thought of as the boundary of U), where $i: Z \rightarrow \bar{U}$ is the complement of U in a compactification $j: U \rightarrow \bar{U}$.

However, the hypothesis $\mathbb{D} \cong \mathbb{Z}_\ell$ of Corollary 3.11 is not satisfied by the other cases of Examples 3.3, which we will tackle by using weighted symplectic structures in §4, leading to Examples 5.13.

Remark 3.14. When \tilde{Y} is a smooth (underived) analytic space or DM stack, a 0-shifted symplectic structure is just a symplectic structure in the classical sense. In particular, this applies when studying ℓ -adic local systems or G -torsors on a smooth proper curve over an algebraically closed field, giving the symplectic structures studied in [Pap].

3.3. Lagrangian structures. We now assume that we have a morphism $f: \partial U \rightarrow U$ of topologically Noetherian schemes (where the notation ∂U is intended to be suggestive, but does not indicate a specific construction at this point) and a constructible complex \mathbb{D} on U , equipped with a system of trace maps

$$\text{cone}(\mathbf{R}\Gamma(U_{\text{ét}}, \mathbb{D}/\ell^n) \rightarrow \mathbf{R}\Gamma(\partial U_{\text{ét}}, f^{-1}\mathbb{D}/\ell^n))[d - 1] \rightarrow \mathbb{Z}/\ell^n$$

similarly to Proposition 3.2. We will require the resulting composite pairing

$$\mathbf{R}\Gamma(U_{\text{ét}}, L/\ell^n) \otimes \text{cone}(\mathbf{R}\Gamma(U_{\text{ét}}, L^* \otimes \mathbb{D}/\ell^n) \rightarrow \mathbf{R}\Gamma(\partial U_{\text{ét}}, f^{-1}(L^* \otimes \mathbb{D}/\ell^n)))[d - 1] \rightarrow \mathbb{Z}/\ell^n$$

to be perfect for constructible complexes L with dual L^* .

Examples 3.15. Trace maps tr of the form required above arise from the six functors formalism (and specifically Poincaré duality) in the following situations, with $\mathbf{R}\Gamma_c(U, L) \simeq \text{cone}(\mathbf{R}\Gamma(U, L) \rightarrow \mathbf{R}\Gamma(\partial U, f^*L))[-1]$ in each case:

- (1) If F is a non-Archimedean local field with residue characteristic prime to ℓ , and \mathcal{O}_F its ring of integers, then we can look at the morphism $\text{Spec } F \rightarrow \text{Spec } \mathcal{O}_F$. Local Tate duality in the form of [Mil1, Theorem II.1.8(b)] then gives the desired trace pairing, with $\mathbb{D} = \underline{\mathbb{Z}_\ell(1)}$ and $d = 3$.
- (2) More generally, if Z is a smooth proper scheme of dimension m over \mathcal{O}_F , consider the morphism $Z \times_{\text{Spec } \mathcal{O}_F} \text{Spec } F \rightarrow Z$. Local Tate duality combines with Poincaré duality to give the desired trace pairing here, with $\mathbb{D} = \underline{\mathbb{Z}_\ell(m + 1)}$ and $d = 2m + 3$.

- (3) If F is a number field and S a finite set of primes, with $\mathcal{O}_{F,S}$ the localisation of \mathcal{O}_F at S , then consider the morphism $\mathrm{Spec} \prod_{v \in S} F_v \rightarrow \mathrm{Spec} \mathcal{O}_{F,S}$. Poitou–Tate duality in the form of [Mil1, Theorems II.2.3 and II.3.1] then gives the desired trace pairing, with $\mathbb{D} = \underline{\mathbb{Z}_\ell(1)}$ and $d = 3$.
- (4) More generally, if Z is a smooth proper scheme of dimension m over $\mathcal{O}_{F,S}$, consider the morphism $Z \times_{\mathrm{Spec} \mathcal{O}_{F,S}} \mathrm{Spec} \prod_{v \in S} F_v \rightarrow Z$. Poitou–Tate duality combines with Poincaré duality to give the desired trace pairing here, with $\mathbb{D} = \underline{\mathbb{Z}_\ell(m+1)}$ and $d = 2m + 3$.
- (5) There are variations of the previous examples in which we adjoin μ_{ℓ^∞} to the base, resulting in duality one dimension lower via pairings of Cassels–Tate form. Explicitly, for F a non-Archimedean local field with residue characteristic prime to ℓ , we can look at $\mathrm{Spec} F(\mu_{\ell^\infty}) \rightarrow \mathrm{Spec} \mathcal{O}_F(\mu_{\ell^\infty})$. The trace map of Example 3.3.(4) then gives a trace pairing with $\mathbb{D} = \Gamma^*(1)$ and $d = 2$.

Similarly, for F a number field and S a finite set of primes, we can consider the morphism $\mathrm{Spec} \prod_{v \in S} F_v(\mu_{\ell^\infty}) \rightarrow \mathrm{Spec} \mathcal{O}_{F,S}(\mu_{\ell^\infty})$. An identical argument, using Poitou–Tate duality rather than local Tate duality, gives a trace pairing with $\mathbb{D} = \underline{\Gamma^*(1)}$ and $d = 2$.

For smooth proper schemes of dimension m over those bases, we can combine these with Poincaré duality to obtain a trace pairing with $\mathbb{D} = \underline{\Gamma^*(m+1)}$ and $d = 2m + 2$.

For these examples, however, the pairing will seldom be perfect on finite coefficients, since the Pontrjagin dual $F_\Gamma(N)^\vee$ of $F_\Gamma(N) \cong N[t, t^{-1}]/tN[t]$ is $\mathcal{F}_\Gamma(N^\vee) := \Lambda \otimes_{\mathbb{Z}_\ell} N^\vee \cong N^\vee[[t]]$ and the duality morphism $F_\Gamma(\Gamma^* \otimes_{\mathbb{Z}_\ell} N^\vee) \rightarrow \mathcal{F}_\Gamma(N^\vee)[1]$ induced by the trace of Example 3.3.(4) is not an isomorphism. The latter corresponds to the extension $N^\vee((t))$ of $N^\vee[t, t^{-1}]/N^\vee[t]$ by $N^\vee[[t]]$, whose cohomology need not vanish.

For our purposes, it will instead suffice to impose the condition that $\mathrm{Hom}_{\mathbb{Z}_\ell}(\mathbb{Q}_\ell, H^*(F(\mu_{\ell^\infty})_{\acute{e}t}, L \otimes_{\mathbb{Z}_\ell} (\mathbb{Q}_\ell/\mathbb{Z}_\ell)))$, or dually $H_{\mathrm{Iw}}^*(F(\mu_{\ell^\infty})/F, L) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ in the notation of [Nek], is finite-dimensional over \mathbb{Q}_ℓ . This would imply vanishing of $H_{\mathrm{Iw}}^*(F(\mu_{\ell^\infty})/F, L) \otimes_{\mathbb{Z}_\ell[[t]]} \mathbb{Q}_\ell\{\{t\}\}$, for the 2-dimensional local field $\mathbb{Q}_\ell\{\{t\}\}$ (see e.g. [Zhu]), which corresponds to the \mathbb{Q}_ℓ version of the error term above, being the cohomology of $L \otimes_{\mathbb{Z}_\ell[[t]]} \mathbb{Q}_\ell\{\{t\}\}$ regarded as a 2-Tate object in the sense of [BGW, §7]. Such vanishing also identifies the cohomology groups of $F(\mu_{\ell^\infty})$ with Iwasawa cohomology groups one degree higher, tying in with the Cassels–Tate pairing of [Nek, §10] since everything is torsion or cotorsion in that case.

- (6) A situation of a similar flavour is given if U is a smooth variety of dimension m over a separably closed field k which is prime to ℓ , and we have an open immersion $j: U \hookrightarrow X$ into a complete variety, with complement $i: Z \rightarrow X$, so let $\partial U := Z \overset{\leftarrow}{\times}_X U$, the deleted tubular neighbourhood. The projection map $p: \partial U \rightarrow U$ is of the desired form, for $\mathbb{D} = \underline{\mathbb{Z}_\ell(m)}$ and $d = 2m$. This follows because $\mathbf{R}\Gamma(\partial U, p^*L) \simeq \mathbf{R}\Gamma(Z, i^*\mathbf{R}j_*L)$ and hence $\mathbf{R}\Gamma(X, \mathbf{R}j_!L) \simeq \mathrm{cone}(\mathbf{R}\Gamma(U, L) \rightarrow \mathbf{R}\Gamma(\partial U, f^*L))[-1]$.

The same example also works over local (resp. finite) fields k prime to ℓ , with $d = 2m + 2$ (resp. $2m + 1$) and $\mathbb{D} = \underline{\mathbb{Z}_\ell(m+1)}$ (resp. $\underline{\mathbb{Z}_\ell(m)}$), and over $k(\mu_{\ell^\infty})$ with $d = 2m + 1$ and $\mathbb{D} = \underline{\Gamma^*(m+1)}$

Example 3.16. We can also combine the previous types of Examples 3.15, at least at a formal level. Take U to be a smooth separated of dimension m over a either the ring of integers \mathcal{O}_F of a non-Archimedean local field, or over a localisation $\mathcal{O}_{F,S}$ of the ring of integers of a number field (there are also versions attaching μ_{ℓ^∞} to the base, replacing d with $d - 1$). The idea then is to take a deleted tubular neighbourhood $\partial_0 U \rightarrow U$ over the same base, with $\partial_{01} U \rightarrow \partial_1 U$ being the base change of that morphism to F (resp. $\prod_{v \in S} F_v$). Then the morphisms $\partial_{01} U \rightarrow \partial_0 U$ and $\partial_{01} U \rightarrow \partial_1 U$ both give trace pairings of the desired form, with $d = 2m + 2$ and $\mathbb{D} = \mathbb{Z}_\ell(m + 1)$.

More significantly, if we formally construct ∂U to be the pushout $\partial_0 U \cup_{\partial_{01} U} \partial_1 U$, then the morphism $\partial U \rightarrow U$ gives a trace pairing of that form, with $d = 2m + 3$ and $\mathbb{D} = \mathbb{Z}_\ell(m + 1)$.

Explicitly, if we have an open immersion $j: U \hookrightarrow X$ into a flat proper scheme over \mathcal{F} , with complement $i: Z \rightarrow X$, then we let $\partial_0 U := Z \overleftarrow{\times}_X U$, the deleted tubular neighbourhood, while $\partial_1 U = \prod_v U_{F_v}$ (base change along $\mathcal{O}_{F,S} \rightarrow \prod_{v \in S} F_v$, or along $\mathcal{O}_F \rightarrow F$ in the local case). Thus $\partial_{01} U = \prod_v Z_{F_v} \overleftarrow{\times}_{X_{F_v}} U_{F_v}$.

For for the Galois group $G_{F,S}$ associated to $\mathcal{O}_{F,S}$ and G_v the Galois group of F_v , Poitou–Tate duality gives $\prod_{v \in S} \mathbf{R}\Gamma(G_v, -) \simeq \text{cone}(\mathbf{R}\Gamma_c(G_{F,S}, -) \rightarrow \mathbf{R}\Gamma(G_{F,S}, -))$, while the six functors formalism gives $\mathbf{R}\Gamma(Z_{\bar{F}}, i^* \mathbf{R}j_* -) \simeq \text{cone}(\mathbf{R}\Gamma_c(U_{\bar{F}}, -) \rightarrow \mathbf{R}\Gamma(U_{\bar{F}}, -))$. For a constructible complex L on U , this means that for the Galois group $G_{F,S}$ associated to $\mathcal{O}_{F,S}$, we have:

$$\begin{aligned} \mathbf{R}\Gamma(U, L) &\simeq \mathbf{R}\Gamma(G_{F,S}, \mathbf{R}\Gamma(U_{\bar{F}}, L)) \\ \mathbf{R}\Gamma(\partial_0 U, f_0^* L) &\simeq \text{cone}(\mathbf{R}\Gamma(G_{F,S}, \mathbf{R}\Gamma_c(U_{\bar{F}}, L)) \rightarrow \mathbf{R}\Gamma(G_{F,S}, \mathbf{R}\Gamma(U_{\bar{F}}, L))) \\ \mathbf{R}\Gamma(\partial_1 U, f_1^* L) &\simeq \text{cone}(\mathbf{R}\Gamma_c(G_{F,S}, \mathbf{R}\Gamma(U_{\bar{F}}, L)) \rightarrow \mathbf{R}\Gamma(G_{F,S}, \mathbf{R}\Gamma(U_{\bar{F}}, L))). \end{aligned}$$

Moreover

$$\mathbf{R}\Gamma(\partial_{01} U, f_{01}^* L) \simeq \prod_{v \in S} \mathbf{R}\Gamma(Z_{F_v}, i^* \mathbf{R}j_* L),$$

which in turn is quasi-isomorphic to the total cone complex of

$$\begin{aligned} &\mathbf{R}\Gamma_c(G_{F,S}, \mathbf{R}\Gamma_c(U_{\bar{F}}, L)) \\ &\rightarrow \mathbf{R}\Gamma_c(G_{F,S}, \mathbf{R}\Gamma(U_{\bar{F}}, L)) \oplus \mathbf{R}\Gamma(G_{F,S}, \mathbf{R}\Gamma_c(U_{\bar{F}}, L)) \\ &\rightarrow \mathbf{R}\Gamma(G_{F,S}, \mathbf{R}\Gamma(U_{\bar{F}}, L)). \end{aligned}$$

Thus

$$\begin{aligned} \mathbf{R}\Gamma(\partial U, f^* L) &\simeq \text{cone}(\mathbf{R}\Gamma(\partial_0 U, f_0^* L) \oplus \mathbf{R}\Gamma(\partial_1 U, f_1^* L) \rightarrow \mathbf{R}\Gamma(\partial_{01} U, f_{01}^* L))[-1] \\ &\simeq \text{cone}(\mathbf{R}\Gamma_c(G_{F,S}, \mathbf{R}\Gamma_c(U_{\bar{F}}, L)) \rightarrow \mathbf{R}\Gamma(G_{F,S}, \mathbf{R}\Gamma(U_{\bar{F}}, L))), \end{aligned}$$

from which the desired duality for the trace pairing associated to $\partial U \rightarrow U$ follows.

Definition 3.17. Given a quasi-dagger dg algebra A and a presheaf N of \underline{A}_U -modules in chain complexes on $U_{\text{proét}}$, we say that N satisfies weak duality with respect to the trace tr above if $\mathbf{L}f^* N := f^{-1} N \otimes_{f^{-1} \underline{A}_U}^{\mathbf{L}} \underline{A}_{\partial U}$ satisfies weak duality in the sense of Definition 3.5, and for all morphisms $A \rightarrow C$ of quasi-dagger dg algebras, the map

$$\begin{aligned} &\text{cone}(\mathbf{R}\underline{\text{Hom}}_{\underline{A}_U}(N, \underline{C}_U) \rightarrow \mathbf{R}\underline{\text{Hom}}_{\underline{A}_{\partial U}}(\mathbf{L}f^* N, \underline{C}_{\partial U})) \\ &\rightarrow \mathbf{R}\underline{\text{Hom}}_A(\mathbf{R}\Gamma(U, N \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{D})[d - 1], C) \end{aligned}$$

induced by the pairing

$$\begin{aligned} \mathbf{R}\Gamma(U, N \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{D}) \otimes_A^{\mathbf{L}} \operatorname{cone}(\mathbf{R}\underline{\operatorname{Hom}}_{\underline{A}_U}(N, \underline{C}_U) \rightarrow \mathbf{R}\underline{\operatorname{Hom}}_{\underline{A}_{\partial U}}(\mathbf{L}f^*N, \underline{C}_{\partial U})) \\ \rightarrow \operatorname{cone}(\mathbf{R}\Gamma(U, \underline{C}_U \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{D}) \rightarrow \mathbf{R}\Gamma(\partial U, \underline{C}_{\partial U} \hat{\otimes}_{\mathbb{Z}_\ell} f^{-1}\mathbb{D})) \xrightarrow{\text{tr}} C[1-d] \end{aligned}$$

is a quasi-isomorphism, and $\mathbf{R}\Gamma(U, N \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{D})$ is a perfect A -module.

Examples 3.18. Similarly to Examples 3.9, if there exists an ℓ -adically complete Noetherian ring R , a constructible \hat{R}_U -complex L in the sense of [BS, §6.5] and a K -algebra homomorphism $R \rightarrow H_0A$, then an H_0A_U -module N with sheafification $L \otimes_{\hat{R}_U} H_0A_U$ satisfies weak duality whenever $(U, \partial U)$ is one of the pairs of schemes or sites from Examples 3.15.

The proofs of Lemmas 3.6 and 3.7 then adapt to give the same statements verbatim in this setting. Corollary 3.11 adapts to give:

Corollary 3.19. *Take a morphism $\partial U \rightarrow U$ of topologically Noetherian schemes which has a trace pairing as above with $\mathbb{D} = \mathbb{Z}_\ell$, and $F: dg_+ \mathbf{CAlg}_K \rightarrow s\text{Set}$ an n -shifted symplectic derived Artin ∞ -stack. Then for the full subfunctors $F(\partial U_{\text{proét}}, -)^{wd} \subset F(\partial U_{\text{proét}}, -)$ and $F(U_{\text{proét}}, -)^{wd} \subset F(U_{\text{proét}}, -)$ consisting at points ϕ at which the presheaf $\mathbb{L}^{F, \underline{A}_{\partial U}, \phi}$ (resp. $\mathbb{L}^{F, \underline{A}_U, \phi}$) satisfies weak duality in the sense of Definition 3.5 (resp. Definition 3.17), the natural map*

$$F(U_{\text{proét}}, -)^{wd} \rightarrow F(\partial U_{\text{proét}}, -)^{wd}$$

carries a natural Lagrangian structure with respect to the $(n-d+1)$ -shifted symplectic structure on $F(\partial U_{\text{proét}}, -)^{wd}$ given by Corollary 3.11.

Corollary 3.12 then adapts to give:

Corollary 3.20. *In the setting of Corollary 3.19, take a morphism $W \rightarrow Y$ of (underived) dagger Artin analytic ∞ -stacks, equipped with a commutative diagram*

$$\begin{array}{ccc} W & \xrightarrow{\eta'} & \pi^0 F(U_{\text{proét}}, -) \\ \downarrow & & \downarrow f^* \\ Y & \xrightarrow{\eta} & \pi^0 F(\partial U_{\text{proét}}, -) \end{array}$$

of functors $\text{AffdAlg}_K^{\text{loc}, \dagger} \rightarrow s\text{Set}$. Assume moreover that the horizontal maps η, η' are formally étale and that at all points ϕ in the image of η' (resp. η), the presheaf $\mathbb{L}^{F, \underline{A}_{\partial U}, \phi}$ (resp. $\mathbb{L}^{F, \underline{A}_U, \phi}$) satisfies weak duality in the sense of Definition 3.5 (resp. Definition 3.5).

Then the functor $\tilde{W}: A \mapsto W(H_0A) \times_{F(\partial U_{\text{proét}}, H_0A)}^h F(\partial U_{\text{proét}}, A)$ on $dg_+ \text{AffdAlg}_K^{\text{loc}, \dagger}$ is a dg dagger Artin analytic ∞ -stack. It carries a natural Lagrangian structure with respect to the $(n-d+1)$ -shifted symplectic structure on \tilde{Y} given by Corollary 3.12.

Example 3.21. Examples 3.15 and 3.18 now lead to some instances of Lagrangians in $(n+1-d)$ -shifted symplectic moduli stacks when substituted into Corollary 3.20. If we take the derived stack F to be BGL_n , or BG for any other affine algebraic group equipped with a G -equivariant inner product on its Lie algebra, or to be the moduli stack of perfect complexes, then $n=2$, so the corollary produces $(3-d)$ -shifted Lagrangian structures on moduli of G -torsors or of complexes of pro-étale sheaves on U , provided we impose some constructibility constraints.

In particular, if U is a smooth proper scheme of dimension m over a separably closed field prime to ℓ , we have $(n + 1 - 2m)$ -shifted Lagrangian structures on suitable open substacks of the derived moduli stack of F -valued sheaves on $U_{\text{proét}}$, depending on a choice of isomorphism $\mathbb{Z}_\ell(m) \cong \mathbb{Z}_\ell$. These stacks are Lagrangian over the derived moduli stack of F -valued sheaves on the deleted tubular neighbourhood, as in Examples 3.13.

The other cases of Examples 3.15 for which the hypothesis $\mathbb{D} \cong \mathbb{Z}_\ell$ of Corollary 3.19 is satisfied are those for which the base field contains μ_{ℓ^∞} . We will tackle the other cases using weighted symplectic structures in Examples 5.18; each example there has an unweighted analogue incorporating μ_{ℓ^∞} , but with the shift one degree higher and additional restrictions following Example 3.15.(5).

3.4. Infinitesimal formality for moduli of G -torsors.

3.4.1. *Deformation functors.* We can look at infinitesimal neighbourhoods of any K -valued point ϕ in one of our derived moduli functors, giving rise to the functor of derived deformations of ϕ . This is purely algebraic in nature, essentially because Artinian K -algebras have unique EFC-structures and are already dagger affinoid. Explicitly, each such derived deformation functor takes inputs from the category of Artinian local K -CDGAs with residue field K , and sends A to the homotopy fibre

$$F(X_{\text{proét}}, A)_\phi := F(X_{\text{proét}}, A) \times_{F(X_{\text{proét}}, K)}^h \{\phi\}.$$

3.4.2. *DGLAs.* For simplicity, we focus on the functor $(BG)(X_{\text{proét}}, -)$ of moduli of G -torsors², for an affine algebraic group G equipped with a G -invariant inner product on its Lie algebra \mathfrak{g} . In particular, $(BG)(X_{\text{proét}}, K)$ is equivalent to the groupoid of pro-étale $G(K)_X$ -torsors. With reasoning as in [Pri6, §4.5], the functor of derived deformations of a torsor ϕ is governed by a DG Lie algebra of the form

$$\mathbf{R}\Gamma(X_{\text{proét}}, \underline{\mathfrak{g}}_{\phi, X}),$$

where $\underline{\mathfrak{g}}_\phi$ is the sheaf given by the adjoint bundle of ϕ (i.e. $(\phi \times \underline{\mathfrak{g}}_X)/G(K)_X$), and derived global sections are taken in the model category of DGLAs (if using a Čech complex, the simplest models for such homotopy limits involve Thom–Sullivan cochains). Specifically, following [Hin], we say that a derived deformation functor is governed by a DGLA L means if it is equivalent to the functor $A \mapsto \underline{\mathbf{MC}}(L \otimes \mathfrak{m}_A)$, where \mathfrak{m}_A is the maximal dg-ideal of A .³

In our setting, the G -invariant inner product $\langle -, - \rangle$ on the Lie algebra \mathfrak{g} induces a morphism

$$\mathbf{R}\Gamma(X_{\text{proét}}, \underline{\mathfrak{g}}_{\phi, X}) \times \mathbf{R}\Gamma(X_{\text{proét}}, \underline{\mathfrak{g}}_{\phi, X}) \xrightarrow{\langle -, - \rangle} \mathbf{R}\Gamma(X_{\text{proét}}, \mathbb{Q}_\ell)$$

which is compatible with the Lie bracket in the sense that $\langle [x, y], z \rangle = \langle x, [y, z] \rangle$. If the conditions of Proposition 3.2 are satisfied with $\mathcal{D} = \mathbb{Z}_\ell$, then we can moreover compose

²Taking a general homogeneous functor instead of BG , [Pri2, Theorems 2.30 and 4.57] give us a hypersheaf of DGLAs instead of $\underline{\mathfrak{g}}_{\phi, X}$, but shifted symplectic structures are generally encoded by more complicated data than the inner products we will work with on \mathfrak{g} .

³There is also an equivalent form given by the homotopy quotient of $\underline{\mathbf{MC}}(L \otimes \mathfrak{m}_A)$ by the gauge action $*$ of the simplicial group $\underline{\mathbf{Gg}}(L \otimes \mathfrak{m}_A)$ given by $n \mapsto \exp(L \otimes \mathfrak{m}_A \otimes \Omega^\bullet(\Delta^n))^0$, the equivalence following by looking at tangent cohomology. There is a further equivalent form given by the nerve of the simplicial groupoid with objects $\underline{\mathbf{MC}}(L \otimes \mathfrak{m}_A)_0$ and with morphisms $\{g \in \underline{\mathbf{Gg}}(L \otimes \mathfrak{m}_A) : g * x = y\}$ between each pair x, y of objects; for further discussion and references, see [Pri6, Remark 4.7].

this with the trace map $\mathrm{tr}: \mathbf{R}\Gamma(X_{\mathrm{pro\acute{e}t}}, \underline{K}) \rightarrow K[-d]$ to give a shifted inner product

$$\langle -, - \rangle_{\mathrm{tr}}: \mathbf{R}\Gamma(X_{\mathrm{pro\acute{e}t}}, \underline{\mathfrak{g}}_{\phi, X}) \times \mathbf{R}\Gamma(X_{\mathrm{pro\acute{e}t}}, \underline{\mathfrak{g}}_{\phi, X}) \rightarrow K[-d]$$

satisfying $\langle [x, y], z \rangle_{\mathrm{tr}} = \langle x, [y, z] \rangle_{\mathrm{tr}}$.

3.4.3. Shifted symplectic structures. Since a Maurer–Cartan element $\alpha \in \underline{\mathrm{MC}}(\mathbf{R}\Gamma(X_{\mathrm{pro\acute{e}t}}, \underline{\mathfrak{g}}_{\phi, X}) \otimes \mathfrak{m}_A)$ defines an element of $(BG)(X_{\mathrm{pro\acute{e}t}}, A)$, Corollary 3.4 induces a $(2-d)$ -shifted pre-symplectic structure $\omega_A \in F^2\mathrm{DR}(A)$ on A whenever X satisfies the conditions of Proposition 3.2 with $\mathbb{D} = \mathbb{Z}_\ell$.

In order to understand this $(2-d)$ -shifted pre-symplectic structure on A explicitly, note that the tangent complex $T_\alpha(\underline{\mathrm{MC}}(L \otimes -))$ is quasi-isomorphic to $(L \otimes A, \delta + [\alpha, -])[1]$. The morphism $\mathbb{L}\underline{\mathrm{MC}}(L \otimes -)^\alpha \rightarrow \Omega_A^1$ then defines a closed element of degree 0 in the complex

$$\mathbf{R}\underline{\mathrm{Hom}}_A(\mathbb{L}\underline{\mathrm{MC}}(L \otimes -)^\alpha, \Omega_A^1) \simeq (L \otimes \Omega_A^1, \delta + [\alpha, -])[1]$$

represented by $d\alpha \in (L \otimes \Omega_A^1)^1$; this satisfies $\delta(d\alpha) + [\alpha, d\alpha] = 0$ because $\delta\alpha + \frac{1}{2}[\alpha, \alpha] = 0$.

Analysing the proof of Corollary 3.4, it thus follows that ω_A is given by

$$\langle d\alpha, d\alpha \rangle_{\mathrm{tr}} \in \mathbb{Z}^{2-d}(\Omega_A^1),$$

which is closed under d .

Although we have described this construction for Artinian CDGAs A , the same arguments apply to pro-Artinian CDGAs after taking suitable limits. In this infinitesimal setting, we can even work with \mathbb{Z} -graded CDGAs (playing the same role here as the stacky CDGAs of §6.6.1) as in [Man], and then this description gives a $(2-d)$ -shifted pre-symplectic structure on the pro-Artinian CDGA $(\widehat{\mathrm{Symm}}(L[1]^*), \delta + [-, -]^*)$ representing $\underline{\mathrm{MC}}(L \otimes -)$.

3.4.4. Formality. Now take $K = \mathbb{Q}_\ell$ and let X_k be a smooth proper variety of dimension m over an ℓ -coprime finite field k , with $X := X_k \otimes_k \bar{k}$ the base change to the algebraic closure \bar{k} of k . Some power of the Frobenius automorphism of \bar{k} gives rise to a k -linear automorphism of X , so acts on the groupoid $(BG)(X_{\mathrm{pro\acute{e}t}}, \mathbb{Q}_\ell)$ of pro-étale $G(\mathbb{Q}_\ell)$ -torsors. Assume moreover that $\phi \in (BG)(X_{\mathrm{pro\acute{e}t}}, \mathbb{Q}_\ell)$ is fixed by some power of Frobenius and is reductive in the sense that for every algebraic G -representation V , the local system V, X, ϕ is irreducible. Then, generalising [Pri1, §2.3], the argument of [Pri3, Theorem 6.10]⁴ implies that our governing DGLA is formal, in the sense that we have a zigzag of quasi-isomorphisms

$$\mathbf{R}\Gamma(X_{\mathrm{pro\acute{e}t}}, \underline{\mathfrak{g}}_{\phi, X}) \simeq \mathbf{H}^*(X_{\mathrm{pro\acute{e}t}}, \underline{\mathfrak{g}}_{\phi, X}).$$

Proposition 3.22. *The $(2-2m)$ -shifted symplectic structure on $BG(X_{\mathrm{pro\acute{e}t}}, -)_\phi$ given by Examples 3.21 (and depending on a choice of isomorphism $\mathbb{Z}_\ell(m) \cong \mathbb{Z}_\ell$) is also formal, in the sense that it is induced by the perfect pairing*

$$\langle -, - \rangle_{\mathrm{tr}}: \mathbf{H}^*(X_{\mathrm{pro\acute{e}t}}, \underline{\mathfrak{g}}_{\phi, X}) \otimes \mathbf{H}^*(X_{\mathrm{pro\acute{e}t}}, \underline{\mathfrak{g}}_{\phi, X}) \rightarrow \mathbb{Q}_\ell[-2m],$$

so the symplectic structure on an A -valued Maurer–Cartan element $\alpha \in \underline{\mathrm{MC}}(\mathbf{H}^*(X_{\mathrm{pro\acute{e}t}}, \underline{\mathfrak{g}}_{\phi, X}) \otimes A)$ is $\langle d\alpha, d\alpha \rangle_{\mathrm{tr}}$.

⁴or the result itself, applied to groups of the form $R \rtimes \exp(\mathfrak{g} \otimes \mathfrak{m}_A)$, for $R \subset G$ the Zariski closure of $\pi_1(X)$

Proof. The trace map for X is given by $H^{2m}(X_{\text{proét}}, \mathbb{Z}_\ell(m)) \rightarrow \mathbb{Z}_\ell$, which (X being algebraically closed) we can rewrite as $H^{2m}(X_{\text{proét}}, \mathbb{Z}_\ell)(m) \rightarrow \mathbb{Z}_\ell$. Consequently, it has weight $-2m$ with respect to the Frobenius action, as does the shifted symplectic structure it induces.

Now write $L := H^*(X_{\text{proét}}, \mathfrak{g}_{\phi, X})$, so the deformation functor is pro-represented by $A := (\widehat{\text{Symm}}(L[1]^*), [-, -]^*)$. We can thus represent the $(2 - 2m)$ -shifted symplectic structure as an element of

$$Z^{4-2m} F^2 \text{DR}(A)$$

of weight $-2m$ with respect to the Frobenius action.

Now the Frobenius action on L^r has weight r for all r , so weight is at most cochain degree in $\text{Symm}(L[1]^*)$ and hence in A . In $\Omega_A^p[-p] \cong S^p(L[2]^*) \otimes A$, it then follows that elements of weight $-2m$ live in cochain degree at least $2p - 2m$, with equality only on $S^p(L[2]^*) \otimes \mathbb{Q}_\ell$.

Since the space of $(2 - 2m)$ -shifted symplectic structures on A only involves terms in $F^2 \text{DR}(A)$ of cochain degree at most $4 - 2m$, the space of such structures ω of weight $-2m$ with respect to the Frobenius action is equivalent to a subset of $S^2(L^*)^{-2m}$ (regarded as a discrete space). This implies that ω lies in Ω_A^2 , and also that ω is determined by its image in $\Omega_A^2 \otimes_A \mathbb{Q}_\ell$.

To calculate this, take a universal deformation $u \in \underline{\text{MC}}(\mathbf{R}\Gamma(X_{\text{proét}}, \mathfrak{g}_{\phi, X}) \otimes \mathfrak{m}_A)$; this is unique up to a contractible space of choices, and encodes the data of an L_∞ -quasi-isomorphism $\Upsilon: L \rightarrow \mathbf{R}\Gamma(X_{\text{proét}}, \mathfrak{g}_{\phi, X})$. We know from §3.4.3 that the equivalence class of ω is given by $\langle du, du \rangle_{\text{tr}} \in H^{2-2m} \Omega_A^2$, and we need to know its image

$$\langle du + \mathfrak{m}_A, du + \mathfrak{m}_A \rangle_{\text{tr}} \in H^{2-2m}(\Omega_A^2 / \mathfrak{m}_A \cdot \Omega_A^2).$$

Now observe that the element

$$\begin{aligned} [du] &\in H^1(\mathbf{R}\Gamma(X_{\text{proét}}, \mathfrak{g}_{\phi, X}) \otimes (\Omega_A^1 / \mathfrak{m}_A)) \cong H^1(\mathbf{R}\Gamma(X_{\text{proét}}, \mathfrak{g}_{\phi, X}) \otimes L[1]^*) \\ &\cong \text{Ext}^0(L, \mathbf{R}\Gamma(X_{\text{proét}}, \mathfrak{g}_{\phi, X})) \end{aligned}$$

is just the linear part of the L_∞ -quasi-isomorphism Υ , so gives the identity on cohomology. Thus $\langle du + \mathfrak{m}_A, du + \mathfrak{m}_A \rangle_{\text{tr}} \in H^{-2m}(S^2(L^*)) \cong S^2(L^*)^{-2m}$ is the pairing $\langle -, - \rangle_{\text{tr}}$ on cohomology, as required. \square

Remark 3.23. In accordance with the philosophy of weights [Del], the analogue of Proposition 3.22 for derived moduli of G -torsors on the analytic site of a compact Kähler manifold is also true. It has a much simpler proof, since trace maps are compatible with the zigzag of quasi-isomorphisms coming from the dd^c - (or $\partial\bar{\partial}$)-lemma in [GM, 7.6] and [Sim2, Lemma 2.2].

Remark 3.24 (Shifted Poisson structures). Since the shifted symplectic structure of Proposition 3.22 is strict, in the sense that it is a genuinely closed 2-form which induces an isomorphism rather than a quasi-isomorphism, we can simply invert it to give the corresponding shifted Poisson structure.

Explicitly, dualising the perfect pairing $\langle -, - \rangle_{\text{tr}}$ on $L = H^*(X_{\text{proét}}, \mathfrak{g}_{\phi, X})$ gives us a Casimir element $\pi \in (S^2 L)^{2m}$. The Poisson structure is then the biderivation of cochain degree $2m - 2$ on $(\widehat{\text{Symm}}(L[1]^*), [-, -]^*)$ given on generators $L[1]^* \times L[1]^*$ by contraction with π .

4. WEIGHTED SHIFTED SYMPLECTIC STRUCTURES

We now set up the theory to allow us to look into cases where the trace maps $H^d(X_{\text{ét}}, \mathbb{D}/\ell^n) \rightarrow \mathbb{Z}/\ell^n$ exist on cohomology of a non-trivial rank 1 local system \mathbb{D} . Non-triviality means that results such as Corollaries 3.4 and 3.11 do not apply, but with some work, we can still construct a moduli functor with a symplectic structure which is shifted in degree and twisted by a line bundle (i.e. a \mathcal{P} -shifted symplectic structure in the terminology of [BG]).

If we were in the algebraic setting, looking at derived mapping stacks from a scheme or algebraic stack X with dualising line bundle \mathcal{L} , then we could take the \mathbb{G}_m -torsor $P := \mathbf{Spec}_X(\bigoplus_{r \in \mathbb{Z}} \mathcal{L}^{\otimes r})$ on X associated to \mathcal{L} , and look at the derived mapping stack $\mathbf{Rmap}(P, F)$. This derived stack carries a natural \mathbb{G}_m -action, with homotopy invariants

$$\mathbf{Rmap}(P, F)^{h\mathbb{G}_m} \simeq \mathbf{Rmap}([P/\mathbb{G}_m], F) \simeq \mathbf{Rmap}(X, F).$$

Moreover, a trace map on $H^d(X, \mathcal{L}^{\otimes e})$ gives rise to a trace map on $H^d(P, \mathcal{O}_P) \cong \bigoplus_{r \in \mathbb{Z}} H^d(X, \mathcal{L}^{\otimes r})$ via projection to the $r = e$ factor, so an n -shifted pre-symplectic structure on F leads to an $(n - d)$ -shifted pre-symplectic structure on $\mathbf{Rmap}(P, F)$, though the latter is rarely representable because P is not proper.

Example 4.1. If X is the nerve BG of an algebraic group, the line bundle \mathcal{L} corresponds to an algebraic group homomorphism $G \rightarrow \mathbb{G}_m$, and then $P = [\mathbb{G}_m/G]$, which is isomorphic to $B \ker(G \rightarrow \mathbb{G}_m)$ if the homomorphism is surjective. The procedure of replacing X with P is thus analogous to replacing a Galois group $\text{Gal}(F)$ with that of a cyclotomic extension $\text{Gal}(F(\mu_\infty))$, as features in Iwasawa theory. By looking at homomorphisms to \mathbb{G}_m rather than $\hat{\mathbb{Z}}^* \cong \text{Gal}(F)/\text{Gal}(F(\mu_\infty))$, we produce objects which have better finiteness properties than infinite cyclotomic extensions would; even when infinite-dimensional, they are so in a controlled way.

When F is symplectic, this $(n - d)$ -shifted pre-symplectic becomes symplectic under finiteness conditions on $\bigoplus_{r \in \mathbb{Z}} H^*(X, \mathcal{L}^{\otimes r})$, which in particular require almost all of the complexes $\mathbf{R}\Gamma(X, \mathcal{L}^{\otimes r})$ to be acyclic. We can resolve this by looking at a formal neighbourhood of $\mathbf{Rmap}(X, F)$ in $\mathbf{Rmap}(P, F)$, which carries a form of shifted symplectic structure and representability relying only on finiteness conditions on the groups $H^*(X, \mathcal{L}^{\otimes r})$ separately for each r .

More explicitly, a \mathbb{G}_m -equivariant derived affine scheme is given by a CDGA A equipped with a weight decomposition $A = \bigoplus_{r \in \mathbb{Z}} \mathcal{W}_r A$, and then the space of \mathbb{G}_m -equivariant maps from $\text{Spec } A$ to $\mathbf{Rmap}(P, F)$ consists of maps

$$\mathbf{Spec}_X \left(\bigoplus_{r \in \mathbb{Z}} \mathcal{W}_r A \otimes \mathcal{L}^{\otimes r} \right) \rightarrow F.$$

The \mathbb{G}_m -invariant substack $\mathbf{Rmap}(X, F)$ is then recovered by restricting to \mathbb{G}_m -equivariant CDGAs A with $A = \mathcal{W}_0 A$. Its formal neighbourhood in $\mathbf{Rmap}(P, F)$ is recovered by restricting to \mathbb{G}_m -equivariant CDGAs with only finitely many non-zero weights, since in that case the ideal generated by non-zero weights is nilpotent. This last functor has a natural affinoid analogue, since the category of coherent modules contains finite direct sums; we can also study it without needing to worry about an analytic analogue of the group scheme \mathbb{G}_m .

4.1. Weighted structures.

4.1.1. *Weighted dagger dg algebras.* We now introduce the test objects on which we define moduli functors of the form above; there are entirely similar definitions in the algebraic setting.

Definition 4.2. Define a weighted dagger (resp. quasi-dagger, resp. localised dagger) dg algebra to be a dagger (resp. quasi-dagger, resp. localised dagger) dg algebra A_\bullet equipped with a decomposition

$$A_\bullet = \bigoplus_{n \in \mathbb{Z}} \mathcal{W}_n A_\bullet,$$

into chain subcomplexes, with only finitely many non-zero terms, respecting the multiplication in the sense that for $a \in \mathcal{W}_m A_\bullet$ and $b \in \mathcal{W}_n A_\bullet$, the product ab lies in $\mathcal{W}_{m+n} A_\bullet$.

Accordingly, say that an affinoid dagger (resp. quasi-dagger, resp. localised dagger) dg space $X = (X^0, \mathcal{O}_{X,\bullet})$ is weighted if it is equipped with such a decomposition on the sheaf $\mathcal{O}_{X,\bullet}$; this gives a contravariant equivalence between weighted algebras and weighted affinoid spaces.

A morphism $A \rightarrow B$ of weighted quasi-dagger dg algebras is just a homomorphism of dg K -algebras which respects the weight decompositions.

Definition 4.3. We denote the category of weighted localised dagger dg K -algebras by $dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}}$. We then denote its full subcategory of objects which are bounded as chain complexes by $dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b}$.

Definition 4.4. Given a weighted quasi-dagger dg algebra A_\bullet , define the quasi-dagger dg algebra A_\bullet / \mathbb{G}_m to be the quotient of A_\bullet by the dg ideal generated by $\bigoplus_{n \neq 0} \mathcal{W}_n A_\bullet$; all elements of this ideal are nilpotent because the weights are bounded.

For a weighted affinoid quasi-dagger dg space X , then define the space $X^{\mathbb{G}_m}$ of \mathbb{G}_m -invariants by the property that $\Gamma(X^{\mathbb{G}_m}, \mathcal{O}_{X^{\mathbb{G}_m}, \bullet}) \cong \Gamma(X, \mathcal{O}_{X, \bullet}) / \mathbb{G}_m$.

Definition 4.5. Given a weighted quasi-dagger dg algebra $A = \bigoplus_{n \in \mathbb{Z}} \mathcal{W}_n A$, we let $dg\text{Coh}_A^{f\mathbb{G}_m}$ (resp. $dg_+ \text{Coh}_A^{f\mathbb{G}_m}$) be the category of those \mathbb{G}_m -equivariant A -modules $M = \bigoplus_n \mathcal{W}_n M$ in chain complexes (resp. non-negatively graded chain complexes) which are levelwise finitely generated and have only finitely many $\mathcal{W}_n M$ non-zero.

We then define $dg\text{Coh}_A^{\hat{f}\mathbb{G}_m}$ (resp. $dg_+ \text{Coh}_A^{\hat{f}\mathbb{G}_m}$) to be the larger category consisting of those \mathbb{G}_m -equivariant A -modules $M = \bigoplus_n \mathcal{W}_n M$ in chain complexes (resp. non-negatively graded chain complexes) which are generated as bigraded A -modules by sets with finitely many elements for each pair (weight, degree).

In particular, note that given $M \in dg\text{Coh}_A^{\hat{f}\mathbb{G}_m}$, the quotient $M / (\mathcal{W}_{<r} M, \mathcal{W}_{>s} M)$ of M by the A -submodule $(\mathcal{W}_{<r} M, \mathcal{W}_{>s} M)$ generated by $\bigoplus_{n \notin [r, s]} \mathcal{W}_n M$ lies in $dg\text{Coh}_A^{f\mathbb{G}_m}$. Moreover, if the weights of A lie in the interval $[-x, y]$ for $x, y \geq 0$, then the weights of $(\mathcal{W}_{<r} M, \mathcal{W}_{>s} M)$ lie in the range $(-\infty, r + y) \cup (s - x, \infty)$; thus M is isomorphic to the limit $\varprojlim_{r, s} M / (\mathcal{W}_{<-r} M, \mathcal{W}_{>s} M)$ in the category of \mathbb{G}_m -equivariant A -modules, since the inverse system stabilises in each weight.

We then have notions of weighted (quasi-)dagger spaces and analytic stacks, by introducing weights into the definitions of §1.3; we will not reiterate the definitions explicitly here. One difference is that we do not have enough quasi-free objects in our category to allow quasi-free replacement, because a free generator in non-zero weight would generate an algebra neither dagger affinoid nor with bounded weights. As a consequence,

our moduli functors will only be representable by formal spaces or stacks, built from objects of $\text{pro}(dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}})$ as in §4.3 below.

4.1.2. *Alternative formulations and quasi-free objects.* In order to get round the lack of quasi-free objects in $dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}}$, we now introduce slightly larger categories with closely related, but more manageable, homotopy theory.

Definition 4.6. Define $dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}$ to consist of K -CDGAs $A_\bullet = A_{\geq 0}$ equipped with a decomposition

$$A_\bullet = \bigoplus_{n \in \mathbb{Z}} \mathcal{W}_n A_\bullet,$$

into chain subcomplexes (all of which are allowed to be non-zero), respecting the multiplication, such that $\mathcal{W}_0 A_\bullet$ is a quasi-dagger dg algebra, A_\bullet is generated as a $\mathcal{W}_0 A_\bullet$ -algebra by finitely many generators in each chain degree, and the map $\mathcal{W}_0 A_0 \rightarrow \text{H}_0 \mathcal{W}_0 A_\bullet / \mathbb{G}_m$ gives an isomorphism on the associated sets of points (i.e. of maximal ideals).

A morphism $A_\bullet \rightarrow B_\bullet$ in $dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}$ is just a homomorphism of dg K -algebras which respects the weight decompositions.

Remarks 4.7. Note that the finiteness condition in Definition 4.6 only applies in each chain degree, rather than in pairs (degree, weight), so in particular for each chain degree, objects of $dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}$ only have generators in finitely many weights.

The condition that $\mathcal{W}_0 A_0 \rightarrow \text{H}_0 \mathcal{W}_0 A_\bullet / \mathbb{G}_m$ give an isomorphism on the associated sets of points implies that $\mathcal{W}_0 A_0 \rightarrow \text{H}_0 \mathcal{W}_0 A_\bullet$ also gives an isomorphism on the associated sets of points, so $\mathcal{W}_0 A_\bullet$ is a localised dagger dg algebra. Dropping that condition would give us a weighted generalisation of the notion of quasi-dagger dg algebra, and localising at $\text{H}_0 \mathcal{W}_0 A_\bullet / \mathbb{G}_m$ (rather than $\text{H}_0 A_\bullet$ as in Definition 1.14) gives an object $A_\bullet^{\text{loc}} \in dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}$, this functor being left adjoint to the obvious inclusion functor.

Definition 4.8. We say that an object $A_\bullet \in dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}$ is quasi-free if, in the category of \mathbb{G}_m -equivariant graded-commutative algebras which are quasi-dagger in weight and degree 0, the underlying object $A_\#$ given by forgetting the differential is freely generated over a quasi-Washnitzer algebra $K\langle \frac{x_1}{r_1}, \dots, \frac{x_n}{r_n} \rangle^\dagger$ (see Definition 1.9).

Similarly, say that a morphism $f: R_\bullet \rightarrow A_\bullet$ in $dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}$ is quasi-free if $A_\#$ is freely generated in that category over $W \otimes_{R_0} R_\#$ for some relative quasi-Washnitzer algebra $W = R_0\langle \frac{x_1}{r_1}, \dots, \frac{x_n}{r_n} \rangle^\dagger$.

As the following example shows, the condition that $\mathcal{W}_0 A_0$ be a quasi-dagger algebra leads to some slightly unusual behaviour for these quasi-free objects

Example 4.9. The quasi-free object $A_\bullet \in dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}$ generated by $x \in \mathcal{W}_1 A_0$ and $y \in \mathcal{W}_{-1} A_0$ is given by

$$\mathcal{W}_i A_\bullet = \begin{cases} K\langle \frac{xy}{0} \rangle^\dagger x^i & i \geq 0 \\ K\langle \frac{xy}{0} \rangle^\dagger y^{-i} & i \leq 0, \end{cases}$$

with the obvious multiplication.

This has the property that $\text{Hom}_{dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}}(A_\bullet, B_\bullet) \cong \mathcal{W}_1 B_0 \times \mathcal{W}_{-1} B_0$, since for $(u, v) \in \mathcal{W}_1 B_0 \times \mathcal{W}_{-1} B_0$, we have $uv \in \ker(\mathcal{W}_0 B \rightarrow B / \mathbb{G}_m)$, so the localisation condition ensures that it gives rise to a morphism $K\langle \frac{xy}{0} \rangle^\dagger \rightarrow \mathcal{W}_0 B_0$.

The proof of Lemma 1.8 adapts directly to give:

Lemma 4.10. *Every morphism $f: A \rightarrow B$ in $dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}$ admits a factorisation $A \xrightarrow{p} \tilde{B} \xrightarrow{r} B$ with p quasi-free and r a surjective quasi-isomorphism, and a factorisation $A \xrightarrow{q} \tilde{B} \xrightarrow{s} B$ with q a quasi-free quasi-isomorphism and s surjective in strictly positive degrees.*

Lemma 4.11. *There is a functor $A \mapsto \hat{A}$ from $dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}$ to the pro-category $\text{pro}(dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b})$ determined by the property that for $A \in dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}$ and $B \in dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b}$, we have*

$$\text{Hom}_{\text{pro}(dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b})}(\hat{A}, B) \cong \text{Hom}_{dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}}(A, B).$$

This functor sends quasi-isomorphisms to pro-quasi-isomorphisms (i.e. essentially levelwise quasi-isomorphism in the sense of [Isa, §2.1]), so induces a functor on the simplicial localisation at those classes of morphisms, respectively.

Proof. Since B is bounded and exists in finitely many weights, any morphism $A \rightarrow B$ must factor through $A/\tau_{>k}A/(\mathcal{W}_{<-n}, \mathcal{W}_{>n})$ for some n, k . Since A is degree-wise finitely generated over a localised dg dagger algebra, those quotients all live in $dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b}$, so we just set $\hat{A} := \{A/\tau_{>k}A/(\mathcal{W}_{<-n}, \mathcal{W}_{>n})\}_{n,k}$.

It remains to prove the final statement. Since good truncation preserves quasi-isomorphisms, we may assume that A is bounded, in which case \hat{A} just becomes the inverse system $\{A/\tau_{>k}A/(\mathcal{W}_{<-n}, \mathcal{W}_{>n})\}$. Now, since A exists in only finitely many chain degrees, it is finitely generated over A_0 . There thus exists some generating set lying in weights $[-r, r]$, which implies that the ideal $(\mathcal{W}_{\leq -nr}, \mathcal{W}_{\geq nr})$ is contained in $(\mathcal{W}_{\neq 0})^n$, where $(\mathcal{W}_{\neq 0})$ is the ideal generated by elements of non-zero weight. Moreover, any set of n elements in $\mathcal{W}_{\neq 0}$ contains either at least $\lceil n/2 \rceil$ positively weighted elements, or at least $\lceil n/2 \rceil$ negatively weighted elements, from which we conclude that $(\mathcal{W}_{\neq 0})^n \subset (\mathcal{W}_{<-n/2}, \mathcal{W}_{>n/2})$. Thus the systems $\{(\mathcal{W}_{<-n}, \mathcal{W}_{>n})\}_n$ and $(\mathcal{W}_{\neq 0})^n$ of ideals are equivalent, and \hat{A} is isomorphic to the $(\mathcal{W}_{\neq 0})$ -adic completion $\{A/(\mathcal{W}_{\neq 0})^n\}_n$.

Now, the proof of [Pri4, Lemma 2.13] generalises to give a pro-quasi-isomorphism $\{A/I^n\}_n \rightarrow \{B/f(I)^n\}_n$ for any surjective quasi-isomorphism $f: A \rightarrow B$ of finitely generated bounded non-negatively graded CDGAs over a Noetherian \mathbb{Q} -algebra, for any ideal I . Thus the functor $A \mapsto \hat{A}$ sends surjective quasi-isomorphisms to pro-quasi-isomorphisms.

Given an arbitrary quasi-isomorphism $A \rightarrow A'$, Lemma 4.10 gives us a factorisation $A \xrightarrow{q} C \xrightarrow{s} A \times A'$ of its graph, with s surjective in strictly positive degrees and q a quasi-isomorphism. Thus the morphisms $s_1: C \rightarrow A$ and $s_2: C \rightarrow A'$ are surjective quasi-isomorphisms (since they are quasi-isomorphisms and are surjective in strictly positive degrees). This gives us pro-quasi-isomorphisms $\hat{A} \leftarrow \hat{C} \rightarrow \hat{A}'$, and the section $\hat{q}: \hat{A} \rightarrow \hat{C}$ of \hat{s}_1 is thus also a pro-quasi-isomorphism, as is the composite $\hat{f} = \hat{s}_2 \circ \hat{q}: \hat{A} \rightarrow \hat{A}'$. \square

Proposition 4.12. *The inclusion functor $dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b} \hookrightarrow dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}$ induces a fully faithful functor on simplicial categories after simplicial localisation at quasi-isomorphisms. More generally, for $A, B \in dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}}$, we have*

$$\mathbf{Rmap}_{dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}}(A, B) \simeq \mathbf{R} \varprojlim_k \mathbf{Rmap}_{dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b}}(A/\tau_{>k}A, B/\tau_{>k}B).$$

Proof. Given $A \in dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}$, the proof of Proposition 1.25, using Lemma 4.10 in place of of Lemma 1.8, gives us a cosimplicial resolution $\tilde{A}^\bullet \rightarrow A$ which is quasi-free in the sense that the latching maps are all quasi-free, and a resolution in the sense that the degeneracy maps are all quasi-isomorphisms and we have a quasi-isomorphism $\tilde{A}^0 \rightarrow A$. Our definition of $dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}$ ensures that quasi-free morphisms have the left lifting property with respect to acyclic surjections, so the functor $\text{Hom}_{dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}}(\tilde{A}^\bullet, -): dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}} \rightarrow s\text{Set}$ given in level m by $\text{Hom}_{dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}}(\tilde{A}^m, -)$ sends acyclic surjections to trivial Kan fibrations, and in particular to weak equivalences. As in the proof of Lemma 4.11, this implies that it sends all quasi-isomorphisms to weak equivalences. Thus Lemma 1.24 implies that

$$\mathbf{Rmap}_{dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}}(A, -) \simeq \text{Hom}_{dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}}(\tilde{A}^\bullet, -).$$

Now for $B \in dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b}$, Lemma 4.11 gives

$$\text{Hom}_{dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}}(\tilde{A}^m, B) \cong \text{Hom}_{\text{pro}(dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b})}(\widehat{\tilde{A}^m}, B),$$

while for $B \in dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}}$ we have

$$\begin{aligned} \text{Hom}_{dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}}(\tilde{A}^m, B) &\cong \varprojlim_n \text{Hom}_{dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}}(\tilde{A}^m, B/\tau_{>n}B) \\ &\cong \varprojlim_n \text{Hom}_{\text{pro}(dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b})}(\widehat{\tilde{A}^m}, B/\tau_{>n}B); \end{aligned}$$

quasi-freeness of \tilde{A}^\bullet implies that $\text{Hom}_{dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}}(\tilde{A}^\bullet, -)$ maps nilpotent surjections to Kan fibrations so

$$\varprojlim_n \text{Hom}_{\text{pro}(dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b})}(\widehat{\tilde{A}^\bullet}, B/\tau_{>n}B) \simeq \text{holim}_n \text{Hom}_{\text{pro}(dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b})}(\widehat{\tilde{A}^\bullet}, B/\tau_{>n}B)$$

Now, observe that if $A \in dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}}$, the inverse system \hat{A} from Lemma 4.11 is isomorphic to $\{A/\tau_{>n}A\}_n$, so that lemma gives quasi-isomorphisms

$$\widehat{\tilde{A}^m} \rightarrow \{A/\tau_{>n}A\}_n \quad \text{and} \quad \widehat{\tilde{A}^m}/\tau_{>n}\widehat{\tilde{A}^m} \rightarrow A/\tau_{>n}A.$$

In particular, $\widehat{\tilde{A}^\bullet}$ is a cosimplicial resolution of $\{A/\tau_{>n}A\}_n$ in the pro-category; since $\text{Hom}_{\text{pro}(dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b})}(\widehat{\tilde{A}^\bullet}, -)$ sends quasi-isomorphisms in $dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b}$ to weak equivalences, the proof of Lemma 1.24 (embedding the pro-category in the model category of restricted diagrams) then implies that

$$\mathbf{Rmap}_{dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b}}(A/\tau_{>n}A, -) \simeq \text{Hom}_{\text{pro}(dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b})}(\widehat{\tilde{A}^\bullet}/\tau_{>n}, -)$$

on $dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b}$, so for $B \in dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b}$ we have

$$\begin{aligned} \mathbf{Rmap}_{dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}}(A, B) &\simeq \text{holim}_n \text{Hom}_{\text{pro}(dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b})}(\widehat{\tilde{A}^\bullet}, B/\tau_{>n}B) \\ &\cong \text{holim}_n \text{Hom}_{\text{pro}(dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b})}(\widehat{\tilde{A}^\bullet}/\tau_{>n}, B/\tau_{>n}B) \\ &\simeq \text{holim}_n \mathbf{Rmap}_{dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b}}(A/\tau_{>n}A, B/\tau_{>n}B). \end{aligned}$$

□

Lemma 4.13. *The essential image of $\mathrm{Ho}(dg_+\mathrm{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \mathrm{loc}, \flat}) \rightarrow \mathrm{Ho}(dg_+\mathrm{AffdAlg}_K^{\mathbb{G}_m, \dagger, \mathrm{loc}})$ is closed under square-zero extensions by complexes I with homology concentrated in a single chain degree and with finitely many weights.*

Proof. Assume that we have a surjection $A \twoheadrightarrow B$ in $dg_+\mathrm{AffdAlg}_K^{\mathbb{G}_m, \dagger, \mathrm{loc}}$ such that the kernel I , with conditions as above, squares to 0 and B is quasi-isomorphic to an object \bar{B} of $dg_+\mathrm{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \mathrm{loc}, \flat}$. Then we wish to show that A is also quasi-isomorphic to such an object.

There is an obvious K -CDGA structure on $\mathrm{cone}(I \rightarrow A)$, and this is easily seen to lie in $dg_+\mathrm{AffdAlg}_K^{\mathbb{G}_m, \dagger, \mathrm{loc}}$, with the obvious map $\mathrm{cone}(I \rightarrow A) \rightarrow B$ being a quasi-isomorphism. Since I squares to 0, we also have a K -CDGA morphism $\mathrm{cone}(I \rightarrow A) \rightarrow \mathrm{cone}(I \xrightarrow{0} B) = B \oplus I[1]$, and then $A = \mathrm{cone}(I \rightarrow A) \times_{B \oplus I[1]} B$.

If $\tilde{B} \rightarrow \bar{B}$ is a quasi-free replacement for \bar{B} , then as in the proof of Lemma 4.11, the lifting property of quasi-free morphisms with respect to acyclic surjections gives us a map (necessarily a quasi-isomorphism) $\tilde{B} \rightarrow B$, which then further lifts to give a map $\tilde{B} \rightarrow \mathrm{cone}(I \rightarrow A)$ and hence $\tilde{B} \rightarrow B \oplus I[1]$. Assume that H_*I is concentrated in degree n . Without loss of generality, we may then replace I with $\tau_{\geq n}I$ in the statements above, since $A/\tau_{\geq n}I \rightarrow A/I$ is a quasi-isomorphism. Since we are now assuming that I is concentrated in degrees $\geq n$, there is a quasi-isomorphism $I \rightarrow H_n(I)[n]$. In particular, we have $\tilde{B} \rightarrow H_0B \oplus H_n(I)[n+1]$; enlarging \tilde{B} if necessary, we may also assume that this map is surjective, so $A \simeq \tilde{B} \times_{H_0B \oplus H_n(I)[n+1]} H_0B$.

Now, since $H_0B \oplus H_n(I)[n+1] \in dg_+\mathrm{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \mathrm{loc}, \flat}$, Lemma 4.11 gives us a morphism $f: \hat{\tilde{B}} \rightarrow H_0B \oplus H_n(I)[n+1]$ in $\mathrm{pro}(dg_+\mathrm{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \mathrm{loc}, \flat})$. We also have a pro-quasi-isomorphism $\hat{\tilde{B}} \rightarrow \hat{B}$, so we can write $\hat{\tilde{B}}$ as an inverse system of objects mapping quasi-isomorphically to B . Our morphism f thus factors through some such object B' ; since f is surjective, our map $f': B' \rightarrow H_0B \oplus H_n(I)[n+1]$ is also so, and then we have

$$A \simeq B' \times_{H_0B \oplus H_n(I)[n+1]} H_0B,$$

which is an object of $dg_+\mathrm{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \mathrm{loc}, \flat}$. \square

Proposition 4.14. *In the homotopy categories given by localising at quasi-isomorphisms, the essential image of $\mathrm{Ho}(dg_+\mathrm{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \mathrm{loc}, \flat}) \rightarrow \mathrm{Ho}(dg_+\mathrm{AffdAlg}_K^{\mathbb{G}_m, \dagger, \mathrm{loc}})$ consists of objects A for which the homology groups H_iA exist in only finitely many weights, and vanish for $i \gg 0$.*

Proof. The conditions are clearly satisfied by objects in the image and are invariant under quasi-isomorphism, so it suffices to show that every object B satisfying the conditions lies in the image of the relevant functor. To prove this, we will proceed by induction using the Postnikov tower $\{B/\tau_{>n}B\}_n$.

For $n = 0$, we have a quasi-isomorphism $B/\tau_{>0}B \rightarrow H_0B$, which is a weighted localised dagger algebra, so lies in $dg_+\mathrm{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \mathrm{loc}, \flat} \subset dg_+\mathrm{AffdAlg}_K^{\mathbb{G}_m, \dagger, \mathrm{loc}}$. In general, the map $B/\tau_{>n+1}B \rightarrow B/\tau_{>n}B$ factorises as the composite of an acyclic surjection and a square-zero extension, so Lemma 4.13 implies that $B/\tau_{>n+1}B$ lies in the essential image of $dg_+\mathrm{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \mathrm{loc}, \flat}$ whenever $B/\tau_{>n}B$ does so. This completes the inductive proof, since $\tau_{>n}B$ is acyclic for $n \gg 0$. \square

The following is now an immediate consequence of Propositions 4.12 and 4.14:

Corollary 4.15. *The following simplicial categories are quasi-equivalent:*

- (1) *homotopy-preserving functors from $dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b}$ to the category of simplicial sets;*
- (2) *homotopy-preserving nilcomplete functors from $dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}}$ to the category of simplicial sets;*
- (3) *homotopy-preserving nilcomplete functors from the full subcategory of $dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}$ on objects with homologically bounded weights to the category of simplicial sets.*

Definition 4.16. Given $A \in dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}$, we define the A -module Ω_A^1 by the property that $d: A \rightarrow \Omega_A^1$ is the universal K -linear derivation from A to A -modules M in weighted complexes which are levelwise finitely generated.

Explicitly, we can calculate Ω_A^1 in terms of the algebraic \mathbb{G}_m -equivariant cotangent module as $\text{cone}(\Omega_{\mathcal{W}_0 A^{\text{alg}}}^1 \otimes_{\mathcal{W}_0 A} A \rightarrow (\Omega_{\mathcal{W}_0 A}^1 \otimes_{\mathcal{W}_0 A} A) \oplus \Omega_{A^{\text{alg}}}^1)$, for the dagger cotangent module $\Omega_{\mathcal{W}_0 A}^1$ of Definition 2.10.

Remark 4.17 (Weighted EFC-DGAs). We could define a weighted EFC-DGA A_\bullet to be a K -CDGA A_\bullet equipped with a weight decomposition \mathcal{W} and a compatible EFC structure on $\mathcal{W}_0 A_0$. Every object of $dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}$ has an underlying weighted EFC-DGA A_\bullet , which is localised in the sense that the natural map $\mathcal{W}_0 A_0 \rightarrow (\mathcal{W}_0 A_0 / (\mathbb{H}_0 A_\bullet / \mathbb{G}_m))^{\text{loc}}$ is an isomorphism, where A_\bullet / \mathbb{G}_m is the quotient of A_\bullet by the dg EFC-ideal generated by $\mathcal{W}_{\neq 0} A_\bullet$.

We could then develop weighted analogues of the results of §1.2, allowing us to recast weighted structures in terms of EFC-algebras, but we will instead just formulate our weighted results in the dagger affinoid world.

4.1.3. Tangent and cotangent complexes.

Definition 4.18. Given $M \in dg\text{Coh}_A^{f\mathbb{G}_m}$, define $M\{r\} \in dg\text{Coh}_A^{f\mathbb{G}_m}$ to be the tensor product of M with a rank 1 \mathbb{G}_m -representation of weight $-r$, so that $\mathcal{W}_n(M\{r\}) \cong \mathcal{W}_{n+r}M$.

Generalising Definition 1.40, we have:

Definition 4.19. Given a homotopy-preserving homogeneous functor $F: dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}} \rightarrow s\text{Set}$, an object $A \in dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}}$ and a point $x \in F(A)$, define the tangent functor $T_x F$

$$T_x F: dg_+ \text{Coh}_A^{f\mathbb{G}_m} \rightarrow s\text{Set},$$

by $T_x F(M) := F(A \oplus M) \times_{F(A)}^h \{x\}$.

By the argument of [Pri4, Lemma 1.12], the space $T_x F(M[1])$ deloops $T_x F(M)$, so we then define tangent cohomology groups by $D_x^{n-i}(F, M) := \pi_i(F(A \oplus M[n]) \times_{F(A)}^h \{x\})$.

Definition 4.20. In the setting of Definition 4.19, we say that F has a coherent cotangent complex $\mathbb{L}^{F, x}$ at x if there is a weighted A -module $\mathbb{L}^{F, x} \in dg\text{Coh}_A^{f\mathbb{G}_m}$ in chain complexes, bounded below in chain degrees, representing $T_x(F)$ homotopically in the sense that the simplicial mapping space

$$\mathbf{Rmap}_{dg\text{Coh}_A^{f\mathbb{G}_m}}(\mathbb{L}^{F, x}, -)$$

is weakly equivalent to $T_x(F)$ when restricted to $dg_+ \text{Coh}_A^{f\mathbb{G}_m}$.

In particular, this means that

$$\pi_i T_x(F)(M) \cong \mathbb{E}xt_A^{-i}(\mathbb{L}^{F,x}, M)^{\mathbb{G}_m} = \mathcal{W}_0 \mathbb{E}xt_A^{-i}(\mathbb{L}^{F,x}, M),$$

for all $M \in dg_+ \mathcal{C}oh_A^{f\mathbb{G}_m}$.

Remark 4.21. It is important to note that we are allowing the cotangent complex to have infinitely many generators in each degree, provided that for each degree, there are finitely many in each weight. Although the more restrictive definition would be easier to work with, for our applications of interest, it would only allow us to handle pro-étale sheaves on schemes over local fields, not over global fields.

Example 4.22. Given an object $C \in dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}$ which is quasi-free in the sense of Definition 4.8, we can look at the functor $F := \mathbf{Rmap}_{dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \text{loc}, \dagger}}(C, -)$ on $dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}}$ given by taking the mapping space in the simplicial localisation at quasi-isomorphisms.

At a point $x \in F(A)$ given by a morphism $C \rightarrow A$, the functor $T_x(F)$ is given by

$$M \mapsto \mathbf{Rmap}_{dg_+ \text{Mod}_C^{\mathbb{G}_m}}(\Omega_C^1, M) \simeq \tau_{\geq 0} \underline{\text{Hom}}_{dg_+ \text{Mod}_C^{\mathbb{G}_m}}(\Omega_C^1, M),$$

since quasi-freeness implies that Ω_C^1 is cofibrant as a weighted C -module. Explicitly, this implies that

$$\pi_i T_x(F)(M) \cong \mathbb{E}xt_C^{-i}(\Omega_C^1, M)^{\mathbb{G}_m} = \mathcal{W}_0 \mathbb{E}xt_C^{-i}(\Omega_C^1, M),$$

and that F has a cotangent complex at x given by

$$\Omega_C^1 \otimes_C A.$$

Lemma 4.23. *If $F: dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}} \rightarrow s\text{Set}$ is a homotopy-preserving, homogeneous, nilcomplete functor such that for all dagger algebras A (regarded as living in weight 0 and degree 0) and all points $x \in F(A)$, the groups $D_x^i(F, A\{m\})$ are all finitely generated A -modules and all vanish for $i \ll 0$, then F has coherent cotangent complexes $\mathbb{L}^{F,y}$ at all points $y \in F(B)$ for all $B \in dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}}$.*

Proof. This proceeds in much the same way as Lemma 1.44. Observe that because C is a nilpotent extension of C/\mathbb{G}_m , our finiteness hypothesis implies that $D_x^i(F, M)$ is finitely generated for all weighted dagger algebras C , all $z \in F(C)$ and all weighted coherent C -modules M , since the category of weighted coherent C -modules is generated by the modules $(C/\mathbb{G}_m)\{m\}$.

We now outline the other modifications needed to the argument of [Lur1, Theorem 3.6.9] to work with weighted modules. The basic idea is still that delooping allows us to extend T_y to a functor \mathbb{T}_y on bounded below complexes in $dg\mathcal{C}oh_B^{f\mathbb{G}_m}$. Moreover, elements of $\mathcal{W}_n B$ define \mathbb{G}_m -equivariant morphisms $M\{i\} \rightarrow M\{i+n\}$ for $M \in dg\mathcal{C}oh_B^{f\mathbb{G}_m}$, sufficiently canonically that $\bigoplus_n \mathbb{T}_y(M\{n\})$ is naturally a \mathbb{G}_m -equivariant B -module, with $\mathbb{T}_y(M\{n\})$ given weight n .

Given a dualising complex $K_{\mathcal{W}_0 B}$ for $\mathcal{W}_0 B$, we have a dualising complex $K_B := \mathbf{RHom}_{\mathcal{W}_0 B}(B, K_{\mathcal{W}_0 B})$ which carries a natural \mathbb{G}_m -action, so lies in $dg\mathcal{C}oh_B^{f\mathbb{G}_m}$ since B has finitely many weights. Then we set

$$\mathcal{W}_n \mathbb{L}^{F,y} := \mathbf{RHom}_{\mathcal{W}_0 B}(\mathbb{T}_y(K_B\{-n\}), K_{\mathcal{W}_0 B});$$

this has multiplication maps $\mathcal{W}_m B \otimes \mathcal{W}_n \mathbb{L}^{F,y} \rightarrow \mathcal{W}_{m+n} \mathbb{L}^{F,y}$ coming from the maps $\mathcal{W}_m B \otimes \mathbb{T}_y(K_B\{-m-n\}) \rightarrow \mathbb{T}_y(K_B\{-n\})$, and these combine to give $\mathbb{L}^{F,y}$ the structure of a \mathbb{G}_m -equivariant B -module.

More generally, the duality functor $D_{\mathcal{W}_0 B} M := \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{W}_0 B}(M, K_{\mathcal{W}_0 B})$ on $\mathcal{W}_0 B$ -modules gives rise to a duality functor $D_B^{\mathbb{G}_m}$ on \mathbb{G}_m -equivariant B -modules, given by

$$\mathcal{W}_n(D_B^{\mathbb{G}_m} M) := D_{\mathcal{W}_0 B}(\mathcal{W}_{-n} B),$$

with the obvious B -module structure on $\bigoplus_n \mathcal{W}_n(D_B^{\mathbb{G}_m} M)$.

For bounded complexes $M \in dg\mathcal{C}oh_B^{f\mathbb{G}_m}$, the argument of [Lur1, Theorem 3.6.9] then adapts to show that $\mathbf{R}\underline{\mathrm{Hom}}_B(M, \bigoplus_n \mathbb{T}_y(K_B\{n\}))^{\mathbb{G}_m} \simeq \mathbb{T}_y(\mathbf{R}\underline{\mathrm{Hom}}_B(M, K_B))$, for the natural \mathbb{G}_m -action on $\mathbf{R}\underline{\mathrm{Hom}}_B(M, K_B)$, since the statement holds when M is of the form $B\{n\}$. For bounded coherent complexes $N \in dg\mathcal{C}oh_B^{f\mathbb{G}_m}$, we thus have equivalences

$$\begin{aligned} \mathbf{R}\underline{\mathrm{Hom}}_B(\mathbb{L}^{F,y}, N)^{\mathbb{G}_m} &\simeq \mathbf{R}\underline{\mathrm{Hom}}_B(D_B^{\mathbb{G}_m} N, D_B^{\mathbb{G}_m} \mathbb{L}^{F,y})^{\mathbb{G}_m} \\ &\simeq \mathbf{R}\underline{\mathrm{Hom}}_B(D_B^{\mathbb{G}_m} N, \bigoplus_n \mathbb{T}_y(K_B\{n\}))^{\mathbb{G}_m} \\ &\simeq \mathbb{T}_y(\mathbf{R}\underline{\mathrm{Hom}}_B(D_B^{\mathbb{G}_m} N, K_B)) \\ &\simeq \mathbb{T}_y(\mathbf{R}\underline{\mathrm{Hom}}_B(D_B^{\mathbb{G}_m} N, D_B^{\mathbb{G}_m} B)) \\ &\simeq \mathbb{T}_y(\mathbf{R}\underline{\mathrm{Hom}}_B(B, N)) \simeq \mathbb{T}_y(N), \end{aligned}$$

as required, noting that N only contains finitely many weights by hypothesis. The statement then extends to arbitrary $N \in dg_+\mathcal{C}oh_B^{f\mathbb{G}_m}$ by nilcompleteness. \square

4.2. Weighted shifted symplectic structures on weighted dagger dg algebras.

4.2.1. *Weighted pre-symplectic structures.* The category $dg_+\mathrm{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \mathrm{loc}}$ of weighted localised dagger dg K -algebras does not contain any quasi-free objects with non-zero weights, but the results of §4.1.2 (with a similar argument to Example 4.22) imply that the functor $A \mapsto (A, \Omega_A^1)$ and its alternating powers $A \mapsto (A, \Omega_A^n)$ admit left-derived functors taking values in the category of pairs (A, M) for $M \in dg_+\mathcal{C}oh_A^{f\mathbb{G}_m}$.

On objects, we can construct the derived functors explicitly by taking a quasi-free resolution C of A in the larger category $dg_+\mathrm{AffdAlg}_K^{\mathbb{G}_m, \dagger, \mathrm{loc}}$, then setting

$$\mathbf{L}\Omega_A^n := \Omega_C^n \otimes_C A \in dg_+\mathcal{C}oh_A^{f\mathbb{G}_m}.$$

In particular, note that although A has finitely many weights, the resolution C can have generators in infinitely many weights as the degree increases, which is why $\mathbf{L}\Omega_A^1$ lies in $dg_+\mathcal{C}oh_A^{f\mathbb{G}_m}$ rather than $dg_+\mathcal{C}oh_A^{f\mathbb{G}_m}$.

Definition 4.24. Given a weighted algebra $A \in dg_+\mathrm{AffdAlg}_K^{\mathbb{G}_m, \dagger, \mathrm{loc}}$, define the de Rham complex $\mathrm{DR}(A)$ to be the \mathbb{G}_m -equivariant complex $\mathrm{DR}(A) := \bigoplus_n \mathcal{W}_n \mathrm{DR}(A)$ given by setting $\mathcal{W}_n \mathrm{DR}(A)$ to be the product total cochain complex of the double complex

$$\mathcal{W}_n A \xrightarrow{d} \mathcal{W}_n \Omega_A^1 \xrightarrow{d} \mathcal{W}_n \Omega_A^2 \xrightarrow{d} \dots,$$

so the total differential is $d \pm \delta$.

We define the Hodge filtration F on $\mathrm{DR}(A)$ by setting $F^p \mathcal{W}_n \mathrm{DR}(A) \subset \mathcal{W}_n \mathrm{DR}(A)$ to consist of terms $\mathcal{W}_n \Omega_A^i$ with $i \geq p$.

Define $\mathbf{LDR}(A)$ to be $\mathrm{DR}(\tilde{A})$ for any quasi-free replacement of A .

Definition 4.25. Define the space $\mathcal{W}_m \text{PreSp}(A, n)$ of n -shifted pre-symplectic structures of weight m on an object $A \in dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}$ to be the simplicial set given by Dold–Kan denormalisation of the chain complex

$$\tau_{\geq 0}(\mathbf{L}F^2 \mathcal{W}_m \text{DR}(A)^{[n+2]}).$$

Definition 4.26. Given a morphism $A \rightarrow B$ in $dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}$, define the space $\mathcal{W}_m \text{Iso}(A, B; n)$ of n -shifted isotropic structures of weight m on the pair (A, B) to be the simplicial set given by Dold–Kan denormalisation of the chain complex

$$\tau_{\geq 0}(\text{cone}(\mathbf{L}F^2 \mathcal{W}_m \text{DR}(A) \rightarrow \mathbf{L}F^2 \mathcal{W}_m \text{DR}(B))^{[n+1]}).$$

4.2.2. Weighted symplectic structures.

Definition 4.27. Given a \mathbb{G}_m -equivariant CDGA $A_\bullet = \bigoplus_n \mathcal{W}_n A_\bullet$, define the internal Hom functor $\mathcal{H}om$ on the category of \mathbb{G}_m -equivariant A -modules by setting

$$\mathcal{W}_n \mathcal{H}om(M, N) := \underline{\text{Hom}}_A(M, N\{n\})^{\mathbb{G}_m},$$

the complex of A -linear \mathbb{G}_m -equivariant maps from M to $N\{n\}$, for $N\{n\}$ as in Definition 4.18.

Multiplication by elements of $\mathcal{W}_m A$ gives \mathbb{G}_m -equivariant maps $N\{n\} \rightarrow N\{n+m\}$, which combine to give the \mathbb{G}_m -equivariant A -module structure on $\mathcal{H}om(M, N) := \bigoplus_n \mathcal{W}_n \mathcal{H}om(M, N)$.

Definition 4.28. For $A \in dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}$, say that an n -shifted pre-symplectic structure $\omega \in Z^{n+2}(F^2 \mathcal{W}_n \text{DR}(\tilde{A}))$ of weight m is *symplectic* if the induced map

$$\omega_2^\sharp: \mathcal{H}om_{\tilde{A}}(\Omega_{\tilde{A}}^1, \tilde{A}) \rightarrow (\Omega_{\tilde{A}}^1)_{[-n]}\{m\}$$

is a quasi-isomorphism.

We then define the space $\mathcal{W}_m \text{Sp}(A, n)$ of n -shifted symplectic structures of weight m to be the subspace of $\mathcal{W}_m \text{PreSp}(A, n)$ (a union of path components) consisting of symplectic objects.

Definition 4.29. Given a morphism $f: A \rightarrow B$ in $dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}$, we say that an element (ω, λ) of

$$Z^{n+1} \text{cone}(F^2 \mathcal{W}_m \text{DR}(\tilde{A}) \rightarrow \mathbf{L}F^2 \mathcal{W}_m \text{DR}(B))$$

is Lagrangian of weight m if $\omega \in \mathcal{W}_m \text{Sp}(A, n)$ (i.e. is symplectic) and if contraction with the image (ω_2, λ_2) of (ω, λ) in $Z^{n-1} \text{cone}(\mathcal{W}_m \Omega_{\tilde{A}}^2 \rightarrow \mathcal{W}_m \Omega_B^2)$ induces a quasi-isomorphism

$$(f \circ \omega_2^\sharp, \lambda_2^\sharp): \text{cone}(\mathcal{H}om_{\tilde{B}}(\Omega_{\tilde{B}}^1, \tilde{B}) \rightarrow \mathcal{H}om_{\tilde{A}}(\Omega_{\tilde{A}}^1, \tilde{B})) \rightarrow (\Omega_{\tilde{B}}^1)_{[-n]}\{m\}.$$

Set $\mathcal{W}_m \text{Lag}(A, B; n) \subset \mathcal{W}_m \text{Iso}(A, B; n)$ to consist of the Lagrangian structures — this is a union of path-components.

Remark 4.30. For general homotopy-preserving functors $F: dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}} \rightarrow s\text{Set}$, we can adapt Remark 2.18 and let $\mathcal{W}_m \text{PreSp}(F, n)$ be the space $\mathbf{R}\text{map}(F, \mathcal{W}_m \text{PreSp}(-, n))$ of maps of homotopy-preserving presheaves from F to $\mathcal{W}_m \text{PreSp}(-, n)$, thus functorially associating an n -shifted pre-symplectic structure of weight m on A to each point in $F(A)$.

In order to define a subspace of shifted symplectic structures of weight m , we need F to moreover be homogeneous with a cotangent complex, and then we can let $\mathcal{W}_m \text{Sp}(F, n) \subset$

$\mathcal{W}_m \text{PreSp}(F, n)$ consist of the objects ω which are non-degenerate in the sense that the maps

$$(\omega_x)^\sharp_2: \mathcal{H}om_{\tilde{A}}(\Omega_{\tilde{A}}^1, \tilde{A}) \rightarrow (\Omega_{\tilde{A}}^1)_{[-n]}\{m\},$$

for all $x \in F(A)$ are induced by compatible quasi-isomorphisms

$$\mathcal{H}om_{\tilde{A}}(\mathbb{L}^{F,x}, \tilde{A}) \rightarrow (\mathbb{L}^{F,x})_{[-n]}\{m\};$$

when F has an étale cover by objects of $dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}}$, this amounts to saying that those objects carry compatible shifted symplectic structures of weight m .

We can make entirely similar constructions for isotropic and Lagrangian structures, with the former given as a mapping space over the category of arrows in $dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}}$.

4.3. Weighted formal dg dagger spaces. Since our functors F will typically have cotangent complexes with infinitely many weights (i.e. lying in $dg\mathcal{C}oh_B^{f\mathbb{G}_m}$ rather than $dg\mathcal{C}oh_B^{f\mathbb{G}_m}$), we cannot just take weighted dagger dg algebras (or dually weighted dg dagger affinoid spaces) as building blocks. Instead, we work with formal weighted algebras and spaces, so take inverse systems $\{A(\alpha)\}_\alpha$ of localised weighted dagger dg algebras for which the system $\{\mathbb{H}_0 A(\alpha)/\mathbb{G}_m\}_\alpha$ is constant. Equivalently, this means looking at direct systems $X = \{X(\alpha)\}_\alpha$ of weighted localised dg dagger affinoid spaces for which the system $\{\pi^0 X(\alpha)^{\mathbb{G}_m}\}_\alpha$ of \mathbb{G}_m -invariants in the underived truncation is constant; we can then glue these using just the topology on $\pi^0 X^{\mathbb{G}_m}$.

4.3.1. Weighted formal dagger dg algebras.

Definition 4.31. Define a weighted formal localised dagger dg algebra to be an object $A = \{A(i)\}_{i \in I}$ of the pro-category $\text{pro}(dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b})$ for which the pro-object $\mathbb{H}_0 A/\mathbb{G}_m = \{\mathbb{H}_0 A(i)/\mathbb{G}_m\}_{i \in I}$ is isomorphic to a constant filtered system (i.e. lies in the essential image of $\text{AffdAlg}_K \rightarrow \text{pro}(\text{AffdAlg}_K)$).

Example 4.32. If we start with a dagger algebra C in weight 0 and introduce free variables x, y of weights 1, -1 , then the free weighted formal dagger K -algebra D over C generated by x, y is given by the CDGA

$$\mathcal{W}_n D = \begin{cases} x^n C[[xy]] & n \geq 0 \\ y^{-n} C[[xy]] & n \leq 0, \end{cases}$$

regarded as a limit of the weighted dagger K -algebras $D/(\mathcal{W}_{<-m} D, \mathcal{W}_{>m} D)$ given by

$$\mathcal{W}_n(D/(\mathcal{W}_{<-m} D, \mathcal{W}_{>m} D)) = \begin{cases} x^n(C[xy]/(xy)^{m+1-n}) & 0 \leq n \leq m \\ y^{-n}(C[xy]/(xy)^{m+1+n}) & -m \leq n \leq 0 \\ 0 & n \notin [-m, m]. \end{cases}$$

This has the property that

$$\text{Hom}_{\text{pro}(\text{AffdAlg}_K^{f\mathbb{G}_m, \dagger})}(D, E) \cong \text{Hom}_{\text{AffdAlg}_K^\dagger}(C, \mathcal{W}_0 E) \times \mathcal{W}_1 E \times \mathcal{W}_{-1} E$$

for all weighted dagger K -algebras E .

Definition 4.33. Define simplicial mapping spaces $\mathbf{Rmap}_{\text{pro}(dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b})}$ by localisation at pro-quasi-isomorphisms (i.e. essentially levelwise quasi-isomorphism in the

sense of [Isa, §2.1]). Explicitly, for $A = \{A(i)\}_{i \in I}$ and $B = \{B(j)\}_{j \in J}$, we have

$$\mathbf{Rmap}_{\mathrm{pro}(dg_+ \mathrm{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \mathrm{loc}, b})}(A, B) \simeq \mathrm{holim} \lim_{\substack{\leftarrow \\ j \\ \rightarrow \\ i}} \mathbf{Rmap}_{dg_+ \mathrm{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \mathrm{loc}, b}}(A(i), B(j)).$$

Note that Corollary 4.15 implies that the obvious left adjoint functors from $dg_+ \mathrm{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \mathrm{loc}}$ and $dg_+ \mathrm{AffdAlg}_K^{\mathbb{G}_m, \dagger, \mathrm{loc}}$ to $\mathrm{pro}(dg_+ \mathrm{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \mathrm{loc}, b})$ induce fully faithful left-derived simplicial functors on the simplicial localisations. This allows us to regard weighted formal localised dagger dg algebras as giving a natural enlargement of the categories we have been studying so far.

Definition 4.34. Say that a morphism $A \rightarrow B$ in $\mathrm{pro}(dg_+ \mathrm{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \mathrm{loc}, b})$ is *homotopy formally étale* (resp. *homotopy formally smooth*) if it induces weak equivalences (resp. π_0 -surjections)

$$\mathbf{Rmap}(B, C) \rightarrow \mathbf{Rmap}(A, C) \times_{\mathbf{Rmap}(A, D)}^h \mathbf{Rmap}(B, D)$$

for all nilpotent surjections $C \rightarrow D$ in $dg_+ \mathrm{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \mathrm{loc}, b}$, where \mathbf{Rmap} denotes $\mathbf{Rmap}_{\mathrm{pro}(dg_+ \mathrm{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \mathrm{loc}, b})}$.

Remark 4.35. There are many equivalent characterisations of Definition 4.34. Filtering by powers of the kernel, it suffices to know that the condition holds for surjections $C \rightarrow D$ with square-zero kernel.

The constructions of [Qui] are sufficiently general to give a theory of cotangent complexes \mathbb{L}^A for objects A of $\mathrm{pro}(dg_+ \mathrm{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \mathrm{loc}, b})$, existing as Beck modules, meaning that $A \oplus \mathbb{L}^A$ is a group object in the slice category $\mathrm{pro}(dg_+ \mathrm{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \mathrm{loc}, b})$; in particular, we can regard \mathbb{L}^A as an A -module in $\mathrm{pro}(\text{bounded chain complexes over } K)$. Standard arguments then show that for $A \rightarrow B$ to be homotopy formally étale amounts to saying that $\mathbb{L}^{B/A}$ is pro-quasi-isomorphic to 0, while being homotopy formally smooth amounts to saying that $\mathbb{L}^{B/A}$ satisfies the left lifting property with respect to surjections.

There is a form of base change for Beck modules, and we can rewrite the last characterisation of homotopy formally étale maps as saying that $\mathbb{L}^B \simeq \mathbb{L}^A \otimes_A B$. We can reduce this further to say that $\mathbb{L}^B \otimes_B C \simeq \mathbb{L}^A \otimes_A C$ for all $C \in B \downarrow dg_+ \mathrm{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \mathrm{loc}, b}$, or even just for $C \in H_0 B \downarrow \mathrm{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \mathrm{loc}, b}$, via a Postnikov induction argument. These characterisations tie in with Definition 4.20 because for $F := \mathbf{Rmap}_{\mathrm{pro}(dg_+ \mathrm{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \mathrm{loc}, b})}(A, -)$ and $x \in F(C)$, we have $\mathbb{L}^{F, x} \simeq \mathbb{L}^A \otimes_A C$.

4.3.2. Weighted formal dagger spaces.

Definition 4.36. Define a weighted formal K -dagger dg space X to be a pair $(\pi^0 X^{\mathbb{G}_m}, \mathcal{O}_X)$ where $\pi^0 X^{\mathbb{G}_m}$ is a K -dagger space and \mathcal{O}_X is a presheaf of weighted formal localised dagger dg algebras on the site of open affinoid subdomains of $\pi^0 X^{\mathbb{G}_m}$, such that the homology presheaf $H_0 \mathcal{O}_X / \mathbb{G}_m$ is just $\mathcal{O}_{\pi^0 X^{\mathbb{G}_m}}$, and such that for an inclusion $U \subset V$ of open affinoid subdomains of $\pi^0 X^{\mathbb{G}_m}$, the map

$$\mathcal{O}_X(V) \rightarrow \mathcal{O}_X(U)$$

is homotopy formally étale.

Remark 4.37. There are entirely similar definitions for weighted formal K -dagger dg Deligne–Mumford and Artin stacks, using the étale and smooth sites, and for N -stacks, proceeding as in Remark 1.31.

Definition 4.38. A morphism $f: X \rightarrow Y$ of weighted formal K -dagger dg spaces is said to be a quasi-isomorphism if it induces an isomorphism $\pi^0 f^{\mathbb{G}_m}: \pi^0 X^{\mathbb{G}_m} \rightarrow \pi^0 Y^{\mathbb{G}_m}$ of formal weighted dagger spaces and a pro-quasi-isomorphism $f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$.

4.4. Representability. Although not strictly necessary for the study of shifted symplectic structures on analytic moduli functors, we now include a representability result which applies to most examples of interest.

Proposition 4.39. *A homotopy-preserving functor $F: dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b} \rightarrow s\text{Set}$ is a weighted formal dagger-analytic Artin derived n -stack with a coherent cotangent complex if and only if the following conditions hold*

- (1) *The restriction $\pi^0 F^{\mathbb{G}_m}: \text{AffdAlg}_K^\dagger \rightarrow s\text{Set}$ to underived dagger algebras is represented by a dagger-analytic Artin n -stack.*
- (2) *F is homogeneous.*
- (3) *for all weighted dagger algebras $A \in \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger}$, all $x \in F(A)_0$ and all étale morphisms $f: A \rightarrow A'$, the maps*

$$D_x^*(F, A) \otimes_A A' \rightarrow D_{f(x)}^*(F, A')$$

are isomorphisms.

- (4) *for all dagger algebras A (regarded as living in weight 0 and degree 0) and all $x \in F(A)$, the groups $D_x^i(F, A\{m\})$ are all finitely generated A -modules.*

Proof. By Lemma 4.23, the conditions ensure that F has coherent cotangent complexes $\mathbb{L}^{F, y}$ at all points $y \in F(B)$ for all $B \in dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b}$. Reasoning as in the proof of Corollary 1.45, it follows that F is an étale hypersheaf. It then only remains to establish the existence of smooth atlases.

Corollary 1.45 implies that the functor $F^{\mathbb{G}_m}$ given by the restriction of F to $dg_+ \text{AffdAlg}_K^{\dagger, \text{loc}}$ (regarded as objects concentrated in weight 0) is representable. Given a smooth atlas $U_{[0]} \rightarrow F^{\mathbb{G}_m}$, we now inductively construct a sequence

$$U_{[0]} \rightarrow U_{[0,1]} \rightarrow U_{[-1,1]} \rightarrow U_{[-1,2]} \rightarrow U_{[-2,2]} \rightarrow \dots \rightarrow F$$

of morphisms of presheaves $U_I = \mathbf{R}\text{Spec } A_I$ on $dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b}$, for $A_I \in dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}}$ (via §4.1.2), such that:

- each morphism $U_I \rightarrow U_J$ of presheaves is an equivalence when restricted to objects concentrated in weights I , and
- each morphism $U_I \rightarrow F$ of presheaves is formally smooth when restricted to objects concentrated in weights I , in the sense that for each surjection $B \rightarrow C$ of such objects, the map

$$U_I(B) \rightarrow F(B) \times_{F(C)}^h U_I(C)$$

is surjective on π_0 .

Given I , let $I' = I \cup \{n\}$ be the next interval in the sequence, noting that n might be negative. Observe that for any weighted dagger algebra B concentrated in weights I' , the ideal generated by $\mathcal{W}_n B$ squares to 0. We can then construct $U_{I'}$ from U_I as the solution to a deformation problem. One approach is to proceed along similar lines to [PY2, Theorem 7.1], iteratively eradicating unwanted terms in the cotangent complex $(\mathbb{L}^{U_I/F}/\mathcal{W}_{\notin I'} \mathbb{L}^{U_I/F}) \otimes_{A_I}^{\mathbf{L}} \mathbf{H}_0 A_{[0]}$ by taking homotopy fibres of universal derivations; this eventually yields an object $U_{I'}$ with $(\mathbb{L}^{U_{I'}/F}/\mathcal{W}_{\notin I'} \mathbb{L}^{U_{I'}/F}) \otimes_{A_I}^{\mathbf{L}} \mathbf{H}_0 A_{[0]}$ a complex

of weighted projective $H_0A_{[0]}$ -modules in non-positive chain degrees, ensuring formal smoothness.

The pro-object $\{A_I\}_I \in \text{pro}(dg_+\text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}})$ then gives the required atlas as an object $\text{pro}(dg_+\text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b})$, via the derived completion functor $dg_+\text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}} \rightarrow \text{pro}(dg_+\text{AffdAlg}_K^{f\mathbb{G}_m, \dagger, \text{loc}, b})$ of §4.1.2. \square

5. WEIGHTED SHIFTED SYMPLECTIC STRUCTURES ASSOCIATED TO PRO-ÉTALE SHEAVES

We now return to the questions outlined at the start of §4, applying the techniques of that section to our examples of interest.

As in §1.4, assume that the valuation on our base field K is discretely valued, so the ring \mathcal{O}_K is a DVR with maximal ideal \mathfrak{m}_K . Let ℓ be the unique integral prime in \mathfrak{m}_K .

5.1. Weighted pre-symplectic structures.

Definition 5.1. Given a scheme X , a locally free rank 1 ℓ -adic lisse sheaf \mathbb{E} on X , and a graded topological K -vector space $V = \bigoplus_{r \in \mathbb{Z}} \mathcal{W}_r V$, we let $\underline{V}_X(\mathbb{E})$ be the pro-étale sheaf $\bigoplus_{n \in \mathbb{Z}} \underline{\mathcal{W}}_n A_X \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{E}^{\otimes n}$ on X .

This allows the following modification of Definition 1.53:

Definition 5.2. For (X, \mathbb{E}) as above, and a functor $F: dg_+\text{CAlg}_K \rightarrow s\text{Set}$ from differential graded-commutative K -algebras in non-negative chain degrees to simplicial sets, define the functor

$$F(X_{\text{proét}}, \mathbb{E}, -): dg_+\text{AffdAlg}_K^{f\mathbb{G}_m, \text{loc}, \dagger} \rightarrow s\text{Set}$$

from weighted localised dagger dg algebras to simplicial sets by

$$A \mapsto \mathbf{R}\Gamma(X_{\text{proét}}, F(\underline{A}_X(\mathbb{E}))),$$

where $\mathbf{R}\Gamma$ is the right-derived functor of the global sections functor Γ in simplicial sets. Note that the hypotheses on A imply that the direct sum in the definition of $\underline{A}_X(\mathbb{E})$ is finite.

Example 5.3 (G -torsors). If G is an algebraic group over K , then we can let F be the derived stack BG , parametrising G -torsors. The functor $BG(X_{\text{proét}}, \mathbb{E}, -)$ can then be thought of as parametrising G -torsors on $\mathbb{E} \setminus \{0\}$ over $X_{\text{proét}}$. In particular, when A is a weighted dagger algebra (concentrated in chain degree 0), the simplicial set $BG(X_{\text{proét}}, \mathbb{E}, A)$ is the nerve of the groupoid of $G(\underline{A}_X(\mathbb{E}))$ -torsors on $X_{\text{proét}}$.

If X is locally topologically Noetherian and connected, with a geometric point x , then \mathbb{E} corresponds to a continuous \mathbb{Z}_ℓ -representation E of the pro-étale fundamental group $\pi_1^{\text{proét}}(X, x)$ of [BS, §7]. For $A \in \text{AffdAlg}_K^{f\mathbb{G}_m}$, the simplicial set $BG(X_{\text{proét}}, \mathbb{E}, A)$ is then equivalent to the nerve of the groupoid of continuous sections of the group homomorphism

$$\pi_1^{\text{proét}}(X, x) \times G\left(\bigoplus_{r \in \mathbb{Z}} \mathcal{W}_r A \hat{\otimes}_{\mathbb{Z}_\ell} E^{\otimes r}\right) \rightarrow \pi_1^{\text{proét}}(X, x)$$

given by projection to the first factor.

Example 5.4 (Commutative groups). As a special case of Example 5.3, if G is commutative with associated (abelian) Lie algebra \mathfrak{g} , then for any $A \in dg_+ \text{AffdAlg}_K^{\dagger, fG_m}$ we have a group isomorphism

$$G\left(\bigoplus_r \mathcal{W}_r A \hat{\otimes}_{\mathbb{Z}_\ell} E^{\otimes r}\right) \simeq G(\mathcal{W}_0 A) \times N^{-1}(\mathfrak{g} \otimes_K \left(\bigoplus_{r \neq 0} \mathcal{W}_r A \hat{\otimes}_{\mathbb{Z}_\ell} E^{\otimes r}\right))$$

given by deriving the map $(g, v) \mapsto g \exp(v)$ on weighted affinoid algebras, where N^{-1} is Dold–Kan denormalisation. To see that this is an isomorphism, observe that for the nilpotent ideal $I = \mathcal{W}_{\neq 0} A \oplus \mathcal{W}_0 I$ generated by $\mathcal{W}_{\neq 0} A$, we have isomorphisms $G(A)/\exp(\mathfrak{g} \otimes_K I) \cong G(A/I) = G(\mathcal{W}_0 A/\mathcal{W}_0 I) \cong G(\mathcal{W}_0 A)/\exp(\mathfrak{g} \otimes_K \mathcal{W}_0 I)$.

The weighted moduli functor then decouples as

$$BG(X_{\text{proét}}, \mathbb{E}, A) \simeq BG(X_{\text{proét}}, \mathcal{W}_0 A) \times N^{-1} \tau_{\geq 0}(\mathbf{R}\Gamma(X_{\text{proét}}, \bigoplus_{r \neq 0} \mathfrak{g} \otimes_K \mathcal{W}_r A \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{E}^{\otimes r})_{[-1]}),$$

where N^{-1} is Dold–Kan denormalisation and $\tau_{\geq 0}$ denotes good truncation of a chain complex.

For instance, if k is a finite field prime to ℓ then the weighted derived moduli stack of G -torsors (for G commutative) is concentrated in weight 0, in the sense that the natural map $BG((\text{Spec } k)_{\text{proét}}, \mathcal{W}_0 A) \rightarrow BG((\text{Spec } k)_{\text{proét}}, \mathbb{Z}_\ell(1), A)$ is an equivalence, since tangent cohomology $H^*(k, \mathcal{W}_n A(n))$ vanishes in all non-zero weights n .

The next simplest example is for non-Archimedean local field K of residue characteristic prime to ℓ , where for G commutative we have

$$BG((\text{Spec } K)_{\text{proét}}, \mathbb{Z}_\ell(1), A) \simeq BG((\text{Spec } K)_{\text{proét}}, \mathcal{W}_0 A) \times N^{-1} \tau_{\geq 0}(\mathfrak{g} \otimes_K (\mathcal{W}_1 A \oplus \mathcal{W}_1 A_{[1]})),$$

since cohomology vanishes in all other weights.

Corollary 3.4 now generalises to give us the following corollary of Proposition 3.2, applicable to all of the cases in Examples 3.3 (taking $\mathbb{E} = \mathbb{Z}_\ell(1)$):

Corollary 5.5. *If X is a topologically Noetherian scheme and \mathbb{E} a locally free rank 1 ℓ -adic lisse sheaf on X , equipped with compatible trace maps*

$$\text{tr}: H^d(X_{\text{ét}}, \mathbb{E}^{\otimes m}/\ell^n) \rightarrow \mathbb{Z}/\ell^n.$$

satisfying the conditions of Proposition 3.2, then for any n -shifted pre-symplectic (in the terminology of [Pri7]) derived ∞ -geometric Artin stack $F: dg_+ \text{CAlg}_K \rightarrow s\text{Set}$, the functor

$$F(X_{\text{proét}}, \mathbb{E}, -): dg_+ \text{AffdAlg}_K^{\text{loc}, \dagger} \rightarrow s\text{Set}$$

of Definition 1.53 carries a functorial $(n-d)$ -shifted pre-symplectic structure of weight m at all points; in particular, this implies that any formally étale map $Y \rightarrow F(X_{\text{proét}}, -)$ from a dg dagger-analytic Artin ∞ -stack Y induces an $(n-d)$ -shifted pre-symplectic structure of weight m on Y .

Proof. For any $A \in dg_+ \text{AffdAlg}_K^{\text{loc}, \dagger}$, Lemma 3.1 combines with Proposition 3.2 to give us maps

$$\mathbf{R}\Gamma(X_{\text{proét}}, F^2 \mathbf{LDR}(\underline{A}_X(\mathbb{E}))) \rightarrow \mathbf{R}\Gamma(X_{\text{proét}}, F^2 \mathbf{LDR}(\underline{A})_X(\mathbb{E})) \xrightarrow{\text{tr}} \mathcal{W}_m F^2 \mathbf{LDR}(A)[-d],$$

via the projection $F^2 \mathbf{LDR}(\underline{A})_X(\mathbb{E}) \rightarrow \mathcal{W}_m F^2 \mathbf{LDR}(\underline{A})_X \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{E}^{\otimes m}$. Hence we have a map

$$\mathbf{R}\Gamma(X_{\text{proét}}, \text{PreSp}^{\text{alg}}(\underline{A}_X(\mathbb{E})^{\text{alg}}, n)) \rightarrow \mathcal{W}_m \text{PreSp}(A, n-d).$$

Similarly to Corollary 3.4, because $\text{PreSp}^{\text{alg}}(F, n) \simeq \mathbf{Rmap}(F, \text{PreSp}^{\text{alg}}(-, n))$, the maps above combine to give a composite transformation

$$\text{PreSp}^{\text{alg}}(F, n) \times F(X_{\text{proét}}, \mathbb{E}, A) \rightarrow \mathbf{R}\Gamma(X_{\text{proét}}, \text{PreSp}^{\text{alg}}(\underline{A}_X(\mathbb{E})^{\text{alg}}, n)) \rightarrow \mathcal{W}_m \text{PreSp}(-, n-d),$$

natural in A . By adjunction, we can rephrase this as a morphism

$$\text{PreSp}^{\text{alg}}(F, n) \rightarrow \mathbf{Rmap}(F(X_{\text{proét}}, \mathbb{E}, -), \mathcal{W}_m \text{PreSp}(-, n-d)).$$

as required. \square

5.2. Weighted symplectic structures. In order to establish non-degeneracy, we need a weighted version of the notion of weak duality from Definition 3.5. This involves looking at a weighted quasi-dagger dg algebra A and a presheaf N of $\underline{A}_X(\mathbb{E})$ -modules in chain complexes on $X_{\text{proét}}$. The complex $\mathbf{R}\Gamma(X, N)$ is not then an A -module, but the complex $\bigoplus_{r \in \mathbb{Z}} \mathbf{R}\Gamma(X, N \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes -r})$ is, via the map $\underline{A}_X \rightarrow \bigoplus_r \underline{A}_X(\mathbb{E}) \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes -r}$ given by

$$\underline{W}_r \underline{A}_X \rightarrow (\underline{W}_r \underline{A}_X(\mathbb{E}) \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes r}) \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes -r} \subset \underline{A}_X(\mathbb{E}) \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes -r}.$$

Accordingly, we regard $\bigoplus_{r \in \mathbb{Z}} N \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes -r}$ as a \mathbb{G}_m -equivariant module, with the term $N \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes -r}$ having weight r .

Definition 5.6. Given a weighted quasi-dagger dg algebra A and a presheaf N of $\underline{A}_X(\mathbb{E})$ -modules in chain complexes on $X_{\text{proét}}$, we say that N satisfies weak duality with respect to the trace tr if for all morphisms $A \rightarrow C$ of weighted quasi-dagger dg algebras, the map

$$\mathbf{R}\underline{\text{Hom}}_{\underline{A}_X(\mathbb{E})}(N, \underline{C}_X(\mathbb{E})) \rightarrow \mathbf{R}\underline{\text{Hom}}_A\left(\bigoplus_r \mathbf{R}\Gamma(X, N \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{D} \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes -r})[d], \bigoplus_r \underline{W}_r C\right)^{\mathbb{G}_m}$$

induced by the pairing

$$\begin{aligned} & \left(\bigoplus_r \mathbf{R}\Gamma(X, N \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{D} \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes -r})\right) \otimes_A^{\mathbf{L}} \mathbf{R}\underline{\text{Hom}}_{\underline{A}_X(\mathbb{E})}(N, \underline{C}_X(\mathbb{E})) \\ & \rightarrow \mathbf{R}\Gamma(X, \bigoplus_r \underline{C}_X(\mathbb{E}) \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{D} \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes -r}) \rightarrow C[-d] \end{aligned}$$

is a quasi-isomorphism. Here, our pairing factors through the obvious projection maps $\bigoplus_r \underline{C}_X(\mathbb{E}) \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes -r} \rightarrow \bigoplus_r \underline{W}_r \underline{C}_X$.

Note that by taking $C = A \oplus M$, we can deduce a similar quasi-isomorphism for all $M \in \text{dg}_+ \text{Coh}_A^{f\mathbb{G}_m}$ in place of C .

Remark 5.7. When A lives in weight 0, we have $\underline{A}_X(\mathbb{E}) = \underline{A}_X$, but beware that Definition 5.6 is then still a more general statement than Definition 3.5, because the algebras C can still have non-zero weights. Explicitly, the condition reduces to

$$\bigoplus_r \mathbf{R}\underline{\text{Hom}}_{\underline{A}_X}(N, \underline{W}_r \underline{C}_X \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes r}) \simeq \bigoplus_r \mathbf{R}\underline{\text{Hom}}_A(\mathbf{R}\Gamma(X, N \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{D} \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes -r})[d], \underline{W}_r C)$$

for all C , so it amounts to saying that the \underline{A}_X -modules $N \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes r}$ all satisfy weak duality in the sense of Definition 3.5.

There are then immediate weighted analogues of Lemmas 3.6 and 3.7; the latter becomes:

Lemma 5.8. *Given a weighted quasi-dagger dg algebra A , a module $N \in dg_+ \text{Mod}_{\underline{A}_X(\mathbb{E})}$ satisfies weak duality in the sense of Definition 5.6 if and only if the presheaves $(\underline{H}_0 A / \underline{\mathbb{G}}_m)_X \otimes_{\underline{A}_X(\mathbb{E})}^{\mathbf{L}} N \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes r}$ of $\underline{H}_0 A / \underline{\mathbb{G}}_m$ -modules all satisfy weak duality in the sense of Definition 3.5.*

Examples 5.9. Roughly speaking, a sufficient condition for an $\underline{A}_X(\mathbb{E})$ -module N on an ℓ -coprime proper scheme X to satisfy weak duality is that its sheafification is constructible in an appropriate sense; by the analogue of Lemma 3.7, we can reduce to looking at the $(\underline{H}_0 A / \underline{\mathbb{G}}_m)_X$ -module $N \otimes_{\underline{A}_X(\mathbb{E})}^{\mathbf{L}} (\underline{H}_0 A / \underline{\mathbb{G}}_m)_X$ (noting that $(\underline{H}_0 A / \underline{\mathbb{G}}_m)_X = (\underline{H}_0 A / \underline{\mathbb{G}}_m)_X(\mathbb{E})$ since $\underline{H}_0 A / \underline{\mathbb{G}}_m$ lives in weight 0). Also note that the category of modules satisfying weak duality is triangulated and idempotent-complete.

In the setting of Lemma 3.8, with a constructible \hat{R}_X -complex L and a homomorphism $R \rightarrow \mathcal{W}_0 A$, consider the $\underline{A}_X(\mathbb{E})$ -complex $N := L \otimes_{\hat{R}_X}^{\mathbf{L}} \underline{A}\{m\}_X(\mathbb{E})$ (for $\underline{A}\{m\}$ as in Definition 4.18). This (and hence any object of the triangulated category generated by such objects) satisfies weak duality, with the following reasoning. Since

$$\underline{A}\{m\}_X(\mathbb{E}) = \bigoplus_r \underline{\mathcal{W}}_{m+r} A \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes r} = \underline{A}_X(\mathbb{E}) \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes -m},$$

Lemma 3.8 implies that

$$\mathbf{R}\Gamma(X_{\text{proét}}, N \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes -r}) \simeq \bigoplus_s \mathbf{R}\Gamma(X_{\text{proét}}, L \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes s-r-m}) \otimes_R^{\mathbf{L}} \mathcal{W}_s A,$$

so

$$\begin{aligned} & \mathbf{R}\underline{\text{Hom}}_A \left(\bigoplus_r \mathbf{R}\Gamma(X, N \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{D} \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes -r}), \bigoplus_r \mathcal{W}_r C \right)^{\mathbb{G}_m} \\ & \simeq \mathbf{R}\underline{\text{Hom}}_A \left(\bigoplus_r \mathbf{R}\Gamma(X_{\text{proét}}, L \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{D} \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes -r-m}) \otimes_R^{\mathbf{L}} A, \bigoplus_r \mathcal{W}_r C \right)^{\mathbb{G}_m} \\ & \simeq \bigoplus_r \mathbf{R}\underline{\text{Hom}}_R \left(\mathbf{R}\Gamma(X_{\text{proét}}, L \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{D} \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes -r-m}), \mathcal{W}_r C \right), \end{aligned}$$

noting that the sum is finite by the hypothesis on C .

Similarly, Lemma 3.8 gives

$$\begin{aligned} \mathbf{R}\underline{\text{Hom}}_{\underline{A}_X(\mathbb{E})}(N, \underline{C}_X(\mathbb{E})) & \simeq \mathbf{R}\underline{\text{Hom}}_{\underline{A}_X(\mathbb{E})}(L \otimes_{\hat{R}_X}^{\mathbf{L}} \underline{A}_X(\mathbb{E}), \underline{C}_X(\mathbb{E}) \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes m}) \\ & \simeq \bigoplus_r \mathbf{R}\underline{\text{Hom}}_{\hat{R}_X}(L, \hat{R}_X \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes m+r}) \otimes_R^{\mathbf{L}} \mathcal{W}_r C, \end{aligned}$$

The natural map

$$\mathbf{R}\underline{\text{Hom}}_{\underline{A}_X(\mathbb{E})}(N, \underline{C}_X(\mathbb{E})) \rightarrow \mathbf{R}\underline{\text{Hom}}_A \left(\bigoplus_r \mathbf{R}\Gamma(X, N \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{D} \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes -r})[d], \bigoplus_r \mathcal{W}_r C \right)^{\mathbb{G}_m}$$

is thus a quasi-isomorphism, since it reduces to the Verdier duality quasi-isomorphisms

$$\mathbf{R}\underline{\text{Hom}}_{\hat{R}_X}(L, \hat{R}_X \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes m+r}) \rightarrow \mathbf{R}\underline{\text{Hom}}_R(\mathbf{R}\Gamma(X_{\text{proét}}, L \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{D} \hat{\otimes}_{\underline{\mathbb{Z}}_\ell} \mathbb{E}^{\otimes -r-m}), R).$$

Lemma 5.10. *Let X be a topologically Noetherian scheme satisfying the conditions of Proposition 3.2, and $F: dg_+ \text{CAlg}_K \rightarrow \text{sSet}$ a derived ∞ -geometric Artin stack. At any point $\phi \in F(X_{\text{proét}}, \mathbb{E}, A)$ at which the presheaf $\mathbb{L}^{F, \underline{A}_X(\mathbb{E}), \phi}$ of $\underline{A}_X(\mathbb{E})$ -modules satisfies weak duality in the sense of Definition 3.5, the functor*

$$T_\phi(F(X_{\text{proét}}, \mathbb{E}, -), -): M \mapsto F(X_{\text{proét}}, \mathbb{E}, A \oplus M) \times_{F(X_{\text{proét}}, \mathbb{E}, A)}^h \{\phi\}$$

on weighted coherent A -modules M is represented by the weighted complex

$$\bigoplus_r \mathbf{R}\Gamma(X, \mathbb{L}^{F, \underline{A}_X(\mathbb{E}), \phi} \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{D} \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{E}^{\otimes -r})[d].$$

Proof. This follows in much the same way as Lemma 3.10. Explicitly, we have

$$\begin{aligned} & F(X_{\text{proét}}, \mathbb{E}, A \oplus M) \times_{F(X_{\text{proét}}, \mathbb{E}, A)}^h \{\phi\} \\ & \simeq N^{-1} \tau_{\geq 0} \mathbf{R}\underline{\text{Hom}}_{A_X(\mathbb{E})}(\mathbb{L}^{F, \underline{A}_X(\mathbb{E}), \phi}, \underline{M}_X(\mathbb{E})). \end{aligned}$$

By weak duality, this in turn is equivalent to

$$N^{-1} \tau_{\geq 0} \mathbf{R}\underline{\text{Hom}}_A(\bigoplus_r \mathbf{R}\Gamma(X, \mathbb{L}^{F, \underline{A}_X(\mathbb{E}), \phi} \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{D} \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{E}^{\otimes -r})[d], M)^{\mathbb{G}_m}$$

so is represented by $\bigoplus_r \mathbf{R}\Gamma(X, \mathbb{L}^{F, \underline{A}_X(\mathbb{E}), \phi} \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{D} \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{E}^{\otimes -r})[d]$. \square

Corollaries 3.11, 3.12 now have the following immediate generalisations:

Corollary 5.11. *Let X be a topologically Noetherian scheme satisfying the conditions of Proposition 3.2 with $\mathbb{D} = \mathbb{E}^{\otimes m}$, and $F: dg_+ \text{CAlg}_K \rightarrow s\text{Set}$ an n -shifted symplectic derived Artin ∞ -stack. Then there is a natural $(n - d)$ -shifted symplectic structure of weight m (in the sense of Remark 4.30) on the full subfunctor $F(X_{\text{proét}}, \mathbb{E}, -)^{wd} \subset F(X_{\text{proét}}, \mathbb{E}, -)$ consisting of points ϕ at which the presheaf $\mathbb{L}^{F, \underline{A}_X(\mathbb{E}), \phi}$ satisfies weak duality.*

Corollary 5.12. *Under the conditions of Corollary 5.11, take an (underived) dagger Artin analytic ∞ -stack Y equipped with a formally étale morphism*

$$\eta: Y \rightarrow \pi^0 F(X_{\text{proét}}, -)$$

of functors $\text{AffdAlg}_K^{\text{loc}, \dagger} \rightarrow s\text{Set}$, such that at all points ϕ in the image of η , the \underline{A}_X -modules $\mathbb{L}^{F, \underline{A}_X(\mathbb{E}), \phi} \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{E}^{\otimes r}$ satisfy weak duality for all r .

Then the functor $\tilde{Y}: A \mapsto Y(\mathbb{H}_0 A / \mathbb{G}_m) \times_{F(X_{\text{proét}}, \mathbb{H}_0 A / \mathbb{G}_m)}^h F(X_{\text{proét}}, \mathbb{E}, A)$ on $dg_+ \text{AffdAlg}_K^{f\mathbb{G}_m, \text{loc}, \dagger}$ is a formal weighted dg dagger Artin analytic ∞ -stack carrying a natural $(n - d)$ -shifted symplectic structure of weight m .

Examples 5.13. Examples 3.3 and 5.9 now lead to many instances of weighted $(n - d)$ -shifted symplectic moduli stacks when substituted into Corollary 5.12, taking in most of the cases we had to exclude from Examples 3.13. If we take the derived stack F to be $B\text{GL}_n$, or BG for any other affine algebraic group equipped with a G -equivariant inner product on its Lie algebra, or to be the moduli stack of perfect complexes, then $n = 2$, so the corollary produces $(2 - d)$ -shifted symplectic structures on moduli of G -torsors or of complexes of pro-étale sheaves on X , provided we impose some constructibility constraints.

These all exist on suitable open substacks (given by the formula of Corollary 5.12) of the formal weighted derived moduli stack $F(X_{\text{proét}}, \mathbb{Z}_\ell(1), -)$, which we can think of as parametrising F -valued sheaves on the \mathbb{G}_m -torsor on $X_{\text{proét}}$ given by the Tate motive.

In particular:

- (1) If X is a smooth proper scheme of dimension m over a non-Archimedean local field k , we have $(n - 2m - 2)$ -shifted symplectic structures of weight $m + 1$ on suitable open substacks \tilde{Y} of $F(X_{\text{proét}}, \mathbb{Z}_\ell(1), -)$.

- (2) If X is a smooth proper scheme of dimension m over a finite field k prime to ℓ , we have $(n - 2m - 1)$ -shifted symplectic structures of weight m on suitable open substacks of $F(X_{\text{proét}}, \mathbb{Z}_\ell(1), -)$.
- (3) If U is a smooth scheme of dimension m over one of the local (resp. finite) fields k above, then we have $(n - 2m - 1)$ -shifted (resp. $(n - 2m)$ -shifted) symplectic structures of weight $m + 1$ (resp. weight m) on suitable open substacks of the derived moduli stack

$$A \mapsto \mathbf{R}\Gamma(Z_{\text{proét}}, i^* \mathbf{R}j_* F(\underline{A}_X(\mathbb{E})))$$

of F -valued sheaves on the deleted tubular neighbourhood $Z \overleftarrow{\times}_{\bar{U}} U$ (thought of as the boundary of U), where $i: Z \rightarrow \bar{U}$ is the complement of U in a compactification $j: U \rightarrow \bar{U}$.

Remark 5.14. In the terminology of [BG], we can regard our notion of n -shifted symplectic structure of weight m on a space Y as being an analytic analogue of a \mathcal{P} -shifted symplectic structure on $[Y/\mathbb{G}_m]$ relative to the morphism $f: [Y/\mathbb{G}_m] \rightarrow B\mathbb{G}_m$, taking $\mathcal{P} = f^* \mathcal{O}(m)[n]$. The proof of [BG] adapts to analytic settings, so for negative shifts gives local models for these \mathcal{P} -shifted symplectic derived spaces as products of \mathcal{P} -shifted twisted cotangent bundles with quadratic bundles, the latter being trivial unless $n \equiv 2 \pmod{4}$; their results refine and generalise the (-1) -shifted case addressed in [BBBBJ].

In our setting, an affinoid \mathcal{P} -shifted twisted cotangent bundle is given by taking a formal weighted localised dg dagger algebra B , then forming the formal weighted dg algebra

$$(\text{Sym}_B(\mathbf{R}\text{Hom}_B(\mathbf{L}\Omega_B^1, B)\{-m\}[-n]), \delta + df \lrcorner -),$$

where $f \in Z_{1+n} \mathcal{W}_m B$ is twisting the differential. A quadratic bundle is a formally weighted coherent B -module M with a symmetric pairing $M \otimes_B^{\mathbf{L}} M \simeq B\{-m\}[-n]$ (so étale locally, M can be taken to be trivial unless $n \equiv 2 \pmod{4}$ and m is even); the general local model for an n -shifted symplectic structure of weight m is then given by tensoring the twisted cotangent algebra above with $\text{Sym}_B M$.

5.3. Weighted Lagrangian structures. We now adapt the constructions of §3.3 to incorporate weights and twists. As in that section, we take a morphism $f: \partial U \rightarrow U$ of topologically Noetherian schemes (where the notation ∂U is intended to be suggestive, but does not indicate a specific construction at this point) and a constructible complex \mathbb{D} on U , equipped with a system of trace maps

$$\text{cone}(\mathbf{R}\Gamma(U_{\text{ét}}, \mathbb{D}/\ell^n) \rightarrow \mathbf{R}\Gamma(\partial U_{\text{ét}}, f^{-1} \mathbb{D}/\ell^n))[d-1] \rightarrow \mathbb{Z}/\ell^n$$

inducing a perfect pairing between $\text{cone}(\mathbf{R}\Gamma(U_{\text{ét}}, -) \rightarrow \mathbf{R}\Gamma(\partial U_{\text{ét}}, f^{-1} -))$ and $\mathbf{R}\Gamma(U_{\text{ét}}, -)$ for constructible sheaves and their duals.

Definition 5.15. Given a quasi-dagger dg algebra A and a presheaf N of $\underline{A}_U(\mathbb{E})$ -modules in chain complexes on $U_{\text{proét}}$, we say that N satisfies weak duality with respect to the trace tr above if the $\underline{A}_{\partial U}(\mathbb{E})$ -module $\mathbf{L}f^* N := f^{-1} N \otimes_{f^{-1} \underline{A}_U(\mathbb{E})}^{\mathbf{L}} \underline{A}_{\partial U}(\mathbb{E})$ satisfies weak duality in the sense of Definition 5.6, and if for all morphisms $A \rightarrow C$ of quasi-dagger dg algebras, the map

$$\begin{aligned} & \text{cone}(\mathbf{R}\underline{\text{Hom}}_{\underline{A}_U(\mathbb{E})}(N, \underline{C}_U(\mathbb{E})) \rightarrow \mathbf{R}\underline{\text{Hom}}_{\underline{A}_{\partial U}(\mathbb{E})}(\mathbf{L}f^* N, \underline{C}_{\partial U}(\mathbb{E}))) \\ & \rightarrow \mathbf{R}\underline{\text{Hom}}_A(\bigoplus_r \mathbf{R}\Gamma(U, N \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{D} \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{E}^{\otimes -r})[d-1], \bigoplus_r \mathcal{W}_r C)^{\mathbb{G}_m} \end{aligned}$$

induced by the composite pairing

$$\begin{aligned} & \mathbf{R}\Gamma(U, N \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{D} \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{E}^{\otimes -r}) \otimes_A^{\mathbf{L}} \text{cone}(\mathbf{R}\underline{\text{Hom}}_{A_U(\mathbb{E})}(N, \underline{C}_U(\mathbb{E})) \rightarrow \mathbf{R}\underline{\text{Hom}}_{A_{\partial U}(\mathbb{E})}(\mathbf{L}f^*N, \underline{C}_{\partial U}(\mathbb{E}))) \\ & \rightarrow \bigoplus_r \text{cone}(\mathbf{R}\Gamma(U, \underline{C}_U \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{D} \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{E}^{\otimes -r}) \rightarrow \mathbf{R}\Gamma(\partial U, \underline{C}_{\partial U} \hat{\otimes}_{\mathbb{Z}_\ell} f^{-1} \mathbb{D} \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{E}^{\otimes -r})) \\ & \xrightarrow{\text{tr}} C[1-d] \end{aligned}$$

is a quasi-isomorphism.

Corollaries 3.19 and 3.20 then adapt along the lines of Corollaries 5.11 and 5.12 to give:

Corollary 5.16. *Take a morphism $\partial U \rightarrow U$ of topologically Noetherian schemes which has a trace pairing as above with $\mathbb{D} = \mathbb{E}^{\otimes m}$, and take $F: dg_+ \text{CAlg}_K \rightarrow s\text{Set}$ an n -shifted symplectic derived Artin ∞ -stack. Then for the full subfunctors $F(\partial U_{\text{proét}}, \mathbb{E}, -)^{wd} \subset F(\partial U_{\text{proét}}, \mathbb{E}, -)$ and $F(U_{\text{proét}}, \mathbb{E}, -)^{wd} \subset F(U_{\text{proét}}, \mathbb{E}, -)$ consisting at points ϕ at which the presheaf $\mathbb{L}^{F, A_{\partial U}(\mathbb{E}), \phi}$ (resp. $\mathbb{L}^{F, A_U(\mathbb{E}), \phi}$) satisfies weak duality in the sense of Definition 5.6 (resp. Definition 5.15), the natural map*

$$F(U_{\text{proét}}, \mathbb{E}, -)^{wd} \rightarrow F(\partial U_{\text{proét}}, \mathbb{E}, -)^{wd}$$

carries a natural Lagrangian structure with respect to the $(n-d+1)$ -shifted symplectic structure of weight m on $F(\partial U_{\text{proét}}, -)^{wd}$ given by Corollary 5.11.

Corollary 5.17. *In the setting of Corollary 5.16, take a morphism $W \rightarrow Y$ of (underived) dagger Artin analytic ∞ -stacks, equipped with a commutative diagram*

$$\begin{array}{ccc} W & \xrightarrow{\eta'} & \pi^0 F(U_{\text{proét}}, -) \\ \downarrow & & \downarrow f^* \\ Y & \xrightarrow{\eta} & \pi^0 F(\partial U_{\text{proét}}, -) \end{array}$$

of functors $\text{AffdAlg}_K^{\text{loc}, \dagger} \rightarrow s\text{Set}$. Assume moreover that the horizontal maps η are formally étale and that at all points ϕ in the image of η' (resp. η), the presheaves $\mathbb{L}^{F, A_{\partial U}, \phi} \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{E}^{\otimes r}$ (resp. $\mathbb{L}^{F, A_U, \phi} \hat{\otimes}_{\mathbb{Z}_\ell} \mathbb{E}^{\otimes r}$) satisfy weak duality in the sense of Definition 3.5 (resp. Definition 3.17) for all r .

Then the functor $\tilde{W}: A \mapsto W(\mathbb{H}_0 A / \mathbb{G}_m) \times_{F(\partial U_{\text{proét}}, \mathbb{H}_0 A / \mathbb{G}_m)}^h F(\partial U_{\text{proét}}, A)$ on $dg_+ \text{AffdAlg}_K^{\text{loc}, \dagger}$ is a formal weighted dg dagger Artin analytic ∞ -stack. It carries a natural Lagrangian structure with respect to the $(n-d+1)$ -shifted symplectic structure of weight m on \tilde{Y} given by Corollary 5.12.

Examples 5.18. Examples 3.15 and 3.18 now lead to many instances of Lagrangians in weighted shifted symplectic moduli stacks when substituted into Corollary 3.20. If we take the n -shifted symplectic derived stack F to be BGL_n , or BG for any other affine algebraic group equipped with a G -equivariant inner product on its Lie algebra, or to be the moduli stack of perfect complexes, then $n = 2$, so the corollary produces $(3-d)$ -shifted Lagrangian structures on moduli of G -torsors or of complexes of pro-étale sheaves on U , provided we impose some constructibility constraints.

- (1) If K is a non-Archimedean local field with residue characteristic prime to ℓ , and \mathcal{O}_K its ring of integers, then we can take $\partial U \rightarrow U$ to be the morphism $\text{Spec } K \rightarrow$

$\mathrm{Spec} \mathcal{O}_K$, giving us $(n - 2)$ -shifted Lagrangian structures of weight 1 on suitable open substacks of the derived moduli stack $F((\mathrm{Spec} \mathcal{O}_K)_{\mathrm{pro\acute{e}t}}, \mathbb{Z}_\ell(1), -)$ over $F((\mathrm{Spec} K)_{\mathrm{pro\acute{e}t}}, \mathbb{Z}_\ell(1), -)$, coming from local Tate duality. If we overlook weighted aspects, this morphism of derived stacks essentially maps from moduli of unramified local Galois representations to moduli of all local Galois representations.

- (2) More generally, if Z is a smooth proper scheme of dimension m over \mathcal{O}_K , we have $(n - 2 - 2m)$ -shifted Lagrangian structures of weight $m + 1$ on suitable open substacks of the derived moduli stack $F(Z_{\mathrm{pro\acute{e}t}}, \mathbb{Z}_\ell(1), -)$ over $F((Z \otimes_{\mathcal{O}_K} K)_{\mathrm{pro\acute{e}t}}, \mathbb{Z}_\ell(1), -)$.
- (3) If K is a number field and S a finite set of primes, with $\mathcal{O}_{K,S}$ the localisation of \mathcal{O}_K at S , then we have $(n - 2)$ -shifted Lagrangian structures of weight 1 on suitable open substacks of the derived moduli stack $F((\mathrm{Spec} \mathcal{O}_{K,S})_{\mathrm{pro\acute{e}t}}, \mathbb{Z}_\ell(1), -)$ over $\prod_{v \in S} F((\mathrm{Spec} K_v)_{\mathrm{pro\acute{e}t}}, \mathbb{Z}_\ell(1), -)$, coming from Poitou–Tate duality.
- (4) More generally, if Z is a smooth proper scheme of dimension m over $\mathcal{O}_{K,S}$, then we have $(n - 2 - 2m)$ -shifted Lagrangian structures of weight $m + 1$ on suitable open substacks of the derived moduli stack $F(Z_{\mathrm{pro\acute{e}t}}, \mathbb{Z}_\ell(1), -)$ over $\prod_{v \in S} F((Z \otimes_{\mathcal{O}_{K,S}} K_v)_{\mathrm{pro\acute{e}t}}, \mathbb{Z}_\ell(1), -)$.
- (5) If U is a smooth proper scheme of dimension m over a local (resp. finite) field prime to ℓ , we have $(n - 1 - 2m)$ -shifted (resp. $n - 2m$ -shifted) Lagrangian structures of weight $m + 1$ (resp. m) on suitable open substacks of the derived moduli stack $F(U_{\mathrm{pro\acute{e}t}}, \mathbb{Z}_\ell(1), -)$. These stacks are Lagrangian over the derived moduli stack

$$A \mapsto \mathbf{R}\Gamma(Z_{\mathrm{pro\acute{e}t}}, i^* \mathbf{R}j_* F(\bigoplus_r \underline{\mathcal{W}}_r A_U(r)))$$

of F -valued sheaves on the deleted tubular neighbourhood of U in its compactification, as in Examples 3.13.

- (6) We can also combine these as in Example 3.16. Take U to be a smooth separated scheme of dimension m over either the ring of integers \mathcal{O}_K of a non-Archimedean local field, or over a localisation $\mathcal{O}_{K,S}$ of the ring of integers of a number field. For a compactification $j: U \hookrightarrow \bar{U}$ of U over the same base, with complement $i: Z \rightarrow \bar{U}$, we then have $(n - 2 - 2m)$ -shifted Lagrangian structures of weight $m + 1$ on suitable open substacks of the derived moduli stack

$$F(U_{\mathrm{pro\acute{e}t}}, \mathbb{Z}_\ell(1), -): A \mapsto \mathbf{R}\Gamma(U_{\mathrm{pro\acute{e}t}}, \bigoplus_r \underline{\mathcal{W}}_r A_U(r)).$$

These are Lagrangian over the derived moduli stack sending A to the homotopy fibre product of the diagram

$$\begin{array}{ccc} \mathbf{R}\Gamma(Z_{\mathrm{pro\acute{e}t}}, i^* \mathbf{R}j_* F(\bigoplus_r \underline{\mathcal{W}}_r A_U(r))) & \longrightarrow & \prod_{v \in S} \mathbf{R}\Gamma(Z_{K_v, \mathrm{pro\acute{e}t}}, i^* \mathbf{R}j_* F(\bigoplus_r \underline{\mathcal{W}}_r A_U(r))) \\ \prod_{v \in S} \mathbf{R}\Gamma(U_{K_v, \mathrm{pro\acute{e}t}}, F(\bigoplus_r \underline{\mathcal{W}}_r A_U(r))) & \longrightarrow & \end{array}$$

of F -valued sheaves on the boundary ∂U constructed as in Example 3.16, where we take $\{K_v\}_{v \in S} = \{K\}$ in the local case.

- (7) We also use these examples to give more shifted symplectic structures. Given Z smooth and proper of dimension m over a ring of integers \mathcal{O}_K , then we can take a derived intersection of the Lagrangians of (4) and (2) to give an $(n - 3 - 2m)$ -shifted symplectic structure of weight $m + 1$ on the derived fibre product of the diagram

$$\begin{array}{ccc}
 F((Z \otimes_{\mathcal{O}_K} \mathcal{O}_{K,S})_{\text{proét}}, \mathbb{Z}_\ell(1), -) & \prod_{v \in S} F((Z \otimes_{\mathcal{O}_K} \mathcal{O}_v)_{\text{proét}}, \mathbb{Z}_\ell(1), -) & \\
 \searrow & \downarrow & \\
 & \prod_{v \in S} F((Z \otimes_{\mathcal{O}_K} K_v)_{\text{proét}}, \mathbb{Z}_\ell(1), -), &
 \end{array}$$

giving the generalisation of Selmer groups envisaged in [Kim, §10]. As phrased, this assumes that the primes in S do not divide ℓ , but primes dividing ℓ can be allowed if we incorporate crystalline data as in Example 5.20 below.

Example 5.19 (Moduli of \mathbb{G}_m -torsors). In the toy example $F = B\mathbb{G}_m$, the description of Example 5.4 combines with local Tate duality to say that for a non-Archimedean local field K_v the functor $B\mathbb{G}_m((\text{Spec } K_v)_{\text{proét}}, \mathbb{Z}_\ell(1), -)$ is the cotangent stack of $B\mathbb{G}_m((\text{Spec } K_v)_{\text{proét}}, -)$ with cotangent space having weight 1, equipped with its natural 0-shifted symplectic structure. Then the Lagrangian $B\mathbb{G}_m((\text{Spec } \mathcal{O}_v)_{\text{proét}}, \mathbb{Z}_\ell(1), -)$ is identified with the open derived substack $B\mathbb{G}_m((\text{Spec } \mathcal{O}_v)_{\text{proét}}, -)$ of the canonical Lagrangian $B\mathbb{G}_m((\text{Spec } v)_{\text{proét}}, -)$.

Similarly, for a localisation $\mathcal{O}_{K,S}$ of the ring of integers of a number field, with the primes in S not dividing ℓ , Example 5.4 combines with Poitou–Tate duality to express $B\mathbb{G}_m((\text{Spec } \mathcal{O}_{K,S})_{\text{proét}}, \mathbb{Z}_\ell(1), -)$ as a product $\mathcal{N}^* \times \prod_{r>1} \mathcal{T}_r^*$ of the derived conormal stack \mathcal{N}^* of $B\mathbb{G}_m((\text{Spec } \mathcal{O}_{K,S})_{\text{proét}}, -) \rightarrow \prod_{v \in S} F((\text{Spec } K_v)_{\text{proét}}, -)$ (where the conormal space is placed in weight 1) with the (-1) -shifted cotangent spaces \mathcal{T}_r^* of the weight r affine derived spaces $N^{-1}\tau_{\geq 0}(\mathbf{R}\Gamma((\text{Spec } \mathcal{O}_{K,S})_{\text{proét}}, -(r))_{[-1]})$ for $r > 1$ (where the cotangent space is placed in weight $1 - r$); roughly speaking, $\mathcal{T}_r^* \simeq T^*[-1](H^1(G_{K,S}, \mathbb{Q}_\ell(r) \oplus \mathbb{Q}_\ell(1 - r)))$.

The derived intersection of these as in Example 5.18.(7) is then the (-1) -shifted cotangent stack of the weighted derived stack given by the product of the unweighted derived moduli stack and the derived affine spaces $N^{-1}\tau_{\geq 0}(\mathbf{R}\Gamma((\text{Spec } \mathcal{O}_{K,S})_{\text{proét}}, -(r))_{[-1]})$ in weight r for $r > 1$.

Example 5.20 (Crystalline constructions). There are analogues of Examples 5.18.(1&2) when the residue characteristic p equals ℓ , but involving crystalline rather than unramified representations. We now indicate how to proceed in this direction, glossing over several technicalities.

Analogues of filtered Φ -modules. Our moduli functor will be defined on dagger affinoids over \mathbb{Q}_p , and we start by looking at derived moduli of Galois representations of a non-Archimedean local field K of residue characteristic p , with ring of integers \mathcal{O}_K and residue field k . Write $K_0 = W(k)[p^{-1}]$, and take period rings B_{cris} and B_{dR} as in [Ill1]. The category of crystalline Galois representations of K is then equivalent to a full subcategory of Fontaine’s category $\text{MF}_K(\Phi)$ of filtered Φ -modules, which consists of triples (M, Φ, Fil) , where M is a K_0 -vector space, Φ is a Frobenius-semilinear injective endomorphism of M , and Fil is a decreasing filtration on $M \otimes_{K_0} K$ which is exhaustive and Hausdorff. The natural cohomology theory for this category associates to such data

the cochain complex

$$\mathrm{Fil}^0(M \otimes_{K_0} K) \oplus M \xrightarrow[(x-y, \Phi(y)-y)]{(x,y) \mapsto} (M \otimes_{K_0} K) \oplus M;$$

in particular, note that H^0 of the complex is $\mathrm{Hom}_{\mathrm{MF}_K(\Phi)}(K_0, M)$, where K_0 is given trivial Frobenius action and filtration.

We can generalise this to give a notion of filtered Φ -objects taking values in any derived stack F , which can be thought of as generalising the constructions $U(B_f^\bullet)$ of [Bet, §6.1]. First, for any \mathbb{Q}_p -CDGA A we simply define crystalline and de Rham functors associated to $\mathrm{Spec} K$ by $F_{\mathrm{cris}}(K, A) := F(A \otimes_{\mathbb{Q}_p} K_0)$ and $F_{\mathrm{dR}}(A) := F(A \otimes_{\mathbb{Q}_p} K)$; these generalise the moduli of K_0 - and K -vector spaces, respectively. Via Rees constructions as in [Sim1, §5], a filtered K -vector space is equivalent to a vector bundle on the stack $[\mathbb{A}_K^1/\mathbb{G}_m]$, so we can generalise this to define a Hodge functor $F_{\mathrm{Hod}}(K, -)$ parametrising filtered K -linear objects of F by

$$F_{\mathrm{Hod}}(K, A) := \mathbf{R}\mathrm{map}([\mathbb{A}_{K \otimes_{\mathbb{Q}_p} A}^1/\mathbb{G}_m], F) = \mathrm{holim}_{n \in \Delta} F(A \otimes_{\mathbb{Q}_p} K[s, u_1^\pm, \dots, u_n^\pm]).$$

We can then set $F_{\mathrm{MF}(\Phi)}(K, A)$ to be the homotopy equaliser

$$F_{\mathrm{Hod}}(K, A) \times F_{\mathrm{cris}}(K, A) \xrightarrow[(x,y) \mapsto (y,y)]{(x,y) \mapsto (x, \sigma^* y)} F_{\mathrm{dR}}(K, A) \times F_{\mathrm{cris}}(K, A)$$

where σ denotes Frobenius. Taking F to be the derived stack $B\mathrm{GL}_n$, it follows that the space $(B\mathrm{GL}_n)_{\mathrm{MF}(\Phi)}(K, \mathbb{Q}_p)$ of \mathbb{Q}_p -valued points is equivalent to the nerve of the maximal subgroupoid of $\mathrm{MF}_K(\Phi)$ on objects which have rank n and for which the map $\Phi: \sigma^*M \rightarrow M$ is an isomorphism.

Before proceeding further, we also need a weighted version $F_{\mathrm{MF}(\Phi)}(K, \mathbb{Z}_p(1), A)$ for \mathbb{G}_m -equivariant CDGAs A ; this is given by the same formula, the only difference being that \mathbb{G}_m and Φ are now acting non-trivially on A in the obvious way. For stacks of modules, this corresponds to taking an $(\bigoplus_r \mathcal{W}_r A(r)) \otimes_{\mathbb{Q}_p} K_0$ -module M with a Frobenius semilinear endomorphism $\Phi: M \rightarrow M$ inducing an isomorphism $\sigma^*M \rightarrow M$, together with a filtration on $M \otimes_{K_0} K$ compatible with the grading on A ; note that Frobenius σ is here acting on $(\bigoplus_r \mathcal{W}_r A(r))$ non-trivially via Tate twists.

The functors $F_{\mathrm{MF}(\Phi)}(K, -)$ and $F_{\mathrm{MF}(\Phi)}(K, \mathbb{Z}_p(1), -)$ are manifestly homogeneous.

Tangent complexes. We can also describe tangent complexes of these functors. Given $x \in F_{\mathrm{MF}(\Phi)}(K, \mathbb{Z}_p(1), A)$, we have an underlying point $x_{\mathrm{cris}} \in F_{\mathrm{cris}}(K, A)$, and hence a cotangent complex $\mathbb{L}^{F, A \otimes_{K_0}, x_{\mathrm{cris}}}$, which is a complex of $A \otimes_{K_0}$ -modules equipped with a quasi-automorphism Φ and a homotopy filtration Fil on $\mathbb{L}^{F, A \otimes_{K_0}, x_{\mathrm{cris}}} \otimes_{K_0} K$. The tangent complex of $F_{\mathrm{MF}(\Phi)}(K, \mathbb{Z}_p(1), -)$ at x then sends a graded A -module M to the cocone of

$$\begin{aligned} & \mathbf{R}\mathrm{Hom}_{A \otimes_{K_0}}(\mathbb{L}^{F, A \otimes_{K_0}, x_{\mathrm{cris}}}, M \otimes_{K_0}) \oplus \mathbf{R}\mathrm{Fil}^0 \mathbf{R}\mathrm{Hom}_{A \otimes_{K_0}}(\mathbb{L}^{F, A \otimes_{K_0}, x_{\mathrm{cris}}} \otimes_{K_0} K, M \otimes_{K_0}) \\ & \rightarrow \mathbf{R}\mathrm{Hom}_{A \otimes_{K_0}}(\mathbb{L}^{F, A \otimes_{K_0}, x_{\mathrm{cris}}}, M \otimes_{K_0}) \oplus \mathbf{R}\mathrm{Hom}_{A \otimes_{K_0}}(\mathbb{L}^{F, A \otimes_{K_0}, x_{\mathrm{cris}}} \otimes_{K_0} K, M \otimes_{K_0}), \end{aligned}$$

where as usual the map is $(y, z) \mapsto (\sigma^*y - y, y - z)$.

Note that if F is l.f.p., this is the same as the complex calculating the cohomology $H_f^*(K, -)$ from [BK], with coefficients in the tangent complex $\mathbf{R}\mathrm{Hom}_{A \otimes_{K_0}}(\mathbb{L}^{F, A \otimes_{K_0}, x_{\mathrm{cris}}}, M \otimes_{K_0})$ of F at x_{cris} (or rather in the corresponding Galois representation, if it exists).

Analogues of (Φ, Γ) -modules. The functor $F_{\text{MF}(\Phi)}(K, \mathbb{Z}_p(1), -)$ does not map directly to $F((\text{Spec } \mathcal{O}_K)_{\text{proét}}, \mathbb{Z}_p(1), -)$, so we now need to introduce an intermediate functor which receives maps from both functors, the first Lagrangian and the second étale, at least on suitable subfunctors.

Since the period ring B_{dR} carries a filtration, we can form its Rees construction $\xi(B_{\text{dR}}, \text{Fil})$, which is a graded $K_0[s]$ -algebra. For $\mathbb{E} := \mathbb{Z}_p(1)$, we can then set

$$F_{\text{Hod}}^{\Phi, \Gamma}(K, \mathbb{E}, A) := \mathbf{R}\Gamma((\text{Spec } K)_{\text{proét}}, \mathbf{R}\text{map}([\text{Spec } (\xi(B_{\text{dR}}, \text{Fil}) \hat{\otimes}_{\mathbb{Q}_p} A(\mathbb{E})) / \mathbb{G}_m], F)),$$

$$F_{\text{cris}}^{\Phi, \Gamma}(K, \mathbb{E}, A) := \mathbf{R}\Gamma((\text{Spec } K)_{\text{proét}}, F(\underline{B_{\text{cris}} \hat{\otimes}_{\mathbb{Q}_p} A(\mathbb{E})})),$$

$$F_{\text{dR}}^{\Phi, \Gamma}(K, \mathbb{E}, A) := \mathbf{R}\Gamma((\text{Spec } K)_{\text{proét}}, F(\underline{B_{\text{dR}} \hat{\otimes}_{\mathbb{Q}_p} A(\mathbb{E})})),$$

and then define $F^{\Phi, \Gamma}(K, \mathbb{E}, A)$ to be the homotopy equaliser

$$F_{\text{Hod}}^{\Phi, \Gamma}(K, \mathbb{E}, A) \times F_{\text{cris}}^{\Phi, \Gamma}(K, \mathbb{E}, A) \xrightarrow[\begin{smallmatrix} (x, y) \mapsto (y, y) \\ (x, y) \mapsto (x, \sigma^* y) \end{smallmatrix}]{\cong} F_{\text{dR}}^{\Phi, \Gamma}(K, \mathbb{E}, A) \times F_{\text{cris}}^{\Phi, \Gamma}(K, \mathbb{E}, A).$$

There is then an obvious natural map $F_{\text{MF}(\Phi)}(K, \mathbb{Z}_p(1), -) \rightarrow F^{\Phi, \Gamma}(K, \mathbb{Z}_p(1), -)$. When F is n -shifted symplectic, we can use the long exact sequence of [BK, Proposition 3.8] to see that the map is $(n - 2)$ -shifted Lagrangian of weight 1 at points x where $\mathbb{L}^{F, A \otimes K_0, x_{\text{cris}}}$ is an extension of weakly admissible filtered Φ -modules.

There is also a natural map

$$F((\text{Spec } K)_{\text{proét}}, \mathbb{Z}_p(1), -) \rightarrow F^{\Phi, \Gamma}(K, \mathbb{Z}_p(1), -)$$

coming from the expression of \mathbb{Q}_p as the homotopy equaliser

$$\text{Fil}^0 B_{\text{dR}} \oplus B_{\text{cris}} \xrightarrow[\begin{smallmatrix} (x, y) \mapsto (y, y) \\ (x, y) \mapsto (x, \sigma^* y) \end{smallmatrix}]{\cong} B_{\text{dR}} \oplus B_{\text{cris}};$$

since \mathbb{Q}_p is a homotopy equaliser, it follows that the map is formally étale.

Lagrangians. Taking the homotopy pullback

$$F((\text{Spec } K)_{\text{proét}}, \mathbb{Z}_p(1), -) \times_{F^{\Phi, \Gamma}(K, \mathbb{Z}_p(1), -)}^h F_{\text{MF}(\Phi)}(K, \mathbb{Z}_p(1), -)$$

gives us a functor we can think of as derived moduli of associations, i.e. associated Galois and crystalline data, or simply as moduli of F -valued Galois representations. This functor is formally étale over $F_{\text{MF}(\Phi)}(K, \mathbb{Z}_p(1), -)$ and consequently Lagrangian over $F((\text{Spec } K)_{\text{proét}}, \mathbb{Z}_p(1), -)$ on any suitable open subfunctor where duality holds. It thus plays the same rôle as unramified Galois representations $F((\text{Spec } \mathcal{O}_K)_{\text{proét}}, \mathbb{Z}_\ell(1), -)$ in the $\ell \neq p$ case.

Higher-dimensional analogues. Finally, there exist versions for a smooth proper variety Z over K admitting a model \mathcal{Z} over $W(k)$, given by combining the duality above with the Poincaré duality of [Ber, §VII.2] and the étale-crystalline comparison theorem of [Fal, 5.6]. For these functors, we can set (for $\mathcal{Z}_k := \mathcal{Z} \otimes_{W(k)} k$)

$$F_{\text{cris}}(Z, A) := \mathbf{R}\Gamma(\mathcal{Z}_{k, \text{indcris}} / W(k), F(\mathcal{O}_{\text{cris}} \otimes_{\mathbb{Z}_p} A)), \quad F_{\text{dR}}(Z, A) := \mathbf{R}\Gamma(Z_{\text{dR}}^{\text{an}}, F(\mathcal{O}_{Z, \text{dR}} \otimes_{\mathbb{Q}_p} A)),$$

where $\mathcal{Z}_{k, \text{indcris}}$ is the variant of the crystalline site allowing formal schemes as thickenings (so the tensor product $\mathcal{O}_{\text{cris}} \otimes_{\mathbb{Z}_p} A$ is non-trivial), and where $Z_{\text{dR}}^{\text{an}}$ is the analytification of the de Rham stack of [Sim1]. We also use Simpson's Hodge stack Z_{Hod} in place of \mathbb{A}_K^1 , so take

$$F_{\text{Hod}}(Z, A) := \mathbf{R}\Gamma([Z_{\text{Hod}} / \mathbb{G}_m], F(\mathcal{O} \otimes_{\mathbb{Q}_p} A)).$$

The other formulae then adapt to this generality.

In particular, this allows us the set S in Example 5.18.7 to include primes dividing ℓ in the derived Lagrangian intersection, by including moduli stacks of crystalline objects at those places to accompany the moduli stacks of unramified representations at the places.

6. WEIGHTED SHIFTED POISSON STRUCTURES

In this section, we introduce shifted Poisson structures on weighted dg algebras, twisted by a weight, and establish a comparison with weighted symplectic structures. For these purposes, there is nothing special about the dagger affinoid setting in which we are working, and obvious analogues of these results hold in particular in derived algebraic geometry.

6.1. Polyvectors. In this section, we fix an object A of the pro-category $\text{pro}(dg_+ \text{AffdAlg}_K^{\mathbb{G}_m, \dagger, \text{loc}})$, which in applications will be quasi-free. The following is adapted from [Pri7, Definition 1.1]:

Definition 6.1. Define the graded cochain complex of m -twisted n -shifted multiderivations on A by

$$\widehat{\text{Pol}}(A, n, m) := \prod_p \mathcal{H}om_A(\text{CoS}_A^p((\Omega_A^1\{m\})_{[-n-1]}), A),$$

for the internal Hom-complex $\mathcal{H}om$ of Definition 4.27 and twist $\{m\}$ as in Definition 4.18; note that the only effect of the twisting is to affect the weights, but that the product has to be taken in each weight separately. This has a \mathbb{G}_m -equivariant graded-commutative multiplication following the usual conventions for symmetric powers. (Here, $\text{CoS}_A^p(M) = \text{CoSymm}_A^p(M) = (M^{\otimes Ap})^{\Sigma_p}$.)

The Lie bracket on $\mathcal{H}om_A(\Omega_{A/R}^1, A) = \mathcal{H}om_A(\Omega_{A/R}^1\{m\}, A)\{m\}$ then extends to give a \mathbb{G}_m -equivariant bracket (the Schouten–Nijenhuis bracket)

$$[-, -]: \widehat{\text{Pol}}(A, n, m)\{m\} \times \widehat{\text{Pol}}(A, n, m)\{m\} \rightarrow \widehat{\text{Pol}}(A, n)^{[-1-n]}\{m\},$$

determined by the property that it is a bi-derivation with respect to the multiplication operation.

Note that the cochain differential δ on $\widehat{\text{Pol}}(A, n, m)$ can be written as $[\delta, -]$, where $\delta \in \mathcal{W}_m \widehat{\text{Pol}}(A, n, m)^{n+2}$ is the element defined by the derivation δ on A .

Definition 6.2. Define a decreasing filtration F on $\widehat{\text{Pol}}(A, n, m)$ by

$$F^i \widehat{\text{Pol}}(A, n, m) := \prod_{j \geq i} \mathcal{H}om_A(\text{CoS}_A^j((\Omega_{A/R}^1\{m\})_{[-n-1]}), A);$$

this has the properties that $\widehat{\text{Pol}}(A, n, m) = \varprojlim_i \widehat{\text{Pol}}(A, n, m)/F^i$, with $[F^i, F^j] \subset F^{i+j-1}$, $\delta F^i \subset F^i$, and that $F^i F^j \subset F^{i+j}$.

Observe that this filtration makes $F^2 \widehat{\text{Pol}}(A, n, m)^{[n+1]}\{m\}$ into a \mathbb{G}_m -equivariant pro-nilpotent DGLA, and hence makes $\mathcal{W}_m F^2 \widehat{\text{Pol}}(A, n, m)^{[n+1]}$ into a pro-nilpotent DGLA.

6.1.1. Poisson structures.

Definition 6.3. Given a DGLA L , define the the Maurer–Cartan set by

$$\mathrm{MC}(L) := \{\omega \in L^1 \mid \delta\omega + \frac{1}{2}[\omega, \omega] = 0 \in \bigoplus_n L^2\}.$$

Following [Hin], define the Maurer–Cartan space $\underline{\mathrm{MC}}(L)$ (a simplicial set) of a nilpotent DGLA L by

$$\underline{\mathrm{MC}}(L)_n := \mathrm{MC}(L \otimes_{\mathbb{Q}} \Omega^\bullet(\Delta^n)),$$

where

$$\Omega^\bullet(\Delta^n) = \mathbb{Q}[t_0, t_1, \dots, t_n, \delta t_0, \delta t_1, \dots, \delta t_n] / (\sum t_i - 1, \sum \delta t_i)$$

is the commutative dg algebra of de Rham polynomial forms on the n -simplex, with the t_i of degree 0.

Definition 6.4. If A is quasi-free, define an n -shifted Poisson structure of weight m on A to be an element of

$$\mathrm{MC}(\mathcal{W}_m F^2 \widehat{\mathrm{Pol}}(A, n, m)^{[n+1]}),$$

and the space $\mathcal{P}(A, n, m)$ of n -shifted Poisson structures of weight m on A to be given by the simplicial set

$$\mathcal{P}(A, n, m) := \varprojlim_p \underline{\mathrm{MC}}(\mathcal{W}_m F^2 \widehat{\mathrm{Pol}}(A, n, m)^{[n+1]} / F^{p+2}).$$

Remark 6.5. Observe that elements of $\mathcal{P}_0(A, n, m) = \mathrm{MC}(\mathcal{W}_m F^2 \widehat{\mathrm{Pol}}(A, n, m)^{[n+1]})$ consist of infinite sums $\pi = \sum_{i \geq 2} \pi_i$ with \mathbb{G}_m -equivariant maps

$$\pi_i: \mathrm{CoS}_A^i((\Omega_{A/R}^1)_{[-n-1]}\{m\}) \rightarrow A_{[-n-2]}\{m\}$$

satisfying $\delta(\pi_i) + \frac{1}{2} \sum_{j+k=i+1} [\pi_j, \pi_k] = 0$. This is precisely the condition which ensures that π defines an L_∞ -algebra structure on $A_{[-n]}\{m\}$.

Definition 6.6. We say that an n -shifted Poisson structure $\pi = \sum_{i \geq 2} \pi_i$ of weight m on A is non-degenerate if $\pi_2: \mathrm{CoS}_A^2((\Omega_A^1\{m\})_{[-n-1]}) \rightarrow A_{[-n-2]}\{m\}$ induces a quasi-isomorphism

$$\pi_2^\sharp: (\Omega_A^1)_{[-n]}\{m\} \rightarrow \mathcal{H}om_A(\Omega_A^1, A).$$

Define $\mathcal{P}(A, n, m)^{\mathrm{nondeg}} \subset \mathcal{P}(A, n, m)$ to consist of non-degenerate elements — this is a union of path-components.

Remark 6.7. Beware that this non-degeneracy condition can only be satisfied if $n \leq 0$. The same phenomenon arises for symplectic structures, with derived dagger affinoids (and more generally derived dagger DM stacks) only capable of carrying non-positively shifted structures. For derived Artin stacks, the formulation of shifted Poisson structures is more subtle, allowing for positive shifts — see §6.6 for details.

6.2. Comparison of weighted Poisson and symplectic structures.

Theorem 6.8. *For a quasi-free object A of the pro-category $\mathrm{pro}(dg_+ \mathrm{AffdAlg}_K^{\mathbb{G}_m, \dagger, \mathrm{loc}})$, there are canonical weak equivalences*

$$\mathcal{W}_m \mathrm{Sp}(A, n) \simeq \mathcal{P}(A, n, m)^{\mathrm{nondeg}}$$

of simplicial sets.

In particular, the sets of equivalence classes of n -shifted symplectic structures of weight m and of non-degenerate n -shifted Poisson structures of weight m on A are isomorphic.

Proof. The proof of [Pri7, Corollary 1.38] adapts, *mutatis mutandis*. We now outline the main steps.

Each Poisson structure $\pi \in \mathcal{P}(A, n, m)$ gives rise to a Poisson cohomology complex

$$T_\pi \widehat{\text{Pol}}(A, n, m),$$

defined as the \mathbb{G}_m -equivariant cochain complex given by the derivation $\delta + [\pi, -]$ acting on $\widehat{\text{Pol}}(A, n, m)$. There is also a canonical element $\sigma(\pi) \in Z^{n+2} \mathcal{W}_m T_\pi \widehat{\text{Pol}}(A, n)$ given by

$$\sigma(\pi) = \sum_{i \geq 2} (i-1) \pi_i.$$

The key construction is then given by the compatibility map

$$\begin{aligned} \mu(-, \pi): \text{DR}(A) &\rightarrow T_\pi \widehat{\text{Pol}}_\pi(X, n, m) \\ \text{ad}f_1 \wedge \dots \wedge \text{d}f_p &\mapsto a[\pi, f_1] \dots [\pi, f_p] \end{aligned}$$

of filtered \mathbb{G}_m -equivariant cochain complexes. When π is non-degenerate, this map is necessarily a \mathbb{G}_m -equivariant quasi-isomorphism, and the symplectic structure associated to π is given by

$$\mu(\pi, -)^{-1} \sigma(\pi) \in H^{n+2} F^2 \mathcal{W}_m \text{DR}(A).$$

By analogy with [KV], we can regard the inverse map $\mu(\pi, -)^{-1}$ as a homotopy Legendre transform.

Establishing that this gives an equivalence between symplectic and Poisson structures relies on obstruction theory associated to filtered DGLAs, building the Poisson form $\pi = \pi_2 + \pi_3 + \dots$ inductively from the symplectic form $\omega = \omega_2 + \omega_3 + \dots$ by solving the equation $\mu(\omega, \pi) \simeq \sigma(\pi)$ up to coherent homotopy; for a readable summary of the argument from [Pri7], see [Saf, §2.5]. \square

Remark 6.9 (\mathcal{P} -shifted Poisson structures). The \mathcal{P} -shifted symplectic structures of [BG] are much more general than we consider here, since they take \mathcal{P} to be a shifted line bundle with a flat connection, then define shifted symplectic structures in terms of the de Rham complex of \mathcal{P} . Under additional finiteness conditions, our weighted shifted symplectic structures on X fit into this picture by looking at \mathcal{P} -shifted symplectic structures relative to the morphism $f: [X/\mathbb{G}_m] \rightarrow B\mathbb{G}_m$, taking $\mathcal{P} = f^* \mathcal{O}(m)[n]$.

Our results so far easily adapt to \mathcal{P} -shifted symplectic structures whenever \mathcal{P} is the pullback of a shifted line bundle on the base, but here seems a suitable place to indicate how to formulate \mathcal{P} -shifted symplectic structures for more general \mathcal{P} . We work with the filtered DGLA

$$\widehat{\text{Pol}}(A, \mathcal{P}) := \prod_p \underline{\text{Hom}}_A(\text{CoS}_A^p(\Omega_A^1 \otimes_A \mathcal{P}), \mathcal{P}),$$

with Lie bracket given in terms of the connection $\nabla: \mathcal{P} \rightarrow \Omega_A^1 \otimes_A \mathcal{P}$ by $[f, g] = f \circ (\nabla \circ g) \mp g \circ (\nabla \circ f)$, where the first \circ is interpreted as a sum over all substitutions. This is a module over the graded algebra $\prod_p \underline{\text{Hom}}_A(\text{CoS}_A^p(\Omega_A^1 \otimes_A \mathcal{P}), A)$, and the bracket is a biderivation with respect to this multiplication.

The generalisation of Theorem 6.8 then holds to give

$$\underline{\mathrm{MC}}(\mathrm{DR}(\mathcal{P}))^{\mathrm{nondeg}} \simeq \underline{\mathrm{MC}}(\widehat{\mathrm{Pol}}(A, \mathcal{P}))^{\mathrm{nondeg}}$$

so we regard Maurer–Cartan elements $\pi \in F^2\widehat{\mathrm{Pol}}(A, \mathcal{P})$ as \mathcal{P} -shifted Poisson structures. In particular a \mathcal{P} -shifted Poisson structure π gives rise to an element $\sigma(\pi) \in T_\pi F^2\widehat{\mathrm{Pol}}(A, \mathcal{P})$ and a filtered compatibility map

$$\mu(-, \pi): \mathrm{DR}(\mathcal{P}) \rightarrow T_\pi \widehat{\mathrm{Pol}}(A, \mathcal{P}),$$

analogous to the map $\mathcal{W}_m \mathrm{DR}(A) \rightarrow T_\pi \mathcal{W}_m \widehat{\mathrm{Pol}}_\pi(X, n, m)$; these give rise to the comparison.

Finally, composition with the connection $\nabla: \mathcal{P} \rightarrow \Omega_A^1 \otimes_A \mathcal{P}$ allows us to interpret elements of $\underline{\mathrm{Hom}}_A(\mathrm{CoS}_A^j(\Omega_A^1 \otimes_A \mathcal{P}), \mathcal{P})$ as j -ary operations on \mathcal{P} . Thus a \mathcal{P} -shifted Poisson structure is the same as an L_∞ -algebra structure $\{[-]_i\}$ on \mathcal{P} for which each operation $[-]_i$ is a ∇ -multiderivation in the sense that each operation $[p_1, p_2, \dots, p_{i-1}, -]_i: \mathcal{P} \rightarrow \mathcal{P}$ acts as $v \lrcorner \nabla$ for some tangent vector $v \in T_A$.

6.3. Co-isotropic structures. In the algebraic setting of [MS1], an n -shifted co-isotropic structure on a morphism $f: A \rightarrow B$ is defined to consist of an n -shifted Poisson structure on A , an $(n-1)$ -shifted Poisson structure π on B and a lift of f to a strong homotopy P_{n+1} -algebra morphism $A \rightarrow (\widehat{\mathrm{Pol}}(B, n-1), \delta + [\pi, -])$ to the complex of polyvectors with differential twisted by $[\pi, -]$. There is an interpretation in terms of additivity showing that the latter data are equivalent to a strongly homotopy associative action of A on the P_n -algebra B .

Unfortunately, the usual formulation of s.h. P_{n+1} -algebra morphisms involves cofibrant replacement of A as a P_{n+1} -algebra, which does not have a natural analogue in analytic settings. However, suitable spaces of morphisms should be constructible by applying operadic Koszul duality and working with the coloured operad of polydifferential operators, incorporating weights in the weighted case.

Given such a setup, the equivalence between (weighted) shifted Lagrangians and non-degenerate (weighted) shifted co-isotropic structures established in [MS2] should have a fairly straightforward analytic analogue, phrased in terms of structures on (weighted) EFC-algebras.

6.4. Global Poisson structures. Functoriality for shifted Poisson structures is fairly subtle, but for homotopy formally étale morphisms $A \rightarrow B$, the constructions of [Pri7, §2] adapt to our setting. As in [Pri7, §2.1], we have a natural notion of a space $\mathcal{P}(A \rightarrow B, n, m)$ of n -shifted Poisson structures of weight m on the diagram $A \rightarrow B$, with a natural equivalence $\mathcal{P}(A \rightarrow B, n, m) \rightarrow \mathcal{P}(A, n, m)$ and a natural map $\mathcal{P}(A \rightarrow B, n, m) \rightarrow \mathcal{P}(B, n, m)$; this leads to an ∞ -functor on the category of homotopy formally étale morphisms of formal weighted dg dagger affinoids.

Passing to homotopy limits as in [Pri7, Definition 2.16], we can then define the space of n -shifted symplectic structures on a weighted formal dagger dg space (or DM ∞ -stack) by setting

$$\mathrm{Sp}(X, n, m) := \mathbf{R}\Gamma((\pi^0 X^{\mathrm{G}m})_{\mathrm{ét}}, \mathcal{P}(\tilde{\mathcal{O}}_X, n, m)).$$

The equivalence generalising Theorem 6.8 is then sufficiently canonical to give us an equivalence

$$\mathrm{Sp}(X, n, m) \simeq \mathcal{P}(X, n, m)^{\mathrm{nondeg}}.$$

For Artin stacks, we encounter the obstacle that Poisson structures are not functorial with respect to smooth morphisms, which we will address in §6.6.

Examples 6.10. Whenever Y is a K -dagger space (or even a K -dagger DM ∞ -stack), Theorem 6.8 gives us r -shifted Poisson structures of weight m associated to each r -shifted symplectic structure of weight m on the functors

$$\tilde{Y}: A \mapsto Y(\mathbb{H}_0 A / \mathbb{G}_m) \times_{F(X_{\text{proét}}, \mathbb{H}_0 A / \mathbb{G}_m)}^{\hbar} F(X_{\text{proét}}, \mathbb{Z}_\ell(1), A)$$

in Examples 5.13.

Moreover, the comparison outlined in §6.3 will give $(r - 1)$ -shifted Poisson structures of weight m on \tilde{W} associated to each r -shifted Lagrangian structure of weight m on $F(U_{\text{proét}}, \mathbb{E}, -)^{wd}$ in Examples 5.18, whenever W is a K -dagger DM ∞ -stack over $\pi^0 F(X_{\text{proét}}, -)$.

These results will also extend to cases where Y is a K -dagger Artin ∞ -stack, relying on the formulation of shifted Poisson structures on derived Artin stacks in §6.6 below, as in Examples 6.19.

6.5. Quantisations. As outlined in [Pri15, §4.4] and [Pri10, Remarks 1.17 and 1.28], we can formulate shifted quantisations in derived analytic settings using polydifferential operators, and algebraic quantisation results will then all adapt.

To quantise Poisson structures of weight m , the results also adapt, with the same proofs. The only modification we have to make is to work in a \mathbb{G}_m -equivariant setting and to set the formal variable \hbar to have weight m . For positively shifted structures, quantisations take the form of E_{n+1} -algebra deformations, whose existence can be inferred directly from Kontsevich formality. Our examples of interest tend to carry non-positively shifted structures, which are more subtle:

- Given a 0-shifted Poisson structure of weight m on Z , the proofs of [Pri9, Pri13] adapt to give us a quantisation in the form of a curved almost commutative \mathbb{G}_m -equivariant A_∞ -deformation $\tilde{\mathcal{O}}_Z$ of $\{\mathcal{O}_Z[\hbar]/\hbar^k\}_k$ over $\{K[\hbar]/\hbar^k\}_k$, with \hbar of weight m . The curvature can be interpreted as giving an algebroid quantisation, and leads to a deformation (over $\{K[\hbar]/\hbar^k\}_k$) of the category of \mathbb{G}_m -equivariant line bundles on X .
- Given a (-1) -shifted symplectic structure of weight m on W , the proof of [Pri11] adapts to give us a quantisation of the square root \mathcal{L} of the canonical bundle on W (or of any line bundle with a right \mathcal{D} -module structure on its square) in the form of a \mathbb{G}_m -equivariant differential operator Δ on $\{\mathcal{L}[\hbar]/\hbar^k\}_k$ over $\{K[\hbar]/\hbar^k\}_k$, with \hbar of weight m . This operator has constraints on the orders of its coefficients, and corresponds to a homotopy BV -algebra structure when $\mathcal{L} = \mathcal{O}_W$ and $\Delta(1) = 0$. On inverting \hbar , Δ gives a complex related to vanishing cycles as in Example 6.11 below. Euler characteristics of such complexes are used in complex-analytic settings to recover the Behrend function used in enumerative questions such as Donaldson–Thomas theory.
- Given a 0-shifted Lagrangian structure of weight m on W over Z , the previous two examples combine compatibly as in [Pri16], with the \mathcal{O}_Z -module structure on \mathcal{L} extending to give an $\tilde{\mathcal{O}}_Z$ -module structure on $(\{\mathcal{L}[\hbar]/\hbar^k\}_k, \delta + \Delta)$. On taking the limit over k and inverting \hbar , the resulting structures resemble objects of an algebraic Fukaya category.

- If the structure sheaf carries a right \mathcal{D} -module structure, then there is also a notion of quantisation for (-2) -shifted symplectic structures, given by solutions of a quantum master equation as in [Pri8]. The only modification necessary in the weighted case is to stipulate that the solution has weight m with respect to the \mathbb{G}_m -action.

Example 6.11. The prototypical example of a (-1) -shifted symplectic structure is given by a derived critical locus, and there is a similar construction for weighted structures as a special case of Remark 5.14. Let $Y = (Y^{\mathbb{G}_m}, \mathcal{O}_Y)$ be a formally smooth weighted formal K -dagger space, and $f \in \Gamma(Y, \mathcal{W}_m \mathcal{O}_Y)$ a function of weight m . For the tangent sheaf \mathcal{T}_Y on Y , the derived critical locus W of f is then a weighted formal K -dagger dg space with $\pi^0 W^{\mathbb{G}_m}$ the vanishing locus of df on $Y^{\mathbb{G}_m}$, and $\mathcal{O}_W = (\text{Sym}_{\mathcal{O}_Y}(\mathcal{T}_Y\{-m\}_{[-1]}), \delta)$, where the differential δ given on generators as contraction with df (a map $\mathcal{T}_Y\{-m\} \rightarrow \mathcal{O}_Y$).

The (-1) -shifted symplectic structure of weight m on W is then given by the closed 2-form arising from the identification of \mathcal{T}_Y with the dual of Ω_Y^1 . If \mathcal{O}_Y has local co-ordinates y_i , then \mathcal{O}_W has co-ordinates $y_i, \eta_i = \partial_{y_i}$, and the 2-form on W is $\omega = \sum_i dy_i \wedge d\eta_i \in \ker d \cap \ker \delta$. If $y_i \in \mathcal{W}_{n_i} \mathcal{O}_Y$, then $\eta_i \in \mathcal{W}_{m-n_i} \mathcal{O}_W$, so

$$\omega \in \mathcal{W}_m(\Omega_W^2 \cap \ker \delta \cap \ker d)_1 \subset \mathcal{W}_m Z^1 F^2 \text{DR}(W).$$

Since this is non-degenerate, it is thus a (-1) -shifted symplectic structure of weight m .

Now, the square root \mathcal{L} of the dualising bundle on W is simply given by the pullback of $\Omega_Y^{\dim Y}$ to W , which is the complex

$$(\Omega_Y^\# \{m\# \}, df \wedge) \{-m \dim Y\}[\dim Y].$$

Reasoning as in [Pri11, §4.2], a differential operator quantising $\{\mathcal{L}[\hbar]/\hbar^k\}_k$ is just the rescaled de Rham differential $\hbar d$, giving us the twisted de Rham complex

$$(\Omega_Y^\# \{m\#\}[\hbar]/\hbar^k, \hbar d + df \wedge) \{-m \dim Y\}[\dim Y].$$

When $m \neq 0$, the \mathbb{G}_m -equivariant limit over all k is

$$(\Omega_Y^\# \{m\#\}[\hbar], \hbar d + df \wedge) \{-m \dim Y\}[\dim Y] \cong \hbar^{\dim Y} (\hbar^{-\#} \Omega_Y^\#[\hbar], d + \hbar^{-1} df \wedge) [\dim Y].$$

In particular, if $m = \pm 1$ then on inverting \hbar we just have a copy of the twisted de Rham complex $(\Omega_Y^\#, d + df \wedge)$ in every weight. Because f has non-zero weight, its exponential does not lie in \mathcal{O}_Y , so this is not isomorphic to the de Rham complex of Y . It behaves more like an algebraic twisted de Rham complex, computing rapid decay cohomology.

When $m = 0$, the complex given by inverting \hbar is $(\Omega_Y^\#((\hbar)), d + \hbar^{-1} df \wedge)$, which computes vanishing cycles.

Remarks 6.12. There are similar approaches to quantisation for shifted weighted symplectic structures on the derived Artin analytic stacks of the next section. We can use the toy example of derived moduli of \mathbb{G}_m -torsors as described in Examples 5.4, 5.19 as a test case for the programme outlined in [Kim, §10].

For derived 0-shifted cotangent stacks, there is a canonical quantisation given by rings of differential operators. The (unweighted) moduli stack of \mathbb{G}_m -representations of a local Galois group is just the product of a smooth one-dimensional analytic space with $B\mathbb{G}_m$, so we can describe the dg category of perfect complexes over the quantisation of the weighted derived moduli stack by substituting in [BPS, Definition 3.12 and Example 3.15], but taking \hbar to have weight 1 so power series do not appear.

For the derived intersection of Example 5.18.(7), corresponding to the major focus of [Kim, §10], Example 5.19 gives a description as a weighted (-1) -shifted derived

cotangent stack, so we can describe the quantisation as in Example 6.11 (with $f = 0$). However, this has the complication that there are infinitely many terms in the product of stacks, so naïvely inverting \hbar will not simply give us the de Rham complex of the unweighted moduli stack shifted by its virtual dimension; although the other terms in the product are derived affine spaces, so have trivial de Rham cohomology, their combined contribution will annihilate this localised complex by shifting it to infinite degree. This suggests that extracting useful invariants will be a delicate process, since in general it will not be possible simply to quotient out the higher weights.

6.6. Poisson structures for derived Artin stacks.

6.6.1. *Stacky derived affines.* The lack of functoriality of tangent spaces and Poisson structures with respect to smooth morphisms makes their definition for derived Artin stacks fairly subtle. Instead of working with functors on CDGAs, the solution is to work with algebras in double complexes, where the chain direction encodes the derived structure and the cochain direction encodes the stacky structure. For details, see [Pri12] (and in particular §4.2). We can then incorporate weights in much the same way as the incorporation of $\mathbb{Z}/2\mathbb{Z}$ -gradings in [Pri10].

The following is adapted from [Pri7, Definition 3.2]:

Definition 6.13. We define a stacky quasi-dagger dg algebra to be a chain cochain complex $A = A_{\geq 0}^{\leq 0}$ of K -vector spaces equipped with a commutative product $A \otimes A \rightarrow A$ and unit $K \rightarrow A$, such that A_0^0 is a quasi-dagger algebra, and the A_0^0 -modules A_j^i are all finite.

A morphism $A \rightarrow B$ of stacky quasi-dagger dg algebra is then said to be a weak equivalence if the morphisms $A^i \rightarrow B^i$ are all quasi-isomorphisms.

There are obvious formal and weighted generalisations, which we will not write down.

On double complexes V_{\bullet} combining both chain and cochain gradings, we denote the chain and cochain differentials by δ and ∂ respectively, regarding the cochain differential ∂ as stacky structure and the chain differential δ as derived structure.

Definition 6.14. Given a chain cochain complex V , define the cochain complex $\widehat{\text{Tot}} V \subset \text{Tot}^{\text{II}} V$ as a subset of the product total complex by

$$(\widehat{\text{Tot}} V)^m := \left(\bigoplus_{i < 0} V_{i-m}^i \right) \oplus \left(\prod_{i \geq 0} V_{i-m}^i \right)$$

with differential $\partial \pm \delta$. This is sometimes referred to as the Tate realisation.

In order to pass from double complexes to complexes in a fashion which behaves well with respect to weak equivalences and tensor operations, we use the following, which appear as [Pri7, Definitions 3.7 and 3.8].

Definition 6.15. Given a chain cochain complex V , define the cochain complex $\widehat{\text{Tot}} V \subset \text{Tot}^{\text{II}} V$ by

$$(\widehat{\text{Tot}} V)^m := \left(\bigoplus_{i < 0} V_{i-m}^i \right) \oplus \left(\prod_{i \geq 0} V_{i-m}^i \right)$$

with differential $\partial \pm \delta$.

Definition 6.16. Given A -modules M, N in chain cochain complexes, we define internal Hom spaces $\mathcal{H}om_A(M, N)$ by

$$\mathcal{H}om_A(M, N)_j^i = \text{Hom}_{A_{\#}}(M_{\#}^{\#}, N_{\#[j]}^{\#[i]}),$$

with differentials $\partial f := \partial_N \circ f \pm f \circ \partial_M$ and $\delta f := \delta_N \circ f \pm f \circ \delta_M$, where $V_{\#}^{\#}$ denotes the bigraded vector space underlying a chain cochain complex V .

We then define the Hom complex $\widehat{\mathbf{H}\mathbf{om}}_A(M, N)$ by

$$\widehat{\mathbf{H}\mathbf{om}}_A(M, N) := \widehat{\mathbf{Tot}} \mathcal{H}\mathit{om}_A(M, N).$$

6.6.2. Poisson structures. We can then extend the definition of shifted Poisson structures to stacky quasi-dagger dg algebras, by using $\widehat{\mathbf{H}\mathbf{om}}$ in place of $\mathbf{H}\mathbf{om}$. Unwinding the definitions, this means that a Poisson structure on a stacky dg algebra A gives rise to a sequence π_2, π_3, \dots of multiderivations on the product total complex $\widehat{\mathbf{Tot}} A = \mathbf{Tot}^{\mathbb{I}} A$, defining a shifted L_{∞} -algebra structure. However, since the multiderivations lie in the spaces defined using $\widehat{\mathbf{H}\mathbf{om}}$, they come with boundedness restrictions from the original cochain direction: if we filter $\widehat{\mathbf{Tot}} A$ by setting $\mathbf{Fil}^p \widehat{\mathbf{Tot}} A := \mathbf{Tot}^{\mathbb{I}} A^{\geq p}$, then each component π_k must be Fil-bounded in the sense that for some integer r , each \mathbf{Fil}^p is mapped to \mathbf{Fil}^{p+r} .

There is a notion of homotopy étale for morphisms $A \rightarrow B$ of stacky dg algebras, which essentially amounts to saying that $\widehat{\mathbf{Tot}}(\mathbf{L}\Omega_A^1 \otimes_A^{\mathbf{L}} B^0) \rightarrow \widehat{\mathbf{Tot}}(\mathbf{L}\Omega_B^1 \otimes_B^{\mathbf{L}} B^0)$ is a quasi-isomorphism. This gives sufficient flexibility to ensure functoriality of shifted Poisson structures with respect to such morphisms, as in [Pri7, §3.4].

In particular, the analogue of Theorem 6.8 still holds, giving a correspondence between n -shifted symplectic structures of weight m and non-degenerate n -shifted Poisson structures of weight m on formal weighted stacky localised dagger dg algebras. The comparison arguments outlined in §6.3 between n -shifted Lagrangian structures of weight m and non-degenerate n -shifted co-isotropic structures of weight m also extend to stacky algebras.

6.6.3. Denormalisation. The denormalisation functor D from cochain complexes to cosimplicial vector spaces combines with the Eilenberg–Zilber shuffle product to give a functor from stacky dg algebras to cosimplicial dg algebras; for explicit formulae, see [Pri12, §1.3]. The iterated codegeneracy maps $D^n A \rightarrow A^0$ are n -nilpotent, which in particular implies that this functor sends stacky (quasi-)dagger dg algebras A to cosimplicial (quasi-)dagger dg algebras DA .

Definition 6.17. Given a functor F from (formal weighted) (quasi-)dagger dg algebras simplicial sets, we define a functor D_*F on (formal weighted) stacky (quasi-)dagger dg algebras as the homotopy limit

$$D_*F(B) := \mathop{\mathrm{holim}}_{n \in \Delta} F(D^n B).$$

Thus D_* naturally extends all of our moduli functors to give functors on suitable stacky dg algebras. The real power of this construction is that derived Artin stacks, and more generally homogeneous functors F , admit homotopy étale atlases $\mathbf{R}\mathbf{Spec} A \rightarrow D_*F$ by stacky affine objects; see [Pri7, §3.4.2] for derived Artin stacks and [Pri14, §2.3] for the generalisation to homogeneous functors (the arguments there are phrased in the non-commutative setting, but other settings work in the same way, and in fact more easily).

The following is adapted from [Pri17, Definition 3.20]:

Definition 6.18. Given a (weighted) stacky (quasi-)dagger dg algebra $B \in DG^+ dg_+ \mathbf{Alg}(R)$ for which the chain complexes $(\mathbf{L}\Omega_B^1 \otimes_B^{\mathbf{L}} B^0)^i$ are acyclic for all $i > q$,

and a simplicial functor F on (weighted) (quasi-)dagger dg algebra which is homogeneous with a cotangent complex $\mathbb{L}^{F,x}$ at a point $x \in F(B^0)$, we say that a point $y \in D_*F(B)$ lifting $x \in F(B^0)$ is *rigid* if the induced morphism

$$\mathbb{L}^{F,x} \rightarrow \mathrm{Tot} \sigma^{\leq q}(\mathbf{L}\Omega_B^1 \otimes_B^{\mathbf{L}} B^0)$$

is a quasi-isomorphism of B^0 -modules. We denote by $(D_*F)_{\mathrm{rig}}(B) \subset D_*F(B)$ the space of rigid points (a union of path components).

In other words, a point $y \in (D_*F)_{\mathrm{rig}}(B)$ corresponds to a homotopy étale morphism $\mathbf{R}\mathrm{Spec} B \rightarrow D_*F$. The reason we think of the pair (B, y) as being rigid is that it does not deform: for any nilpotent surjection $e: C \rightarrow B$ with a point $z \in D_*F(C)$ lifting y , the map e has an essentially unique section s with $s(y) \simeq z$.

6.6.4. Global Poisson structures. On any homogeneous functor F , we can now define the space of n -shifted Poisson structures of weight m by

$$\mathcal{P}(F, n, m) := \mathbf{R}\mathrm{map}((D_*F)_{\mathrm{rig}}, \mathcal{P}(-, n, m)),$$

where the mapping space is taken in the category of simplicial homotopy preserving functors on a category of (weighted) stacky (quasi-)dagger dg algebras and homotopy étale morphisms.

The arguments of [Pri7, Pri12, Pri14] ensure this is consistent with our earlier definitions when $\pi^0 F^{\mathbb{G}_m}$ is a dagger analytic space or even DM ∞ -stack.

The generalisation of Theorem 6.8 to stacky algebras gives us an equivalence

$$\mathcal{P}(F, n, m)^{\mathrm{nondeg}} \simeq \mathbf{R}\mathrm{map}((D_*F)_{\mathrm{rig}}, \mathcal{W}_m \mathrm{Sp}(-, n))$$

between non-degenerate Poisson structures and symplectic structures. Moreover, there is a natural map

$$\mathbf{R}\mathrm{map}(F, \mathcal{W}_m \mathrm{PreSp}(-, n)) \rightarrow \mathbf{R}\mathrm{map}((D_*F)_{\mathrm{rig}}, \mathcal{W}_m \mathrm{PreSp}(-, n)),$$

where the first mapping space is taken in the ∞ -category of functors on localised weighted dagger dg algebras. Thus pre-symplectic structures on F give rise to pre-symplectic structures on D_*F for all functors F . In particular, we can apply to all of our examples from §3 without having to impose any representability conditions.

There is a similar global definition of co-isotropic structures, and global comparisons between Lagrangians and non-degenerate co-isotropic structures.

6.6.5. Global Poisson structures associated to pro-étale sheaves. Combining the Poisson/symplectic correspondence for stacky algebras outlined above into Corollary 5.11 gives a non-degenerate $(n-d)$ -shifted Poisson structure of weight m on $F(X_{\mathrm{pro\acute{e}t}}, \mathbb{E}, -)^{wd}$ whenever F is an n -shifted symplectic derived Artin ∞ -stack, and X a topologically Noetherian scheme with dualising complex $\mathbb{E}^{\otimes m}[d]$.

Examples 6.19. Generalising Examples 6.10, Theorem 6.8 now gives us r -shifted Poisson structures of weight m associated to each r -shifted symplectic structure of weight m on

$$F(X_{\mathrm{pro\acute{e}t}}, \mathbb{Z}_\ell(1), -)^{wd}$$

in Examples 5.13.

Moreover, the comparison outlined in §6.3 will give $(r-1)$ -shifted Poisson structures of weight m associated to each r -shifted Lagrangian structure of weight m on $F(U_{\mathrm{pro\acute{e}t}}, \mathbb{E}, -)^{wd}$ in Examples 5.18.

Note that in this form, these results do not assume that the functors F^{wd} are representable, because the use of stacky dg algebras allows us to formulate Poisson structures for any homogeneous functor.

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