TANNAKA DUALITY FOR ENHANCED TRIANGULATED CATEGORIES II: t-STRUCTURES AND HOMOTOPY TYPES

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Abstract. We consider the effect of t-structures on the Tannaka duality theory for dg categories developed in [Pri6]. We associate non-negative dg coalgebras C to dg functors on the hearts of t-structures, and relate dg C-comodules to the original dg category. We give several applications for pro-algebraic homotopy types associated to various cohomology theories, and for motivic Galois groups.

Introduction

Tannaka duality in Joyal and Street's formulation ([JS, §7, Theorem 3]) characterises abelian k-linear categories $\mathcal{A}$ with exact faithful k-linear functors ω to finite-dimensional k-vector spaces as categories of finite-dimensional comodules of coalgebras C. When $\mathcal{A}$ is a rigid tensor category and ω monoidal, C becomes a Hopf algebra (so $\operatorname{Spec} C$ is a group scheme), giving the duality theorem of [DMOS, Ch. II].

In [Pri6], these duality theorems are extended to dg categories. Given a k-linear dg category $\mathcal{A}$ and a k-linear dg functor ω to finite-dimensional complexes, there is natural a dg coalgebra structure C on the Hochschild homology complex

$$\omega^\vee \otimes^{\mathbf{L}}_{\mathcal{A}} \omega \simeq \mathrm{CC}_\bullet(\mathcal{A}, \omega^\vee \otimes_k \omega).$$

Then [Pri6, Theorem 2.9] shows that the dg derived category $\mathcal{D}_{\mathrm{dg}}(C)$ of C-comodules is quasi-equivalent to a derived quotient $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})/(\ker\omega)$ of the dg derived category $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ generated by $\mathcal{A}$. In particular, when ω is faithful, this gives a quasi-equivalence $\mathcal{D}_{\mathrm{dg}}(\mathcal{A}) \simeq \mathcal{D}_{\mathrm{dg}}(C)$, which is a derived analogue of Joyal and Street's Tannaka duality.

The main drawback of the Hochschild construction for the dg coalgebra in [Pri6] is that it always creates terms in negative cochain degrees. This means that quasi-isomorphisms of such dg coalgebras might not be derived Morita equivalences, and that we cannot rule out negative homotopy groups for dg categories of cohomological origin.

In Section 2, we give an alternative presentation of the Hochschild construction which associates non-negative dg coalgebras to hearts of t-structures (Corollary 2.19, Propositions 2.7, 2.9). In this setting, the correspondence between dg categories and dg coalgebras can be understood as a form of Koszul duality (Proposition 2.16). Via duality of the commutative and Lie operads, dg tensor categories then correspond to dg Hopf algebras (Corollary 2.29, Proposition 2.27). Example 2.20 then explains how these results combine with Ayoub's calculations to show that existence of a motivic t-structure would characterise Voevodsky's motives over a number field as the derived category of Nori's abelian category of mixed motives, implying the $K(\pi, 1)$ conjecture. §2.5 explains how our constructions generalise Moriya's Tannakian dg categories.

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Section 3 is mostly concerned with applications to the real relative Malcev homotopy types of a manifold X. Lemma 3.4 equates the dg category of derived connections on X with the pre-triangulated category generated by the de Rham dg category of semisimple local systems. Corollary 3.9 then equates this with the dg category of representations of the schematic homotopy type $G(X,x)^{\mathrm{alg}}$. §3.3 looks at the universal bialgebra, which avoids choices of basepoint and can be thought of as the sheaf of functions on the space of algebraic paths. In §3.4, we establish analogues for $\mathbb{Q}_\ell$ relative Malcev homotopy types of a scheme, and §3.5 discusses motivic generalisations.

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Notational conventions. Fix a commutative ring k. When the base is not specified, $\otimes$ will mean $\otimes_k$. When k is a field, we write Vect_k for the category of all vector spaces over k, and FDVect_k for the full subcategory of finite-dimensional vector spaces.

We will always use the symbol $\cong$ to denote isomorphism, while $\simeq$ will be equivalence, quasi-isomorphism or quasi-equivalence.

Contents

1. Background from [Pri6]

We now recall some conventions, definitions and results from [Pri6].

1.1. Conventions for DG categories.

Definition 1.1. A k-linear dg category $\mathcal{C}$ is a category enriched in cochain complexes of k-modules, so has objects $\mathrm{Ob}\mathcal{C}$, cochain complexes $\underline{\mathrm{Hom}}_{\mathcal{C}}(x,y)$ of morphisms, associative multiplication

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(y,z)\otimes_k \underline{\mathrm{Hom}}_{\mathcal{C}}(x,y)\to \underline{\mathrm{Hom}}_{\mathcal{C}}(x,z)$$

and identities $\mathrm{id}_x\in \underline{\mathrm{Hom}}_{\mathcal{C}}(x,x)^0$.

Given a dg category $\mathcal{C}$, we will write $\mathrm{Z}^0\mathcal{C}$ and $\mathrm{H}^0\mathcal{C}$ for the categories with the same objects as $\mathcal{C}$ and with morphisms

$$\mathrm{Hom}_{\mathrm{Z}^0\mathcal{C}}(x,y) := \mathrm{Z}^0\underline{\mathrm{Hom}}_{\mathcal{C}}(x,y),$$
$$\mathrm{Hom}_{\mathrm{H}^0\mathcal{C}}(x,y) := \mathrm{H}^0\underline{\mathrm{Hom}}_{\mathcal{C}}(x,y).$$

When we refer to limits or colimits in a dg category $\mathcal{C}$, we will mean limits or colimits in the underlying category $\mathrm{Z}^0\mathcal{C}$.

Definition 1.2. Given a dg category $\mathcal{C}$ and objects x, y, write $\mathcal{C}(x,y) := \underline{\mathrm{Hom}}_{\mathcal{C}}(y,x)$.

Definition 1.3. A dg functor $F\colon \mathcal{A} \to \mathcal{B}$ is said to be a quasi-equivalence if $\mathrm{H}^0F\colon \mathrm{H}^0\mathcal{A} \to \mathrm{H}^0\mathcal{B}$ is an equivalence of categories, with $\mathcal{A}(X,Y) \to \mathcal{B}(FX,FY)$ a quasi-isomorphism for all objects $X, Y \in \mathcal{A}$.

Definition 1.4. We follow [Kel2, 2.2] in writing $\mathcal{C}_{\mathrm{dg}}(k)$ for the dg category of chain complexes over k, where $\underline{\mathrm{Hom}}(U,V)^i$ consists of graded k-linear morphisms $U \to V[i]$, and the differential is given by $df = d \circ f \mp f \circ d$.

We write $\mathrm{per}_{\mathrm{dg}}(k)$ for the full dg subcategory of finite rank cochain complexes of projective k-modules. Beware that this category is not closed under quasi-isomorphisms, so does not include all perfect complexes in the usual sense.

Following the conventions of [Kel2, 3.1], we will write $\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ for the dg category of k-linear dg functors $\mathcal{A}^{\mathrm{opp}} \to \mathcal{C}_{\mathrm{dg}}(k)$ to chain complexes over k. Observe that when $\mathcal{A}$ has a single object $*$ with $\mathcal{A}(*,*) = A$, $\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ is equivalent to the category of A-modules in complexes. We write $\mathcal{C}(\mathcal{A})$ for the (non-dg) category $\mathrm{Z}^0\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ of dg $\mathcal{A}$-modules.

An object P of $\mathcal{C}(\mathcal{A})$ is cofibrant (for the projective model structure) if every surjective quasi-isomorphism $L \to P$ has a section. The full dg subcategory of $\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ on cofibrant objects is denoted $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$. This is the idempotent-complete pre-triangulated category (in the sense of [BK2, Definition 3.1]) generated by $\mathcal{A}$ and closed under filtered colimits. We write $\mathcal{D}(\mathcal{A})$ for the derived category $\mathrm{H}^0\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ of dg $\mathcal{A}$-modules — this is equivalent to the localisation of $\mathcal{C}(\mathcal{A})$ at quasi-isomorphisms. Thus $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ is a dg enhancement of the triangulated category $\mathcal{D}(A)$.

Definition 1.5. Define $\mathrm{per}_{\mathrm{dg}}(\mathcal{A}) \subset \mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ to be the full subcategory on compact objects, i.e those X for which

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(X,-)$$

preserves filtered colimits. Explicitly, $\mathrm{per}_{\mathrm{dg}}(\mathcal{A})$ consists of objects arising as direct summands of finite complexes of objects of the form $h_X[n]$, for $X \in \mathcal{A}$, where h is the Yoneda embedding.

When $\mathcal{A}$ has a single object $*$ with $\mathcal{A}(*,*) = A$, then $h_*[n]$ corresponds to the A-module $A[n]$. Since projective modules are direct summands of free modules, Definitions 1.5 and 1.4 are thus consistent.

As explained in [Kel2, 4.5], $\mathrm{per}_{\mathrm{dg}}(\mathcal{A})$ is the idempotent-complete pre-triangulated envelope or hull of $\mathcal{A}$, in the sense of [BK2, §3]. Note that in [Kel1, §2], pre-triangulated categories are called exact DG categories.

By [Tab, Theorem 5.1], there is a Morita model structure on k-linear dg categories. Weak equivalences are dg functors $\mathcal{A} \to \mathcal{B}$ which are derived Morita equivalences in the sense that

$$\mathcal{D}_{\mathrm{dg}}(\mathcal{A}) \to \mathcal{D}_{\mathrm{dg}}(\mathcal{B})$$

is a quasi-equivalence. The dg functor $\mathcal{A} \to \mathrm{per}_{\mathrm{dg}}(\mathcal{A})$ is fibrant replacement in this model structure.

Note that a dg category $\mathcal{A}$ is an idempotent-complete pre-triangulated category if and only if the natural embedding $\mathcal{A} \to \mathrm{per}_{\mathrm{dg}}(\mathcal{A})$ is a quasi-equivalence. This is equivalent to saying that $\mathcal{A}$ is Morita fibrant (i.e. fibrant in the Morita model structure), or triangulated in the terminology of [TV, Definition 2.4].

1.2. **Hochschild homology of a DG category.** The following is adapted from [Mit, §12] and [Kel1, 1.3]:

Definition 1.6. Take a small k-linear dg category $\mathcal{A}$ and an $\mathcal{A}$-bimodule

$$F\colon\ \mathcal{A} \times \mathcal{A}^{\mathrm{opp}} \to \mathcal{C}_{\mathrm{dg}}(k),$$

(i.e. a k-bilinear dg functor). Define the homological Hochschild complex

$$\underline{\mathrm{CC}}_\bullet(\mathcal{A}, F)$$

(a simplicial diagram of cochain complexes) by

$$\underline{\mathrm{CC}}_n(\mathcal{A}, F) := \bigoplus_{X_0,\ldots,X_n \in \mathrm{Ob}\mathcal{A}} \mathcal{A}(X_0, X_1) \otimes_k \mathcal{A}(X_1, X_2) \otimes_k \ldots \otimes_k \mathcal{A}(X_{n-1}, X_n) \otimes_k F(X_n, X_0),$$

with face maps

$$\partial_i(a_1 \otimes \ldots a_n \otimes f) = \begin{cases} a_2 \otimes \ldots a_n \otimes (f \circ a_1) & i = 0 \\ a_1 \otimes \ldots a_{i-1} \otimes (a_i \circ a_{i+1}) \otimes a_{i+2} \otimes \ldots \otimes a_n \otimes f & 0 < i < n \\ a_1 \otimes \ldots a_{n-1} \otimes (a_n \circ f) & i = n \end{cases}$$

and degeneracies

$$\sigma_i(a_1 \otimes \ldots a_n \otimes f) = (a_1 \otimes \ldots \otimes a_i \otimes \mathrm{id} \otimes a_{i+1} \otimes \ldots \otimes a_n \otimes f).$$

Definition 1.7. Define the total Hochschild complex

$$\mathrm{CC}(\mathcal{A}, F)$$

by first regarding $\underline{\mathrm{CC}}_\bullet(\mathcal{A}, F)$ as a chain cochain complex with chain differential $\sum_i (-1)^i \partial_i$, then taking the total complex

$$(\mathrm{Tot}\, \underline{\mathrm{CC}}_\bullet(\mathcal{A}, F)^\bullet)^n = \bigoplus_i \underline{\mathrm{CC}}_i(\mathcal{A}, F)^{n+i},$$

with differential given by the cochain differential $\pm$ the chain differential.

There is also a quasi-isomorphic normalised version

$$N\mathrm{CC}(\mathcal{A}, F),$$

given by replacing $\underline{\mathrm{CC}}_i$ with $\underline{\mathrm{CC}}_i / \sum_j \sigma_j \underline{\mathrm{CC}}_{i-1}$.

Remark 1.8. Note that $\mathrm{H}^i \mathrm{CC}(\mathcal{A}, F)^\bullet = \mathrm{HH}_{-i}(\mathcal{A}, F)$, which is a Hochschild *homology* group. We have, however, chosen cohomological gradings because our motivating examples will all have $\mathrm{H}^{<0} = 0$.

1.2.1. *The Tannakian envelope.* Fix a small k-linear dg category $\mathcal{A}$ and a k-linear dg functor $\omega\colon \mathcal{A} \to \mathrm{per}_{\mathrm{dg}}(k)$.

Definition 1.9. Define the Tannakian dual $C_\omega(\mathcal{A})$ by

$$C_\omega(\mathcal{A}) := \mathrm{CC}(\mathcal{A}, \omega \otimes \omega^\vee),$$

where the $\mathcal{A}$-bimodule

$$\omega \otimes \omega^\vee\colon \mathcal{A} \times \mathcal{A}^{\mathrm{opp}} \to \mathrm{per}_{\mathrm{dg}}(k)$$

is given by

$$\omega \otimes \omega^\vee(x, y) = (\omega x) \otimes_k (\omega y)^\vee.$$

Similarly, write $NC_\omega(\mathcal{A}) := N\mathrm{CC}(\mathcal{A}, \omega \otimes \omega^\vee)$.

By [Pri6, Proposition 1.10], The cochain complexes $C_\omega(\mathcal{A}), NC_\omega(\mathcal{A})$ have the natural structure of coassociative counital dg coalgebras over k.

1.2.2. *The universal coalgebra and tilting modules.* Take a k-linear dg category $\mathcal{A}$, and $D \in \mathcal{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}} \otimes \mathcal{A})$ a coassociative $\otimes_{\mathcal{A}}$-coalgebra, with the co-unit $D \to \mathrm{id}_{\mathcal{A}}$ a quasi-isomorphism. We regard this as being a universal coalgebra associated to $\mathcal{A}$.

Example 1.10. As in [Pri6, Example 1.14], if the k-complexes $\mathcal{A}(X, Y)$ are all cofibrant (automatic when k is a field), canonical choices for D are the unnormalised and normalised Hochschild complexes

$$\mathrm{CC}(\mathcal{A}, h_{\mathcal{A}^{\mathrm{opp}}} \otimes h_{\mathcal{A}})$$
$$N\mathrm{CC}(\mathcal{A}, h_{\mathcal{A}^{\mathrm{opp}}} \otimes h_{\mathcal{A}})$$

of the Yoneda embedding $h_{\mathcal{A}^{\mathrm{opp}}} \otimes h_{\mathcal{A}}\colon \mathcal{A}^{\mathrm{opp}} \otimes \mathcal{A} \to \mathcal{C}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}} \otimes \mathcal{A})$. D

Given $\omega\colon \mathcal{A} \to \mathrm{per}_{\mathrm{dg}}(k)$, define the tilting module P by $P := D \otimes_{\mathcal{A}} \omega \in \mathcal{C}(\mathcal{A}^{\mathrm{opp}})$; this is cofibrant and has a natural quasi-isomorphism $P \to \omega$. Also set $Q \in \mathcal{C}(\mathcal{A})$ by $Q := \omega^\vee \otimes_{\mathcal{A}} D$ and set $C := \omega^\vee \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega$. Note that the natural transformation $\mathrm{id}_{\mathcal{A}} \to \omega \otimes_k \omega^\vee$ makes C into a dg coalgebra over k. Likewise, P becomes a right C-comodule and Q a left C-comodule.

Also note that because D is a cofibrant replacement for $\mathrm{id}_{\mathcal{A}}$, we have

$$C \simeq \omega^\vee \otimes^{\mathbf{L}}_{\mathcal{A}} \mathrm{id}_{\mathcal{A}} \otimes^{\mathbf{L}}_{\mathcal{A}} \omega \simeq \omega^\vee \otimes^{\mathbf{L}}_{\mathcal{A}} \omega.$$

1.2.3. *Monoidal categories.* For the purposes of this subsection $(\mathcal{A}, \boxtimes, \mathbb{1})$ is a strictly monoidal dg category, so we have k-linear dg functors $\mathbb{1}\colon k \to \mathcal{A}$ and $\boxtimes\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, such that if we also write $\mathbb{1}$ for the image of the unique object in k,

$$(X \boxtimes Y) \boxtimes Z = X \boxtimes (Y \boxtimes Z), \quad \mathbb{1} \boxtimes X = X, \quad X \boxtimes \mathbb{1} = \mathbb{1}.$$

Definition 1.11. Say that a dg functor $\omega\colon \mathcal{A} \to \mathrm{per}_{\mathrm{dg}}(k)$ is lax monoidal if it is equipped with natural transformations

$$\mu_{XY}\colon \omega(X) \otimes \omega(Y) \to \omega(X \boxtimes Y), \quad \eta\colon k \to \omega(\mathbb{1})$$

satisfying associativity and unitality conditions.

It is said to be strict (resp. strong, resp. quasi-strong) if μ and η are equalities (resp. isomorphisms, resp. quasi-isomorphisms).

By [Pri6, Proposition 1.23], if $\omega\colon \mathcal{A} \to \mathrm{per}_{\mathrm{dg}}(k)$ is strongly monoidal, the monoidal structures endow the dg coalgebras $C_\omega(\mathcal{A}), NC_\omega(\mathcal{A})$ with the natural structure of unital dg bialgebras. These are graded-commutative whenever $\boxtimes$ and ω are symmetric.

The monoidal structure $\boxtimes$ on $\mathcal{A}$ induces a monoidal structure on $\mathcal{A}^{\mathrm{opp}}$, which we also denote by $\boxtimes$. There is also a monoidal structure $\boxtimes^2$ on $\mathcal{A}^{\mathrm{opp}} \otimes \mathcal{A}$, given by $(X \otimes Y) \boxtimes^2 (X' \otimes Y') := (X \boxtimes X') \otimes (Y \boxtimes Y')$. As in [Pri6, Definition 1.29], these extend to dg functors

$$\begin{aligned} \boxtimes\colon \mathcal{D}_{\mathrm{dg}}(\mathcal{A}) \otimes \mathcal{D}_{\mathrm{dg}}(\mathcal{A}) &\to \mathcal{D}_{\mathrm{dg}}(\mathcal{A}), \\ \boxtimes\colon \mathcal{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}}) \otimes \mathcal{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}}) &\to \mathcal{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}}), \\ \boxtimes^2\colon \mathcal{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}} \otimes \mathcal{A}) \otimes \mathcal{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}} \otimes \mathcal{A}) &\to \mathcal{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}} \otimes \mathcal{A}) \end{aligned}$$

extending the dg functors to finite complexes, filtered colimits and direct summands.

Definition 1.12. As in [Pri6, Definition 1.33], we say that a universal coalgebra D (in the sense of §1.2.2) is a universal bialgebra with respect to $\boxtimes$ if is equipped with an associative multiplication $D \boxtimes^2 D \to D$ and a unit $\mathbb{1} \otimes \mathbb{1} \to D$, both compatible with the coalgebra structure.

As in [Pri6, Example 1.35], the Hochschild complexes of Example 1.10 are universal bialgebras whenever k is a field.

By [Pri6, Lemma 1.36], for any universal bialgebra D and a strong monoidal dg functor ω, the dg coalgebra $C := \omega^\vee \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega$ becomes a unital associative dg bialgebra, which is commutative whenever D is commutative and ω symmetric.

Definition 1.13. Let $\mathcal{C}_{\mathrm{dg}}(C)$ be the dg category of right C-comodules in cochain complexes over k. Write $\mathcal{C}(C)$ for the underlying category $\mathrm{Z}^0\mathcal{C}_{\mathrm{dg}}(C)$ of right C-comodules in cochain complexes, and $\mathcal{D}(\mathcal{C})$ for the homotopy category given by formally inverting quasi-isomorphisms. We then write $\mathcal{D}_{\mathrm{dg}}(C)$ for the full dg subcategory of $\mathcal{C}_{\mathrm{dg}}(C)$ on fibrant objects, for model structure "of the first kind" described in [Pos, Remark 8.2].

1.2.4. *Tannakian comparison.*

Definition 1.14. Write $\ker\omega$ for the full dg subcategory of $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ consisting of objects X with $\omega(X) := X \otimes_{\mathcal{A}} \omega$ quasi-isomorphic to 0.

Recall from [Dri, §12.6] that the right orthogonal complement $(\ker\omega)^\perp \subset \mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ is the full dg subcategory consisting of those X for which $\underline{\mathrm{Hom}}_{\mathcal{A}^{\mathrm{opp}}}(M, X) \simeq 0$ for all $M \in \ker\omega$.

The following is [Pri6, Theorem 2.9]:

Theorem 1.15. *For the constructions of $C \simeq \omega^\vee \otimes^{\mathbf{L}}_{\mathcal{A}} \omega$ and the tilting module P of §1.2.2, the derived adjunction $(- \otimes_{\mathcal{A}} P) \dashv \mathbf{R}\underline{\mathrm{Hom}}_C(P, -)$ gives rise to a quasi-equivalence between the dg categories $(\ker\omega)^\perp$ and $\mathcal{D}_{\mathrm{dg}}(C)$. Moreover, the map $(\ker\omega)^\perp \to \mathcal{D}_{\mathrm{dg}}(\mathcal{A})/(\ker\omega)$ to the dg quotient is a quasi-equivalence.*

2. Dense subcategories and semisimplicity

The beauty of Theorem 1.15 is that it describes the derived category $\mathcal{D}(\mathcal{A})$ in terms of a fibre functor on $\mathcal{A}$, so is invariant under Morita equivalences. In particular, for any derived Morita equivalence $\mathcal{B} \to \mathcal{A}$, we have a quasi-equivalence $\mathcal{D}_{\mathrm{dg}}(\mathcal{B}) \to \mathcal{D}_{\mathrm{dg}}(\mathcal{A})$. This becomes particularly important when we can find a Morita equivalent dg category

$\mathcal{B}$ for which the category $\mathrm{Z}^0\mathcal{B}$ is semisimple, since the representing coalgebra then admits a particularly simple description.

In [Pri6, Remark 2.10], it was observed that different choices of universal coalgebra will give dg coalgebras which are derived Morita equivalent. A quasi-isomorphism $C \to C'$ of dg coalgebras need not be a derived Morita equivalence, in general. However, for the Tannakian dg coalgebras constructed in this section, it turns out that quasi-isomorphisms will be derived Morita equivalences (see §2.2 below).

2.1. **Non-negatively graded dg categories.**

Definition 2.1. Let $DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}_k$ denote the category of dg k-coalgebras C in non-negative cochain degrees, satisfying the additional property that the map $\mathrm{H}^0C \to C$ of coalgebras is ind-conilpotent. This means that we can write C as a nested union $C = \varinjlim_\alpha C_\alpha$ of dg coalgebras with $\mathrm{H}^0C = \mathrm{H}^0C_\alpha$ for all α and C_α/H^0C conilpotent in the sense that the comultiplication

$$C_\alpha/\mathrm{H}^0C \to (C_\alpha/\mathrm{H}^0C)^{\otimes m}$$

is 0 for some $m \geq 2$.

For any $C \in DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}_k$, the maximal cosemisimple subcoalgebra ([Pos, 4.3]) $C_{\mathrm{red}} := (\mathrm{H}^0C)_{\mathrm{red}} \subset \mathrm{H}^0C$ thus gives an ind-conilpotent map $C_{\mathrm{red}} \to C$. Since C_{red} is cosemisimple, the ind-conilpotent morphism $C_{\mathrm{red}} \to C_0$ admits a retraction, so C is of the form $C = C_{\mathrm{red}} \oplus N$, for N an ind-conilpotent dg coalgebra with a compatible C_{red}-bicomodule structure.

Proposition 2.2. *Take a k-linear dg category $\mathcal{A}$ with $\mathcal{A}(X,Y)$ concentrated in non-negative degrees, $d\mathcal{A}^0(X,Y) = 0$ for all X, Y, and with $\mathcal{A}^0$ a semisimple abelian category. Assume that we have a k-linear functor $\omega\colon \mathcal{A}^0 \to \mathrm{FDVect}_k$. Then there is a model for the coalgebra $C \simeq \omega^\vee \otimes^{\mathbf{L}}_{\mathcal{A}} \omega$ of §1.2.2 with $C \in DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}_k$.*

Proof. For $i\colon \mathcal{A}^0 \hookrightarrow \mathcal{A}$, we set D to be the direct sum total complex $N\mathrm{CC}(\mathcal{A}/\mathcal{A}^0, i^{\mathrm{opp}} \otimes i)$ of the normalisation $N\mathrm{CC}_\bullet(\mathcal{A}/\mathcal{A}^0, i^{\mathrm{opp}} \otimes i)$ of the simplicial cochain complex $\underline{\mathrm{CC}}_\bullet(\mathcal{A}/\mathcal{A}^0, i^{\mathrm{opp}} \otimes i)$ given by

$$\underline{\mathrm{CC}}_n(\mathcal{A}/\mathcal{A}^0, i^{\mathrm{opp}} \otimes i) := \mathcal{A}(-,-) \otimes_{\mathcal{A}^0} \underbrace{\mathcal{A}(-,-) \otimes_{\mathcal{A}^0} \ldots \otimes_{\mathcal{A}^0} \mathcal{A}(-,-)}_{n} \otimes_{\mathcal{A}^0} \mathcal{A}(-,-).$$

Equivalently, $N\mathrm{CC}_\bullet(\mathcal{A}/\mathcal{A}^0, i^{\mathrm{opp}} \otimes i)$ is the total complex of

$$n \mapsto \mathcal{A}(-,-) \otimes_{\mathcal{A}^0} \underbrace{\mathcal{A}^{>0}(-,-) \otimes_{\mathcal{A}^0} \ldots \otimes_{\mathcal{A}^0} \mathcal{A}^{>0}(-,-)}_{n} \otimes_{\mathcal{A}^0} \mathcal{A}(-,-).$$

The comultiplication and counit are given by the formulae of [Pri6, Proposition 1.10], so we need only show that the counit $D \to \mathrm{id}_{\mathcal{A}}$ is a quasi-isomorphism and that D is a cofibrant module.

The identity $\mathrm{id}_X \in \mathcal{A}(X,X)$ gives a contracting homotopy of the complex $D(X,Y) \to \mathcal{A}(X,Y)$, which ensures that the counit is a quasi-isomorphism. To see that D is cofibrant, we just note that

$$\mathcal{A}(-,iX) \otimes_k \underbrace{\mathcal{A}^{>0}(-,-) \otimes_{\mathcal{A}^0} \ldots \otimes_{\mathcal{A}^0} \mathcal{A}^{>0}(-,-)}_{n} \otimes_k \mathcal{A}(iY,-)$$

is a cofibrant module for all $X, Y \in \mathcal{A}^0$, and that semisimplicity of $\mathcal{A}^0$ ensures that taking $\mathcal{A}^0$-coinvariants is an exact functor, so preserves cofibrancy.

Now, observe that $C := \omega^\vee \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega$ is the direct sum total complex of

$$n \mapsto \omega^\vee \otimes_{\mathcal{A}^0} \underbrace{\mathcal{A}^{>0}(-,-) \otimes_{\mathcal{A}^0} \ldots \otimes_{\mathcal{A}^0} \mathcal{A}^{>0}(-,-)}_{n} \otimes_{\mathcal{A}^0} \omega,$$

which has no negative terms, since ω is concentrated in degree 0 and $(\mathcal{A}^{>0})^{\otimes n}$ in degrees $\geq n$.

Finally, observe that the morphism $\omega^\vee \otimes_{\mathcal{A}^0} \omega \to C$ is an ind-conilpotent extension (cofiltering by copowers of $\omega^\vee \otimes_{\mathcal{A}^0} \mathcal{A}^{>0} \otimes_{\mathcal{A}^0} \omega$) and that $\omega^\vee \otimes_{\mathcal{A}^0} \omega \subset \mathrm{H}^0 C$, so the morphism $\mathrm{H}^0 C \to C$ is necessarily also conilpotent. □

Remark 2.3. Like the construction of Example 1.10, the universal coalgebra of Proposition 2.2 can be written as a Čech nerve. Set $L := \mathcal{A}(-,-) \otimes_{\mathcal{A}^0} \mathcal{A}(-,-)$, which is a $\otimes_{\mathcal{A}}$-coalgebra in $\mathcal{C}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}} \otimes \mathcal{A})$. We then have

$$\underline{\mathrm{CC}}_n(\mathcal{A}/\mathcal{A}^0, i^{\mathrm{opp}} \otimes i) = \underbrace{L \otimes_{\mathcal{A}} L \otimes_{\mathcal{A}} \ldots \otimes_{\mathcal{A}} L}_{n+1},$$

giving the Čech nerve of the $\otimes_{\mathcal{A}}$-comonoid L.

Remark 2.4. If k is algebraically closed, then the complex $CC_\bullet(\mathcal{A}/\mathcal{A}^0, i^{\mathrm{opp}} \otimes i)$ admits a simpler description. Let $\{V_\alpha\}_\alpha$ be a set of irreducible objects of $\mathcal{A}^0$, with one in each isomorphism class. Since k is algebraically closed, $\mathrm{End}_{\mathcal{A}^0}(V_\alpha) \cong k$, and we get

$$\underline{\mathrm{CC}}_n(\mathcal{A}/\mathcal{A}^0, i^{\mathrm{opp}} \otimes i) \cong \bigoplus_{\alpha_0,\ldots,\alpha_n} \mathcal{A}(-, V_{\alpha_0}) \otimes_k \mathcal{A}(V_{\alpha_0}, V_{\alpha_1}) \otimes_k \ldots \otimes_k \mathcal{A}(V_{\alpha_{n-1}}, V_{\alpha_n}) \otimes_k \mathcal{A}(V_{\alpha_n}, -).$$

Writing $\mathcal{A}_s \subset \mathcal{A}$ for the full dg subcategory on objects $\{V_\alpha\}_\alpha$, this gives an isomorphism

$$\mathrm{CC}_\bullet(\mathcal{A}_s, i^{\mathrm{opp}} \otimes i) \cong \mathrm{CC}_\bullet(\mathcal{A}/\mathcal{A}^0, i^{\mathrm{opp}} \otimes i).$$

Thus the quasi-isomorphism $\mathrm{CC}_\bullet(\mathcal{A}/\mathcal{A}^0, i^{\mathrm{opp}} \otimes i) \to \mathrm{CC}_\bullet(\mathcal{A}, h_{\mathcal{A}^{\mathrm{opp}}} \otimes h_{\mathcal{A}})$ is a consequence of the derived Morita equivalence $\mathcal{A}_s \to \mathcal{A}$.

Corollary 2.5. *Take a k-linear dg category $\mathcal{A}$ with $\mathcal{A}(X,Y)$ concentrated in non-negative degrees, and $\mathrm{H}^0\mathcal{A}$ a semisimple abelian category. Assume that we have a k-linear functor $\omega\colon \mathcal{A}^0 \to \mathrm{FDVect}_k$. Then there is a dg coalgebra $C \in DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}_k$ with $C \simeq \omega^\vee \otimes^{\mathbf{L}}_{\mathcal{A}} \omega$, together with quasi-equivalences $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})/(\ker \omega) \simeq (\ker \omega)^\perp \simeq \mathcal{D}_{\mathrm{dg}}(C)$.*

Proof. Consider the morphism $d\colon \mathcal{A}^0(-,-) \to \mathcal{A}^1(-,-)$ of $\mathrm{H}^0\mathcal{A}$-bimodules. Since $\mathrm{H}^0\mathcal{A}$ is semisimple, there exists a $\mathrm{H}^0\mathcal{A}$-bimodule decomposition

$$\mathcal{A}^1(-,-) = d\mathcal{A}^0(-,-) \oplus \mathcal{B}^1(-,-).$$

We may therefore define a dg subcategory $\mathcal{B} \subset \mathcal{A}$ by

$$\mathcal{B}^n(-,-) := \begin{cases} \mathcal{A}^n(-,-) & n \neq 0,1 \\ \mathcal{B}^1(-,-) & n = 1 \\ \mathrm{H}^0\mathcal{A}(-,-) & n = 0. \end{cases}$$

Then $\mathcal{B} \to \mathcal{A}$ is a quasi-equivalence, and $\mathcal{B}$ satisfies the conditions of Proposition 2.2, giving a dg coalgebra C concentrated in non-negative degrees. We then apply Theorem 1.15. □

Definition 2.6. For a dg coalgebra $C \in DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}_k$, define $\mathcal{D}^+_{\mathrm{dg}}(C) \subset \mathcal{D}_{\mathrm{dg}}(C)$ to be the full dg subcategory on cochain complexes V for which $\mathrm{H}^*(V)$ is bounded below. Write $\mathcal{D}^+(C) := \mathrm{H}^0\mathcal{D}^+_{\mathrm{dg}}(C)$

For a k-linear dg category $\mathcal{A}$ with $\mathcal{A}(X,Y)$ concentrated in non-negative degrees, define $\mathcal{D}^+_{\mathrm{dg}}(\mathcal{A}) \subset \mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ to be the full dg subcategory consisting of dg functors F for which $\prod_{X\in\mathcal{A}} \mathrm{H}^*F(X)$ is bounded below. Write $\mathcal{D}^+(\mathcal{A}) := \mathrm{H}^0\mathcal{D}^+_{\mathrm{dg}}(\mathcal{A})$.

Proposition 2.7. *Under the conditions of Corollary 2.5, if the functor* $\omega\colon \mathrm{H}^0\mathcal{A} \to \mathrm{FDVect}_k$ *is faithful, then we have a quasi-equivalence* $\mathcal{D}^+_{\mathrm{dg}}(\mathcal{A}) \simeq \mathcal{D}^+_{\mathrm{dg}}(C)$.

Proof. We first replace $\mathcal{A}$ with the dg category $\mathcal{B}$ from the proof of Corollary 2.5, so $\mathcal{B}^0 = \mathrm{H}^0\mathcal{B}$. Since ω is additive and $\mathcal{B}^0$ semisimple, it follows that $\omega|_{\mathcal{B}^0}$ is exact, and hence represented by some $T \in \mathrm{ind}((\mathcal{B}^0)^{\mathrm{opp}})$, with $\omega|_{\mathcal{B}^0} = - \otimes_{\mathcal{B}^0} T$. We may write T as a filtered colimit $T = \varinjlim T_\alpha$ for $T_\alpha \in (\mathcal{B}^0)^{\mathrm{opp}}$, and because $\mathcal{B}^0$ is abelian we may assume that each T_α is a subobject of T. Since $\mathcal{B}^0$ is semisimple, this means that T_α is a direct summand of T.

Because $\omega|_{\mathcal{B}^0}$ is faithful, it follows that the set $\{T_\alpha\}_\alpha$ generates $\mathcal{B}^0$. Now, the dg functor $\omega\colon \mathcal{D}_{\mathrm{dg}}(\mathcal{B}) \to \mathcal{C}_{\mathrm{dg}}$ is $(- \otimes_{\mathcal{B}} \mathcal{B}^0(-,-) \otimes_{\mathcal{B}^0} T)$, so if $\omega(M) \simeq 0$ then

$$M \otimes_{\mathcal{B}} \mathcal{B}^0(-,T_\alpha) = M \otimes_{\mathcal{B}} \mathcal{B}^0(-,-)(T_\alpha) \simeq 0$$

for all α (T_α being a direct summand of T). Since $\{T_\alpha\}_\alpha$ generates $\mathcal{B}^0$, it follows that $M \otimes_{\mathcal{B}} \mathcal{B}^0(-,-)(X) \simeq 0$ for all objects $X \in \mathcal{B}$, which is precisely the same as saying that $M \otimes_{\mathcal{B}} \mathcal{B}^0(-,-) \simeq 0$.

For the natural projection $\pi\colon \mathcal{B} \to \mathcal{B}^0$, this says that $\pi^*M \simeq 0$; for any $\mathcal{B}^0$-module N, we then have $\underline{\mathrm{Hom}}_{\mathcal{B}}(M, \pi_*N) \simeq 0$, so any complex quasi-isomorphic to π_*N lies in $(\ker\omega)^\perp$. Since $(\ker\omega)^\perp$ is closed under extensions and homotopy limits, and any $R \in \mathcal{D}^+_{\mathrm{dg}}(\mathcal{B})$ can be recovered from the $\mathcal{B}^0$-modules H^iR via these operations, it follows that $R \in (\ker\omega)^\perp$.

Thus Corollary 2.5 shows that the dg functor $- \otimes_{\mathcal{B}} P$ gives a quasi-equivalence from $\mathcal{D}^+_{\mathrm{dg}}(\mathcal{B})$ to a full dg subcategory of $\mathcal{D}^+_{\mathrm{dg}}(C)$. It remains to show that for any $N \in \mathcal{D}^+_{\mathrm{dg}}(C)$, we have $\underline{\mathrm{Hom}}_C(P,N) \in \mathcal{D}^+_{\mathrm{dg}}(\mathcal{B})$; without loss of generality we may assume $\mathrm{H}^{<0}N = 0$. Now,

$$\prod_{X\in\mathcal{B}} \underline{\mathrm{Hom}}_C(P,N)(X) = \prod \underline{\mathrm{Hom}}_C(X \otimes_{\mathcal{B}} P, N),$$

and $\mathrm{H}^*(X \otimes_{\mathcal{B}} P) \cong \mathrm{H}^*(\omega X)$, which is concentrated in degree 0. By applying the cobar resolution in the proof of [Pos, Theorem 4.4] to $\tau^{\geq 0}N$, it follows that that N is quasi-isomorphic to a C-comodule N' concentrated in non-negative degrees and fibrant in the coderived model structure. Then [Pos, Theorem 4.3.1] implies that $\underline{\mathrm{Hom}}_C(-,N) \simeq \underline{\mathrm{Hom}}_C(-,N')$, so

$$\underline{\mathrm{Hom}}_C(X \otimes_{\mathcal{B}} P, N) \simeq \underline{\mathrm{Hom}}_C(\mathrm{H}^0(X \otimes_{\mathcal{B}} P), N'),$$

which is concentrated in non-negative degrees. □

Definition 2.8. For any dg coalgebra C, define $\mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(C)$ to be the full dg subcategory of $\mathcal{C}_{\mathrm{dg}}(C)$ on objects K for which the graded module $K^\#$ underlying K is injective as a comodule over the graded coalgebra $C^\#$ underlying C.

Note that from the properties of the model structure of [Pos, Theorem 8.2], the homotopy category $\mathrm{H}^0\mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(C)$ is equivalent to Positselski's coderived category $\mathcal{D}^{\mathrm{co}}(C)$.

Weak equivalences with respect to this model structure are morphisms whose cone L is coacyclic in the sense of [Pos, 4.2] — this is a stronger condition than acyclicity, and is equivalent to saying that $\underline{\mathrm{Hom}}_C(L,K)$ is acyclic for all $K \in \mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(C)$.

Proposition 2.9. *The equivalence of Proposition 2.7 induces a quasi-equivalence between* $\mathrm{per}_{\mathrm{dg}}(\mathcal{A})$ *(see Definition 1.5) and the full dg subcategory* $\mathcal{F}_{\mathrm{dg}}(C)$ *of* $\mathcal{D}_{\mathrm{dg}}(C)$ *on fibrant replacements of* C*-comodules in finite-dimensional cochain complexes. This gives a quasi-equivalence from* $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ *to* $\mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(C)$.

Proof. Observe that when filtered colimits exist in $\mathcal{D}^+_{\mathrm{dg}}(\mathcal{A})$, they are homotopy colimits, and that $\mathrm{Hom}_{\mathcal{A}}(K,-)$ commutes with such limits for all $K \in \mathrm{per}_{\mathrm{dg}}(\mathcal{A})$. Since every object of $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ can be written as a filtered colimit of perfect complexes, it follows that $\mathrm{per}_{\mathrm{dg}}(\mathcal{A})$ consists of the homotopy-compact objects of $\mathcal{D}^+_{\mathrm{dg}}(\mathcal{A})$ (i.e. the objects K for which $\mathrm{Hom}_{\mathcal{D}^*}(\mathcal{A})(K,-)$ commutes with filtered homotopy colimits, when they exist). They therefore correspond under Proposition 2.7 to the homotopy-compact objects of $\mathcal{D}^+_{\mathrm{dg}}(C)$.

By [Pos, Theorem 4.3.1(a)], $\mathcal{D}^+_{\mathrm{dg}}(C)$ is quasi-equivalent to the full subcategory of $\mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(C)$ consisting of cochain complexes which are bounded below. By [Pos, 5.5], C-comodules in finite-dimensional cochain complexes are compact generators of $D^{\mathrm{co}}(C)$, and hence of $\mathcal{D}^+(C)$. Thus the essential image of

$$- \otimes_{\mathcal{A}} P\colon\ \mathrm{per}_{\mathrm{dg}}(\mathcal{A}) \to \mathcal{D}_{\mathrm{dg}}(C)$$

is just $\mathcal{F}_{\mathrm{dg}}(C)$.

Moreover, since the finite-k-dimensional C-comodules generate $\mathcal{D}^{\mathrm{co}}(C)$ and the latter is closed under arbitrary direct sums, we have a quasi-equivalence

$$\varinjlim\colon\ \mathrm{ind}(\mathcal{F}_{\mathrm{dg}}(C)) \to \mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(C),$$

which combines with the quasi-equivalence $\mathrm{per}_{\mathrm{dg}}(\mathcal{A}) \to \mathcal{F}_{\mathrm{dg}}(C)$ above to give a quasi-equivalence $\mathcal{D}_{\mathrm{dg}}(\mathcal{A}) \to \mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(C)$ on the associated ind-categories. □

Example 2.10. Observe that if $\ker \omega|_{\mathrm{H}^0(\mathcal{A})} = 0$, we need not have $\ker \omega = 0$ on $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$. We see this by considering an example which is in some respects dual to [Pri6, Example 2.13]. For t of degree 1, we can take $k\langle t\rangle$ to be the free non-commutative graded algebra generated by t and let $\mathcal{A}$ be the full dg subcategory of $\mathcal{D}_{\mathrm{dg}}(k\langle t\rangle)$ on objects $k\langle t\rangle^n$. Then $\mathcal{D}_{\mathrm{dg}}(\mathcal{A}) \simeq \mathcal{D}_{\mathrm{dg}}(k\langle t\rangle)$, and the dg fibre functor $\omega(M) := k \otimes_{k\langle t\rangle} M$ is faithful on $\mathcal{D}^+(\mathcal{A})$. However, it is not faithful on $\mathcal{D}(\mathcal{A})$, since $k\langle t, t^{-1}\rangle$ lies in the kernel.

The associated dg coalgebra C is Morita equivalent to $k[\epsilon]^\vee$, for ϵ of degree 0 with $\epsilon^2 = 0$, so a Corollary 2.5 in this case gives

$$\mathcal{D}_{\mathrm{dg}}(k\langle t\rangle)/(k\langle t, t^{-1}\rangle) \simeq \mathcal{D}_{\mathrm{dg}}(k[\epsilon]),$$

while Proposition 2.9 gives an equivalence between $\mathrm{per}_{\mathrm{dg}}(k\langle t\rangle)$ and the dg derived category of finite $k[\epsilon]$-modules. The difference between derived and coderived categories in this case can also be seen by noting that the $k[\epsilon]$-module k is not perfect, but is compact in the coderived category.

2.2. **Koszul duality.** The correspondence of Proposition 2.9 is a manifestation of Koszul duality between modules and comodules, and can be regarded as a partial generalisation of [Pos, Theorem 6.3.a]. In particular, $\mathcal{A} \rightsquigarrow C$ is a cobar construction and $- \otimes_{\mathcal{A}} P$ can be thought of as $\omega \otimes^{\tau,S} C$ for the canonical twisting cochain τ.

Rather than fixing a dg category, we now use Koszul duality to give an equivalence between certain homotopy categories of dg categories and of dg coalgebras. A consequence is that quasi-isomorphisms in the category $DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}$ (see Definition 2.1) induce quasi-isomorphisms of the associated categories, so are necessarily derived Morita equivalences.

Fix a cosemisimple coalgebra S, and set $\mathcal{S}$ to be the category of finite-dimensional S-comodules, with ω the forgetful functor to vector spaces. We can then interpret $\mathcal{S}$-bimodules as S-bicomodules, observing that $S = \omega^\vee \otimes_{\mathcal{S}} \omega$. Note that a non-counital coassociative dg $\otimes_{\mathcal{S}}$-coalgebra $\mathcal{B}$ in $\mathcal{S}$-bimodules then corresponds to the non-counital coassociative dg $\otimes^S$-coalgebra $B := \omega^\vee \otimes_{\mathcal{S}} \mathcal{B} \otimes_{\mathcal{S}} \omega$ in S-bicomodules. This is equivalent to a coassociative dg coalgebra structure on $S \oplus B$, for which $S \to S \oplus B \to S$ are morphisms of dg coalgebras.

Definition 2.11. Let $DG^{>0}\mathrm{Cat}(\mathcal{S})$ be the category of dg categories $\mathcal{A}$ in non-negative degrees with $\mathcal{A}^0 = \mathcal{S}$ and $d\mathcal{A}^0 = 0$, as considered in Proposition 2.2.

This is equivalent to the category of associative $\otimes_{\mathcal{S}}$-algebras $\mathcal{A}^{>0}(-,-)$ in cochain complexes of $\mathcal{S}$-bimodules in strictly positive degrees. Applying [Hir, Theorem 11.3.2] to the forgetful functor mapping to $\mathcal{S}$-bimodules, it follows that $DG^{>0}\mathrm{Cat}(\mathcal{S})$ has a cofibrantly generated model structure in which weak equivalences are quasi-isomorphisms and fibrations are surjections.

Definition 2.12. Dually, let $DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}(\mathcal{S})$ be the category of non-counital ind-conilpotent coassociative dg $\otimes_{\mathcal{S}}$-coalgebras B in complexes of $\mathcal{S}$-bimodules in non-negative cochain degrees.

By analogy with [Hin, Theorem 3.1] and [Pri4, Proposition 1.26], $DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}(\mathcal{S})$ has a fibrantly cogenerated model structure in which weak equivalences are quasi-isomorphisms and cofibrations are injective in degrees > 0.

Definition 2.13. Write $\beta(\mathcal{A})$ for the cofree ind-conilpotent graded $\otimes_{\mathcal{S}}$-coalgebra on generators $\mathcal{A}^{>0}(-,-)[1]$; thus

$$\beta(\mathcal{A}) = \bigoplus_{n>0} \underbrace{\mathcal{A}^{>0}(-,-) \otimes_{\mathcal{S}} \ldots \otimes_{\mathcal{S}} \mathcal{A}^{>0}(-,-)}_{n}[n].$$

We make this a dg coalgebra by defining the differential on cogenerators to be

$$d_{\beta(\mathcal{A})} = (d_{\mathcal{A}}, \circ)\colon\ (\mathcal{A}^{>0}(-,-)[1]) \oplus (\mathcal{A}^{>0}(-,-) \otimes_{\mathcal{S}} \mathcal{A}^{>0}(-,-)[2]) \to \mathcal{A}^{>0}(-,-)[2];$$

Note that the dg coalgebra C of Proposition 2.2 is just the dg coalgebra $S \oplus (\omega^\vee \otimes_{\mathcal{S}} \beta(\mathcal{A}) \otimes_{\mathcal{S}} \omega)$. This cobar construction defines a functor $\beta\colon DG^{>0}\mathrm{Cat}(\mathcal{S}) \to DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}(\mathcal{S})$, with Proposition 2.2 saying that $\mathcal{D}_{\mathrm{dg}}(\mathcal{A}) \simeq \mathcal{D}_{\mathrm{dg}}(S \oplus (\omega^\vee \otimes_{\mathcal{S}} \beta(\mathcal{A}) \otimes_{\mathcal{S}} \omega))$.

Definition 2.14. Let β^* be the left adjoint to β. This is the bar construction sending C to the tensor algebra

$$\beta^*(C)(X,Y) = \bigoplus_{n\geq 0} \underbrace{C(X,-) \otimes_{\mathcal{S}} \ldots \otimes_{c\mathcal{S}} C(-,Y)}_{n}[-n],$$

with differential defined on generators by $d_C + \Delta_C$.

Remark 2.15. A key observation is that the filtration of $\beta(C)$ by powers of $C[-1]$ gives a convergent spectral sequence

$$\mathrm{H}^q(C^{\otimes p}) \implies \mathrm{H}^{p+q}(\beta^* C).$$

Note that convergence of this spectral sequence relies on $\mathrm{H}^{<0}(C)$ vanishing, and allows us to use quasi-isomorphisms for our notion of dg coalgebra weak equivalences where [Hin, Theorem 3.1] used β^* to reflect weak equivalences.

Proposition 2.16. *The functors $\beta^* \dashv \beta$ are a pair of Quillen equivalences between the categories $DG^{>0}\mathrm{Cat}(\mathcal{S}), DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}(\mathcal{S})$.*

Proof. We need to show that for any $C \in DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}(\mathcal{S})$, the unit $C \to \beta\beta^* C$ of the adjunction is a quasi-isomorphism, and that for any $\mathcal{A} \in DG^{>0}\mathrm{Cat}(\mathcal{S})$, the co-unit $\beta^*\beta\mathcal{A} \to \mathcal{A}$ is a quasi-isomorphism.

We begin by noting that [LV, Proposition 11.4.4] says that for C fibrant, the unit $C \to \beta\beta^*(C)$ gives a quasi-isomorphism on tangent spaces, where $\tan(C) = \ker(\Delta\colon C \to C \otimes C)$. Now, $\tan\beta(\mathcal{A}) = \mathcal{A}^{>0}(-,-)$, so setting $C = \beta(\mathcal{A})$, it follows that the unit $C \to \beta\beta^*(C)$ gives a quasi-isomorphism $\mathcal{A}^{>0} \to \beta^*\beta(\mathcal{A})^{>0}$, from which it follows that the co-unit is a quasi-isomorphism.

Now for C fibrant, filtration by the subspaces $\ker(C \to C^{\otimes n})$ gives a convergent spectral sequence

$$E_1^{pq} = \mathrm{H}^{p+q}((\tan C)^{\otimes -p}) \implies \mathrm{H}^{p+q}(C),$$

so cotangent quasi-isomorphisms of fibrant objects are always quasi-isomorphisms, and in particular the unit $C \to \beta\beta^*(C)$ is a quasi-isomorphism for fibrant C. Since the functor $\beta\beta^*$ preserves quasi-isomorphisms, the unit must be a quasi-isomorphism for all C. □

Remark 2.17. The key step in Proposition 2.16 invokes Koszul duality in the form of [LV, Proposition 11.4.4]. When the field k has characteristic 0, there are therefore analogues for any Koszul-dual pair of operads, with cochain dg $\mathcal{P}$-algebras in strictly positive degrees corresponding to conilpotent $\mathcal{P}^{\mathrm{i}}$-coalgebras in non-negative degrees.

2.3. **Hearts of t-structures.** For a Morita fibrant dg category $\mathcal{D}$ to admit a compatible t-structure amounts to the existence of a full generating dg subcategory $\mathcal{A}$ with $\mathrm{H}^i\mathcal{A}(X,Y) = 0$ for all $i < 0$ and all X, Y. The objects of $\mathcal{A}$ are given by any choice of generators for the heart $\mathrm{H}^0\mathcal{D}^\heartsuit$ of the t-structure, and in particular we can take $\mathcal{A}$ to be the full dg subcategory of $\mathcal{D}$ on the semisimple objects of $\mathrm{H}^0\mathcal{D}^\heartsuit$, in which case $\mathrm{H}^0\mathcal{A}$ will be abelian semisimple.

We now show how to extend the results of §2.1 to this generality.

Proposition 2.18. *Take a k-linear dg category $\mathcal{A}$ with the category $\mathrm{H}^0\mathcal{A}$ abelian semisimple and $\mathrm{H}^{<0}\mathcal{A}(X,Y) = 0$ for all objects X, Y. Then $\mathcal{A}$ is quasi-isomorphic to a dg category $\mathcal{B}$ concentrated in non-negative degrees, with $d\mathcal{B}^0(X,Y) = 0$ for all X, Y.*

Proof. First, observe that the good truncation filtration $\tau_n = \tau^{\leq n}$ for $n \geq 0$ gives quasi-isomorphisms

$$\bigoplus_n \mathrm{gr}_n^\tau \mathcal{A}(X,Y) \to \bigoplus_n \mathrm{H}^n\mathcal{A}(X,Y)[-n]$$

for all $X, Y \in \mathcal{A}$, giving $\mathbb{G}_m$-equivariant quasi-isomorphisms of the corresponding dg categories, where gr_n^τ is assigned weight n.

Now, a dg category $\mathcal{B}$ over $\tau_0\mathcal{A}$ on the same objects corresponds to the associative unital $\otimes_{\tau_0\mathcal{A}}$-algebra $\mathcal{B}(-,-)$ in $\mathcal{C}_{\mathrm{dg}}((\tau_0\mathcal{A})^{\mathrm{opp}}\otimes\tau_0\mathcal{A})$. Thus the quasi-isomorphism $\tau_0\mathcal{A}\to \mathrm{H}^0\mathcal{A}$ ensures that $\mathcal{A}$ is quasi-isomorphic to some dg category $\mathcal{A}'$ over H^0A.

Consider the polynomial ring $k[t]$ with t in degree 0 but equipped with a $\mathbb{G}_m$-action of weight 1. The Rees construction gives us a dg category $\zeta(\mathcal{A},\tau)$ with the same objects as $\mathcal{A}$ and morphisms

$$\zeta(\mathcal{A},\tau)(X,Y) := \bigoplus_n \tau_n\mathcal{A}(X,Y)t^n \cong \bigoplus_n \tau_n\mathcal{A}(X,Y),$$

which then becomes a $\mathbb{G}_m$-equivariant dg category, flat over $k[t]$. This has the properties that $\zeta(\mathcal{A},\tau)/t \cong \mathrm{gr}^\tau\mathcal{A}$ and $\zeta(\mathcal{A},\tau)/(t-1)\cong\mathcal{A}$.

Applying the argument above, we see that the $\mathbb{G}_m$-equivariant dg category $\zeta(\mathcal{A},\tau)$ over $\tau_0\mathcal{A}[t]$ must be quasi-isomorphic to some $\mathbb{G}_m$-equivariant dg category $\mathcal{Z}$ over $\mathrm{H}^0\mathcal{A}[t]$ (flat over $k[t]$). Note that $\mathcal{Z}/t$ and $\mathcal{Z}/(t-1)$ are then quasi-isomorphic to $\mathrm{gr}^\tau\mathcal{A}$ and $\mathcal{A}$, respectively.

Now, set $\mathcal{G}^0=\mathrm{H}^0\mathcal{A}$, then take a cofibrant replacement $\mathcal{G}^{>0}(-,-)$ of $\mathrm{H}^{>0}\mathcal{A}(-,-)$ as a $\otimes_{\mathcal{G}^0}$-algebra in $\mathcal{G}$-bimodules. This can be constructed canonically as a bar-cobar resolution as in Proposition 2.16, with the output concentrated in strictly positive degrees (this relies on the semisimplicity of $\mathrm{H}^0\mathcal{A}$). Moreover, the $\mathbb{G}_m$-action on $\mathrm{H}^*(\mathcal{A}(X,Y))$ (with H^n of weight n) transfers equivariantly to $\mathcal{G}:=\mathcal{G}^0\oplus\mathcal{G}^{>0}$.

Since $\mathcal{G}$ is cofibrant and $\mathrm{gr}^\tau\mathcal{A}$ is quasi-isomorphic to $\mathrm{H}^*\mathcal{A}$, we may lift the quasi-isomorphism $\mathcal{G}\to\mathrm{H}^*\mathcal{A}$ to give a $\mathbb{G}_m$-equivariant quasi-isomorphism

$$\mathcal{G}\to\mathcal{Z}/t.$$

We now mimic [Pri7, Propositions 14.11,14.12]. Since $\mathcal{G}(-,-)$ is cofibrant as a unital $\otimes_{\mathcal{G}^0}$-algebra, forgetting the differential gives a retract $\mathcal{G}^*$ of a freely generated $\mathbb{G}_m$-equivariant $\otimes_{\mathcal{G}^0}$-algebra. Since $\mathcal{Z}\to\mathcal{Z}/t$ is surjective, we may lift the map $\mathcal{G}\to\mathcal{Z}/t$ to give a $\mathbb{G}_m$-equivariant map $f\colon \mathcal{G}^*\to\mathcal{Z}$ of graded categories, and hence $\mathcal{G}^*[t]\to\mathcal{Z}$.

We then consider possible differentials on $\mathcal{G}^{>0}[t]$ making f into a map of $\otimes_{\mathcal{G}^0}$-algebras. The proof of [Pri5, Proposition 3.7] characterises obstructions to passing from a differential on $\mathcal{G}^{>0}[t]/t^r$ to one on $\mathcal{G}^{>0}[t]/t^{r+1}$ in terms of elements in Hochschild cohomology, which necessarily vanish because the lift $\mathcal{Z}/t^{r+1}\to\mathcal{Z}/t^r$ exists. In the $\mathbb{G}_m$-equivariant category, $\mathcal{G}^*[t]$ is the limit $\varprojlim_r \mathcal{G}^*[t]/t^r$, so a suitable differential δ exists on $\mathcal{G}^*[t]$, set to 0 on $\mathcal{G}^0[t]$.

Writing $\mathcal{R}=(\mathcal{G}^*[t],\delta)$ with its differential, we thus have a $\mathbb{G}_m$-equivariant quasi-isomorphism

$$\mathcal{R}\to\mathcal{Z}$$

over $k[t]$ (deformations of quasi-isomorphisms being quasi-isomorphisms), and hence a quasi-isomorphism

$$\mathcal{R}/(t-1)\to\mathcal{Z}/(t-1)\simeq\mathcal{A},$$

so $\mathcal{B}:=\mathcal{R}/(t-1)$ has the required properties. □

Corollary 2.19. *Take a k-linear dg category $\mathcal{A}$ with the category $\mathrm{H}^0\mathcal{A}$ abelian semisimple and $\mathrm{H}^{<0}\mathcal{A}(X,Y)=0$ for all objects X,Y. Assume that we have a k-linear dg functor $\omega\colon \mathcal{A}\to\mathcal{C}_{\mathrm{dg}}(k)$ with $\mathrm{H}^*\omega(X)$ finite-dimensional and concentrated in degree 0 for all $X\in\mathcal{A}$. Then there is a dg coalgebra $C\in DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}_k$ with $C\simeq\omega^\vee\otimes^{\mathbf{L}}_{\mathcal{A}}\omega$, together with quasi-equivalences $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})/(\ker\omega)\simeq(\ker\omega)^\perp\simeq\mathcal{D}_{\mathrm{dg}}(C)$.*

Proof. By Proposition 2.18, there is a quasi-isomorphism $q\colon \mathcal{B} \to \mathcal{A}$ with $\mathcal{B}$ satisfying the conditions of Proposition 2.2. It therefore suffices to replace $\omega \circ q\colon \mathcal{B} \to \mathcal{C}_{\mathrm{dg}}(k)$ with a quasi-isomorphic dg functor taking values in FDVect_k. Now, since $\mathcal{B}^0 \simeq \mathrm{H}^0\mathcal{A}$ is semisimple, we may decompose the $\mathcal{B}^0$-module $(\omega \circ q)^0$ as $d(\omega \circ q)^{-1} \oplus M^0$ for some $\mathcal{B}^0$-module M^0. Setting $M^i = (\omega \circ q)^i$ for $i > 0$ and $M^{<0} = 0$ then gives a quasi-isomorphism $M \to \omega \circ q$ of $\mathcal{B}$-modules. Next, choose a decomposition $M^0 = \mathrm{Z}^0 M \oplus N^0$ of $\mathcal{B}^0$-modules, and set $N^i = M^i$ for $i > 0$. Since $\mathrm{H}^i M = 0$ for $i \neq 0$, it follows that N is acyclic, so $M \to M/N$ is a quasi-isomorphism, and $M/N \cong \mathrm{H}^0(\omega \circ q)$ takes values in FDVect_k. □

Example 2.20 (The motivic coalgebra and mixed motives). As in [Pri6, §2.5], consider the dg functor $E^\vee\colon \mathcal{M}_{\mathrm{dg},\mathbb{A}^1,c}(F,\mathbb{Q}) \to \mathcal{C}_{\mathrm{dg}}(\mathbb{Q})$ associated to a mixed Weil homology theory, and let C be the associated dg coalgebra, with quasi-equivalences

$$\mathcal{D}_{\mathrm{dg}}(C) \simeq \mathcal{M}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q})/(\ker E) \simeq \mathcal{M}_{\mathrm{dg}}(F,\mathbb{Q})/(\ker E).$$

Similarly, let C^{eff} be the dg coalgebra associated to the the restriction of E to effective Beilinson motives, giving

$$\mathcal{D}_{\mathrm{dg}}(C^{\mathrm{eff}}) \simeq \mathcal{M}^{\mathrm{eff}}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q})/(\ker E) \simeq \mathcal{M}^{\mathrm{eff}}_{\mathrm{dg}}(F,\mathbb{Q})/(\ker E),$$

where $\mathcal{M}^{\mathrm{eff}}_{\mathrm{dg}}(F,\mathbb{Q})$ is the dg category of cofibrant presheaves of $\mathbb{Q}$-complexes on smooth F-schemes.

When E is Betti cohomology, it is shown in [Ayo2, Corollary 2.105] that $\mathrm{H}^{>0}C = 0$ and $\mathrm{H}^{>0}C^{\mathrm{eff}} = 0$. By [Ayo2, Lemma 2.145], existence of a motivic t-structure would also imply that $\mathrm{H}^{<0}C = 0$, so we would have quasi-isomorphisms $C \to \tau^{\geq 0}C \leftarrow \mathrm{H}^0 C$ of dg coalgebras, but it is not immediate that these are Morita equivalences.

However, if a motivic t-structure exists, then $\mathcal{M}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q}) \subset (\ker E)^\perp$ and applying Corollary 2.19 to the full dg subcategory $\mathcal{A}$ of $\mathcal{M}_{\mathrm{dg},\mathbb{A}^1,c}(F,\mathbb{Q})$ or $\mathcal{M}^{\mathrm{eff}}_{\mathrm{dg},\mathbb{A}^1,c}(F,\mathbb{Q})$ on semisimple objects in the heart of the t-structure would yield N or N^{eff} in $DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}_k$, Morita equivalent to C or C^{eff}, so by Propositions 2.7 and 2.9,

$$\mathcal{M}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q}) \simeq \mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(N), \quad \mathcal{M}^{\mathrm{eff}}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q}) \simeq \mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(N^{\mathrm{eff}}),$$
$$\mathcal{M}^{+}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q}) \simeq \mathcal{D}^{+}_{\mathrm{dg}}(N), \quad \mathcal{M}^{\mathrm{eff},+}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q}) \simeq \mathcal{D}^{+}_{\mathrm{dg}}(N^{\mathrm{eff}}),$$

where $\mathcal{M}^{+}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q})$ is the full dg subcategory of $\mathcal{M}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q})$ consisting of objects in $\bigcup_n \mathcal{M}_{\mathbb{A}^1}(F,\mathbb{Q})^{\geq n}$ for the motivic t-structure.

By [Ayo2, Corollary 2.105], the morphisms $\mathrm{H}^0 N \to N$ and $\mathrm{H}^0 N^{\mathrm{eff}} \to N^{\mathrm{eff}}$ would then be quasi-isomorphisms, hence Morita equivalences by Proposition 2.16, and we would have

$$\mathcal{M}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q}) \simeq \mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(\mathrm{H}^0 N), \quad \mathcal{M}^{\mathrm{eff}}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q}) \simeq \mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(\mathrm{H}^0 N^{\mathrm{eff}}).$$

Letting $\mathcal{MM}_F$ and $\mathcal{MM}^{\mathrm{eff}}_F$ be the categories of $\mathrm{H}^0 N$- and $\mathrm{H}^0 N^{\mathrm{eff}}$-comodules in finite-dimensional vector spaces, and $\mathcal{D}_{\mathrm{dg}}(\mathcal{MM}_F)$ and $\mathcal{D}_{\mathrm{dg}}(\mathcal{MM}^{\mathrm{eff}}_F)$ the dg enhancements of their derived categories, we would then have

$$\mathcal{M}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q}) \simeq \mathcal{D}_{\mathrm{dg}}(\mathcal{MM}_F), \quad \mathcal{M}^{\mathrm{eff}}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q}) \simeq \mathcal{D}_{\mathrm{dg}}(\mathcal{MM}^{\mathrm{eff}}_F),$$

so existence of a motivic t-structure would automatically realise $\mathcal{M}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q})$ as the dg derived category of an abelian category of mixed motives, implying the $K(\pi,1)$ conjecture of [BK1, (1.2)].

Moreover, the full dg subcategory $\mathcal{M}_{\mathrm{dg}}^{\mathrm{eff},\heartsuit} \subset \mathcal{M}_{\mathrm{dg}}^{\mathrm{eff}}$ consisting of objects M with $\mathrm{H}^* E(M)$ concentrated in degree 0 contains $\ker E$, so the heart of the t-structure would be

$$\mathcal{M}_{\mathrm{dg},\mathbb{A}^1}^{\mathrm{eff}}(F,\mathbb{Q})^\heartsuit \simeq \mathcal{M}_{\mathrm{dg}}^{\mathrm{eff}}(F,\mathbb{Q})^\heartsuit / \ker E = \mathcal{M}_{\mathrm{dg}}^{\mathrm{eff}}(F,\mathbb{Q})^\heartsuit/(\ker \mathrm{H}^0 E).$$

Passing to the associated triangulated categories $\mathcal{M} := \mathrm{H}^0\mathcal{M}_{\mathrm{dg}}$ would give

$$\mathcal{MM}_F^{\mathrm{eff}} \simeq \mathcal{M}_{\mathbb{A}^1}^{\mathrm{eff}}(F,\mathbb{Q})^\heartsuit \simeq \mathcal{M}^{\mathrm{eff}}(F,\mathbb{Q})^\heartsuit/(\ker \mathrm{H}^0 E).$$

Nori's abelian category $\mathcal{MM}_{\mathrm{Nori}}^{\mathrm{eff}}$ featuring in [Pri6, Remark 2.17] is defined by forming a diagram D^{eff} of good pairs, and taking the universal $\mathbb{Q}$-linear abelian category under D^{eff} on which $\mathrm{H}^0 E$ is faithful. The proof of [HMS, Corollary 1.7] gives a functor from D^{eff} to $\mathrm{H}^0\mathcal{G}^{\mathrm{eff}} \subset \mathcal{M}_c^{\mathrm{eff}}(\mathbb{Q},\mathbb{Q})^\heartsuit$ and [HMS, Proposition D.3] gives a quasi-inverse to the induced functor

$$\mathcal{MM}_{\mathrm{Nori}}^{\mathrm{eff}} \to \mathcal{M}_c^{\mathrm{eff}}(\mathbb{Q},\mathbb{Q})^\heartsuit/(\ker \mathrm{H}^0 E),$$

so existence of a motivic t-structure would also imply

$$\mathcal{MM}_{\mathrm{Nori}}^{\mathrm{eff}} \simeq \mathcal{MM}_{\mathbb{Q}}^{\mathrm{eff}}.$$

Hanamura has shown in [Han] that the existence of such a t-structure would follow from Grothendieck's standard conjectures, Murre's conjectures and a generalisation of the Beilinson–Soulé vanishing conjecture.

Following [Pri6, Example 2.22], all these coalgebras can be promoted to bialgebras, with the equivalences of categories preserving monoidal structures. In particular, $\mathrm{H}^0 C$ is a naturally a Hopf algebra and $\mathrm{H}^0 C^{\mathrm{eff}}$ a bialgebra. Writing $G_{\mathrm{mot}}(F) := \operatorname{Spec} \mathrm{H}^0 C$ would then give a motivic Galois group, and letting $\mathcal{MM}_F$ be the abelian tensor category of finite-dimensional $G_{\mathrm{mot}}(F)$-representations, the equivalence

$$\mathrm{H}^0\mathcal{M}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q}) \simeq \mathcal{D}(\mathcal{MM}_F),$$

would become monoidal, as would the equivalence

$$\mathcal{MM}_{\mathbb{Q}} \simeq \mathcal{MM}_{\mathrm{Nori}}$$

induced from $\mathcal{MM}_{\mathbb{Q}}^{\mathrm{eff}} \simeq \mathcal{MM}_{\mathrm{Nori}}^{\mathrm{eff}}$ by stabilisation. In particular, this would imply that $G_{\mathrm{mot}}(\mathbb{Q})$ is Nori's motivic Galois group from [HMS, Theorem 1.14]. This last result has since been proved in [CdS] without assuming existence of a motivic t-structure.

2.4. **Tensor categories.** For the remainder of this section, we require that the field k be of characteristic 0.

2.4.1. *Non-negatively graded dg tensor categories.* Assume that $\mathcal{T}$ is a rigid tensor dg category over k (i.e. a symmetric monoidal dg category with strong duals), with $\mathcal{S} := \mathcal{T}^0$ a rigid tensor subcategory. Define the dg $\mathcal{S}^{\mathrm{opp}}$-module $B \in \mathcal{C}(\mathcal{S}^{\mathrm{opp}})$ by $B(U) := \mathcal{T}(U, \mathbb{1})$. Note that tensor properties ensure that $\mathcal{T}(U,V) = \mathcal{T}(U \boxtimes V^\vee, \mathbb{1}) = B(U \boxtimes V^\vee)$.

Thus $B\colon \mathcal{S} \to \mathcal{C}(k)$ is a symmetric lax monoidal functor, or equivalently a unital commutative algebra object in $\mathcal{C}(\mathcal{S}^{\mathrm{opp}})$. This is the same as saying that B is a DGA over $\mathcal{S}$ in the sense of [Pri2, Definition 3.2].

If $\mathcal{S}$ is a semisimple abelian category and $\omega\colon \mathcal{S} \to \mathrm{FDVect}$ a faithful symmetric monoidal functor, then [DMOS, Ch. II] shows that $\mathcal{S}$ is equivalent to the category $\mathrm{Rep}(R)$ of finite-dimensional R-representations for the pro-reductive affine group scheme

$$R := \operatorname{Spec} \omega^\vee \otimes_{\mathcal{S}} \omega.$$

Equivalently, this is the category of finite-dimensional $O(R)$-comodules, where $O(R)$ is the Hopf algebra $\omega^\vee \otimes_{\mathcal{S}} \omega$.

As observed in [Pri2, Remark 3.15], the category of DGAs over $\mathcal{S}$ is equivalent to the category of R-equivariant commutative dg algebras. Under this correspondence, B corresponds to $A := B(O(R))$ (regarding the right R-representation $O(R)$ as an object of $\mathrm{ind}(\mathcal{S})$, with the R-action on A coming from the left action on $O(R)$). When this is concentrated in non-negative cochain degrees, note that it defines a schematic homotopy type in the sense of [KPT]. For the inverse construction, we have $B(V) = A \otimes^R V$ and $\mathcal{T}(U,V) = \mathcal{S}(U \otimes A, V) = \mathrm{Hom}_R(V, U \otimes_k A)$.

Definition 2.21. Define $DG^{\geq 0}\mathrm{Hopf}_n\mathrm{Alg}_k$ to consist of (commutative but not necessarily cocommutative) dg Hopf algebras C for which the underlying dg k-coalgebra lies in the category $DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}_k$ of Definition 2.1. In other words, C is concentrated in non-negative cochain degrees, with $\mathrm{H}^0 C \to C$ ind-conilpotent.

Proposition 2.22. *Take a k-linear rigid tensor dg category $\mathcal{T}$ with $\mathcal{T}(X,Y)$ concentrated in non-negative degrees, $d\mathcal{T}^0(X,Y) = 0$ for all X, Y, and $\mathcal{T}^0$ a semisimple rigid tensor subcategory. Assume that we have a strong monoidal k-linear functor $\omega\colon \mathcal{T}^0 \to \mathrm{FDVect}_k$. Then there is a model for the bialgebra $C \simeq \omega^\vee \otimes^{\mathbf{L}}_{\mathcal{T}} \omega$ of §1.2.3 with $C \in DG^{\geq 0}\mathrm{Hopf}_n\mathrm{Alg}_k$.*

Proof. We just take the coalgebra C from the proof of Proposition 2.2, and observe that the formulae of [Pri6, §1.3.1] adapt to define a coproduct Δ and antipode ρ on C, making it into a dg Hopf algebra.

Explicitly, writing $\mathcal{S} = \mathcal{T}^0$, the expression $\mathcal{T}(U,V) = \mathcal{S}(U \otimes A, V)$ above allows us to rewrite

$$D = \bigoplus_{n \geq 0} A^{\otimes n+2} \otimes O(R) \quad C = \bigoplus_{n \geq 0} A^{\otimes n} \otimes O(R),$$

with

$$\Delta(a_1 \otimes \ldots \otimes a_n \otimes 1) = \sum_{0 \leq r \leq n} (a_1 \otimes \ldots \otimes a_r \otimes 1) \otimes (a_{r+1} \otimes \ldots \otimes a_n \otimes 1)$$

and

$$\rho(a_1 \otimes \ldots \otimes a_n \otimes 1) = (-1)^n (a_n \otimes \ldots \otimes a_1 \otimes 1),$$

and with multiplication given by the shuffle product. The coalgebra structure on C is then given as the semidirect tensor product of $\bigoplus_{n \geq 0} A^{\otimes n}$ and $O(R)$. □

Definition 2.23. Given a commutative unital dg algebra A in R-representations, define the dg category $\mathrm{Rep}(R, A)$ to have R-representations in finite-dimensional vector spaces as objects, with morphisms given by

$$\mathrm{Rep}(R,A)(U,V) := A \otimes^R (U \otimes_k V^\vee).$$

Multiplication is induced by multiplication in A, with identities $1_A \otimes \mathrm{id}_V \in A \otimes^R (V \otimes_k V^\vee)$.

Remark 2.24. In the notation of [Pri3, Remark 4.35]), the dg Hopf algebra corresponding to $\mathcal{T}$ is given by $O(R \ltimes \bar{G}(A)) = O(R \ltimes \bar{G}(\mathcal{T}(O(R), \mathbb{1})))$.

Then Propositions 2.7 and 2.9 give quasi-equivalences

$$\begin{aligned} \mathcal{D}^+_{\mathrm{dg}}(\mathrm{Rep}(R,A)) &\simeq \mathcal{D}^+_{\mathrm{dg}}(O(R \ltimes \bar{G}(A))), \\ \mathcal{D}_{\mathrm{dg}}(\mathrm{Rep}(R,A)) &\simeq \mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(O(R \ltimes \bar{G}(A))). \end{aligned}$$

2.4.2. *Koszul duality for tensor categories.* Now fix a pro-reductive affine group scheme R.

Definition 2.25. Let $DG^{>0}\mathrm{Cat}^{\otimes}(R)$ be the category of rigid tensor dg categories $\mathcal{T}$ in non-negative degrees with $\mathcal{T}^0$ the category of finite-dimensional R-representations and $d\mathcal{T}^0 = 0$.

Note that under the discussion of §2.4.1, $DG^{>0}\mathrm{Cat}^{\otimes}(R)$ is equivalent to the category $DG^{>0}\mathrm{Comm}(R)$ of commutative R-equivariant dg algebras A in strictly positive cochain degrees. There is a model structure on $DG^{>0}\mathrm{Cat}^{\otimes}(R)$ in which weak equivalences are quasi-isomorphisms and fibrations are surjections.

Definition 2.26. Dually, let $DG^{\geq 0}\mathrm{Hopf}_n\mathrm{Alg}(R)$ be the category of R-equivariant ind-conilpotent dg Hopf algebras N in non-negative cochain degrees.

Note that these correspond to Hopf algebras C equipped with maps $O(R) \to C \to O(R)$, such that $O(R) \to C$ is ind-conilpotent. The correspondence sends N to the tensor product $O(R) \otimes N$, with comultiplication given by semidirect product.

Moreover, as in [Qui, Theorem B.4.5], ind-conilpotent dg Hopf algebras N correspond to ind-conilpotent dg Lie coalgebras L, with $L = \tan(N)$. Thus $DG^{\geq 0}\mathrm{Hopf}_n\mathrm{Alg}(R)$ is equivalent to the category $DG^{\geq 0}\mathrm{Co}_n\mathrm{Lie}(R)$ of ind-conilpotent dg Lie coalgebras L in non-negative cochain degrees. It is thus equivalent to the category $dg\hat{\mathcal{N}}(R)$ of [Pri3, Definition 4.1], so is a model for relative Malcev homotopy types over R.

By analogy with [Hin, Theorem 3.1] and [Pri4, Proposition 1.26], there is then a model structure on $DG^{\geq 0}\mathrm{Hopf}_n\mathrm{Alg}_k$ in which weak equivalences are quasi-isomorphisms and cofibrations induce injections on tangent spaces in degrees > 0.

Now, the functor β of Definition 2.13 preserves tensor structures, so induces $\beta\colon DG^{>0}\mathrm{Cat}^{\otimes}(R) \to DG^{\geq 0}\mathrm{Hopf}_n\mathrm{Alg}(R)$. Equivalently, we have $\beta_{\otimes}\colon DG^{>0}\mathrm{Comm}(R) \to DG^{\geq 0}\mathrm{Co}_n\mathrm{Lie}(R)$ given by setting $\beta_{\otimes}(A)$ to be the cofree ind-conilpotent graded Lie coalgebra on cogenerators $A[1]$, with differential defined on cogenerators by

$$d_{\beta_{\otimes}(A)} = (d_A, \circ)\colon\ A[1] \oplus (\mathrm{Symm}^2 A)[2] \to A[2],$$

noting that $(\mathrm{Symm}^2 A)[2] = \bigwedge^2(A[1])$.

Moreover, the commutative and Lie operads are Koszul duals, and β has a left adjoint $\beta^*_{\otimes}$, given by

$$\beta^*_{\otimes}(L) = \bigoplus_{n>0} \mathrm{Symm}^n(L[-1]),$$

with differential given on the generators by $d_L \oplus \Delta\colon\ L[-1] \to L \oplus (\bigwedge^2 L)[-1]$ where Δ is the Lie cobracket.

Thus Remark 2.17 and the comparisons above allow us to adapt Proposition 2.16 to give:

Proposition 2.27. *The functors $\beta^*_{\otimes} \dashv \beta_{\otimes}$ give a pair of Quillen equivalences between the categories $DG^{>0}\mathrm{Cat}^{\otimes}(R), DG^{\geq 0}\mathrm{Hopf}_n\mathrm{Alg}(R)$.*

Using the characterisation of $DG^{>0}\mathrm{Cat}^{\otimes}(R)$ in terms of commutative dg algebras and $DG^{\geq 0}\mathrm{Hopf}_n\mathrm{Alg}(R)$ in terms of dg Lie algebras, this result is effectively one of the equivalences of [Pri3, Theorem 4.41].

2.4.3. *Hearts of tensor t-structures.*

Proposition 2.28. *Take a k-linear rigid tensor dg category $\mathcal{T}$ with the category $\mathrm{H}^0\mathcal{T}$ abelian semisimple and $\mathrm{H}^{<0}\mathcal{T}(X,Y) = 0$ for all objects X, Y. Then $\mathcal{T}$ is quasi-isomorphic to a rigid tensor dg category $\mathcal{B}$ concentrated in non-negative degrees, with $d\mathcal{B}^0(X,Y) = 0$ for all X, Y.*

Proof. If we write $\mathcal{S} := \mathrm{H}^0\mathcal{T}$, we see that the dg tensor categories $\mathcal{S}$ and $\tau^{\leq 0}\mathcal{T}$ are quasi-isomorphic, from which it follows that $\mathcal{T}$ is quasi-isomorphic to some rigid tensor dg category $\mathcal{T}'$ over $\mathcal{S}$. Via the discussion of §2.4.1, this is equivalent to giving a commutative dg algebra A in $\mathcal{C}(\mathcal{S}^{\mathrm{opp}})$ with $\mathrm{H}^0 A = k$.

We now apply the Rees algebra construction $\zeta(A,\tau)$ to the good truncation filtration on A, regarding the Rees algebra as a deformation of $\zeta(A,\tau)/t = \mathrm{H}^*(A)$. The bar-cobar resolution $\beta_{\otimes}^*\beta_{\otimes}(\mathrm{H}^{>0}A)$ of Proposition 2.27 gives a cofibrant replacement for $\mathrm{H}^{>0}A$ concentrated in strictly positive degrees. The proof of Proposition 2.18 now adapts, substituting André–Quillen cohomology for Hochschild cohomology. □

Corollary 2.29. *Take a k-linear rigid tensor dg category $\mathcal{T}$ with the category $\mathrm{H}^0\mathcal{T}$ abelian semisimple and $\mathrm{H}^{<0}\mathcal{T}(X,Y) = 0$ for all objects X, Y. Assume that we have a lax monoidal k-linear dg functor $\omega\colon \mathcal{T} \to \mathcal{C}_{\mathrm{dg}}(k)$ with $\mathrm{H}^i\omega(X) = 0$ for all $i \neq 0$, $\mathrm{H}^0\omega(X)$ finite-dimensional for all $X \in \mathcal{T}$, and quasi-strong in the sense that the structure maps*

$$\omega(X) \otimes_k \omega(Y) \to \omega(X \otimes Y), \quad k \to \omega(\mathbb{1})$$

quasi-isomorphisms for all $X, Y \in \mathcal{T}$.

Then there is a dg Hopf algebra $C \in DG^{\geq 0}\mathrm{Hopf}_n\mathrm{Alg}_k$ with $C \simeq \omega^\vee \otimes^{\mathbf{L}}_{\mathcal{T}} \omega$, together with a tensor dg functor $\mathcal{D}_{\mathrm{dg}}(\mathcal{T}) \to \mathcal{C}_{\mathrm{dg}}(C)$ inducing quasi-equivalences $\mathcal{D}_{\mathrm{dg}}(\mathcal{T})/\ker\omega \simeq (\ker\omega)^\perp \simeq \mathcal{D}_{\mathrm{dg}}(C)$. Here, $\mathcal{C}_{\mathrm{dg}}(C)$ and $\mathcal{D}_{\mathrm{dg}}(C)$ are defined using the coalgebra (not the algebra) structure of C.

If the functor $\mathrm{H}^0\omega\colon \mathrm{H}^0\mathcal{T} \to \mathrm{FDVect}_k$ is faithful, then these induce quasi-equivalences $\mathcal{D}^+_{\mathrm{dg}}(\mathcal{T}) \simeq \mathcal{D}^+_{\mathrm{dg}}(C)$ and $\mathcal{D}_{\mathrm{dg}}(\mathcal{T}) \simeq \mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(C)$.

Proof. By Proposition 2.28, there is a tensor quasi-isomorphism $q\colon \mathcal{B} \to \mathcal{A}$ with $\mathcal{B}$ satisfying the conditions of Proposition 2.2. Write $\mathcal{S} := \mathcal{B}^0 \simeq \mathrm{H}^0\mathcal{A}$.

We now show how to replace $\omega\circ q\colon \mathcal{B} \to \mathcal{C}_{\mathrm{dg}}(k)$ with $\mathrm{H}^0(\omega\circ q)$. If A is the commutative dg algebra in $\mathcal{C}(\mathcal{S}^{\mathrm{opp}})$ corresponding to $\mathcal{B}$, then $\omega \circ q$ corresponds to an A-algebra M in $\mathcal{C}(\mathcal{S}^{\mathrm{opp}})$. Since $\mathrm{H}^*M(X)$ is concentrated in degree 0 for all X, we have quasi-isomorphisms $M \leftarrow \tau^{\leq 0}M \to \mathrm{H}^0M$ of algebras in $\mathcal{C}(\mathcal{S}^{\mathrm{opp}})$. As A is cofibrant, the map $A \to M$ is homotopic to a map taking values in $\tau^{\leq 0}M$, so M and H^0M are quasi-isomorphic as A-algebras.

Since we now have a monoidal functor $\mathrm{H}^0(\omega\circ q)\colon \mathcal{B}^0 \to \mathrm{FDVect}_k$, the first statement of the corollary follows from Corollary 2.5, with [Pri6, Proposition 2.21] ensuring that the tensor structure is preserved. The second statement is then an immediate consequence of Propositions 2.7 and 2.9. □

2.5. **Comparison with Moriya.**

Definition 2.30. For a dg category $\mathcal{A}$ satisfying the conditions of Corollary 2.5, write $\widehat{\mathrm{H}^0\mathcal{A}} \subset \mathcal{D}(\mathcal{A})$ for the subcategory generated by $\mathrm{H}^0\mathcal{A}$ under finite extensions (but not by suspensions). This is the completion functor of [Mor, §2.2].

Note that all objects of $\widehat{\mathrm{H}^0\mathcal{A}}$ are perfect, so we also have a natural embedding $\mathrm{ind}(\widehat{\mathrm{H}^0\mathcal{A}}) \to \mathcal{D}(\mathcal{A})$. Under the conditions of Proposition 2.7, recall that Proposition 2.9

gives an equivalence between $\mathcal{D}^+(\mathcal{A})$ and $\mathcal{D}^+(C)$, with C concentrated in non-negative degrees, and that $\mathrm{H}^0\mathcal{A}$ is equivalent to the category of semisimple H^0C-comodules in finite-dimensional vector spaces. It then follows that $\widehat{\mathrm{H}^0\mathcal{A}}$ is equivalent to the category of all H^0C-comodules in finite-dimensional vector spaces, and $\mathrm{ind}(\widehat{\mathrm{H}^0\mathcal{A}})$ is equivalent to the category of all H^0C-comodules in vector spaces.

In [Mor, Definition 3.1.1], Moriya gives a notion of Tannakian dg category. Given a rigid dg tensor category $\mathcal{A}$ satisfying the conditions of Corollary 2.5 with a tensor functor $\omega\colon \mathcal{A}^0 \to \mathrm{FDVect}$ such that $\omega|_{\mathrm{H}^0\mathcal{A}}$ is faithful (as in Proposition 2.7), we could form a Tannakian dg category in Moriya's sense by taking $\hat{\mathcal{A}}$ to be the full dg subcategory of $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ on objects $\widehat{\mathrm{H}^0\mathcal{A}}$.

Note that because $\hat{\mathcal{A}}$ is a full generating subcategory of $\mathrm{per}_{\mathrm{dg}}(\mathcal{A})$, it follows that $\mathrm{per}_{\mathrm{dg}}(\mathcal{A}) \to \mathrm{per}_{\mathrm{dg}}(\hat{\mathcal{A}})$ is a quasi-equivalence, so Theorem 1.15 gives the same output for both $\mathcal{A}$ and $\hat{\mathcal{A}}$. In topological contexts, this just amounts to saying that semisimple local systems generate all local systems under extension.

In [Mor, §3.3], functors T^{ss} and T are defined from commutative unital dg algebras A in R-representations to dg categories. In fact, $T^{\mathrm{ss}}(R,A) = \mathrm{Rep}(R,A)$ as given in Definition 2.23, and $T(R,A) = \widehat{T^{\mathrm{ss}}(R,A)}$. We have therefore defined Morita equivalences

$$\mathrm{Rep}(R,A) \to T(R,A) \to \mathrm{per}_{\mathrm{dg}}(\mathrm{Rep}(R,A)),$$

so Rep and T give rise to the same theory. However, $T(R,A)$ feels like a halfway house between the minimal choice $\mathrm{Rep}(R,A)$ and the fibrant replacement $\mathrm{per}_{\mathrm{dg}}(\mathrm{Rep}(R,A))$.

Moriya's analogue of the construction in Theorem 1.15 is the construction A_{red} of [Mor, Definition 3.3.3], but this has only limited functoriality, which is a well-known limitation of working with equivariant DGAs. By allowing dg coalgebras to have negative terms, [Pri6, Example 1.35] gives us a completely functorial choice of the dg Hopf algebra $C(\mathcal{T},\omega)$ corresponding under [Pri3, Theorem 4.41] to Moriya's $A_{\mathrm{red}}(\mathcal{T},\omega)$.

3. Schematic and Relative Malcev homotopy types

3.1. **de Rham homotopy types.** Take a pointed connected manifold (X,x), and choose a full rigid tensor subcategory $\mathcal{S}$ of the category of real finite-dimensional semisimple local systems on X. Let $\mathcal{T}$ be the real dg tensor category with the same objects as $\mathcal{S}$, but with morphisms

$$\mathcal{T}(U,V) = A^\bullet(X, U \otimes V^\vee),$$

where $A^\bullet(X,-)$ is the de Rham complex. Note that $\mathrm{H}^0\mathcal{T} \simeq \mathcal{S}$.

Now, the basepoint x defines a fibre functor $x^*\colon \mathcal{T}^0 \to \mathrm{FDVect}_{\mathbb{R}}$ sending U to U_x. We are therefore in the setting of Corollary 2.29. Moreover, $x^*\colon \mathcal{S} \to \mathrm{FDVect}$ is faithful because X is connected, so the conditions of Proposition 2.7 are satisfied.

Let R be the real pro-algebraic group $\mathrm{Spec}\,((x^*)^\vee \otimes_{\mathcal{S}} x^*)$; as in §2.4, we have an equivalence $x^*\colon \mathcal{S} \to \mathrm{Rep}(R)$ of tensor categories. Equivalently, we have a Zariski-dense group homomorphism $\rho\colon \pi_1(X,x) \to R(\mathbb{R})$. As in [Pri3], write $\mathbb{O}(R)$ for the local system corresponding to the right R ind-representation $O(R)$. The R-equivariant dg algebra A from §2.4 is then just the dg algebra

$$A^\bullet(X, \mathbb{O}(R))$$

of equivariant cochains from [Pri3, Definition 3.51], so $\mathcal{T}$ is equivalent to $\mathrm{Rep}(R, A^\bullet(X,\mathbb{O}(R)))$.

Definition 3.1. Write $\mathscr{A}_X^\bullet$ for the sheaf of real $\mathcal{C}^\infty$ differential forms on X, regarded as a sheaf of dg algebras with standard differential $\mathscr{A}_X^n \to \mathscr{A}_X^{n+1}$. Note that $A^\bullet(X,U) = \Gamma(X, U \otimes_{\mathbb{R}} \mathscr{A}_X^\bullet)$.

Definition 3.2. Define the dg category $\mathcal{P}(X)$ to consist of locally perfect $\mathscr{A}_X^\bullet$-modules in complexes of sheaves on X. Define $\mathcal{P}(X,\mathcal{S})$ to be the full dg subcategory of $\mathcal{P}(X)$ generated under shifts and extensions by objects of the form $U \otimes_{\mathbb{R}} \mathscr{A}_X^\bullet$ for $U \in \mathcal{S}$.

Note that because $\mathscr{A}_X^\bullet$ is a flabby resolution of $\mathbb{R}$, the dg category $\mathcal{P}(X)$ is quasi-equivalent to the category of locally constant hypersheaves in real complexes on X.

Lemma 3.3. *When $\mathcal{S}$ consists of all semisimple local systems, we have $\mathcal{P}(X,\mathcal{S}) = \mathcal{P}(X)$.*

Proof. Given $\mathscr{V}^\bullet \in \mathcal{P}(X)$, the sheaf $\bar{\mathscr{V}}^\bullet := \mathscr{V}^\bullet \otimes_{\mathscr{A}_X^\bullet} \mathscr{A}_X^0$ is a finite rank complex of $\mathcal{C}^\infty$-vector bundles on X. We then form the good truncation filtration $\{\tau^{\leq n}\bar{\mathscr{V}}^\bullet\}_n$. Now, the morphisms $\mathscr{A}_X^0 \to \mathscr{A}_X^* \to \mathscr{A}_X^0$ of sheaves of dg algebras give an isomorphism

$$\mathscr{V}^* \cong \bar{\mathscr{V}}^* \otimes_{\mathscr{A}_X^0} \mathscr{A}_X^*$$

of graded $\mathscr{A}_X^*$-modules (where we write U^* for the graded object underlying a complex $U^\bullet$). We then define an increasing filtration $\{W_n\mathscr{V}^*\}_n$ on $\mathscr{V}^*$ by

$$W_n\mathscr{V}^* = (\tau^{\leq n}\bar{\mathscr{V}}^*) \otimes_{\mathscr{A}_X^0} \mathscr{A}_X^*.$$

Writing ∇ for the differential on $\mathscr{V}^\bullet$ and δ for the differential on $\bar{\mathscr{V}}^\bullet$, we can set $\nabla = \delta + D$, for some $D\colon \mathscr{V}^n \to \bigoplus_{i\geq 0} \mathscr{V}^{n-i} \otimes_{\mathscr{A}_X^0} \mathscr{A}_X^{i+1}$. By construction, $\delta W_n \subset W_n$, and we automatically have $DW_n \subset W_n$, so $\{W_n\mathscr{V}^\bullet\}_n$ defines a filtration on $\mathscr{V}^\bullet$.

Let $\mathscr{U}^\bullet$ be the quotient $W_n\mathscr{V}^\bullet/W_{n-1}\mathscr{V}^\bullet$; the only non-zero terms of $\bar{\mathscr{U}}$ are $\bar{\mathscr{U}}^{n-1}, \bar{\mathscr{U}}^n$. For $\nabla = D + \delta$ as above, we have flat connections $D\colon \bar{\mathscr{U}}^i \to \bar{\mathscr{U}}^i \otimes_{\mathscr{A}_X^0} \mathscr{A}^1$ and a map $\delta\colon \mathscr{U}^{n-1} \to \mathscr{U}^n$ commuting with D. Thus there exist local systems U^{n-1}, U^n with $(\bar{\mathscr{U}}^i, D) = (U^i \otimes_{\mathbb{R}} \mathscr{A}_X^0, \mathrm{id}_U \otimes d)$. Because any local system is an extension of semisimple local systems, it follows that the $\mathscr{F}_i := (\bar{\mathscr{U}}^i \otimes_{\mathscr{A}_X^0} \mathscr{A}_X^\bullet, \mathrm{id}\otimes d \pm D \otimes \mathrm{id})$ lie in $\mathcal{P}(X,\mathcal{S})$. Since $\mathscr{U}^\bullet$ is an extension of $\mathscr{F}_{n-1}[1-n]$ by $\mathscr{F}_n[-n]$, we also have $W_n\mathscr{V}^\bullet/W_{n-1}\mathscr{V}^\bullet \in \mathcal{P}(X,\mathcal{S})$. As the filtration is finite ($\bar{\mathscr{V}}^*$ being of finite rank), this means that $\mathscr{V}^\bullet \in \mathcal{P}(X,\mathcal{S})$, as required. □

Lemma 3.4. *The dg categories $\mathrm{per}_{\mathrm{dg}}(\mathcal{T})$ and $\mathcal{P}(X,\mathcal{S})$ are quasi-equivalent.*

Proof. We define a dg functor $\iota\colon \mathcal{T} \to \mathcal{P}(X,\mathcal{S})$ by sending U to $U \otimes_{\mathbb{R}} \mathscr{A}_X^\bullet$. The dg category $\mathcal{P}(X,\mathcal{S})$ is closed under shifts, extensions and direct summands, so $\mathcal{P}(X,\mathcal{S})$ is Morita fibrant — in other words, $\mathcal{P}(X,\mathcal{S}) \to \mathrm{per}_{\mathrm{dg}}(\mathcal{P}(X,\mathcal{S}))$ is a quasi-equivalence.

Now, the dg functor ι is clearly full and faithful, since the maps $\mathcal{T}(U,V) \to \mathcal{P}(X)(\iota U, \iota V)$ are isomorphisms. The definition of $\mathcal{P}(X,\mathcal{S})$ ensures that it is generated by $\iota\mathcal{T}$, so $\mathrm{per}_{\mathrm{dg}}(\mathcal{T}) \to \mathrm{per}_{\mathrm{dg}}(\mathcal{P})$ must be a quasi-equivalence. □

Note that Moriya's category $\hat{\mathcal{T}}$ (see §2.5) embeds in $\mathcal{P}(X)$ as the full dg subcategory on objects $\mathscr{V}^\bullet$ with $\bar{\mathscr{V}}^\bullet$ concentrated in degree 0 — these correspond to flabby resolutions of local systems.

3.2. **Betti homotopy types.** We now let k be a field of characteristic 0.

Definition 3.5. As in [Pri3, Definition 3.11], define the relative Malcev homotopy type $G(X,x)^{R,\mathrm{Mal}}$ of a pointed connected topological space (X,x) with respect to a Zariski-dense representation $\rho\colon \pi_1(X,x) \to R(k)$ as follows. First form the reduced simplicial set $\mathrm{Sing}(X,x)$ of singular simplices based at x, then apply Kan's loop group functor from [Kan] to give a simplicial group $G(X,x) := G(\mathrm{Sing}(X,x))$. Note that $\pi_0 G(X,x) = \pi_1(X,x)$, and apply the relative Malcev completion construction of [Hai] levelwise to $G(X,x) \to R(k)$, obtaining a simplicial affine group scheme

$$G(X,x)^{R,\mathrm{Mal}},$$

with each $G(X,x)_n^{R,\mathrm{Mal}}$ a pro-unipotent extension of R.

In other words, $G(X,x)_n \to (G(X,x)^{R,\mathrm{Mal}})_n(k) \xrightarrow{f(k)} R(k)$ is the universal diagram with f a pro-unipotent extension.

To a relative Malcev homotopy type $G(X,x)^{R,\mathrm{Mal}}$ are associated relative Malcev homotopy groups $\varpi_n(X^{R,\mathrm{Mal}},x) := \pi_{n-1}G(X,x)^{R,\mathrm{Mal}}$. These are affine group schemes, with $\varpi_1(X^{R,\mathrm{Mal}},x) = \pi_1(X,x)^{R,\mathrm{Mal}}$. The higher homotopy groups are pro-finite dimensional vector spaces, and are often just $\pi_n(X,x) \otimes_{\mathbb{Z}} k$ — see [Pri3, Theorem 3.21], [Pri7, Theorem 4.15] and [Pri8, Theorem 3.40].

Examples 3.6. When $\mathcal{S}$ is the category of all semisimple local systems in k-vector spaces on X, we write $G(X,x)^{R,\mathrm{Mal}} = G(X,x)^{\mathrm{alg}}$. Note that [Pri3, Corollary 3.57] shows that $G(X,x)^{\mathrm{alg}}$ is a model for Toën's schematic homotopy types.

When $\mathcal{S}$ is the category of constant local systems on X, note that $R = 1$ and that $G(X,x)^{1,\mathrm{Mal}}$ is the nilpotent k-homotopy type, so Quillen's rational homotopy type from [Qui] when $k = \mathbb{Q}$.

We now specialise to the setting of the previous section, with $k = \mathbb{R}$.

Proposition 3.7. *When X is a manifold and $\mathcal{T} = \mathrm{Rep}(R, A^\bullet(X,\mathbb{O}(R)))$, the dg Hopf algebra $C \simeq (x^*)^\vee \otimes^{\mathbf{L}}_{\mathcal{T}} x^*$ of Corollary 2.29 associated to the fibre functor $x^*\colon \mathcal{T} \to$* FDVect *is a model for the relative Malcev homotopy type $G(X,x)^{R,\mathrm{Mal}}$ of (X,x) under the equivalences of* [Pri3, Theorem 4.41].

Proof. We need to show that the Dold–Kan denormalisation functor D ([Pri3, Definition 4.24]) from dg Hopf algebras to cosimplicial Hopf algebras sends C to a model for the ring of functions on the simplicial group scheme $G(X,x)^{R,\mathrm{Mal}}$. By Remark 2.24, the dg Hopf algebra C is given by $O(R \ltimes \bar{G}(A))$, for $A = A^\bullet(X,\mathbb{O}(R))$. Applying D then gives

$$DC = O(R \ltimes \bar{G}(DA)),$$

where $\bar{G}$ is now the functor on cosimplicial algebras defined in [Pri3, Definition 3.46]. [Equivalently, this is a weak equivalence

$$B\mathrm{Spec}\, DC \simeq [(\mathrm{Spec}\, DA)/R]$$

of affine stacks in the sense of [Toë], where B is the nerve.]

By [Pri3, Proposition 4.50], the simplicial group scheme $G(X,x)^{R,\mathrm{Mal}}$ is quasi-isomorphic to $R \ltimes \bar{G}(DA)$, so we have shown

$$\mathrm{Spec}\, DC \simeq G(X,x)^{R,\mathrm{Mal}}.$$

□

Remark 3.8. For any reduced simplicial set X and Zariski-dense representation $\rho\colon \pi_1(X) \to R(k)$, there is a relative Malcev homotopy type $G(X)^{R,\mathrm{Mal}}$. By [Pri3, Theorem 3.55], this homotopy type corresponds (via [Pri3, Theorem 4.41]) to the R-equivariant cosimplicial algebra

$$\mathrm{C}^\bullet(X, \rho^{-1}O(R))$$

of equivariant singular cochains with coefficients in the local coefficient system $\rho^{-1}O(R)$ (with right multiplication).

A model for the corresponding R-equivariant dg algebra is given by applying the Thom–Sullivan functor Th. The corresponding dg tensor category $\mathcal{T}$ has finite-dimensional R-representations as objects, and morphisms

$$\mathcal{T}(U,V) = \mathrm{Th}\,\mathrm{C}^\bullet(X, \rho^{-1}(U \otimes V^\vee)).$$

When $R = \pi_1(X)^{\mathrm{red}}$ is the reductive pro-algebraic fundamental group of X, the quasi-isomorphism between cosimplicial and cocubical cochains gives a quasi-isomorphism between $\hat{\mathcal{T}}$ and the dg category $T_{\mathrm{dR}}(X)$ of [Mor, Theorem 1.0.4].

Corollary 3.9. *The dg category $\mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(O(G(X,x)^{R,\mathrm{Mal}}))$ of Definition 2.8 is quasi-equivalent to $\mathrm{ind}(\mathcal{P}(X,\mathrm{Rep}(R)))$, for $\mathcal{P}(X,\mathrm{Rep}(R))$ the dg category of derived connections from Definition 3.2. Under this equivalence, $\mathcal{P}(X,\mathrm{Rep}(R))$ corresponds to the full dg subcategory of $\mathcal{D}_{\mathrm{dg}}(O(G(X,x)^{R,\mathrm{Mal}}))$ on fibrant replacements of finite-dimensional comodules. The equivalence respects the tensor structures.*

Proof. Remark 2.24 gives a quasi-equivalence

$$\mathcal{D}_{\mathrm{dg}}(\mathrm{Rep}(R,A)) \simeq \mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(O(R \ltimes \bar{G}(A))),$$

which is compatible with tensor structures by Corollary 2.29. Proposition 3.7 gives $\mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(O(R\ltimes\bar{G}(A))) \simeq \mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(O(G(X,x)^{R,\mathrm{Mal}}))$, while Lemma 3.4 gives $\mathrm{per}_{\mathrm{dg}}(\mathrm{Rep}(R,A)) \simeq \mathcal{P}(X,\mathrm{Rep}(R))$ and hence $\mathcal{D}_{\mathrm{dg}}(\mathrm{Rep}(R,A)) \simeq \mathrm{ind}(\mathcal{P}(X,\mathrm{Rep}(R)))$. Combining these gives the tensor quasi-equivalence $\mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(O(G(X,x)^{R,\mathrm{Mal}})) \simeq \mathrm{ind}(\mathcal{P}(X,\mathrm{Rep}(R)))$.

For the characterisation of $\mathcal{P}(X,\mathrm{Rep}(R))$, we appeal to Proposition 2.9. □

Remark 3.10. As in [Pri6, Remark 2.14], we can also consider a finite set T of basepoints. Proposition 3.7 then adapts to show that the dg Hopf algebroid C_T given by $C_T(x,y) \simeq (x^*)^\vee \otimes^{\mathbf{L}}_T y^*$ is a for the unpointed relative Malcev homotopy type $G(X;T)^{R,\mathrm{Mal}}$ of X, where $G(X;T)$ is the restriction of Dwyer and Kan's loop groupoid $G(X)$ (from [DK]) to the set T of objects.

Because these dg Hopf algebroids are all equivalent as T varies (or equivalently, because the fibre functors are all quasi-isomorphic), taking the colimit over all finite T gives a model for $G(X)^{R,\mathrm{Mal}}$.

Points of X also give a set $\{\bar{x}^*\}$ of fibre functors on the category of all k-linear sheaves, not just on locally constant sheaves. Any such set T of points yields a dg bialgebroid C'_T, but the dg derived category of C'_T-comodules is then just monoidally quasi-equivalent to the dg derived category of k-linear sheaves supported on T. Because the site has enough points, the set of all points gives a jointly faithful set of fibre functors on the category of k-linear sheaves. However, [Pri6, Remark 2.14] only applies to finite sets of fibre functors, so only finitely supported k-linear sheaves arise as comodules of the associated dg bialgebroid $C' = \varinjlim_T C'_T$.

3.3. **The universal Hopf algebra.** An unfortunate feature of relative Malcev homotopy types is that they rely on a choice of basepoint(s). However, the constructions of §1.2.3 give us a universal bialgebra construction $D(X,\mathcal{S})$ associated to a topological space X and a tensor category $\mathcal{S}$ of semisimple local systems. This should be regarded as the ring of functions on the space of algebraic paths generated by $\mathcal{S}$, while $G(X,x)^{R,\mathrm{Mal}}$ is the loop group at a fixed basepoint.

In order to understand $D(X,\mathcal{S})$, we must first understand the category $\mathcal{D}_{\mathrm{dg}}(\mathcal{T}\otimes\mathcal{T}^{\mathrm{opp}})$ in which it lives. When X is a manifold, recall that $\mathcal{T}=\mathrm{Rep}(R,A^\bullet(X,\mathbb{O}(R)))$ and $\mathrm{Rep}(R)\simeq\mathcal{S}=\mathrm{H}^0\mathcal{T}$. By Lemma 3.4, we have a quasi-equivalence

$$\begin{aligned}\mathrm{per}_{\mathrm{dg}}(\mathcal{T}\otimes\mathcal{T}^{\mathrm{opp}})&\simeq\mathcal{P}(X^2,\mathrm{Rep}(R^2)),\\ (U,V)&\mapsto\iota_{X^2}((\mathrm{pr}_1^{-1}U)\otimes_k(\mathrm{pr}_2^{-1}V^\vee)),\end{aligned}$$

and hence

$$\mathcal{D}_{\mathrm{dg}}(\mathcal{T}\otimes\mathcal{T}^{\mathrm{opp}})\simeq\mathrm{ind}(\mathcal{P}(X^2,\mathrm{Rep}(R^2))).$$

Understanding $\mathcal{C}_{\mathrm{dg}}(\mathcal{T}\otimes\mathcal{T}^{\mathrm{opp}})$ is harder, but observe that there is a dg functor r from the dg category of $\mathscr{A}^\bullet_{X^2}$-modules to $\mathcal{C}_{\mathrm{dg}}(\mathcal{T}\otimes\mathcal{T}^{\mathrm{opp}})$, given by

$$(rM)(U,V)=\underline{\mathrm{Hom}}_{\mathscr{A}^\bullet_{X^2}}(\iota_{X^2}((\mathrm{pr}_1^{-1}U^\vee)\otimes_k(\mathrm{pr}_2^{-1}V)),M).$$

Now, note that for $K,L\in\mathcal{P}(X)$, we have

$$\underline{\mathrm{Hom}}_{\mathcal{P}(X)}(K,L)\cong\underline{\mathrm{Hom}}_{\mathscr{A}^\bullet_{X^2}}((\mathrm{pr}_1^*K)\otimes_{\mathscr{A}^\bullet_{X^2}}(\mathrm{pr}_2^*L^\vee),\Delta_*\iota_X(k)),$$

where $\Delta\colon X\to X\times X$ is the diagonal morphism. Thus

$$\mathrm{id}_{\mathcal{A}}=r\Delta_*\iota_X(k)\in\mathcal{C}_{\mathrm{dg}}(\mathcal{T}\otimes\mathcal{T}^{\mathrm{opp}}).$$

Likewise,

$$\begin{aligned}&\underline{\mathrm{Hom}}_{\mathscr{A}^\bullet_{X^2}}(\iota_{X^2}((\mathrm{pr}_1^{-1}U)\otimes_k(\mathrm{pr}_2^{-1}V^\vee)),M\otimes_{\mathcal{T}}N)\\ =\;&M(U,-)\otimes_{\mathcal{T}}N(-,V)\\ \cong\;&\underline{\mathrm{Hom}}_{\mathscr{A}^\bullet_{X^3}}(\iota_{X^2}((\mathrm{pr}_1^{-1}U)\otimes_k(\mathrm{pr}_3^{-1}V^\vee)),(\mathrm{pr}_{12}^*M)\otimes_{\mathscr{A}^\bullet_{X^3}}(\mathrm{pr}_{23}^*N))\\ \cong\;&\underline{\mathrm{Hom}}_{\mathscr{A}^\bullet_{X^2}}(\iota_{X^2}((\mathrm{pr}_1^{-1}U)\otimes_k(\mathrm{pr}_2^{-1}V^\vee)),\mathrm{pr}_{12*}((\mathrm{pr}_{12}^*M)\otimes_{\mathscr{A}^\bullet_{X\times X\times X}}(\mathrm{pr}_{23}^*N)),\end{aligned}$$

so we have

$$M\otimes_{\mathcal{T}}N=r\mathrm{pr}_{13*}((\mathrm{pr}_{12}^*M)\otimes_{\mathscr{A}^\bullet_{X^3}}(\mathrm{pr}_{23}^*N)).$$

Combining these results gives:

Lemma 3.11. *A universal bialgebra $D(X,\mathcal{S})$ corresponds under the equivalence $\mathcal{D}_{\mathrm{dg}}(\mathcal{T}\otimes\mathcal{T}^{\mathrm{opp}})\simeq\mathrm{ind}(\mathcal{P}(X^2,\mathrm{Rep}(R^2)))$ above to a sheaf $\mathbb{D}\in\mathrm{ind}(\mathcal{P}(X^2,\mathrm{Rep}(R^2)))$ equipped with a commutative unital multiplication*

$$\mathbb{D}\otimes_k\mathbb{D}\to\mathbb{D}$$

and a coassociative $\mathscr{A}^\bullet_{X^2}$-linear comultiplication

$$\mathbb{D}\mathrm{pr}_{13*}((\mathrm{pr}_{12}^*\mathbb{D})\otimes_{\mathscr{A}^\bullet_{X^3}}(\mathrm{pr}_{23}^*\mathbb{D}))$$

with $\mathscr{A}^\bullet_{X^2}$-linear counit

$$\mathbb{D}\to\Delta_*\iota_X(k).$$

Beware that although the co-unit $\mathbb{D} \to r\Delta_*\iota_X(k)$ is a quasi-isomorphism of sheaves, the induced map $\mathbb{D} \to \Delta_*\iota_X(k)$ is far from being so, with the object on the left locally constant and that on the right supported on the diagonal. In some sense, $\mathbb{D}$ is the universal coalgebra under Δ_*k generated by $\mathcal{S} \otimes \mathcal{S}^{\mathrm{opp}}$. In the same way that a path space in topology is a fibrant replacement for the diagonal, $\mathbb{D}$ is a cofibrant replacement for functions on the diagonal, which is why we think of it as functions on the space of algebraic paths generated by $\mathcal{S}$.

Example 3.12. Note that the construction of Propositions 2.2 and 2.22 gives an efficient choice $\iota_{X^2}NCC(\mathcal{T}/\mathcal{S}, i^{\mathrm{opp}} \otimes i)$ for the dg bialgebra $\mathbb{D}$, in which case it becomes a dg Hopf algebra. Explicitly, we have

$$\iota_{X^2}NCC(\mathcal{T}/\mathcal{S}, i^{\mathrm{opp}} \otimes i) = NCC(\mathcal{T}/\mathcal{S}, (\mathrm{pr}_1^{-1}\iota_X i^{\mathrm{opp}}) \otimes (\mathrm{pr}_2^{-1}\iota_X i)) \otimes_{[(\mathrm{pr}_1^{-1}\mathscr{A}_X^\bullet)\otimes(\mathrm{pr}_2^{-1}\mathscr{A}_X^\bullet)]} \mathscr{A}_{X^2}^\bullet,$$

where

$$\begin{aligned}&\underline{\mathrm{CC}}_n(\mathcal{T}/\mathcal{S}, (\mathrm{pr}_1^{-1}\iota_X i^{\mathrm{opp}}) \otimes (\mathrm{pr}_2^{-1}\iota_X i)) = \\ &(\mathrm{pr}_1^{-1}\iota_X \mathbb{O}(R)) \otimes^R \underbrace{A^\bullet(X, \mathbb{O}(R)) \otimes^R \ldots \otimes^R A^\bullet(X, \mathbb{O}(R))}_{n} \otimes^R (\mathrm{pr}_2^{-1}\iota_X \mathbb{O}(R)).\end{aligned}$$

Thus $\underline{\mathrm{CC}}_0(\mathcal{T}/\mathcal{S}, i^{\mathrm{opp}} \otimes i) = \iota_{X^2}(\mathbb{O}(R) \otimes^R \mathbb{O}(R))$, which is quasi-isomorphic to the local system given by the $\pi_1(X)^2$-representation $O(R)$, with the two copies of $\pi_1(X)$ acting as left and right multiplication.

Example 3.13. Following Remark 2.4, for $i\colon \mathcal{S} \to \mathcal{T}$ we may describe $NCC(\mathcal{T}/\mathcal{S}, i^{\mathrm{opp}} \otimes i)$ in terms of irreducibles. Let $\{V_\alpha\}_\alpha$ be a set of irreducible objects of complex R-representations, with one in each isomorphism class. Complex conjugation $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on this set, and then we have

$$\begin{aligned}&\underline{\mathrm{CC}}_n(\mathcal{T}/\mathcal{S}, (\mathrm{pr}_1^{-1}\iota_X i^{\mathrm{opp}}) \otimes (\mathrm{pr}_2^{-1}\iota_X i)) \otimes_{\mathbb{R}} \mathbb{C} \cong \\ &\bigoplus_{\alpha_0,\ldots,\alpha_n} (\mathrm{pr}_1^{-1}\mathscr{A}_X^\bullet(V_{\alpha_0}^\vee)) \otimes_{\mathbb{C}} A^\bullet(X, V_{\alpha_0} \otimes_{\mathbb{C}} V_{\alpha_1}^\vee) \otimes_{\mathbb{C}} \ldots \otimes_{\mathbb{C}} A^\bullet(X, V_{\alpha_{n-1}} \otimes_{\mathbb{C}} V_{\alpha_n}^\vee) \otimes_{\mathbb{C}} (\mathrm{pr}_2^{-1}\mathscr{A}_X^\bullet(V_{\alpha_n})),\end{aligned}$$

with $\underline{\mathrm{CC}}_n(\mathcal{T}/\mathcal{S}, (\mathrm{pr}_1^{-1}\iota_X i^{\mathrm{opp}}) \otimes (\mathrm{pr}_2^{-1}\iota_X i))$ given by taking $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariants.

When $\mathcal{S}$ is the category of constant local systems, corresponding to the real (nilpotent) homotopy type as in Examples 3.6, this simplifies to

$$\begin{aligned}&\iota_{X^2}\underline{\mathrm{CC}}_n(\mathcal{T}/\mathcal{S}, i^{\mathrm{opp}} \otimes i) \\ &\cong [\mathrm{pr}_1^{-1}\mathscr{A}_X^\bullet \otimes_{\mathbb{R}} (A^\bullet(X, \mathbb{R}))^{\otimes n} \otimes_{\mathbb{R}} \mathrm{pr}_2^{-1}\mathscr{A}_X^\bullet] \otimes_{[(\mathrm{pr}_1^{-1}\mathscr{A}_X^\bullet)\otimes(\mathrm{pr}_2^{-1}\mathscr{A}_X^\bullet)]} \mathscr{A}_{X^2}^\bullet \\ &= (A^\bullet(X, \mathbb{R}))^{\otimes n} \otimes_{\mathbb{R}} \mathscr{A}_{X^2}^\bullet.\end{aligned}$$

In other words,

$$\iota_{X^2}\underline{\mathrm{CC}}_\bullet(\mathcal{T}/\mathcal{S}, i^{\mathrm{opp}} \otimes i) = \underline{\mathrm{CC}}_\bullet(A^\bullet(X, \mathbb{R}), \mathscr{A}_{X^2}^\bullet).$$

Remark 3.14. Consider the case of a group G acting on a manifold X, with $\mathcal{S}$ a G-equivariant rigid tensor subcategory of semisimple local systems (so $g^*U \in \mathcal{S}$ whenever $U \in \mathcal{S}$ and $g \in G$). Then we have an action of G on $\mathbb{D}$ over $X \times X$, with respect to the diagonal action of G on $X \times X$. This is because G-equivariance of $\mathcal{S}$ gives an action of G on $\mathbb{O}(R)$ over X (i.e. compatible isomorphisms $\mathbb{O}(R) \cong g^*\mathbb{O}(R)$ for all $g \in G$), and hence an action on $A^\bullet(X, \mathbb{O}(R))$. For well-behaved G-actions, this allows us to regard $\mathbb{D}$ as a sheaf of dg algebras on the quotient $(X \times X)/G$. When $x \in X$ is a fixed point for the G-action, note that the dg Hopf algebra $C = (x, x)^*\mathbb{D}$ inherits a G-action from $\mathbb{D}$.

Of course, in order to define a G-action on $\mathbb{D}$, it suffices to have G-actions on $\mathcal{S}$ and on the relative Malcev homotopy type $A^\bullet(X, \mathbb{O}(R))$. When X is a compact Kähler manifold, [Pri7, Theorem 5.17] show that the Tannakian fundamental group $\Pi(\mathrm{MTS})$ of the category of mixed twistor structures acts algebraically on $A^\bullet(X, \mathbb{O}(R))$ for all R, with trivial action on $\mathcal{S}$ (and hence $\mathbb{O}(R)$). This gives an algebraic action of $\Pi(\mathrm{MTS})$ on $\mathbb{D}$, so would allow us to regard $\mathbb{D}$ as an object of the derived category of mixed twistor modules, compatibly with the Hopf algebra structure. When the local systems in $\mathcal{S}$ all underlie variations of Hodge structure, there is also an algebraic circle action on $\mathcal{S}$ and on $A^\bullet(X, \mathbb{O}(R))$, combining with the $\Pi(\mathrm{MTS})$-action to give an action of the Tannakian fundamental group $\Pi(\mathrm{MHS})$ of the category of mixed Hodge structures. Then $\mathbb{D}$ would lie in the derived category of mixed Hodge modules.

Remark 3.15. Note that the cohomology sheaf $\mathscr{H}^0\mathbb{D}$ is the local system $\mathbb{O}(\varpi_1 X^{R,\mathrm{Mal}})$ on $X \times X$ defined in [Pri7, Corollary 8.7] as corresponding to $O(\varpi_1(X^{R,\mathrm{Mal}}, x))$ with its left and right actions by $\pi_1(X, x)$. All of the cohomology sheaves of $\mathbb{D}$ are necessarily local systems.

The pullback $\Delta^*\mathbb{D}$ to the diagonal is a sheaf of Hopf algebras (the ring of functions on the space of algebraic loops generated by $\mathcal{S}$). Then the higher cohomology sheaves of the sheaf of primitive elements of $\Delta^*\mathbb{D}$ are dual to the local systems $\Pi^n(X^{\rho,\mathrm{Mal}})$ of [Pri7, Corollary 8.7] corresponding to the relative Malcev homotopy groups $\varpi_n(X^{R,\mathrm{Mal}}, x)$ with their adjoint actions by $\pi_1(X, x)$.

When $f\colon X \to Y$ is a fibration with section p, choose R so that $\mathrm{Rep}(R)$ contains the semisimplifications of the local systems $\mathbf{R}^n f_*\mathbb{R}$ for all n. Then observe that we have a decomposition

$$p^*\Delta^*\varpi_1 X^{R,\mathrm{Mal}} \cong \pi_1^{\mathrm{dR}}(X/Y, p) \rtimes \Delta^*\varpi_1 Y^{R,\mathrm{Mal}},$$

where $\pi_1^{\mathrm{dR}}(X/Y, p)$ is Lazda's relative fundamental group from [Laz].

3.4. **$\mathbb{Q}_\ell$-homotopy types.** Take a connected algebraic space X and choose a full rigid tensor subcategory $\mathcal{S}$ of the category of semisimple lisse $\mathbb{Q}_\ell$-sheaves on X. Let $\mathcal{T}$ be the cosimplicial tensor category with the same objects as $\mathcal{S}$, but with morphisms

$$\mathcal{T}(U, V) = \mathrm{C}^\bullet(X, U \otimes V^\vee),$$

where $\mathrm{C}^\bullet(X, -)$ is the ℓ-adic Godement resolution of [Pri1, Definition 2.3]. Note that $\mathrm{H}^0\mathcal{T} \simeq \mathcal{S}$, and that

$$\mathrm{H}^i\mathcal{T}(U, V) \cong \mathrm{H}^i_{\text{ét}}(X, U \otimes V^\vee),$$

the ℓ-adic étale cohomology groups. As in Remark 3.8, we may apply the Thom–Sullivan functor Th to obtain a dg category $\mathrm{Th}\,(\mathcal{T})$.

Now, any geometric point $\bar{x}$ defines a fibre functor $\bar{x}^*\colon \mathcal{T}^0 \to \mathrm{FDVect}_{\mathbb{Q}_\ell}$ sending U to $U_{\bar{x}}$. As in §3.1, we may then construct an affine group scheme $R := \mathrm{Spec}\,(\bar{x}^*)^\vee \otimes_{\mathcal{S}} \bar{x}^*$ over $\mathbb{Q}_\ell$, with an equivalence $\bar{x}^*\colon \mathcal{S} \to \mathrm{Rep}(R)$ of tensor categories. Equivalently, we have a Zariski-dense continuous group homomorphism $\rho\colon \pi_1^{\text{ét}}(X, \bar{x}) \to R(\mathbb{Q}_\ell)$. The R-equivariant dg algebra A from §2.4 is then just the dg algebra

$$\mathrm{Th}\,\mathrm{C}^\bullet(X, \mathbb{O}(R))$$

of equivariant cochains from [Pri8, Definition 1.21], so $\mathcal{T}$ is equivalent to $\mathrm{Rep}(R, \mathrm{C}^\bullet(X, \mathbb{O}(R)))$.

Proposition 3.16. *The dg Hopf algebra $C \simeq (\bar{x}^*)^\vee \otimes^{\mathbf{L}}_{\mathrm{Th}\,(\mathcal{T})} \bar{x}^*$ of Corollary 2.29 associated to the fibre functor $\bar{x}^*\colon \mathcal{T} \to \mathrm{FDVect}$ is a model for the relative Malcev homotopy type $G(X, \bar{x})^{R,\mathrm{Mal}}$ of $(X, \bar{x})$ under the equivalences of* [Pri3, Theorem 4.41].

Proof. The proof of Proposition 3.7 carries over, replacing [Pri3, Proposition 4.50] with [Pri8, Theorem 3.30]. □

Now, we may regard $\mathrm{per}_{\mathrm{dg}}(\mathcal{T})$ as the dg subcategory of generated by $\mathcal{S}$ in the dg category of $\mathscr{C}^{\bullet}_X(\mathbb{Q}_\ell)$-modules in complexes of $\mathbb{Q}_\ell$-sheaves, where $\mathscr{C}^{\bullet}_X$ is the sheaf version of the Godement resolution. Since $\mathscr{C}^{\bullet}_X(\mathbb{Q}_\ell)$ is a flabby resolution of $\mathbb{Q}_\ell$, this means that $\mathcal{D}(\mathcal{T})$ is the derived category of $\mathbb{Q}_\ell$-hypersheaves generated by $\mathcal{S}$ under extensions, shifts and direct sums.

If X is defined over a separably closed field $\bar{F}$, then $(X \times_{\bar{F}} X)_{\text{ét}} \cong X_{\text{ét}} \times X_{\text{ét}}$, which means that a universal bialgebra D for $(X, \mathcal{S})$ corresponds to a bialgebra $\mathbb{D}$ in the category of $\mathbb{Q}_\ell$-hypersheaves on $X \times_{\bar{F}} X$.

Remark 3.17. Although we are working with étale homotopy types rather than Betti homotopy types, the argument of Remark 3.14 carries over to say that symmetries of X transfer to the universal bialgebra. In particular, this applies to Galois actions.

Explicitly, take an an algebraic space X_0 over a field F with separable closure $\bar{F}$, and set $X = X_0 \otimes_F \bar{F}$. Assume that $\mathcal{S}$ is generated by pullbacks of lisse sheaves on X_0 — this is equivalent to saying that $\mathcal{S}$ is $\mathrm{Gal}(F)$-equivariant with finite orbits. Then [Pri8, Theorem 3.32] ensures that the relative Malcev homotopy type $G(X)^{R,\mathrm{Mal}}$ carries a continuous Galois action, so we may regard the universal bialgebra $\mathbb{D}$ as a Galois-equivariant $\mathbb{Q}_\ell$-hypersheaf on $X \times_{\bar{F}} X$, or equivalently as a $\mathbb{Q}_\ell$-hypersheaf on $X_0 \times_F X_0$. For any basepoint $x \in X_0(F')$, this gives an action of $\mathrm{Gal}(F')$ on the dg bialgebra $(\bar{x}, \bar{x})^*\mathbb{D}$, but the $\mathrm{Gal}(F)$-action on the universal bialgebra $\mathbb{D}$ does not require $X(F)$ to be non-empty.

Remark 3.18. As in [Pri6, Remark 2.14] and Remark 3.10, we can consider multiple basepoints instead, and taking the set of all geometric points gives a dg Hopf algebroid C with $C(\bar{x}, \bar{y}) \simeq (\bar{x}^*)^\vee \otimes^{\mathbf{L}}_{\mathrm{Th}(\mathcal{T})} \bar{y}^*$ as a model for the relative Malcev homotopy type $G(X_{\text{ét}})^{R,\mathrm{Mal}}$.

3.5. **Motivic homotopy types.** There is nothing special about the de Rham, Betti and ℓ-adic cohomology theories considered so far in this section. Each construction of pro-algebraic homotopy types has only relied on a suitable sheaf of dg algebras, and a category of projective modules over it. There are thus analogues for any mixed Weil cohomology theory in the sense of [CD], or if we are willing to replace Hopf algebras with coalgebras, for any stable cohomology theory.

3.5.1. *Nilpotent homotopy types.* We now look at the simplest relative Malcev homotopy types, when $R = 1$, as in Examples 3.6. A mixed Weil cohomology theory E has an associated sheaf E_X of commutative dg algebras on each scheme X over our base field F, and we write $E(X) := \Gamma(X, E_X)$. Set $\mathcal{S} = \mathrm{FDVect}_k$ and $\mathcal{T} = E(X) \otimes \mathcal{S}$, so $\mathcal{T}$ has the same objects as $\mathcal{S}$, but $\mathcal{T}(U, V) = E(X) \otimes \mathcal{S}(U, V)$. We may then embed $\mathcal{D}_{\mathrm{dg}}(\mathcal{T}^{\mathrm{opp}} \otimes \mathcal{T}) = \mathcal{D}_{\mathrm{dg}}(E(X)^{\mathrm{opp}} \otimes E(X))$ into the category of E_{X^2}-modules by setting

$$\iota_{X^2}(U, V) = U \otimes_k E_{X^2} \otimes_k V^\vee,$$

with the left and right actions of $E(X)$ on E_{X^2} coming from the projections $X^2 \to X$.

As in §3.3, we may now construct a universal Hopf algebra $\mathbb{D}$ on $X^2 = X \times X$, and regard it as the ring of functions on the space of nilpotent algebraic paths. For an explicit model, we follow Examples 1.10 and 3.13, setting

$$\mathbb{D} := \iota_{X^2}\mathrm{CC}(\mathcal{T}/\mathcal{S}, i^{\mathrm{opp}} \otimes i) = \mathrm{CC}(E(X), E_{X^2}),$$

for $i\colon \mathcal{S} \to \mathcal{T}$. This is the Hochschild homology complex of the DGA $E(X)$ with coefficients in the $E(X)$-bimodule E_{X^2} in sheaves on X^2.

As before, we have

$$\mathbb{D} = \mathrm{CC}(E(X), (\mathrm{pr}_1^{-1}E_X) \otimes_k (\mathrm{pr}_2^{-1}E_X)) \otimes_{[(\mathrm{pr}_1^{-1}E_X)\otimes_k(\mathrm{pr}_2^{-1}E_X)]} E_{X^2},$$

which is defined in terms of the $(\mathrm{pr}_1^{-1}E_X) \otimes_k (\mathrm{pr}_2^{-1}E_X)$-modules

$$\underline{\mathrm{CC}}_n(\mathcal{T}/\mathcal{S}, (\mathrm{pr}_1^{-1}\iota_X i^{\mathrm{opp}}) \otimes (\mathrm{pr}_2^{-1}\iota_X i)) \cong \mathrm{pr}_1^{-1}E_X \otimes_k E(X)^{\otimes n} \otimes_k \mathrm{pr}_2^{-1}E_X.$$

on X^2. Since E is a mixed Weil cohomology theory, this is quasi-isomorphic to

$$\mathrm{pr}_1^{-1}E_X \otimes_k E(X^n) \otimes_k \mathrm{pr}_2^{-1}E_X.$$

Writing $h(X/Y)$ for the cohomological motive $M_k(X/Y)^{\mathrm{opp}} \in \mathcal{M}_{\mathbb{A}^1}(Y)^{\mathrm{opp}}$ of X over a base scheme Y and $h(X) := h(X/F)$, we see that the sheaf $\mathbb{D}$ comes from applying E to the simplicial motive

$$n \mapsto \mathrm{pr}_1^{-1}h(X/X) \otimes_k h(X^n) \otimes_k \mathrm{pr}_2^{-1}h(X/X)$$

on X^2.

A choice of basepoint $a\colon \operatorname{Spec} F \to X$ gives a fibre functor $a^*\colon E(X) \to k$, and hence $\mathcal{T} \to \mathrm{FDVect}_k$. The associated dg Hopf algebra is

$$C := (a,a)^*\mathbb{D} \cong E(X^\bullet),$$

with the outer boundary maps coming from pulling back by $(a,\mathrm{id})^*, (\mathrm{id},a)^*\colon X^n \to X^{n+1}$.

Thus C comes from applying the cohomology theory E to the simplicial motive

$$\begin{tikzcd} h(F) \arrow[r, shift left=2, dotted] \arrow[r, shift right=2, dotted] & h(X) \arrow[l, shift left=2, "a^*"] \arrow[l, shift right=2, "a^*"'] \arrow[r, shift left=3, dotted] \arrow[r, shift right=3, dotted] \arrow[r, dotted] & h(X\times X) \arrow[l, shift left=3, "(\mathrm{id},a)^*"] \arrow[l, shift right=3, "(a,\mathrm{id})^*"'] \arrow[l, "\Delta^*"] \ldots, \end{tikzcd}$$

which is just $h(\mathcal{P}_a(X))$, for Wojtkowiak's cosimplicial loop space $\mathcal{P}_a(X)$ from [Woj]. The motivic fundamental group of [EL] is then essentially just

$$\operatorname{Spec} \mathrm{H}^0 h(\mathcal{P}_a(X)),$$

so becomes a special case of our Hochschild homology construction for Tannakian duals.

In fact, we can say much more. Following [EL, §6], we define a cosimplicial scheme $X^{[0,1]}$ by $(X^{[0,1]})^n = X^{\Delta^1_n} \cong X^{n+2}$. The vertices of Δ^1 give a cosimplicial map from $X^{[0,1]}$ to the constant cosimplicial scheme X^2, with $\mathcal{P}_a(X)$ the fibre over (a,a). Now, observe that the ring of functions $\mathbb{D}$ on the space of nilpotent algebraic paths is just given by applying our chosen cohomology theory to the simplicial cohomological motive $\mathscr{D} := h(X^I/X^2)$ over X^2.

3.5.2. *Relative Malcev homotopy types.* Rather than just looking at nilpotent homotopy types, we could consider more general motivic homotopy types by choosing a set S of rigid cohomological motives over X, the nilpotent case being $S = \{h(X/X)\}$. Taking $\mathcal{T}$ to be the full dg category of E_X-modules on objects $E_X(M)$ for $M \in S$, we find that the universal coalgebra $\mathbb{D}$ (thought of as the sheaf of functions on the space of algebraic

paths generated by S) is the normalised total complex of the simplicial diagram given in level n by

$$\underline{\mathrm{CC}}_n(\mathcal{T}, (\mathrm{pr}_1^{-1}\iota_X h_{\mathcal{T}}^{\mathrm{opp}}) \otimes (\mathrm{pr}_2^{-1}\iota_X h_{\mathcal{T}})) \cong$$
$$\bigoplus_{M_0,\ldots,M_n \in S} \mathrm{pr}_1^{-1} E_X(M_0^\vee) \otimes_k E(M_0 \otimes_X M_1^\vee) \otimes_k \ldots \otimes_k E(M_{n-1} \otimes_X M_n^\vee) \otimes_k \mathrm{pr}_2^{-1} E_X(M_n).$$

Here $M^\vee$ denotes the dual motive to M over X, which is just $M(-d)[-2d]$ when $M = h(Y/X)$ is the motive of a smooth and proper morphism $Y \to X$ of relative dimension d. We write $\otimes_X$ for the derived tensor product of motives over X (i.e. with respect to $k_X := h(X/X)$), and we set $E(N) := \Gamma(X, E_X(N))$.

Beware that the duals and tensor products in this expression are only defined up to homotopy, so we have only described $\mathbb{D}$ as a coalgebra in the derived category of E-modules over $X \times X$, with respect to the tensor product

$$(F, G) \mapsto \mathrm{pr}_{13*}((\mathrm{pr}_{12}^* F) \otimes_{E_{X^3}} (\mathrm{pr}_{23}^* G)).$$

Now, $\mathbb{D}$ arises by applying E to the simplicial cohomological motive $\mathscr{D}$ over X^2 given by

$$n \mapsto \bigoplus_{M_0,\ldots,M_n \in S} M_0^\vee \otimes_k (M_0 \otimes_X M_1^\vee) \otimes_k \ldots \otimes_k (M_{n-1} \otimes_X M_n^\vee) \otimes_k M_n,$$

where the $h(X^2)$-module structure comes from the $h(X)$-module structures of $M_0^\vee$ and M_n. When the set S is closed under the tensor product $\otimes_X$, the universal coalgebra $\mathscr{D}$ becomes a $\otimes_X$-bialgebra over X^2.

To understand the relation between $\mathscr{D}$ and the universal coalgebras of §1.2.2, observe that the six functors formalism of [Ayo1] makes $\mathcal{M}_{\mathbb{A}^1}(X)$ a category enriched in $\mathcal{M}_{\mathbb{A}^1}(F)$ and linear over it. The $\otimes_X$-coalgebra $\mathscr{D}$ on X^2 is then a resolution of the enriched Hom functor on objects in S given by $(N, M) \mapsto \mathbf{R}f_*(M \otimes_X N^\vee)$, for $f\colon X \to \operatorname{Spec} F$. This construction is thus the direct generalisation of §1.2.2 to enriched categories.

At a basepoint $a \in X$, the E-Malcev homotopy type of (X, a) relative to S is the dg coalgebra $C := (a, a)^*\mathbb{D}$, which just comes from applying E to the simplicial cohomological F-motive $(a, a)^*\mathscr{D}$ given by

$$n \mapsto \bigoplus_{M_0,\ldots,M_n \in S} (M_0^\vee)_a \otimes_k (M_0 \otimes_X M_1^\vee) \otimes_k \ldots \otimes_k (M_{n-1} \otimes_X M_n^\vee) \otimes_k (M_n)_a.$$

In other words, we should think of $\mathscr{D}$ as the motive of $\mathcal{M}_{\mathbb{A}^1}(F)$-valued functions on the space of algebraic paths generated by S. At any basepoint a, the motive $(a, a)^*\mathscr{D}$ is then the geometric motivic homotopy type of (X, a) relative to S. Note that the arithmetic homotopy type would replace the motive $(M_{i-1} \otimes_X M_i^\vee)$ with its motivic cohomology complex.

As in Example 1.10, the motive $L := \bigoplus_{M \in S} M^\vee \otimes_k M$ is a $\otimes_X$-coalgebra over X^2. Then $\mathscr{D}_n = \underbrace{L \otimes_X L \otimes_X \ldots \otimes_X L}_{n+1}$, so $\mathscr{D}$ is just the Čech nerve of the comonoid L. Setting $S = \{k_X\}$ (the nilpotent case), we get $L = k_{X^2} = h(X^2/X^2)$ and recover the description $\mathscr{D} = h(X^I/X^2)$ of §3.5.1.

For rigid motives $P, Q \in \mathcal{M}_{\mathbb{A}^1}(X)^{\mathrm{opp}}$, we have

$$\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{M}_{\mathbb{A}^1}^{\mathrm{opp}}(X \times X)}(\mathrm{pr}_1^* P^\vee \otimes_{X \times X} \mathrm{pr}_2^* Q, \mathscr{D}) \simeq \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{M}_{\mathbb{A}^1}^{\mathrm{opp}}(X)}(P, Q)$$
$$\simeq \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{M}_{\mathbb{A}^1}^{\mathrm{opp}}(X \times X)}(\mathrm{pr}_1^* P^\vee \otimes_{X \times X} \mathrm{pr}_2^* Q, h(X/X^2)),$$

where the morphism $X \to X^2$ is the diagonal map. Thus the universal coalgebra $\mathscr{D}$ is just the universal motive under $h(X/X^2)$ generated by motives in S. Since duals and tensor products are here only defined up to homotopy, we should perhaps think of $h(X/X^2)$ (or at least its induced dg functor on rigid motives) as the fundamental object.

When S is the set of all rigid motives and we have a basepoint $a \in X$, $a^*\mathscr{D} \in \mathcal{D}_{\mathbb{A}^1}(F)$ is just Ayoub's motivic Hopf algebra from [Ayo3, §2.4].

References

[Ayo1] Joseph Ayoub. Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. I. *Astérisque*, (314):x+466 pp. (2008), 2007.

[Ayo2] Joseph Ayoub. L'algèbre de Hopf et le groupe de Galois motiviques d'un corps de caractéristique nulle, I. *J. Reine Angew. Math.*, 693:1–149, 2014.

[Ayo3] Joseph Ayoub. L'algèbre de Hopf et le groupe de Galois motiviques d'un corps de caractéristique nulle, II. *J. Reine Angew. Math.*, 693:151–226, 2014.

[BK1] Spencer Bloch and Igor Kříž. Mixed Tate motives. *Ann. of Math. (2)*, 140(3):557–605, 1994.

[BK2] A. I. Bondal and M. M. Kapranov. Framed triangulated categories. *Mat. Sb.*, 181(5):669–683, 1990.

[CD] Denis-Charles Cisinski and Frédéric Déglise. Mixed Weil cohomologies. *Adv. Math.*, 230(1):55–130, 2012.

[CdS] Utsav Choudry and Martin Gallauer Alves de Souza. An isomorphism of motivic Galois groups. arXiv:1410.6104v1, 2014.

[DK] W. G. Dwyer and D. M. Kan. Homotopy theory and simplicial groupoids. *Nederl. Akad. Wetensch. Indag. Math.*, 46(4):379–385, 1984.

[DMOS] Pierre Deligne, James S. Milne, Arthur Ogus, and Kuang-yen Shih. *Hodge cycles, motives, and Shimura varieties*, volume 900 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1982.

[Dri] Vladimir Drinfeld. DG quotients of DG categories. *J. Algebra*, 272(2):643–691, 2004.

[EL] Hélène Esnault and Marc Levine. Tate motives and the fundamental group. arXiv:0708.4034v1 [math.AG], 2007.

[Hai] Richard M. Hain. The Hodge de Rham theory of relative Malcev completion. *Ann. Sci. École Norm. Sup. (4)*, 31(1):47–92, 1998.

[Han] Masaki Hanamura. Mixed motives and algebraic cycles. III. *Math. Res. Lett.*, 6(1):61–82, 1999.

[Hin] Vladimir Hinich. DG coalgebras as formal stacks. *J. Pure Appl. Algebra*, 162(2-3):209–250, 2001.

[Hir] Philip S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.

[HMS] Annette Huber and Stefan Müller-Stach. On the relation between Nori motives and Kontsevich periods. arXiv:1105.0865v4, 2011.

[JS] André Joyal and Ross Street. An introduction to Tannaka duality and quantum groups. In *Category theory (Como, 1990)*, volume 1488 of *Lecture Notes in Math.*, pages 413–492. Springer, Berlin, 1991.

[Kan] Daniel M. Kan. On homotopy theory and c.s.s. groups. *Ann. of Math. (2)*, 68:38–53, 1958.

[Kel1] Bernhard Keller. On the cyclic homology of exact categories. *J. Pure Appl. Algebra*, 136(1):1–56, 1999.

[Kel2] Bernhard Keller. On differential graded categories. In *International Congress of Mathematicians. Vol. II*, pages 151–190. Eur. Math. Soc., Zürich, 2006.

[KPT] L. Katzarkov, T. Pantev, and B. Toën. Algebraic and topological aspects of the schematization functor. *Compos. Math.*, 145(3):633–686, 2009. arXiv math.AG/0503418 v2.

[Laz] Chris Lazda. Relative fundamental groups and rational points. arXiv:1303.6484v3 [math.NT], 2013.

[LV] Jean-Louis Loday and Bruno Vallette. *Algebraic operads*. Berlin: Springer, 2012.

[Mit] Barry Mitchell. Rings with several objects. *Advances in Math.*, 8:1–161, 1972.

[Mor] Syunji Moriya. The de Rham homotopy theory and differential graded category. *Math. Z.*, 271(3-4):961–1010, 2012.

[Pos] Leonid Positselski. Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence. *Mem. Amer. Math. Soc.*, 212(996):vi+133, 2011.

[Pri1] J. P. Pridham. Deforming l-adic representations of the fundamental group of a smooth variety. *J. Algebraic Geom.*, 15(3):415–442, 2006.

[Pri2] J. P. Pridham. The pro-unipotent radical of the pro-algebraic fundamental group of a compact Kähler manifold. *Ann. Fac. Sci. Toulouse Math. (6)*, 16(1):147–178, 2007. arXiv math.CV/0502451 v5.

[Pri3] J. P. Pridham. Pro-algebraic homotopy types. *Proc. London Math. Soc.*, 97(2):273–338, 2008. arXiv math.AT/0606107 v8.

[Pri4] J. P. Pridham. Unifying derived deformation theories. *Adv. Math.*, 224(3):772–826, 2010. arXiv:0705.0344v6 [math.AG], corrigendum 228 (2011), no. 4, 2554–2556.

[Pri5] J. P. Pridham. Derived moduli of schemes and sheaves. *J. K-Theory*, 10(1):41–85, 2012. arXiv:1011.2189v5 [math.AG].

[Pri6] J. P. Pridham. Tannaka duality for enhanced triangulated categories i: reconstruction. http://www.maths.ed.ac.uk/∼jpridham/HHtannaka1.pdf or §§1–2 of arXiv:1309.0637v5 [math.KT], 2014.

[Pri7] J. P. Pridham. Real non-abelian mixed Hodge structures for quasi-projective varieties: formality and splitting. *Mem. Amer. Math. Soc.*, 243(1150):vi+178 pp., 2016. arXiv: 1104.1409v3 [math.AG].

[Pri8] Jonathan P. Pridham. Galois actions on homotopy groups of algebraic varieties. *Geom. Topol.*, 15(1):501–607, 2011. arXiv:0712.0928v4 [math.AG].

[Qui] Daniel Quillen. Rational homotopy theory. *Ann. of Math. (2)*, 90:205–295, 1969.

[Tab] Gonçalo Tabuada. Invariants additifs de DG-catégories. *Int. Math. Res. Not.*, (53):3309–3339, 2005.

[Toë] Bertrand Toën. Champs affines. *Selecta Math. (N.S.)*, 12(1):39–135, 2006. arXiv math.AG/0012219.

[TV] Bertrand Toën and Michel Vaquié. Moduli of objects in dg-categories. *Ann. Sci. École Norm. Sup. (4)*, 40(3):387–444, 2007.

[Woj] Zdzisław Wojtkowiak. Cosimplicial objects in algebraic geometry. In *Algebraic K-theory and algebraic topology (Lake Louise, AB, 1991)*, volume 407 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 287–327. Kluwer Acad. Publ., Dordrecht, 1993.