

TANNAKA DUALITY FOR ENHANCED TRIANGULATED CATEGORIES I: RECONSTRUCTION

J.P.PRIDHAM

ABSTRACT. We develop Tannaka duality theory for dg categories. To any dg functor from a dg category \mathcal{A} to finite-dimensional complexes, we associate a dg coalgebra C via a Hochschild homology construction. When the dg functor is faithful, this gives a quasi-equivalence between the derived dg categories of \mathcal{A} -modules and of C -comodules. When \mathcal{A} is Morita fibrant (i.e. an idempotent-complete pre-triangulated category), it is thus quasi-equivalent to the derived dg category of compact C -comodules. We give several applications for motivic Galois groups.

INTRODUCTION

Tannaka duality in Joyal and Street's formulation ([JS, §7, Theorem 3]) characterises abelian k -linear categories \mathcal{A} with exact faithful k -linear functors ω to finite-dimensional k -vector spaces as categories of finite-dimensional comodules of coalgebras C . When \mathcal{A} is a rigid tensor category and ω monoidal, C becomes a Hopf algebra (so $\text{Spec } C$ is a group scheme), giving the duality theorem of [DMOS, Ch. II].

The purpose of this paper is to extend these duality theorems to dg categories. Various derived versions of Tannaka duality have already been established, notably [Toë1, Wal, FI, Lur, Iwa]. However, those works usually require the presence of t -structures, and all follow [DMOS, Ch. II] in restricting attention to monoidal derived categories, then take higher stacks as the derived generalisation of group schemes.

Our viewpoint does not require the dg categories to have monoidal structures, and takes dg coalgebras as the dual objects. Arbitrary dg coalgebras are poorly behaved (for instance, quasi-isomorphism does not imply Morita equivalence), but they perfectly capture the behaviour of arbitrary dg categories without t -structures. Even in the presence of monoidal structures, we consider more general dg categories than heretofore, and our dg coalgebras then become dg bialgebras, in which case our results give dg enhancements and strengthenings of Ayoub's weak Tannaka duality from [Ayo3]. A similar strengthening has appeared in [Iwa], but without the full description needed for applications to motives (see Remarks 2.11 and 2.17).

The first crucial observation we make is that in the Joyal–Street setting, the dual coalgebra C to $\omega: \mathcal{A} \rightarrow \text{FDVect}$ is given by the Hochschild homology group

$$\omega^\vee \otimes_{\mathcal{A}} \omega = \text{HH}_0(\mathcal{A}, \omega^\vee \otimes_k \omega),$$

where $\omega^\vee: \mathcal{A}^{\text{opp}} \rightarrow \text{FDVect}$ sends X to the dual $\omega(X)^\vee$. The natural generalisation of the dual coalgebra to dg categories is then clear: given a k -linear dg category \mathcal{A} and a k -linear dg functor ω to finite-dimensional complexes, we put a dg coalgebra structure

This work was supported by the Engineering and Physical Sciences Research Council [grant numbers EP/I004130/1 and EP/I004130/2].

C on the Hochschild homology complex

$$\omega^\vee \otimes_{\mathcal{A}}^{\mathbf{L}} \omega \simeq \mathrm{CC}_\bullet(\mathcal{A}, \omega^\vee \otimes_k \omega).$$

In order to understand the correct generalisation of the dg fibre functor ω , we look to Morita (or Morita–Takeuchi) theory. In the underived setting, if ω is representable by an object $G \in \mathcal{A}$, the condition that ω be exact and faithful amounts to requiring that G be a projective generator for \mathcal{A} . This means that in the dg setting, $\underline{\mathrm{Hom}}(G, -)$ should be a dg fibre functor if and only if G is a derived generator. In other words, $\underline{\mathrm{Hom}}(G, -)$ must reflect acyclicity of complexes, so we consider dg functors ω from \mathcal{A} to finite-dimensional complexes which are faithful in the sense that $\omega(X)$ is acyclic only if X is acyclic, for X in the derived category $\mathcal{D}(\mathcal{A})$.

Corollary 2.12 gives a derived analogue of [JS, §7, Theorem 3]. When ω is faithful, this gives a quasi-equivalence between the dg enhancements $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ and $\mathcal{D}_{\mathrm{dg}}(C)$ of the derived categories (of the first kind) of \mathcal{A} and C . This comparison holds for all dg categories; in particular, replacing \mathcal{A} with any subcategory of compact generators of $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ will yield a dg coalgebra C with the same property. Our derived analogue of an abelian category is a Morita fibrant dg category: when \mathcal{A} is such a dg category, we have a quasi-equivalence between \mathcal{A} and the full dg subcategory of $\mathcal{D}_{\mathrm{dg}}(C)$ on compact objects.

Crucially, Theorem 2.9 gives a further generalisation to non-faithful dg functors ω , showing that the dg derived category $\mathcal{D}_{\mathrm{dg}}(C)$ of C -comodules is quasi-equivalent to a derived quotient $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})/(\ker \omega)$ of the dg derived category $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ generated by \mathcal{A} . This has many useful applications to scenarios where \mathcal{A} arises as a quotient of a much simpler dg category \mathcal{B} , allowing us to compute C directly from \mathcal{B} and ω .

Section 1 contains the key constructions used throughout the paper. After recalling the Hochschild homology complex $\mathrm{CC}_\bullet(\mathcal{A}, F)$ of a dg category \mathcal{A} with coefficients in a \mathcal{A} -bimodule F , we study the dg coalgebra $C_\omega(\mathcal{A}) := \mathrm{CC}_\bullet(\mathcal{A}, \omega^\vee \otimes_k \omega)$.

We then introduce the notion of universal coalgebras of \mathcal{A} , which are certain resolutions D of $\mathcal{A}(-, -)$ as a $\otimes_{\mathcal{A}}$ -coalgebra in \mathcal{A} -bimodules. The canonical choice is the Hochschild complex $\mathrm{CC}_\bullet(\mathcal{A}, h_{\mathcal{A}^{\mathrm{opp}}} \otimes_k h_{\mathcal{A}})$ of the Yoneda embedding. For any universal coalgebra D , a dg fibre functor ω gives a dg coalgebra $C := \omega^\vee \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega$, and a tilting module $P := D \otimes_{\mathcal{A}} \omega$. When (\mathcal{A}, \boxtimes) is a tensor dg category, we consider universal bialgebras, which are universal coalgebras equipped with compatible multiplication with respect to \boxtimes , the Hochschild complex again being one such. In this case, a tensor dg functor ω makes C into a dg bialgebra.

The main results of the paper are in Section 2. For C and P a dg coalgebra and tilting module as above, there is a left Quillen dg functor $-\otimes_{\mathcal{A}} P$ from the category of dg \mathcal{A} -modules to the category of dg C -comodules (Lemma 2.5). The functors $\mathcal{D}(C) \rightarrow \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{D}(C)$ then form a retraction (Proposition 2.7). Theorem 2.9 establishes quasi-equivalences

$$\mathcal{D}_{\mathrm{dg}}(C) \rightarrow (\ker \omega)^\perp \rightarrow \mathcal{D}_{\mathrm{dg}}(\mathcal{A})/(\ker \omega)$$

of dg enhancements of derived categories, which simplify to the equivalences of Corollary 2.12 when ω is faithful. Remark 2.11 relates this to Ayoub’s weak Tannaka duality, with various consequences for describing motivic Galois groups given in §2.5. Proposition 2.21 ensures that the equivalences preserve tensor structures when present, and Example 2.22 applies this to motivic Galois groups.

In the appendix, we give technical details for constructing monoidal dg functors giving rise to the motivic Galois groups of Example 2.22, and show that, in the case of Betti

cohomology, this construction gives a dg functor non-canonically quasi-isomorphic to the usual cohomology dg functor (Corollary A.19).

The main drawback of the Hochschild construction for the dg coalgebra is that it always creates terms in negative cochain degrees. This means that quasi-isomorphisms of such dg coalgebras might not be derived Morita equivalences, and that we cannot rule out negative homotopy groups for dg categories of cohomological origin. This issue is addressed in the sequel [Pri2], by relating vanishing of the dg coalgebra in negative degrees to the existence of a t -structure.

I would like to thank Joseph Ayoub for providing helpful comments and spotting careless errors.

Notational conventions. Fix a commutative ring k . When the base is not specified, \otimes will mean \otimes_k . When k is a field, we write Vect_k for the category of all vector spaces over k , and FDVect_k for the full subcategory of finite-dimensional vector spaces.

We will always use the symbol \cong to denote isomorphism, while \simeq will be equivalence, quasi-isomorphism or quasi-equivalence.

CONTENTS

Introduction	1
Notational conventions	3
1. Hochschild homology of a DG category	3
1.1. The Tannakian envelope	5
1.2. The universal coalgebra and tilting modules	6
1.3. Monoidal categories	9
2. Comodules	15
2.1. The Quillen adjunction	15
2.2. Tannakian comparison	17
2.3. Tannakian equivalence	20
2.4. Homotopy invariance	21
2.5. Example: motives	22
2.6. Monoidal comparisons	24
Appendix A. Formal Weil cohomology theories	26
A.1. Quasi-projective pairs and localisation	26
A.2. Mixed Hodge structures on Betti cohomology	30
A.3. Splittings for Betti cohomology	32
References	33

1. HOCHSCHILD HOMOLOGY OF A DG CATEGORY

Definition 1.1. A k -linear dg category \mathcal{C} is a category enriched in cochain complexes of k -modules, so has objects $\text{Ob}\mathcal{C}$, cochain complexes $\underline{\text{Hom}}_{\mathcal{C}}(x, y)$ of morphisms, associative multiplication

$$\underline{\text{Hom}}_{\mathcal{C}}(y, z) \otimes_k \underline{\text{Hom}}_{\mathcal{C}}(x, y) \rightarrow \underline{\text{Hom}}_{\mathcal{C}}(x, z)$$

and identities $\text{id}_x \in \underline{\text{Hom}}_{\mathcal{C}}(x, x)^0$.

Given a dg category \mathcal{C} , we will write $Z^0\mathcal{C}$ and $H^0\mathcal{C}$ for the categories with the same objects as \mathcal{C} and with morphisms

$$\begin{aligned}\mathrm{Hom}_{Z^0\mathcal{C}}(x, y) &:= Z^0\mathrm{Hom}_{\mathcal{C}}(x, y), \\ \mathrm{Hom}_{H^0\mathcal{C}}(x, y) &:= H^0\mathrm{Hom}_{\mathcal{C}}(x, y).\end{aligned}$$

When we refer to limits or colimits in a dg category \mathcal{C} , we will mean limits or colimits in the underlying category $Z^0\mathcal{C}$.

Definition 1.2. Given a dg category \mathcal{C} and objects x, y , write $\mathcal{C}(x, y) := \mathrm{Hom}_{\mathcal{C}}(y, x)$.

Definition 1.3. A dg functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is said to be a quasi-equivalence if $H^0F: H^0\mathcal{A} \rightarrow H^0\mathcal{B}$ is an equivalence of categories, with $\mathcal{A}(X, Y) \rightarrow \mathcal{B}(FX, FY)$ a quasi-isomorphism for all objects $X, Y \in \mathcal{A}$.

Definition 1.4. We follow [Kel2, 2.2] in writing $\mathcal{C}_{\mathrm{dg}}(k)$ for the dg category of chain complexes over k , where $\mathrm{Hom}(U, V)^i$ consists of graded k -linear morphisms $U \rightarrow V[i]$, and the differential is given by $df = d \circ f \mp f \circ d$.

We write $\mathrm{per}_{\mathrm{dg}}(k)$ for the full dg subcategory of finite rank cochain complexes of projective k -modules. Beware that this category is not closed under quasi-isomorphisms, so does not include all perfect complexes in the usual sense.

The following is adapted from [Mit, §12] and [Kel1, 1.3]:

Definition 1.5. Take a small k -linear dg category \mathcal{A} and an \mathcal{A} -bimodule

$$F: \mathcal{A} \times \mathcal{A}^{\mathrm{opp}} \rightarrow \mathcal{C}_{\mathrm{dg}}(k),$$

(i.e. a k -bilinear dg functor). Define the homological Hochschild complex

$$\underline{\mathrm{CC}}_{\bullet}(\mathcal{A}, F)$$

(a simplicial diagram of cochain complexes) by

$$\underline{\mathrm{CC}}_n(\mathcal{A}, F) := \bigoplus_{X_0, \dots, X_n \in \mathrm{Ob}\mathcal{A}} \mathcal{A}(X_0, X_1) \otimes_k \mathcal{A}(X_1, X_2) \otimes_k \dots \otimes_k \mathcal{A}(X_{n-1}, X_n) \otimes_k F(X_n, X_0),$$

with face maps

$$\partial_i(a_1 \otimes \dots \otimes a_n \otimes f) = \begin{cases} a_2 \otimes \dots \otimes a_n \otimes (f \circ a_1) & i = 0 \\ a_1 \otimes \dots \otimes a_{i-1} \otimes (a_i \circ a_{i+1}) \otimes a_{i+2} \otimes \dots \otimes a_n \otimes f & 0 < i < n \\ a_1 \otimes \dots \otimes a_{n-1} \otimes (a_n \circ f) & i = n \end{cases}$$

and degeneracies

$$\sigma_i(a_1 \otimes \dots \otimes a_n \otimes f) = (a_1 \otimes \dots \otimes a_i \otimes \mathrm{id} \otimes a_{i+1} \otimes \dots \otimes a_n \otimes f).$$

Definition 1.6. Define the total Hochschild complex

$$\mathrm{CC}(\mathcal{A}, F)$$

by first regarding $\underline{\mathrm{CC}}_{\bullet}(\mathcal{A}, F)$ as a chain cochain complex with chain differential $\sum_i (-1)^i \partial_i$, then taking the total complex

$$(\mathrm{Tot} \underline{\mathrm{CC}}_{\bullet}(\mathcal{A}, F))^{\bullet} = \bigoplus_i \underline{\mathrm{CC}}_i(\mathcal{A}, F)^{n+i},$$

with differential given by the cochain differential \pm the chain differential.

There is also a quasi-isomorphic normalised version

$$\mathrm{NCC}(\mathcal{A}, F),$$

given by replacing $\underline{\mathbb{C}\mathbb{C}}_i$ with $\underline{\mathbb{C}\mathbb{C}}_i / \sum_j \sigma_j \underline{\mathbb{C}\mathbb{C}}_{i-1}$.

Remark 1.7. Note that $\mathrm{H}^i \mathbb{C}\mathbb{C}(\mathcal{A}, F)^\bullet = \mathrm{HH}_{-i}(\mathcal{A}, F)$, which is a Hochschild *homology* group. We have, however, chosen cohomological gradings because our motivating examples will all have $\mathrm{H}^{<0} = 0$.

1.1. The Tannakian envelope. Fix a small k -linear dg category \mathcal{A} and a k -linear dg functor $\omega: \mathcal{A} \rightarrow \mathrm{per}_{\mathrm{dg}}(k)$.

Remark 1.8. If k is a field and we instead have a dg functor $\omega: \mathcal{A} \rightarrow \mathrm{hFDCh}_k$ to the dg category of cohomologically finite-dimensional complexes (i.e. perfect complexes in the usual sense), we can reduce to the setting above. We could first take a cofibrant replacement $\tilde{\mathcal{A}} \rightarrow \mathcal{A}$ of \mathcal{A} in Tabuada's model structure ([Tab2], as adapted in [Kel2, Theorem 4.1]) on dg categories. Because k is a field, the inclusion $\mathrm{per}_{\mathrm{dg}}(k) \rightarrow \mathrm{hFDCh}_k$ is a quasi-equivalence, so the composite dg functor $\omega: \tilde{\mathcal{A}} \rightarrow \mathrm{hFDCh}_k$ is homotopy equivalent to a dg functor $\omega': \tilde{\mathcal{A}} \rightarrow \mathrm{per}_{\mathrm{dg}}(k)$.

Definition 1.9. Define the Tannakian dual $C_\omega(\mathcal{A})$ by

$$C_\omega(\mathcal{A}) := \mathbb{C}\mathbb{C}(\mathcal{A}, \omega \otimes \omega^\vee),$$

where the \mathcal{A} -bimodule

$$\omega \otimes \omega^\vee: \mathcal{A} \times \mathcal{A}^{\mathrm{opp}} \rightarrow \mathrm{per}_{\mathrm{dg}}(k)$$

is given by

$$\omega \otimes \omega^\vee(x, y) = (\omega x) \otimes_k (\omega y)^\vee.$$

Similarly, write $NC_\omega(\mathcal{A}) := N\mathbb{C}\mathbb{C}(\mathcal{A}, \omega \otimes \omega^\vee)$.

Proposition 1.10. *The cochain complexes $C_\omega(\mathcal{A}), NC_\omega(\mathcal{A})$ have the natural structure of coassociative counital dg coalgebras over k .*

Proof. We may rewrite

$$\begin{aligned} \underline{\mathbb{C}\mathbb{C}}_n(\mathcal{A}, \omega \otimes \omega^\vee) = \\ \bigoplus_{X_0, \dots, X_n \in \mathrm{Ob} \mathcal{A}} \mathcal{A}(X_0, X_1) \otimes \mathcal{A}(X_1, X_2) \otimes \dots \otimes \mathcal{A}(X_{n-1}, X_n) \otimes \omega X_n \otimes \omega(X_0)^\vee, \end{aligned}$$

as

$$\bigoplus_{X_0, \dots, X_n \in \mathrm{Ob} \mathcal{A}} \omega(X_0)^\vee \otimes \mathcal{A}(X_0, X_1) \otimes \mathcal{A}(X_1, X_2) \otimes \dots \otimes \mathcal{A}(X_{n-1}, X_n) \otimes \omega X_n.$$

There is a comultiplication Δ on the bicomplex

$$\underline{\mathbb{C}\mathbb{C}}_\bullet(\mathcal{A}, \omega \otimes \omega^\vee),$$

with

$$\begin{aligned} \Delta: \underline{\mathbb{C}\mathbb{C}}_{m+n}(\mathcal{A}, \omega \otimes \omega^\vee) \rightarrow \\ \underline{\mathbb{C}\mathbb{C}}_m(\mathcal{A}, \omega \otimes \omega^\vee) \otimes_k \underline{\mathbb{C}\mathbb{C}}_n(\mathcal{A}, \omega \otimes \omega^\vee) \end{aligned}$$

being the map

$$\begin{aligned} (\omega X_0)^\vee \otimes \mathcal{C}(X_0, X_1) \otimes \dots \otimes \mathcal{C}(X_{m+n-1}, X_{m+n}) \otimes (\omega X_{m+n}) \rightarrow \\ [(\omega X_0)^\vee \otimes \mathcal{C}(X_0, X_1) \otimes \dots \otimes \mathcal{C}(X_{m-1}, X_m) \otimes (\omega X_m)] \\ \otimes [(\omega X_m)^\vee \otimes \mathcal{C}(X_m, X_{m+1}) \otimes \dots \otimes \mathcal{C}(X_{m+n-1}, X_{m+n}) \otimes (\omega X_{m+n})] \end{aligned}$$

given by tensoring with

$$\mathrm{id}_X \in (\omega X_m) \otimes (\omega X_m)^\vee.$$

Now,

$$\begin{aligned} & (\partial_i \otimes \mathrm{id}) \circ \Delta_{m+1,n}(x \otimes c_1 \otimes \dots \otimes c_{m+n+1} \otimes y) \\ &= \begin{cases} \Delta_{m,n} \partial_i(x \otimes c_1 \otimes \dots \otimes c_{m+n+1} \otimes y) & i \leq m, \\ x \otimes c_1 \otimes \dots \otimes c_m \otimes (\omega c_{m+1}) \otimes c_{m+2} \otimes \dots \otimes c_{m+n+1} \otimes y & i = m+1; \end{cases} \end{aligned}$$

$$\begin{aligned} & (\mathrm{id} \otimes \partial_i) \circ \Delta_{m,n+1}(x \otimes c_1 \otimes \dots \otimes c_{m+n+1} \otimes y) \\ &= \begin{cases} \Delta_{m,n} \partial_{i+m}(x \otimes c_1 \otimes \dots \otimes c_{m+n} \otimes y) & i > 0, \\ x \otimes c_1 \otimes \dots \otimes c_m \otimes (\omega c_{m+1}) \otimes c_{m+2} \otimes \dots \otimes c_{m+n+1} \otimes y & i = 0. \end{cases} \end{aligned}$$

Thus the differential $d = \sum (-1)^i \partial_i$ has the property that

$$[(d \otimes \mathrm{id} + (-1)^m \mathrm{id} \otimes d) \circ \Delta]_{m,n} = \Delta_{m,n} \circ d.$$

In other words, Δ is a chain map with respect to d , so passes to a comultiplication on $C_\omega(\mathcal{A}) = \mathrm{Tot} \mathbb{C}\mathbb{C}_\bullet(\mathcal{A}, \omega \otimes \omega^\vee)$. The properties of the ∂_i above also ensure that the comultiplication descends to $NC_\omega(\mathcal{A})$. \square

It is immediately clear that the constructions are functorial in the following sense:

Lemma 1.11. *For any k -linear dg functor $F: \mathcal{B} \rightarrow \mathcal{A}$, there is an induced morphism $C_{\omega \circ F}(\mathcal{B}) \rightarrow C_\omega(\mathcal{A})$ of dg coalgebras, which also induces a morphism on the normalisations.*

In §2.4, we will combine this lemma with Theorem 2.9 to show that $C_\omega(\mathcal{A})$ is essentially invariant under quasi-equivalent choices of \mathcal{A} and quasi-isomorphic choices of ω , so that the choice in Remark 1.8 does not affect the output.

1.2. The universal coalgebra and tilting modules.

1.2.1. Background terminology. Following the conventions of [Kel2, 3.1], we will write $\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ for the dg category of k -linear dg functors $\mathcal{A}^{\mathrm{opp}} \rightarrow \mathcal{C}_{\mathrm{dg}}(k)$ to chain complexes over k . Observe that when \mathcal{A} has a single object $*$ with $\mathcal{A}(*, *) = A$, $\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ is equivalent to the category of A -modules in complexes. We write $\mathcal{C}(\mathcal{A})$ for the (non-dg) category $Z^0 \mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ of dg \mathcal{A} -modules.

An object P of $\mathcal{C}(\mathcal{A})$ is cofibrant (for the projective model structure) if every surjective quasi-isomorphism $L \rightarrow P$ has a section. The full dg subcategory of $\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ on cofibrant objects is denoted $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$. This is the idempotent-complete pre-triangulated category (in the sense of [BK3, Definition 3.1]) generated by \mathcal{A} and closed under filtered colimits. We write $\mathcal{D}(\mathcal{A})$ for the derived category $H^0 \mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ of dg \mathcal{A} -modules — this is equivalent to the localisation of $\mathcal{C}(\mathcal{A})$ at quasi-isomorphisms. Thus $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ is a dg enhancement of the triangulated category $\mathcal{D}(\mathcal{A})$.

Definition 1.12. Define $\mathrm{per}_{\mathrm{dg}}(\mathcal{A}) \subset \mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ to be the full subcategory on compact objects, i.e those X for which

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(X, -)$$

preserves filtered colimits. Explicitly, $\mathrm{per}_{\mathrm{dg}}(\mathcal{A})$ consists of objects arising as direct summands of finite complexes of objects of the form $h_X[n]$, for $X \in \mathcal{A}$, where h is the Yoneda embedding.

When \mathcal{A} has a single object $*$ with $\mathcal{A}(*, *) = A$, then $h_*[n]$ corresponds to the A -module $A[n]$. Since projective modules are direct summands of free modules, Definitions 1.12 and 1.4 are thus consistent.

As explained in [Kel2, 4.5], $\text{per}_{\text{dg}}(\mathcal{A})$ is the idempotent-complete pre-triangulated envelope or hull of \mathcal{A} , in the sense of [BK3, §3]. Note that in [Kel1, §2], pre-triangulated categories are called exact DG categories.

By [Tab1, Theorem 5.1], there is a Morita model structure on k -linear dg categories. Weak equivalences are dg functors $\mathcal{A} \rightarrow \mathcal{B}$ which are derived Morita equivalences in the sense that

$$\mathcal{D}_{\text{dg}}(\mathcal{A}) \rightarrow \mathcal{D}_{\text{dg}}(\mathcal{B})$$

is a quasi-equivalence. The dg functor $\mathcal{A} \rightarrow \text{per}_{\text{dg}}(\mathcal{A})$ is fibrant replacement in this model structure.

Note that a dg category \mathcal{A} is an idempotent-complete pre-triangulated category if and only if the natural embedding $\mathcal{A} \rightarrow \text{per}_{\text{dg}}(\mathcal{A})$ is a quasi-equivalence. This is equivalent to saying that \mathcal{A} is Morita fibrant (i.e. fibrant in the Morita model structure), or triangulated in the terminology of [TV, Definition 2.4].

1.2.2. Universal coalgebras.

Definition 1.13. Recall (e.g. from [Toë2, Remark 8.5]) that there is a monoidal structure $\otimes_{\mathcal{A}}$ on the dg category $\mathcal{C}_{\text{dg}}(\mathcal{A}^{\text{opp}} \otimes \mathcal{A})$, given by

$$(F \otimes_{\mathcal{A}} G)(X, Y) = F(X, -) \otimes_{\mathcal{A}} G(-, Y),$$

for $X \in \mathcal{A}, Y \in \mathcal{A}^{\text{opp}}$. The unit of the monoidal structure is the dg functor $\text{id}_{\mathcal{A}}$, given by

$$\text{id}_{\mathcal{A}}(X, Y) = \mathcal{A}(X, Y).$$

Take a k -linear dg category \mathcal{A} , and $D \in \mathcal{D}_{\text{dg}}(\mathcal{A}^{\text{opp}} \otimes \mathcal{A})$ a coassociative $\otimes_{\mathcal{A}}$ -coalgebra, with the co-unit $D \rightarrow \text{id}_{\mathcal{A}}$ a quasi-isomorphism. We regard this as being a universal coalgebra associated to \mathcal{A} .

Example 1.14. If the k -complexes $\mathcal{A}(X, Y)$ are all cofibrant (automatic when k is a field), a canonical choice for D is the Hochschild complex

$$\text{CC}(\mathcal{A}, h_{\mathcal{A}^{\text{opp}}} \otimes h_{\mathcal{A}})$$

of the Yoneda embedding $h_{\mathcal{A}^{\text{opp}}} \otimes h_{\mathcal{A}}: \mathcal{A}^{\text{opp}} \otimes \mathcal{A} \rightarrow \mathcal{C}_{\text{dg}}(\mathcal{A}^{\text{opp}} \otimes \mathcal{A})$. Explicitly, $\text{CC}(\mathcal{A}, h_{\mathcal{A}^{\text{opp}}} \otimes h_{\mathcal{A}})$ is the total complex of the chain complex

$$\bigoplus_{X_0 \in \mathcal{A}} X_0^{\text{opp}} \otimes X_0 \leftarrow \bigoplus_{X_0, X_1 \in \mathcal{A}} X_0^{\text{opp}} \otimes \mathcal{A}(X_0, X_1) \otimes X_1 \leftarrow \dots,$$

where we write X and X^{opp} for the images of X under the Yoneda embeddings $\mathcal{A} \rightarrow \mathcal{C}_{\text{dg}}(\mathcal{A}), \mathcal{A}^{\text{opp}} \rightarrow \mathcal{C}_{\text{dg}}(\mathcal{A}^{\text{opp}})$.

The $\otimes_{\mathcal{A}}$ -coalgebra structure is given by the formulae of Proposition 1.10, noting that

$$Y \otimes_{\mathcal{A}} X^{\text{opp}} = \mathcal{A}(Y, X),$$

so $\text{id}_X \in X \otimes_{\mathcal{A}} X^{\text{opp}}$.

The normalised version of the Hochschild complex $N\text{CC}(\mathcal{A}, h_{\mathcal{A}^{\text{opp}}} \otimes h_{\mathcal{A}})$ provides another choice for D , which is more canonical in some respects.

If we write $L = \bigoplus_{X \in \mathcal{A}} X^{\text{opp}} \otimes_k X$, then L is a $\otimes_{\mathcal{A}}$ -coalgebra in $\mathcal{C}_{\text{dg}}(\mathcal{A}^{\text{opp}} \otimes \mathcal{A})$. The counit is just the composition $\bigoplus_X \mathcal{A}(-, X) \otimes_k \mathcal{A}(X, -) \rightarrow \mathcal{A}(-, -)$, and comultiplication comes from $\text{id}_X \in X \otimes_{\mathcal{A}} X^{\text{opp}}$. Then D is the total complex of the simplicial diagram

given by $\underbrace{L \otimes_{\mathcal{A}} L \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} L}_{n+1}$ in level n , so D is just the Čech nerve of the $\otimes_{\mathcal{A}}$ -comonoid L .

Definition 1.15. Say that a coassociative $\otimes_{\mathcal{A}}$ -coalgebra $D \in \mathcal{D}_{\text{dg}}(\mathcal{A}^{\text{opp}} \otimes \mathcal{A})$ is ind-compact if it can be expressed as a filtered colimit $D \cong \varinjlim_i D_i$ in the underlying category $Z^0 \mathcal{D}_{\text{dg}}(\mathcal{A}^{\text{opp}} \otimes \mathcal{A})$, with each D_i a coassociative $\otimes_{\mathcal{A}}$ -coalgebra which is compact as an object of $\mathcal{D}_{\text{dg}}(\mathcal{A}^{\text{opp}} \otimes \mathcal{A})$.

Note that if \mathcal{A} itself is a field, then the fundamental theorem of coalgebras ([JS, Proposition 7.1]) says that all $\otimes_{\mathcal{A}}$ -coalgebras are filtered colimits (i.e. nested unions) of finite-dimensional coalgebras, so are ind-compact.

Example 1.16. If k is a field, then the $\otimes_{\mathcal{A}}$ -coalgebra $\text{CC}(\mathcal{A}, h_{\mathcal{A}^{\text{opp}}} \otimes h_{\mathcal{A}})$ is ind-compact. We construct the exhaustive system of compact subcoalgebras as follows. The indexing set will consist of triples (S, n, V) with S a finite subset of $\text{Ob} \mathcal{A}$, $n \in \mathbb{N}_0$ and $V(X, Y) \subset \mathcal{A}(X, Y)$ a collection of finite-dimensional cochain complexes for $X, Y \in S$.

For $X', Y' \in S$, we now let $V^{(i)}(X', Y') \subset \mathcal{A}(X', Y')$ be the cochain complex generated by strings of length at most 2^i in elements of V . We now define $D_{(S, n, V)} \subset D$ to be the total complex of

$$\begin{aligned} \bigoplus_{X_0 \in S} X_0^{\text{opp}} \otimes X_0 &\leftarrow \bigoplus_{X_0, X_1 \in S} X_0^{\text{opp}} \otimes V^{(n-1)}(X_0, X_1) \otimes X_1 \\ &\leftarrow \bigoplus_{X_0, X_1, X_2 \in S} X_0^{\text{opp}} \otimes V^{(n-2)}(X_0, X_1) \otimes V^{(n-2)}(X_1, X_2) \otimes X_2 \\ &\leftarrow \dots \leftarrow \bigoplus_{X_0, \dots, X_n \in S} X_0^{\text{opp}} \otimes V^{(0)}(X_0, X_1) \otimes \dots \otimes V^{(0)}(X_{n-1}, X_n) \otimes X_n. \end{aligned}$$

This is indeed a complex because multiplication in \mathcal{A} gives boundary maps $V^{(n-i)}(X, Y) \otimes V^{(n-i)}(Y, Z) \rightarrow V^{(n-i-1)}(X, Z)$, and it is a subcoalgebra because $V^{(n-i-j)} \subset V^{(n-i)} \cap V^{(n-j)}$. The indexing set becomes a poset by saying $(S, m, U) \subset (T, n, V)$ whenever $S \subset T$, $m \leq n$ and $U \subset V$. Thus we have a filtered colimit

$$D = \varinjlim_{(S, n, V)} D_{(S, n, V)},$$

of the required form.

1.2.3. Tilting modules. Given $\omega: \mathcal{A} \rightarrow \text{per}_{\text{dg}}(k)$, define the tilting module P by $P := D \otimes_{\mathcal{A}} \omega \in \mathcal{C}(\mathcal{A}^{\text{opp}})$; this is cofibrant and has a natural quasi-isomorphism $P \rightarrow \omega$. Also set $Q \in \mathcal{C}(\mathcal{A})$ by $Q := \omega^{\vee} \otimes_{\mathcal{A}} D$ and set $C := \omega^{\vee} \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega$. Note that the natural transformation $\text{id}_{\mathcal{A}} \rightarrow \omega \otimes_k \omega^{\vee}$ makes C into a dg coalgebra over k :

$$\begin{aligned} C &= \omega^{\vee} \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega \rightarrow \omega^{\vee} \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega \\ &= \omega^{\vee} \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \text{id}_{\mathcal{A}} \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega \\ &\rightarrow \omega^{\vee} \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega \otimes_k \omega^{\vee} \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega \\ &= C \otimes_k C. \end{aligned}$$

Likewise, P becomes a right C -comodule and Q a left C -comodule.

Also note that because D is a cofibrant replacement for $\text{id}_{\mathcal{A}}$, we have

$$C \simeq \omega^{\vee} \otimes_{\mathcal{A}}^{\mathbf{L}} \text{id}_{\mathcal{A}} \otimes_{\mathcal{A}}^{\mathbf{L}} \omega \simeq \omega^{\vee} \otimes_{\mathcal{A}}^{\mathbf{L}} \omega.$$

For a chosen exhaustive system $D = \varinjlim_i D_i$ of an ind-compact coalgebra, we also write $P_i := D_i \otimes_{\mathcal{A}} \omega$, $C_i := \omega^\vee \otimes_{\mathcal{A}} D_i \otimes_{\mathcal{A}} \omega$ and $Q_i := \omega^\vee \otimes_{\mathcal{A}} D_i$. Each C_i is a dg coalgebra, with P_i (resp. Q_i) a right (resp. left) C_i -comodule.

Example 1.17. When $D = \text{CC}(\mathcal{A}, h_{\mathcal{A}^{\text{opp}}} \otimes h_{\mathcal{A}})$, observe that

$$\begin{aligned} C &= \text{CC}(\mathcal{A}, \omega \otimes \omega), \\ P &= \text{CC}(\mathcal{A}, h_{\mathcal{A}} \otimes \omega), \\ Q &= \text{CC}(\mathcal{A}, \omega^\vee \otimes h_{\mathcal{A}^{\text{opp}}}), \end{aligned}$$

so C is just the dg coalgebra $C_\omega(\mathcal{A})$ of Definition 1.9.

1.2.4. Preduals.

Definition 1.18. Given $M \in \mathcal{C}_{\text{dg}}(\mathcal{A})$, define the predual $M' \in \mathcal{C}_{\text{dg}}(\mathcal{A}^{\text{opp}})$ to be the dg functor $M': \mathcal{A} \rightarrow \mathcal{C}_{\text{dg}}(k)$ given by $M'(Y) = \underline{\text{Hom}}_{\mathcal{C}_{\text{dg}}(\mathcal{A})}(M, Y)$. Note that this construction is only quasi-isomorphism invariant for $M \in \mathcal{D}_{\text{dg}}(\mathcal{A})$.

Observe that for $X \in \mathcal{A}$ and the Yoneda embeddings h , we have $h'_X = h_{X^{\text{opp}}}$, giving isomorphisms $N \otimes_{\mathcal{A}} h_X \cong N(X) \cong \underline{\text{Hom}}_{\mathcal{A}^{\text{opp}}}(h'_X, N)$ for all $N \in \mathcal{C}_{\text{dg}}(\mathcal{A}^{\text{opp}})$. Passing to finite complexes and arbitrary colimits in $\mathcal{C}_{\text{dg}}(\mathcal{A})$, this gives us a natural transformation

$$N \otimes_{\mathcal{A}} M \rightarrow \underline{\text{Hom}}_{\mathcal{A}^{\text{opp}}}(M', N)$$

for all $M \in \mathcal{C}_{\text{dg}}(\mathcal{A})$; this is necessarily an isomorphism when $M \in \text{per}_{\text{dg}}(\mathcal{A})$ because both sides preserve finite complexes and direct summands.

1.3. Monoidal categories. In order to recover the setting of [DMOS, Ch. II], we now introduce monoidal structures. For the purposes of this subsection $(\mathcal{A}, \boxtimes, \mathbb{1})$ is a strictly monoidal dg category, so we have k -linear dg functors $\mathbb{1}: k \rightarrow \mathcal{A}$ and $\boxtimes: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$, such that if we also write $\mathbb{1}$ for the image of the unique object in k ,

$$(X \boxtimes Y) \boxtimes Z = X \boxtimes (Y \boxtimes Z), \quad \mathbb{1} \boxtimes X = X, \quad X \boxtimes \mathbb{1} = X.$$

Definition 1.19. Say that a dg functor $\omega: \mathcal{A} \rightarrow \text{per}_{\text{dg}}(k)$ is lax monoidal if it is equipped with natural transformations

$$\mu_{XY}: \omega(X) \otimes \omega(Y) \rightarrow \omega(X \boxtimes Y), \quad \eta: k \rightarrow \omega(\mathbb{1})$$

satisfying associativity and unitality conditions.

It is said to be strict (resp. strong, resp. quasi-strong) if μ and η are equalities (resp. isomorphisms, resp. quasi-isomorphisms).

Remark 1.20. The hypothesis that \mathcal{A} and ω be strictly monoidal is of course very strong. All the results of this section will be straightforwardly functorial with respect to isomorphisms, though not always with respect to quasi-isomorphisms, so we could replace \mathcal{A} with any equivalent dg category (quasi-equivalent does not suffice). Thus the results will also apply to strongly monoidal dg categories and functors, where the equalities above are replaced by isomorphisms in such a way that $Z^0\mathcal{A}$ becomes a strongly monoidal category and $\omega: Z^0\mathcal{A} \rightarrow Z^0\text{per}_{\text{dg}}(k)$ a strongly monoidal functor. This condition is satisfied by Example 2.22, our main motivating example.

1.3.1. *The Tannakian envelope for strongly monoidal functors.*

Definition 1.21. Given dg functors $F: \mathcal{B} \rightarrow \text{per}_{\text{dg}}(k)$, $G: \mathcal{C} \rightarrow \text{per}_{\text{dg}}(k)$, define the external tensor product

$$F \odot G: \mathcal{B} \otimes \mathcal{C} \rightarrow \text{per}_{\text{dg}}(k)$$

by $(F \odot G)(X \otimes Y) := F(X) \otimes_k G(Y)$.

Lemma 1.22. *For dg categories \mathcal{B}, \mathcal{C} and k -linear dg functors $F: \mathcal{B} \rightarrow \text{per}_{\text{dg}}(k)$, $G: \mathcal{C} \rightarrow \text{per}_{\text{dg}}(k)$, the dg coalgebras of Proposition 1.10 have canonical quasi-isomorphisms*

$$\begin{aligned} C_F(\mathcal{B}) \otimes_k C_G(\mathcal{C}) &\rightarrow C_{F \odot G}(\mathcal{B} \otimes \mathcal{C}), \\ NC_F(\mathcal{B}) \otimes_k NC_G(\mathcal{C}) &\rightarrow NC_{F \odot G}(\mathcal{B} \otimes \mathcal{C}). \end{aligned}$$

These maps are symmetric on interchanging (\mathcal{B}, F) and (\mathcal{C}, G) , and the construction is associative in the sense that it induces a unique map

$$C_F(\mathcal{B}) \otimes_k C_G(\mathcal{C}) \otimes_k C_H(\mathcal{D}) \rightarrow C_{F \odot G \odot H}(\mathcal{B} \otimes \mathcal{C} \otimes \mathcal{D}),$$

and similarly for NC .

Proof. First observe that we have canonical isomorphisms

$$\underline{\text{CC}}_m(\mathcal{B} \otimes \mathcal{C}, (F \odot G)^\vee \otimes (F \odot G)) \cong \underline{\text{CC}}_m(\mathcal{B}, F) \otimes_k \underline{\text{CC}}_m(\mathcal{C}, G)$$

for all m . These isomorphisms are clearly compatible with the comultiplication maps $\Delta: \underline{\text{CC}}_{m+n} \rightarrow \underline{\text{CC}}_m \otimes \underline{\text{CC}}_n$ and with the simplicial operations.

Now the Eilenberg–Zilber shuffle product of [Qui, I.4.2–3] gives a symmetric associative quasi-isomorphism from $C_F(\mathcal{B}) \otimes_k C_G(\mathcal{C})$ to the total complex of the simplicial cochain complex $m \mapsto \underline{\text{CC}}_m(\mathcal{B}, F) \otimes_k \underline{\text{CC}}_m(\mathcal{C}, G)$, which is compatible with the comultiplications. Combined with the isomorphisms above, this gives

$$C_F(\mathcal{B}) \otimes_k C_G(\mathcal{C}) \rightarrow C_{F \odot G}(\mathcal{B} \otimes \mathcal{C}),$$

and similarly on normalisations. \square

Proposition 1.23. *If $\omega: \mathcal{A} \rightarrow \text{per}_{\text{dg}}(k)$ is strongly monoidal, the monoidal structures endow the dg coalgebras $C_\omega(\mathcal{A}), NC_\omega(\mathcal{A})$ with the natural structure of unital dg bialgebras. These are graded-commutative whenever \boxtimes and ω are symmetric.*

Proof. Define a dg functor $\boxtimes_* \omega$ on $\mathcal{A} \otimes \mathcal{A}$ by $(\boxtimes_* \omega)(X \otimes Y) := \omega(X \boxtimes Y)$. We may apply Lemma 1.11 to the dg functor \boxtimes to obtain a morphism

$$C_{\boxtimes_* \omega}(\mathcal{A} \otimes \mathcal{A}) \rightarrow C_\omega(\mathcal{A}).$$

Strong monoidality of ω gives $\boxtimes_* \omega \cong \omega \odot \omega$ and hence $C_{\boxtimes_* \omega}(\mathcal{A} \otimes \mathcal{A}) \cong C_{\omega \odot \omega}(\mathcal{A} \otimes \mathcal{A})$. Lemma 1.22 then provides a dg coalgebra quasi-isomorphism

$$C_\omega(\mathcal{A}) \otimes_k C_\omega(\mathcal{A}) \rightarrow C_{\omega \odot \omega}(\mathcal{A} \otimes \mathcal{A}),$$

which completes the construction of the associative multiplication. This product is moreover commutative whenever \boxtimes and ω are symmetric, and induces a product on $NC_\omega(\mathcal{A})$ similarly.

Applying Lemma 1.11 to the unit $\mathbb{1}$ similarly induces morphisms

$$k = C_{\text{id}}(k) \cong C_{\omega \circ \mathbb{1}}(k) \rightarrow C_\omega(\mathcal{A}),$$

and unitality of ω and $\mathbb{1}$ ensures that this is a unit for the multiplication above. \square

Remark 1.24. In the scenario considered in [DMOS, Ch. II], the tensor category was rigid in the sense that it admitted strong duals, or equivalently internal Homs. Then the Tannaka dual bialgebra $\mathrm{HH}_0(\mathcal{A}, \omega^\vee \otimes_k \omega)$ became a Hopf algebra.

If our dg category \mathcal{A} has strong duals, then we may define an involution ρ on $C_\omega(\mathcal{A})$ by combining the isomorphism $C_\omega(\mathcal{A})^{\mathrm{opp}} \cong C_{\omega^\vee}(\mathcal{A}^{\mathrm{opp}})$ with the map $C_{\omega^\vee}(\mathcal{A}^{\mathrm{opp}}) \rightarrow C_\omega(\mathcal{A})$ induced by applying Lemma 1.11 to the duality functor.

The condition that ρ be an antipode on a bialgebra C is that the diagrams

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \varepsilon \downarrow & & \downarrow (\rho \otimes \mathrm{id}) \\ k & \xrightarrow{1} & C \end{array} \quad \begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \varepsilon \downarrow & & \downarrow (\mathrm{id} \otimes \rho) \\ k & \xrightarrow{1} & C \end{array}$$

commute.

On the bialgebra $\pi_0 \underline{\mathrm{CC}}_\bullet(\mathcal{A}, \omega^\vee \otimes \omega)$, it turns out that ρ defines an antipode, making the bialgebra into a Hopf algebra and recovering the construction of [DMOS, II.2]. However, the dg bialgebras $C_\omega(\mathcal{A}), NC_\omega(\mathcal{A})$ are far from being dg Hopf algebras. This is easily seen by looking at

$$\underline{\mathrm{CC}}_0(\mathcal{A}, \omega^\vee \otimes \omega)^\vee = \prod_{X \in \mathcal{A}} \mathrm{End}(\omega X).$$

The antipodal condition above reduces to saying that for all $f \in \underline{\mathrm{CC}}_0(\mathcal{A}, \omega^\vee \otimes \omega)^\vee$, we require that $\omega(\epsilon_X) \circ f_{X^* \boxtimes X} = f_1 \circ \omega(\epsilon_X)$, for $\epsilon_X: X^* \boxtimes X \rightarrow \mathbb{1}$ the duality transformation. There are few dg categories \mathcal{A} for which this holds, so ρ seldom makes $\underline{\mathrm{CC}}_0(\mathcal{A}, \omega^\vee \otimes \omega)$ into a Hopf algebra. However, the condition above automatically holds for all Hochschild 0-cocycles, which is why $\pi_0 \underline{\mathrm{CC}}_\bullet(\mathcal{A}, \omega^\vee \otimes \omega)$ is a Hopf algebra.

In the sequel [Pri2, §2.4] there appears a context where a variant of the Hochschild complex does have a suitable antipode, and hence the structure of a Hopf algebra.

1.3.2. The Tannakian envelope for lax monoidal functors. If the dg functor ω is only quasi-strong, it is too much to expect that $C_\omega(\mathcal{A})$ will be a bialgebra in general. A dg bialgebra is a monoid in dg coalgebras, and it is not usually possible to strictify algebraic and coalgebraic structures simultaneously, so $C_\omega(\mathcal{A})$ should be a form of strong homotopy monoid in dg coalgebras.

We will now construct the structures enriching $C_\omega(\mathcal{A})$ for any lax monoidal dg functor ω . When ω is quasi-strong, Corollary 2.16 will ensure that this indeed gives a form of strong homotopy monoid.

Definition 1.25. Define I to be the category on two objects $0, 1$ with a unique non-identity morphism $0 \rightarrow 1$. Define K to be the category whose objects are $I, \{0\}, \{1\}$ and whose non-identity morphisms are the inclusions $\{0\}, \{1\} \rightarrow I$. Thus the objects of K are categories in their own right.

As in [Pri1, Definition 1.1], we will write Δ_{**} for the subcategory of the ordinal number category Δ consisting of morphisms which fix the initial and final vertices. This has a monoidal structure given by setting $\mathbf{m} \otimes \mathbf{n} = \mathbf{m} + \mathbf{n}$. As in [Pri1, Definition 1.2], Δ_{**} is opposite to the augmented ordinal number category, so an up-to-homotopy monoid structure on $C \in \mathcal{C}$ in the sense of [Lei] is a colax monoidal functor $M: \Delta_{**} \rightarrow \mathcal{C}$ for which the maps $M(i+j) \rightarrow M(i) \otimes M(j)$ are weak equivalences and $M(\mathbf{1}) = C$.

We now adapt [Pri1, Definition 3.10]:

Definition 1.26. Define $\mathbb{K}: \Delta_{**} \rightarrow \text{Cat}$ to be the category-valued lax monoidal functor given on objects by $\mathbf{0} \mapsto *$, $\mathbf{n} \mapsto K^{n-1}$ and on morphisms by

$$\begin{aligned}\mathbb{K}(\partial^i)(k_1, \dots, k_n) &= (k_1, \dots, k_{i-1}, \{1\}, k_i, \dots, k_n); \\ \sigma^i(k_1, \dots, k_n) &= (k_1, \dots, k_{i-1}, m(k_i, k_{i+1}), k_{i+2}, \dots, k_n); \\ \sigma^0(k_1, \dots, k_n) &= (k_2, \dots, k_n); \\ \sigma^n(k_1, \dots, k_n) &= (k_1, \dots, k_{n-1}),\end{aligned}$$

where $m: K^2 \rightarrow K$ is the symmetric functor determined by $m(\{0\}, k) = \{0\}$, $m(\{1\}, k) = k$, $m(I, I) = I$. The monoidal structure on \mathbb{K} is given by the maps

$$K^{m-1} \times K^{n-1} \cong K^{m-1} \times \{0\} \times K^{n-1} \subset K^{m+n-1}.$$

Proposition 1.27. *If $\omega: \mathcal{A} \rightarrow \text{per}_{\text{dg}}(k)$ is lax monoidal with $\omega(\mathbb{1}) = k$, the monoidal structures give rise to a colax monoidal functor $\mathbb{C}_\omega(\mathcal{A})$ from the opposite $\Gamma(\mathbb{K}^{\text{opp}})^{\text{opp}}$ of the Grothendieck construction $\Gamma(\mathbb{K}^{\text{opp}})$ of \mathbb{K}^{opp} to the category of dg coalgebras over k , with $\mathbb{C}_\omega(\mathcal{A})(\mathbf{1}) = C_\omega(\mathcal{A})$. There is a similar construction for $NC_\omega(\mathcal{A})$.*

Proof. We prove this for $C_\omega(\mathcal{A})$, the proof for $NC_\omega(\mathcal{A})$ being entirely similar. On the category $\mathcal{A}^{\otimes 2}$, ω induces dg fibre functors $\omega \odot \omega$ and $\boxtimes_* \omega$. The transformation between them gives a dg fibre functor $\underline{\omega}$ on $\mathcal{A}^{\otimes 2} \times I$ with $(X, Y, 0) \mapsto \omega(X) \otimes \omega(Y)$ and $(X, Y, 1) \mapsto \omega(X \boxtimes Y)$.

Iterating this construction gives us dg fibre functors $\underline{\omega}: \mathcal{A}^{\otimes n} \times I^{n-1} \rightarrow \text{per}_{\text{dg}}(k)$ for all n . At a vertex of I^{n-1} , the corresponding fibre functor is given by writing ω for each 0 co-ordinate and \boxtimes_* for each 1 co-ordinate, then appending a final ω and introducing \odot s as separators. For $I^{n-1} \in K^{n-1}$, we define

$$\mathbb{C}_\omega(\mathcal{A})(n, I^{n-1}) := C_{\underline{\omega}}(\mathcal{A} \times I^{n-1}).$$

Now, any object $k \in K^{n-1}$ is a subcategory of I^{n-1} , and we may define $\mathbb{C}_\omega(\mathcal{A})(n, k) := C_{\underline{\omega}|_k}(\mathcal{A} \times k)$. Lemma 1.11 then gives morphisms $\mathbb{C}_\omega(\mathcal{A})(n, k) \rightarrow \mathbb{C}_\omega(\mathcal{A})(n, l)$ for each morphism $k \rightarrow l$ in K^{n-1} . This defines $\mathbb{C}_\omega(\mathcal{A})$ on the subcategories $\mathbb{K}(n) \subset \Gamma(\mathbb{K}^{\text{opp}})^{\text{opp}}$, and it remains to define the images of the cosimplicial morphisms ∂^i, σ^i and the monoidal structure.

The fibred dg functor $(\mathcal{A} \otimes \mathcal{A}, \omega \boxtimes \omega) \xrightarrow{\boxtimes} (\mathcal{A}, \omega)$ induces fibred dg functors $(\mathcal{A}^{\otimes n+1} \times \partial^i(k), \underline{\omega}|_{\partial^i(k)}) \rightarrow (\mathcal{A}^{\otimes n} \times k, \underline{\omega}|_k)$ for all i, n and $k \in K^{n-1}$. By Lemma 1.11, this gives a morphism

$$\partial_i: \mathbb{C}_\omega(\mathcal{A})(n+1, \partial^i(k)) \rightarrow \mathbb{C}_\omega(\mathcal{A})(n, k)$$

of dg coalgebras, which we define to be the image of $\partial_i: (\mathbf{n} + \mathbf{1}, \partial^i(k)) \rightarrow (\mathbf{n}, k)$.

Substituting the unit $\mathbb{1} \in \mathcal{A}$ in either factor induces fibred dg functors $(\mathcal{A}, \omega) \rightarrow (\mathcal{A} \otimes \mathcal{A}, \omega \boxtimes \omega)$ and $(\mathcal{A}, \omega) \rightarrow (\mathcal{A} \otimes \mathcal{A}, \omega \odot \omega)$. Similar arguments show that these induce morphisms

$$\sigma_i: \mathbb{C}_\omega(\mathcal{A})(n-1, \sigma^i(k)) \rightarrow \mathbb{C}_\omega(\mathcal{A})(n, k).$$

To define the colax monoidal structure, we need morphisms

$$\mathbb{C}_\omega(\mathcal{A})(m, k) \otimes \mathbb{C}_\omega(\mathcal{A})(n, l) \rightarrow \mathbb{C}_\omega(\mathcal{A})(m+n, k \times \{0\} \times l),$$

but these are just given by Lemma 1.22. \square

Remark 1.28. We can give $\Gamma(\mathbb{K}^{\text{opp}})$ the structure of a relative category by setting a morphism to be a weak equivalence when its image in Δ_{**} is the identity. Since K has a final object, its nerve is contractible, so the projection map $\Gamma(\mathbb{K}^{\text{opp}}) \rightarrow \Delta_{**}$ is a weak equivalence of relative categories in the sense of [BK1].

If ω is quasi-strong, then Corollary 2.16 will imply that \mathbb{C}_ω sends all weak equivalences to derived Morita equivalences. If we let \mathcal{C} be the relative category of dg k -coalgebras with weak equivalences given by derived Morita equivalences, then Proposition 1.27 gives a colax monoidal functor $(\int \mathbb{K}^{\text{opp}})^{\text{opp}} \rightarrow \mathcal{C}$ of relative categories. By contrast, an up-to-homotopy monoid is just a colax monoidal functor $\Delta_{**}^{\text{opp}} \rightarrow \mathcal{C}$ of relative categories. Since $\int \mathbb{K}^{\text{opp}}$ is weakly equivalent to Δ_{**} , Proposition 1.27 can thus be regarded as giving $\mathbb{C}_\omega(\mathcal{A})$ the structure of a monoid up to coherent homotopy whenever ω is quasi-strong.

If the monoidal structure on the pair (\mathcal{A}, ω) is moreover symmetric, then structures above can be adapted by replacing Δ with the category of finite sets, thus incorporating symmetries by not restricting to non-decreasing maps, and giving rise to a homotopy coherent symmetric monoid.

1.3.3. The universal bialgebra. The monoidal structure \boxtimes on \mathcal{A} induces a monoidal structure on \mathcal{A}^{opp} , which we also denote by \boxtimes . There is also a monoidal structure \boxtimes^2 on $\mathcal{A}^{\text{opp}} \otimes \mathcal{A}$, given by $(X \otimes Y) \boxtimes^2 (X' \otimes Y') := (X \boxtimes X') \otimes (Y \boxtimes Y')$.

Definition 1.29. Define the dg functor

$$\boxtimes: \mathcal{D}_{\text{dg}}(\mathcal{A}) \otimes \mathcal{D}_{\text{dg}}(\mathcal{A}) \rightarrow \mathcal{D}_{\text{dg}}(\mathcal{A})$$

as follows. The dg functor $\boxtimes: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ induces a dg functor $\mathcal{D}_{\text{dg}}(\mathcal{A} \otimes \mathcal{A}) \rightarrow \mathcal{D}_{\text{dg}}(\mathcal{A})$, which we compose with the dg functor $\mathcal{D}_{\text{dg}}(\mathcal{A}) \otimes \mathcal{D}_{\text{dg}}(\mathcal{A}) \xrightarrow{\circ} \mathcal{D}_{\text{dg}}(\mathcal{A} \otimes \mathcal{A})$ given by $(M \circ N)(X \otimes Y) := M(X) \otimes_k N(Y)$ for $X, Y \in \mathcal{A}$. Define $\boxtimes: \mathcal{D}_{\text{dg}}(\mathcal{A}^{\text{opp}}) \otimes \mathcal{D}_{\text{dg}}(\mathcal{A}^{\text{opp}}) \rightarrow \mathcal{D}_{\text{dg}}(\mathcal{A}^{\text{opp}})$ and $\boxtimes^2: \mathcal{D}_{\text{dg}}(\mathcal{A}^{\text{opp}} \otimes \mathcal{A}) \otimes \mathcal{D}_{\text{dg}}(\mathcal{A}^{\text{opp}} \otimes \mathcal{A}) \rightarrow \mathcal{D}_{\text{dg}}(\mathcal{A}^{\text{opp}} \otimes \mathcal{A})$ similarly.

In simpler terms, \boxtimes is just given by extending the dg functor on \mathcal{A} to finite complexes, filtered colimits and direct summands.

Definition 1.30. Define $\boxtimes_*: \mathcal{C}_{\text{dg}}(\mathcal{A}) \rightarrow \mathcal{C}_{\text{dg}}(\mathcal{A} \otimes \mathcal{A})$ by setting

$$(\boxtimes_* M)(X \otimes Y) = M(X \boxtimes Y)$$

for $X, Y \in \mathcal{A}$.

Define $\boxtimes_*: \mathcal{C}_{\text{dg}}(\mathcal{A}^{\text{opp}}) \rightarrow \mathcal{C}_{\text{dg}}(\mathcal{A}^{\text{opp}} \otimes \mathcal{A}^{\text{opp}})$ and $\boxtimes_*^2: \mathcal{C}_{\text{dg}}(\mathcal{A}^{\text{opp}} \otimes \mathcal{A}) \rightarrow \mathcal{C}_{\text{dg}}(\mathcal{A}^{\text{opp}} \otimes \mathcal{A} \otimes \mathcal{A}^{\text{opp}} \otimes \mathcal{A})$ similarly.

Remark 1.31. For $S, T \in \mathcal{D}_{\text{dg}}(\mathcal{A})$ and $M \in \mathcal{C}_{\text{dg}}(\mathcal{A})$, note that we have a natural isomorphism

$$\underline{\text{Hom}}_{\mathcal{C}_{\text{dg}}(\mathcal{A})}(S \boxtimes T, M) \cong \underline{\text{Hom}}_{\mathcal{C}_{\text{dg}}(\mathcal{A} \otimes \mathcal{A})}(S \otimes_k T, \boxtimes_* M).$$

This isomorphism is tautological when $S = h_X, T = h_Y$ for $X, Y \in \mathcal{A}$, noting that

$$h_{X \otimes Y}(U \otimes V) = \mathcal{A}(X, U) \otimes_k \mathcal{A}(Y, V) = h_X \otimes_k h_Y.$$

The general case follows by passing to complexes and direct summands.

The same observation holds for any monoidal dg category, and hence to $(\mathcal{A}^{\text{opp}}, \boxtimes)$ and $(\mathcal{A}^{\text{opp}} \otimes \mathcal{A}, \boxtimes^2)$.

Lemma 1.32. *The unit $\text{id}_{\mathcal{A}} \in \mathcal{C}(\mathcal{A} \otimes \mathcal{A}^{\text{opp}})$ is equipped with a canonical associative multiplication*

$$\text{id}_{\mathcal{A}} \otimes_k \text{id}_{\mathcal{A}} \rightarrow \boxtimes_*^2 \text{id}_{\mathcal{A}},$$

which is commutative whenever \boxtimes is symmetric. The unit for this multiplication is

$$\text{id}_{\mathbb{1}} \in \mathcal{A}(\mathbb{1}, \mathbb{1}) = \text{id}_{\mathcal{A}}(\mathbb{1}, \mathbb{1}) = \mathbb{1}_*^2 k,$$

for $\mathbb{1}_^2: \mathcal{C}_{\text{dg}}(k) \rightarrow \mathcal{C}_{\text{dg}}(\mathcal{A}^{\text{opp}} \otimes \mathcal{A})$.*

Proof. Evaluated at $X \otimes Y \otimes X' \otimes Y' \in \mathcal{A} \otimes \mathcal{A}^{\text{opp}} \otimes \mathcal{A} \otimes \mathcal{A}^{\text{opp}}$, this is just the map

$$\mathcal{A}(X, Y) \otimes_k \mathcal{A}(X', Y') \rightarrow \mathcal{A}(X \boxtimes X', Y \boxtimes Y')$$

induced by the bilinearity of \boxtimes . \square

Definition 1.33. We say that a universal coalgebra D (in the sense of §1.2.2) is a universal bialgebra with respect to \boxtimes if it is equipped with an associative multiplication $D \otimes_k D \rightarrow \boxtimes_*^2 D$ and unit $k \rightarrow D(\mathbb{1}, \mathbb{1})$. These are required to be compatible with the coalgebra structure, in the sense that the comultiplication and co-unit

$$D \rightarrow D \otimes_{\mathcal{A}} D \quad D \rightarrow \text{id}_{\mathcal{A}}$$

must be morphisms of associative unital \boxtimes^2 -algebras.

When \boxtimes is symmetric, we say that D is a universal commutative bialgebra if the multiplication $D \otimes_k D \rightarrow \boxtimes_*^2 D$ is commutative.

Remark 1.34. Since universal coalgebras are required to be objects of $\mathcal{D}_{\text{dg}}(\mathcal{A}^{\text{opp}} \otimes \mathcal{A})$, we may apply Remark 1.31 to rephrase the algebra structure on D to be an associative multiplication $D \boxtimes^2 D \rightarrow D$ and a unit $\mathbb{1} \otimes \mathbb{1} \rightarrow D$.

Example 1.35. Under the conditions of Example 1.16 (e.g. when k is a field), the Hochschild complexes

$$\text{CC}(\mathcal{A}, h_{\mathcal{A}^{\text{opp}}} \otimes h_{\mathcal{A}}) \quad \text{NCC}(\mathcal{A}, h_{\mathcal{A}^{\text{opp}}} \otimes h_{\mathcal{A}})$$

associated to the Yoneda embedding $h_{\mathcal{A}^{\text{opp}}} \otimes h_{\mathcal{A}}: \mathcal{A}^{\text{opp}} \otimes \mathcal{A} \rightarrow \mathcal{C}_{\text{dg}}(\mathcal{A}^{\text{opp}} \otimes \mathcal{A})$ are universal bialgebras, commutative whenever \boxtimes is symmetric.

The coalgebra structure is given in Example 1.16, and the multiplication and unit are given by the formulae of Proposition 1.23.

Lemma 1.36. *Given a universal bialgebra D and a strong monoidal dg functor ω , the dg coalgebra $C := \omega^{\vee} \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega$ becomes a unital associative dg bialgebra, which is commutative whenever D is commutative and ω symmetric.*

Proof. Since ω, ω^{\vee} are strong monoidal dg functors, we have an isomorphism

$$\omega^{\vee} \otimes_{\mathcal{A}} (D \boxtimes^2 D) \otimes_{\mathcal{A}} \omega \cong (\omega^{\vee} \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega) \otimes_k (\omega^{\vee} \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega),$$

so the multiplication $D \boxtimes^2 D \rightarrow D$ gives $C \otimes_k C \rightarrow C$. Likewise,

$$\omega^{\vee} \otimes_{\mathcal{A}} (\mathbb{1} \otimes \mathbb{1}) \otimes_{\mathcal{A}} \omega = \omega^{\vee}(\mathbb{1}) \otimes_k \omega(\mathbb{1}) \cong k,$$

so the unit gives $k \rightarrow C$. Compatibility of the algebra and coalgebra structures follows from the corresponding results for D . \square

Remark 1.37. When \mathcal{A} is a neutral Tannakian category, taking duals gives an equivalence $\mathcal{A}^{\text{opp}} \simeq \mathcal{A}$. Then $\text{id}_{\mathcal{A}} \in \mathcal{C}(\mathcal{A}^{\text{opp}} \otimes \mathcal{A})$ corresponds to the ring of functions on Deligne's fundamental groupoid $G(\mathcal{A}) \in \mathcal{C}(\mathcal{A} \otimes \mathcal{A})^{\text{opp}}$ from [Del1, 6.13]. Since $\text{id}_{\mathcal{A}} = H^0(D)$, we thus think of D as being the ring of functions on the path space of \mathcal{A} .

1.3.4. Tilting modules.

Lemma 1.38. *Given a universal bialgebra D and a strong monoidal dg functor ω , the tilting module $P := D \otimes_{\mathcal{A}} \omega$ becomes a monoid in $\mathcal{D}_{\text{dg}}(\mathcal{A}^{\text{opp}})$ with respect to \boxtimes , which is commutative whenever D is commutative and ω symmetric.*

Moreover, the co-action $P \rightarrow P \otimes_k C$ of §1.2.3 is an algebra morphism in the sense that the diagram

$$\begin{array}{ccc} P \boxtimes P & \longrightarrow & (P \otimes_k C) \boxtimes (P \otimes_k C) \quad \equiv \quad (P \boxtimes P) \otimes_k (C \otimes_k C) \\ \downarrow & & \downarrow \\ P & \longrightarrow & (P \otimes_k C) \end{array}$$

commutes, where the horizontal maps are co-action and the vertical maps are multiplication.

Proof. Since ω is a strong monoidal dg functor, we have an isomorphism

$$(D \boxtimes^2 D) \otimes_{\mathcal{A}} \omega \cong (D \otimes_{\mathcal{A}} \omega) \boxtimes (D \otimes_{\mathcal{A}} \omega),$$

which gives the required multiplication, with existence of the unit coming from the isomorphism $(\mathbb{1} \otimes \mathbb{1}) \otimes_{\mathcal{A}} \omega \cong \mathbb{1} \otimes_k \omega(\mathbb{1}) \cong \mathbb{1}$.

The final statement follows from compatibility of the algebra and coalgebra structures for D . \square

2. COMODULES

From now on, k will be a field. Throughout this section, we will fix a small k -linear dg category \mathcal{A} , a k -linear dg functor $\omega: \mathcal{A} \rightarrow \text{per}_{\text{dg}}(k)$, and a universal coalgebra $D \in \mathcal{D}_{\text{dg}}(\mathcal{A}^{\text{opp}} \otimes \mathcal{A})$ in the sense of §1.2.2. We write $C := \omega^{\vee} \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega$ and $P := D \otimes_{\mathcal{A}} \omega$ for the associated dg coalgebra and tilting module.

2.1. The Quillen adjunction.

2.1.1. Model structure on dg comodules.

Definition 2.1. Let $\mathcal{C}_{\text{dg}}(C)$ be the dg category of right C -comodules in cochain complexes over k . Write $\mathcal{C}(C)$ for the underlying category $Z^0 \mathcal{C}_{\text{dg}}(C)$ of right C -comodules in cochain complexes, and $\mathcal{D}(C)$ for the homotopy category given by formally inverting quasi-isomorphisms.

Proposition 2.2. *There is a closed model structure on $\mathcal{C}(C)$ in which weak equivalences are quasi-isomorphisms and cofibrations are injections. Fibrations are surjections with kernel K such that*

- (1) *the graded module $K^{\#}$ underlying K is injective as a comodule over the graded coalgebra $C^{\#}$ underlying C , and*
- (2) *for all acyclic N , $\underline{\text{Hom}}_C(N, K)$ is acyclic.*

Proof. This is described in [Pos, Remark 8.2], as the model structure “of the first kind”. For ease of reference, we summarise the arguments here.

As in [Pos, Theorem 8.1], the lifting properties follow from the statement that $\underline{\text{Hom}}_C(E, I) \simeq 0$ whenever I is fibrant and either E or I is acyclic. For E , this is tautologous. For I , note that the identity morphism in $\underline{\text{Hom}}_C(I, I)$ is then a coboundary, so we have a contracting homotopy h with $[d, h] = \text{id}$, implying that $\underline{\text{Hom}}_C(E, I) \simeq 0$ for all E .

To establish factorisation, we first observe that we can embed any comodule M into a quasi-isomorphic $C^{\#}$ -injective comodule using a bar resolution

$$\bigoplus_{n \geq 0} M \otimes C^{\otimes n+1}[-n].$$

Fibrant replacement then follows from a triangulated argument, [Pos, Lemma 1.3]. The key step is given in [Pos, Lemma 5.5], where Brown representability gives a right adjoint to the functor from the coderived category to the derived category. \square

We then write $\mathcal{D}_{\text{dg}}(C)$ for the full dg subcategory of $\mathcal{C}_{\text{dg}}(C)$ on fibrant objects.

Remark 2.3. We might sometimes wish to consider multiple dg fibre functors. Given a set $\{\omega_x\}_{x \in X}$ of dg fibre functors, we can consider the coalgebroid C on objects X given by $C(x, y) := \omega_x^\vee \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega_y$, with comultiplication $C(x, y) \rightarrow C(x, z) \otimes_k C(z, y)$ and counit $C(x, x) \rightarrow k$ defined by the usual formulae.

There is also a category $\mathcal{C}(C)$ of right C -comodules in cochain complexes, with such a comodule M consisting of cochain complexes $M(x)$ for each $x \in X$, together with a distributive action $M(y) \rightarrow M(x) \otimes C(x, y)$. The proof of Proposition 2.2 then adapts to give a closed model structure on the category $\mathcal{C}(C)$, noting that bar resolutions

$$y \mapsto \bigoplus_{x \in X^{n+1}} M(x_0) \otimes C(x_0, x_1) \otimes \dots \otimes C(x_{n-1}, x_n) \otimes C(x_n, y)[-n]$$

still exist in this setting.

Definition 2.4. Given a left C -comodule M and a right C -comodule N , set the cotensor product $N \otimes^C M$ to be kernel of the map

$$(\mu_N \otimes_k \text{id}_M - \text{id}_N \otimes_k \mu_M): N \otimes_k M \rightarrow N \otimes_k C \otimes_k M,$$

where μ denotes the C -coaction. Note that this is denoted by $N \square_C M$ in [Pos, 2.1].

2.1.2. *The Quillen adjunction.*

Lemma 2.5. *The adjunction*

$$\mathcal{C}(\mathcal{A}) \begin{array}{c} \xrightarrow{-\otimes_{\mathcal{A}} P} \\ \perp \\ \xleftarrow{\text{Hom}_C(P, -)} \end{array} \mathcal{C}(C)$$

is a Quillen adjunction.

Proof. It suffices to show that $-\otimes_{\mathcal{A}} P$ sends (trivial) generating cofibrations to (trivial) cofibrations. Generating cofibrations are of the form $X \otimes_k U \rightarrow X \otimes_k V$ for $X \in \mathcal{A}$ and $U \hookrightarrow V$ finite-dimensional cochain complexes. Now, $(X \otimes_k U) \otimes_{\mathcal{A}} P = U \otimes_k P(X)$, and $\otimes_k P(X)$ preserves both injections and quasi-isomorphisms. \square

Definition 2.6. Denote the co-unit of the Quillen adjunction by

$$\varepsilon_N: \underline{\text{Hom}}_C(P, N) \otimes_{\mathcal{A}} P \rightarrow N.$$

2.1.3. *The retraction.* From now on, we assume that our chosen $\otimes_{\mathcal{A}}$ -coalgebra D is ind-compact.

Proposition 2.7. *The counit*

$$\varepsilon_N: \mathbf{R}\underline{\text{Hom}}_C(P, N) \otimes_{\mathcal{A}} P \rightarrow N$$

of the derived adjunction $(-\otimes_{\mathcal{A}} P) \dashv \mathbf{R}\underline{\text{Hom}}_C(P, -)$ is an isomorphism in the derived category $\mathcal{D}(C)$ for all N .

Proof. For any C -comodule N , we have the following isomorphisms

$$\begin{aligned} \underline{\mathrm{Hom}}_C(P, N) \otimes_{\mathcal{A}} P &= \underline{\mathrm{Hom}}_C(P, N) \otimes_{\mathcal{A}} (\varinjlim_i P_i) \\ &\cong \varinjlim_i (\underline{\mathrm{Hom}}_C(P, N) \otimes_{\mathcal{A}} P_i) \\ &\cong \varinjlim_i \underline{\mathrm{Hom}}_{\mathcal{A}}(P'_i, \underline{\mathrm{Hom}}_C(P, N)) \\ &\cong \varinjlim_i \underline{\mathrm{Hom}}_C(P'_i \otimes_{\mathcal{A}} P, N), \end{aligned}$$

where P'_i is the predual of P_i , as in Definition 1.18.

The co-unit ε_N induces C -comodule morphisms

$$\underline{\mathrm{Hom}}_C(P'_i \otimes_{\mathcal{A}} P, N) = \underline{\mathrm{Hom}}_C(P, N) \otimes_{\mathcal{A}} P_i \rightarrow N$$

for all i , and hence C_i -comodule morphisms

$$\underline{\mathrm{Hom}}_C(P'_i \otimes_{\mathcal{A}} P, N) \rightarrow N \otimes^C C_i.$$

Now, since C_i is finite-dimensional, we have $N \otimes^C C_i \cong \underline{\mathrm{Hom}}_C(C_i^\vee, N)$, so ε induces C -bicomodule morphisms

$$\alpha_i: C_i^\vee \rightarrow P'_i \otimes_{\mathcal{A}} P,$$

compatible with the transition maps $C_i \rightarrow C_j$, $P_i \rightarrow P_j$.

As P'_i is cofibrant, the quasi-isomorphism $P \rightarrow \omega$ induces a quasi-isomorphism $\beta_i: P'_i \otimes_{\mathcal{A}} P \rightarrow P'_i \otimes_{\mathcal{A}} \omega = C_i^\vee$ of cochain complexes. Now, α_i is equivalent to the coaction map $P_i \rightarrow P_i \otimes_k C_i \rightarrow P \otimes_k C_i$, so $\beta_i \circ \alpha_i$ is equivalent to the coaction map $P_i \rightarrow \omega \otimes_k C_i$. This is equivalent to the isomorphism $P'_i \otimes_{\mathcal{A}} \omega = C_i^\vee$, so $\beta_i \circ \alpha_i$ is the identity.

Therefore the α_i are all quasi-isomorphisms, so for N fibrant, the map

$$\underline{\mathrm{Hom}}_C(P, N) \otimes_{\mathcal{A}} P_i \rightarrow N \otimes^C C_i$$

is a quasi-isomorphism. Since filtered colimits commute with finite limits, this gives a quasi-isomorphism

$$\begin{aligned} \underline{\mathrm{Hom}}_C(P, N) \otimes_{\mathcal{A}} P &= \varinjlim_i \underline{\mathrm{Hom}}_C(P, N) \otimes_{\mathcal{A}} P_i \\ &\rightarrow \varinjlim_i N \otimes^C C_i \\ &\cong N \otimes^C (\varinjlim_i C_i) \\ &= N \otimes^C C = N. \end{aligned}$$

□

2.2. Tannakian comparison. We now show how for our chosen ind-compact universal coalgebra $D \in \mathcal{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}} \otimes \mathcal{A})$ and dg functor $\omega: \mathcal{A} \rightarrow \mathrm{per}_{\mathrm{dg}}(k)$, the tilting module $P = D \otimes_{\mathcal{A}} \omega$ can give rise to a comparison between the derived category of \mathcal{A} -modules and the derived category of comodules of $C = \omega^\vee \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega$. This is analogous to derived Morita theory (comparing two derived categories of modules) or Morita–Takeuchi theory (comparing two derived categories of comodules).

Definition 2.8. Write $\ker \omega$ for the full dg subcategory of $\mathcal{D}_{\text{dg}}(\mathcal{A})$ consisting of objects X with $\omega(X) := X \otimes_{\mathcal{A}} \omega$ quasi-isomorphic to 0.

Recall from [Dri, §12.6] that the right orthogonal complement $(\ker \omega)^{\perp} \subset \mathcal{D}_{\text{dg}}(\mathcal{A})$ is the full dg subcategory consisting of those X for which $\underline{\text{Hom}}_{\mathcal{A}^{\text{opp}}}(M, X) \simeq 0$ for all $M \in \ker \omega$.

The following theorem contrasts strongly with [CLM], which shows that if we work with coderived categories of comodules instead of derived categories, and take a specific model for C , then we would not need to take the dg quotient by $\ker \omega$. As we will see, there are many applications in which the quotient is more interesting than the original dg category.

Theorem 2.9. *For the constructions of $C \simeq \omega^{\vee} \otimes_{\mathcal{A}}^{\mathbf{L}} \omega$ and the tilting module P above, the derived adjunction $(- \otimes_{\mathcal{A}} P) \dashv \mathbf{R}\underline{\text{Hom}}_C(P, -)$ gives rise to a quasi-equivalence between the dg categories $(\ker \omega)^{\perp}$ and $\mathcal{D}_{\text{dg}}(C)$. Moreover, the map $(\ker \omega)^{\perp} \rightarrow \mathcal{D}_{\text{dg}}(\mathcal{A})/(\ker \omega)$ to the dg quotient is a quasi-equivalence.*

Proof. Functorial cofibrant and fibrant replacement give us composite dg functors $U: \mathcal{D}_{\text{dg}}(C) \xrightarrow{\underline{\text{Hom}}_C(P, -)} \mathcal{C}_{\text{dg}}(\mathcal{A}) \rightarrow \mathcal{D}_{\text{dg}}(\mathcal{A})$ and $F: \mathcal{D}_{\text{dg}}(\mathcal{A}) \xrightarrow{\otimes_{\mathcal{A}} P} \mathcal{C}_{\text{dg}}(C) \rightarrow \mathcal{D}_{\text{dg}}(C)$, and these will yield the quasi-equivalence.

First observe that for $K \in \ker \omega$, we have quasi-isomorphisms

$$K \otimes_{\mathcal{A}} P \simeq \omega(K) \simeq 0$$

of cochain complexes, since P is a resolution of ω and K is cofibrant.

For any $N \in \mathcal{D}_{\text{dg}}(C)$, Proposition 2.7 gives that the counit

$$\varepsilon_N: \underline{\text{Hom}}_C(P, N) \otimes_{\mathcal{A}} P \rightarrow N$$

of the adjunction $(- \otimes_{\mathcal{A}} P) \dashv \underline{\text{Hom}}_C(P, -)$ is a quasi-isomorphism. Thus for any $K \in \ker \omega$, we have

$$\mathbf{R}\underline{\text{Hom}}_{\mathcal{A}}(K, \mathbf{R}\underline{\text{Hom}}_C(P, N)) \simeq \mathbf{R}\underline{\text{Hom}}_C(K \otimes_{\mathcal{A}} P, N) \simeq \mathbf{R}\underline{\text{Hom}}_C(0, N) \simeq 0,$$

so UN (the cofibrant replacement of $\mathbf{R}\underline{\text{Hom}}_C(P, N)$) lies in $(\ker \omega)^{\perp}$.

Thus F provides a retraction of $(\ker \omega)^{\perp}$ onto $\mathcal{D}_{\text{dg}}(C)$, and in particular $U: \mathcal{D}_{\text{dg}}(C) \rightarrow (\ker \omega)^{\perp}$ is a full and faithful dg functor.

For any $M \in \mathcal{D}_{\text{dg}}(\mathcal{A})$, we now consider the unit

$$\eta_M: M \rightarrow \mathbf{R}\underline{\text{Hom}}_C(P, M \otimes_{\mathcal{A}} P) = \underline{\text{Hom}}_C(P, FM)$$

of the adjunction. On applying $\otimes_{\mathcal{A}} P$, this becomes a quasi-isomorphism, with quasi-inverse $\varepsilon_{M \otimes_{\mathcal{A}} P}$, so $\omega \otimes_{\mathcal{A}}^{\mathbf{L}} (\eta_M)$ is a quasi-isomorphism. Since M is cofibrant, the map η_M lifts to a map

$$\tilde{\eta}_M: M \rightarrow UFM$$

of cofibrant objects, with

$$\text{cone}(\tilde{\eta}_M) \in \ker \omega, \quad \text{i.e. } F\text{cone}(\tilde{\eta}_M) \simeq 0.$$

The dg subcategory $\ker \omega$ is thus right admissible in the sense of [Dri, §12.6], because we have the morphism

$$M \xrightarrow{\tilde{\eta}_M} UFM$$

for all $M \in \mathcal{D}_{\text{dg}}(\mathcal{A})$, with $UFM \in (\ker \omega)^{\perp}$ and $\text{cocone}(\tilde{\eta}_M) \in \ker \omega$.

In particular, this implies that if $M \in (\ker \omega)^\perp$, the map $\tilde{\eta}_M: M \rightarrow UFM$ is a quasi-isomorphism, so $U: \mathcal{D}_{\text{dg}}(C) \rightarrow (\ker \omega)^\perp$ is essentially surjective and hence a quasi-equivalence.

As observed in [Dri, §12.6], the results of [BK2, §1] and [Ver, §I.2.6] show that right admissibility is equivalent to saying that $(\ker \omega)^\perp \rightarrow \mathcal{D}_{\text{dg}}(\mathcal{A})/(\ker \omega)$ is an equivalence. \square

Remark 2.10. Note that Theorem 2.9 implies that for any choices D, D' of ind-compact $\otimes_{\mathcal{A}}$ -coalgebra resolution of $\text{id}_{\mathcal{A}}$, the associated coalgebras C, C' are derived Morita equivalent. Given a quasi-isomorphism $D \rightarrow D'$, we then have a derived Morita equivalence $C \rightarrow C'$, which is *a fortiori* a quasi-isomorphism.

It might therefore seem curious that $D \rightarrow \text{id}_{\mathcal{A}}$ is only required to be a quasi-isomorphism. However, any quasi-isomorphism to the trivial coalgebra $\text{id}_{\mathcal{A}}$ is automatically a Morita equivalence. The reason for this is that fibrant replacement in the category of D -comodules is given by the coaction $M \rightarrow M \otimes D$, so the forgetful dg functor from D -comodules to $\text{id}_{\mathcal{A}}$ -comodules is a quasi-equivalence.

Remark 2.11. In [Ayo3], Ayoub establishes a weak Tannaka duality result for any monoidal functor $f: \mathcal{M} \rightarrow \mathcal{E}$ of monoidal categories equipped with a (non-monoidal) right adjoint g . He sets $H := fg(\mathbb{1})$, shows (Theorem 1.21) that H has the natural structure of a biunital bialgebra, and then (Propositions 1.28 and 1.55) proves that f factors through the category of H -comodules, and that H is universal with this property.

We may compare this with our setting by taking $\mathcal{M} = \mathcal{D}(\mathcal{A})$ and $\mathcal{E} = \mathcal{D}(k)$, the derived categories of \mathcal{A} and k . In this case, Ayoub's formula for the coalgebra underlying H is defined provided \mathcal{A} and f are k -linear, without requiring that $\mathcal{D}(\mathcal{A})$ be monoidal.

We can take f to be $\otimes_{\mathcal{A}}^{\mathbf{L}} \omega$, which has right adjoint $\mathbf{R}\underline{\text{Hom}}_k(\omega, -)$ (with the same reasoning as Lemma 2.5). Thus

$$H \simeq \mathbf{R}\underline{\text{Hom}}_k(\omega, k) \otimes_{\mathcal{A}}^{\mathbf{L}} \omega = \omega^\vee \otimes_{\mathcal{A}}^{\mathbf{L}} \omega,$$

which is the image $[C] \in \mathcal{D}(k)$ of our dg coalgebra $C \in \mathcal{C}_{\text{dg}}(k)$.

One reason our duality results in Theorem 2.9 give a comparison rather than just universality is that we use the dg category of C -comodules in $\mathcal{C}_{\text{dg}}(k)$. Instead, [Ayo3, Proposition 1.55] just looks at C -comodules in the derived category $\mathcal{D}(k)$ — in other words, (weak) homotopy comodules without higher coherence data. Likewise, his bialgebra H is only defined as a (weak) homotopy bialgebra.

To recover Ayoub's weak universality from Theorem 2.9 in this setting, first observe that there is a forgetful functor from $\mathcal{D}(C)$ to the category $\text{CoMod}([C])$ of $[C]$ -comodules in $\mathcal{D}(k)$. The equivalence $\mathcal{D}(\mathcal{A})/(\ker \omega) \simeq \mathcal{D}(C)$ then ensures that $\omega: \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{D}(k)$ factors through $\text{CoMod}([C])$. Likewise, if $\omega: \text{per}(\mathcal{A}) \rightarrow \mathcal{D}(k)$ factored through $\text{CoMod}(B)$ for some other coalgebra $B \in \mathcal{D}(k)$, Theorem 2.9 would give an exact functor $\mathcal{D}(C) \rightarrow \text{CoMod}(B)$ fibred over $\mathcal{D}(k)$. The image of C would be a B -comodule structure on $[C] \in \mathcal{D}(k)$, compatible with the coalgebra structure of $[C]$ via the image of the comultiplication $C \otimes C \rightarrow C$, giving a morphism $[C] \rightarrow B$ of coalgebras in $\mathcal{D}(k)$.

In [Iwa, Theorem 4.14], Iwanari effectively gives a refinement of Ayoub's Tannaka duality. Starting from a monoidal ∞ -functor of stable ∞ -categories (a generalisation of dg categories), he constructs a derived affine group scheme G (thus incorporating higher coherence data), and shows that its ∞ -category $\text{Rep}(G)$ of representations has a universal property, but without the characterisation $\text{Rep}(G) \simeq (\ker \omega)^\perp$ of Theorem

2.9 above — this characterisation will be essential in our comparisons of categories of motives in Remark 2.17, and in Example 2.22 and [Pri2, Example 2.20].

2.3. Tannakian equivalence. When ω is faithful, we now have statements about the idempotent-complete pre-triangulated envelope $\text{per}_{\text{dg}}(\mathcal{A})$ of \mathcal{A} , and its closure $\mathcal{D}_{\text{dg}}(\mathcal{A}) = \text{ind}(\text{per}_{\text{dg}}(\mathcal{A}))$ under filtered colimits:

Corollary 2.12. *Assume that $\omega: \mathcal{D}_{\text{dg}}(\mathcal{A}) \rightarrow \mathcal{C}_{\text{dg}}(k)$ is faithful in the sense that $\ker \omega$ is the category of acyclic \mathcal{A} -modules, and take the tilting module P and dg coalgebra $C \simeq \omega^\vee \otimes_{\mathcal{A}}^{\mathbf{L}} \omega$ as above. Then the derived adjunction $(- \otimes_{\mathcal{A}} P) \dashv \mathbf{R}\underline{\text{Hom}}_C(P, -)$ of Theorem 2.9 gives rise to a quasi-equivalence between the dg categories $\mathcal{D}_{\text{dg}}(\mathcal{A})$ and $\mathcal{D}_{\text{dg}}(C)$.*

Moreover, the idempotent-complete pre-triangulated category $\text{per}_{\text{dg}}(\mathcal{A})$ generated by \mathcal{A} is quasi-equivalent to the full dg subcategory of $\mathcal{D}_{\text{dg}}(C)_{\text{cpt}} \subset \mathcal{D}_{\text{dg}}(C)$ on objects which are compact in $\mathcal{D}(C)$. If \mathcal{A} is Morita fibrant, this gives a quasi-equivalence $\mathcal{A} \simeq \mathcal{D}_{\text{dg}}(C)_{\text{cpt}}$.

Proof. Since $\ker \omega$ consists only of acyclic modules, $(\ker \omega)^\perp = \mathcal{D}_{\text{dg}}(\mathcal{A})$, and we apply Theorem 2.9. For the second part, note that the quasi-equivalence $- \otimes_{\mathcal{A}} P: \mathcal{D}_{\text{dg}}(\mathcal{A}) \rightarrow \mathcal{D}_{\text{dg}}(C)$ preserves filtered colimits, so $- \otimes_{\mathcal{A}} P: \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{D}(C)$ preserves and reflects compact objects. Since $\text{H}^0 \text{per}_{\text{dg}}(\mathcal{A}) \subset \mathcal{D}(\mathcal{A})$ is the full subcategory on compact objects, the same must be true of its image in $\mathcal{D}(C)$. Finally, if \mathcal{A} is Morita fibrant, then $\mathcal{A} \rightarrow \text{per}_{\text{dg}}(\mathcal{A})$ is a quasi-equivalence. \square

Example 2.13. Let $A = k[\epsilon]$ with $\epsilon^2 = 0$ (the dual numbers), and let ω be the A -module $k = k[\epsilon]/\epsilon$. This is faithful because for any cofibrant complex M of A -modules, we have $\omega(M) = M/\epsilon M$, and a short exact sequence

$$0 \rightarrow \omega(M) \xrightarrow{\epsilon} M \rightarrow \omega(M) \rightarrow 0.$$

A model for the cofibrant dg \otimes_A -coalgebra D in A -bimodules is given by

$$D^{-n} = A \otimes (k\xi_n) \otimes A$$

for $n \geq 0$, with comultiplication given by

$$\Delta(a \otimes \xi_n \otimes b) = \sum_{i+j=n} a \otimes \xi_i \otimes 1 \otimes \xi_j \otimes b \in A \otimes (k\xi_i) \otimes A \otimes (k\xi_j) \otimes A,$$

and counit $a \otimes \xi_0 \otimes b \mapsto ab \in A$. The differential is determined by

$$d(1 \otimes \xi_1 \otimes 1) = (\epsilon \otimes \xi_0 \otimes 1) - (1 \otimes \xi_0 \otimes \epsilon).$$

We therefore get $C := k \otimes_A D \otimes_A k = k\langle \xi \rangle$, the free dg coalgebra on generator $\xi = \xi_1$ in degree -1 , with $d\xi = 0$. The tilting module P is given by $A\langle \xi \rangle$, with left multiplication by A , right comultiplication by C , and $d\xi = \epsilon$.

Thus the dg category of cofibrant A -modules is equivalent to the dg category of fibrant $k\langle \xi \rangle$ -comodules. Contrast this with [Kel1, Example 2.5], which uses the tilting module P to give a derived Morita equivalence between the category of all finitely generated A -modules and the dg category of perfect $k\langle \xi \rangle^\vee$ -modules.

Remark 2.14. If we have a finite set $\{\omega_x: \mathcal{A} \rightarrow \text{per}_{\text{dg}}(k)\}_{x \in X}$ of dg fibre functors, we can form a dg coalgebroid on objects X by $C(x, y) = \omega_x \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega_y$, and then Theorem 2.9 adapts to give an equivalence between dg C -comodules and $(\bigcap_{x \in X} \ker \omega_x)^\perp$, using Remark 2.3. When the ω_x are jointly faithful, Corollary 2.12 will thus adapt to give a quasi-equivalence between $\mathcal{D}_{\text{dg}}(\mathcal{A})$ and $\mathcal{D}_{\text{dg}}(C)$.

Beware that if we had infinitely many dg fibre functors, the proof of Theorem 2.9 would no longer adapt, because the expression $N \otimes^C C_i$ in the proof of Proposition 2.7 would then be an infinite limit.

This also raises the question of a generalisation to faithful dg fibre functors $\omega: \mathcal{A} \rightarrow \mathcal{C}$ to more general categories. The obvious level of generality would replace $\text{per}_{\text{dg}}(k)$ with some rigid tensor category \mathcal{C} over k . In order to proceed further, we would need an extension of Theorem 2.9 to deal with \mathcal{C} -coalgebras. In particular, generalisations would be required of the relevant model structures on comodules in [Pos, 8.2].

2.4. Homotopy invariance.

Corollary 2.15. *Given a k -linear dg functor $\omega: \mathcal{A} \rightarrow \text{per}_{\text{dg}}(k)$ and a k -linear quasi-equivalence $F: \mathcal{B} \rightarrow \mathcal{A}$, the morphism*

$$C_{(\omega \circ F)}(\mathcal{B}) \rightarrow C_\omega(\mathcal{A})$$

of dg coalgebras induced by F is a derived Morita equivalence, so a fortiori a quasi-isomorphism.

Proof. By Theorem 2.9, the dg functor $\mathcal{D}_{\text{dg}}(C_{(\omega \circ F)}(\mathcal{B})) \rightarrow \mathcal{D}_{\text{dg}}(C_\omega(\mathcal{A}))$ is quasi-equivalent to $\mathcal{D}_{\text{dg}}(\mathcal{B})/(\ker(\omega \circ F)) \rightarrow \mathcal{D}_{\text{dg}}(\mathcal{A})/(\ker \omega)$, which is a quasi-equivalence because F is so. Therefore we have a derived Morita equivalence of dg coalgebras. \square

Corollary 2.16. *Given a natural quasi-isomorphism $\eta: \omega \rightarrow \omega'$ of k -linear dg functors $\omega, \omega': \mathcal{A} \rightarrow \text{per}_{\text{dg}}(k)$, there is a span of morphisms between $C_\omega(\mathcal{A})$ and $C_{\omega'}(\mathcal{A})$ which are derived Morita equivalences.*

Proof. If we let I be the category with objects $0, 1$ and a unique non-identity morphism $\partial: 0 \rightarrow 1$, then η defines a k -linear dg functor

$$\underline{\eta}: \mathcal{A} \times I \rightarrow \text{per}_{\text{dg}}(k)$$

determined by $\underline{\eta}|_{\mathcal{A} \times 0} = \omega$, $\underline{\eta}|_{\mathcal{A} \times 1} = \omega'$ and $\underline{\eta}(\partial: X \times 0 \rightarrow X \times 1) = \eta_X: \omega(X) \rightarrow \omega'(X)$.

Lemma 1.11 combined with the functors $0, 1 \rightarrow I$ then gives us morphisms

$$C_\omega(\mathcal{A}) \rightarrow C_{\underline{\eta}}(\mathcal{A} \times I) \leftarrow C_{\omega'}(\mathcal{A})$$

of dg coalgebras; we need to show these are derived Morita equivalences.

By Theorem 2.9, this is equivalent to showing that the functors

$$\mathcal{D}_{\text{dg}}(\mathcal{A})/(\ker \omega) \xrightarrow{0^*} \mathcal{D}_{\text{dg}}(\mathcal{A} \times I)/(\ker \underline{\eta}) \xleftarrow{1^*} \mathcal{D}_{\text{dg}}(\mathcal{A})/(\ker \omega')$$

are quasi-equivalences.

For all $X \in \mathcal{A}$, the cone c_X of $h_{(X,0)} \rightarrow h_{(X,1)}$ lies in $\ker \underline{\eta}$ because η is a quasi-isomorphism. For $M \in (c_X)^\perp$, this implies that the maps $M(X, 1) \rightarrow M(X, 0)$ are all quasi-isomorphisms. For $M, N \in \mathcal{D}_{\text{dg}}(\mathcal{A} \times I)$, the complex $\underline{\text{Hom}}_{\mathcal{A} \times I}(M, N)$ is quasi-isomorphic to the cocone of

$$\underline{\text{Hom}}_{\mathcal{A}}(M(0), N(0)) \times \underline{\text{Hom}}_{\mathcal{A}}(M(1), N(1)) \rightarrow \underline{\text{Hom}}_{\mathcal{A}}(M(1), N(0)),$$

so the dg functors

$$0^*, 1^*: \mathcal{D}_{\text{dg}}(\mathcal{A}) \rightarrow \mathcal{D}_{\text{dg}}(\mathcal{A} \times I)/(\{c_X : X \in \mathcal{A}\})$$

are quasi-equivalences, as is their retraction 0_* given by $M \mapsto M(0)$.

We now just observe (by checking on representables $h_{(X,i)}$) that there is a natural quasi-isomorphism from $\underline{\eta}$ to the dg functor $M \mapsto \text{cone}(\omega M(1) \rightarrow \omega M(0) \oplus \omega' M(1))$.

Since $\omega \rightarrow \omega'$ is a quasi-isomorphism, this is quasi-isomorphic to the dg functor $\omega \circ 0_*$, so $\ker \underline{\eta} = \ker(\omega \circ 0_*)$. \square

In particular, this implies that the choice in Remark 1.8 does not affect the output, and that the constructions of §1.3.2 associate strong homotopy monoids to quasi-strong monoidal functors.

2.5. Example: motives. Our main motivating example comes from the derived category of motives.

As explained in [Ayo2, §3], there is a projective model structure on the category \mathcal{M} of symmetric T -spectra in presheaves of k -linear complexes on the category Sm/S of smooth S -schemes. By [Ayo1, Definitions 4.3.6 et 4.5.18], this has a left Bousfield localisation $\mathcal{M}_{\mathbb{A}^1}$, the projective $(\mathbb{A}^1, \text{ét})$ -local model structure, whose homotopy category is Voevodsky's triangulated category of motives over S whenever S is normal. These model categories are defined in terms of cochain complexes, so have the natural structure of dg model categories.

Write $\mathcal{M}_{\text{dg}}, \mathcal{M}_{\text{dg}, \mathbb{A}^1}$ for the full dg subcategories on fibrant cofibrant objects — this ensures that $\text{Ho}(\mathcal{M}) \simeq \text{H}^0 \mathcal{M}_{\text{dg}}$ and similarly for $\mathcal{M}_{\mathbb{A}^1}$. Take $\mathcal{M}_{\text{dg}, c}, \mathcal{M}_{\text{dg}, \mathbb{A}^1, c}$ to be the full subcategories of $\mathcal{M}_{\text{dg}}, \mathcal{M}_{\text{dg}, \mathbb{A}^1}$ on homotopically compact objects. For similar constructions along these lines, see [BV].

By [CD1, Proposition 2.1.6], any choice of k -linear stable cohomology theory over S gives a commutative ring object \mathcal{E} in \mathcal{M} , with the property that

$$E := \underline{\text{Hom}}_{\mathcal{M}}(-, \mathcal{E}) : \mathcal{M}_{\text{dg}}^{\text{opp}} \rightarrow \mathcal{D}_{\text{dg}}(k)$$

represents the cohomology theory. In particular, the functor is stable and \mathbb{A}^1 -invariant, so by [CD1, Theorem 1], E gives a dg functor from $\mathcal{M}_{\text{dg}, \mathbb{A}^1, c}^{\text{opp}}$ to cohomologically finite complexes.

Remark 1.8 allows us to replace this with a dg functor

$$\tilde{E} : \tilde{\mathcal{M}}_{\text{dg}, \mathbb{A}^1, c}^{\text{opp}} \rightarrow \text{per}_{\text{dg}}(k)$$

for some cofibrant replacement $\tilde{\mathcal{M}}_{\text{dg}, \mathbb{A}^1, c}$ of $\mathcal{M}_{\text{dg}, \mathbb{A}^1, c}$, and we can then form the dg coalgebra $C := C_{\tilde{E}^\vee}(\tilde{\mathcal{M}}_{\text{dg}, \mathbb{A}^1, c})$. By Corollary 2.16, this construction is essentially independent of the choice \tilde{E} of replacement.

Theorem 2.9 then gives a quasi-equivalence

$$\mathcal{D}_{\text{dg}}(C) \simeq \mathcal{M}_{\text{dg}, \mathbb{A}^1} / (\ker E)$$

between the dg category of C -comodules and the dg enhancement of the triangulated category of motives modulo homologically acyclic motives. If E' is the composition of E with the derived localisation dg functor $\mathcal{M}_{\text{dg}, c} \rightarrow \tilde{\mathcal{M}}_{\text{dg}, \mathbb{A}^1, c}$, and $C' := C_{\tilde{E}'^\vee}(\tilde{\mathcal{M}}_{\text{dg}, c})$ this also gives

$$\mathcal{D}_{\text{dg}}(C) \simeq \mathcal{M}_{\text{dg}} / (\ker E') \simeq \mathcal{D}_{\text{dg}}(C').$$

One consequence of the existence of a motivic t -structure over S would be that $\mathcal{M}_{\text{dg}, \mathbb{A}^1, c}$ lies in the right orthogonal complement $(\ker E)^\perp$, in which case $\mathcal{M}_{\text{dg}, \mathbb{A}^1, c}$ would be quasi-equivalent to a full dg subcategory of $\mathcal{D}_{\text{dg}}(C)$.

These constructions can all be varied by replacing \mathcal{M} with the category \mathcal{M}^{eff} of presheaves of k -linear complexes on Sm/S with its projective model structure. This has a left Bousfield localisation $\mathcal{M}_{\mathbb{A}^1}$, by [Ayo1, Definition 4.4.33], and the homotopy category of $\mathcal{M}_{\mathbb{A}^1}^{\text{eff}}$ is Voevodsky's triangulated category of effective motives when S is

normal, as in [Ayo2, Appendix B]. There are natural dg enhancements, and we denote their restrictions to fibrant cofibrant objects by $\mathcal{M}_{\mathrm{dg}}^{\mathrm{eff}}, \mathcal{M}_{\mathrm{dg}, \mathbb{A}^1}^{\mathrm{eff}}$.

Composing E with the suspension functor $\Sigma^\infty: \mathcal{M}_{\mathrm{dg}, \mathbb{A}^1}^{\mathrm{eff}} \rightarrow \mathcal{M}_{\mathrm{dg}, \mathbb{A}^1}$ defines a dg functor $E: (\mathcal{M}_{\mathrm{dg}, \mathbb{A}^1}^{\mathrm{eff}})^{\mathrm{opp}} \rightarrow \mathcal{D}_{\mathrm{dg}}(k)$, which is simply given by $\underline{\mathrm{Hom}}_{\mathcal{M}^{\mathrm{eff}}}(-, \mathcal{E}(0))$. We now get a dg coalgebra $C^{\mathrm{eff}} := C_{\tilde{E}^\vee}(\tilde{\mathcal{M}}_{\mathrm{dg}, \mathbb{A}^1, c}^{\mathrm{eff}})$ with

$$\mathcal{D}_{\mathrm{dg}}(C^{\mathrm{eff}}) \simeq \mathcal{M}_{\mathrm{dg}, \mathbb{A}^1}^{\mathrm{eff}} / (\ker E) \simeq \mathcal{M}_{\mathrm{dg}}^{\mathrm{eff}} / (\ker E').$$

A set of compact generators of $\mathcal{M}_{\mathrm{dg}}^{\mathrm{eff}}$ is given by the set of presheaves $k(X)$ for smooth S -varieties X . If \mathcal{A} is the full subcategory on these generators and E' has finite-dimensional values on \mathcal{A} , then we get a Morita equivalence between C and $C_{E'}(\mathcal{A})$. Note that the latter is just given by the total complex of

$$n \mapsto \bigoplus_{X_0, \dots, X_n} E'(X_0) \otimes k(X_0(X_1) \times X_1(X_2) \times \dots \times X_{n-1}(X_n)) \otimes E'(X_n)^\vee,$$

where we write $X(Y) = \mathrm{Hom}_S(Y, X)$, with the dg coalgebra structure of Proposition 1.10.

Example 2.17 (Ayoub and Nori's motivic Galois groups). We now compare this with Ayoub's construction of a motivic Galois group from [Ayo3, §2]. For a number field F , he applies his Tannaka duality construction to the Betti realisation functor

$$\mathrm{Ho}(E)^\vee: \mathrm{H}^0 \mathcal{M}_{\mathrm{dg}, \mathbb{A}^1}(F, \mathbb{Q}) \rightarrow \mathcal{D}(\mathbb{Q})$$

associated to an embedding $\sigma: F \rightarrow \mathbb{C}$, giving a Hopf algebra $\mathcal{H}_{\mathrm{mot}}(F, \sigma) \in \mathcal{D}(\mathbb{Q})$. Replacing $\mathcal{M}_{\mathrm{dg}, \mathbb{A}^1}(F, \mathbb{Q})$ with $\mathcal{M}_{\mathrm{dg}, \mathbb{A}^1}^{\mathrm{eff}}(F, \mathbb{Q})$ gives a bialgebra $\mathcal{H}_{\mathrm{mot}}^{\mathrm{eff}}(F, \sigma) \in \mathcal{D}(\mathbb{Q})$.

From Remark 2.11, it follows that $\mathcal{H}_{\mathrm{mot}}(F, \sigma)$ and $\mathcal{H}_{\mathrm{mot}}^{\mathrm{eff}}(F, \sigma)$ are just the homotopy classes of our dg coalgebras C, C^{eff} above, equipped with their natural multiplications (and in the former case, antipode) in the homotopy category coming from the rigid monoidal structure of $\mathrm{Ho}(E)$.

A variant of the construction above is given by considering generators of $\mathcal{M}_{\mathrm{dg}}^{\mathrm{eff}}(\mathbb{Q})$ given by $\mathrm{cone}(\mathbb{Q}(Y) \rightarrow \mathbb{Q}(X))[i]$, for Nori's good pairs (X, Y, i) as in [HMS, Definition 1.1]. These have the property that their Betti realisations are cohomologically concentrated in degree 0. Writing $\mathcal{A}_{\mathrm{Nori}}$ for the full dg subcategory on these generators, we have $C^{\mathrm{eff}} \simeq C_{E'}(\mathcal{A}_{\mathrm{Nori}})$.

We also have $C_{E'}(\mathbb{Z}^0 \mathcal{A}_{\mathrm{Nori}}) \simeq C_{\mathrm{H}^0 E'}(\mathbb{Z}^0 \mathcal{A}_{\mathrm{Nori}})$ and $\mathrm{H}^0 C_{\mathrm{H}^0 E'}(\mathbb{Z}^0 \mathcal{A}_{\mathrm{Nori}}) \cong \mathrm{H}^0 C_{\mathrm{H}^0 E'}(\mathrm{H}^0 \mathcal{A}_{\mathrm{Nori}})$, and (by [JS, Theorem 7.3]) comodules of the latter are precisely Nori's abelian category $\mathcal{MM}_{\mathrm{Nori}}^{\mathrm{eff}}$ of effective mixed motives as in [HMS, Definition 1.3], since the diagram D^{eff} of good pairs generates $\mathbb{Z}^0 \mathcal{A}$. Then $\mathrm{Spec} \mathrm{H}^0 C_{E'}(\mathbb{Z}^0 \mathcal{A}_{\mathrm{Nori}})$ is a pro-algebraic monoid whose group of units is Nori's Galois group G_{Nori} . The inclusion $\mathrm{H}^0 C_{E'}(\mathbb{Z}^0 \mathcal{A}_{\mathrm{Nori}}) \hookrightarrow \mathrm{H}^0 C^{\mathrm{eff}}$ thus induces a surjection $\mathrm{Spec} \mathrm{H}^0 C \twoheadrightarrow G_{\mathrm{Nori}}$.

Contrast this with [Iwa, Remark 5.20], highlighting that a comparison between Nori's and Voevodsky's motives is beyond the reach of Iwanari's Tannakian formulation.

We now introduce alternative simplifications of the dg coalgebra in special cases.

Remark 2.18. When S is the spectrum of a field, the comparison of [Ayo2, Appendix B] combines with the results of [VSF] or [CD2, Corollary 4.4.3] to show that a set of generators of $\mathcal{M}_{\mathrm{dg}, \mathbb{A}^1}$ is given by the motives of the form $M_k(X)(r)$ for X smooth and projective over S , and $r \in \mathbb{Z}$. Thus the set of motives of the form $M_{k,r}(X) :=$

$M_k(X)(r)[2r]$ is another generating set. For X of dimension d over S , the dual of $M_{k,r}(X)$ is $M_{k,d-r}(X)$, so this set of generators is closed under duals, and

$$\mathcal{M}_{\mathrm{dg},\mathbb{A}^1}(M_{k,r}(X), M_{k,s}(Y)) \simeq \mathcal{M}_{\mathrm{dg},\mathbb{A}^1}(M_k(S), M_{k,d+s-r}(X \times_S Y)).$$

When S is the spectrum of a perfect field, this implies that

$$\mathrm{H}^i \mathcal{M}_{\mathbb{A}^1}(M_{k,r}(X), M_{k,s}(Y)) \cong \mathrm{CH}^{d+s-r}(X \times_S Y, -i) \otimes_{\mathbb{Z}} k,$$

so these generators have no positive Ext groups between them, and we can replace the full dg category on these generators with its good truncation \mathcal{B} in non-positive degrees, given by

$$\mathcal{B}(M_{k,r}(X), M_{k,s}(Y)) := \tau^{\leq 0} \mathcal{M}_{\mathrm{dg},\mathbb{A}^1}(M_{k,r}(X), M_{k,s}(Y));$$

we then have $\mathcal{D}_{\mathrm{dg}}(\mathcal{B}) \simeq \mathcal{M}_{\mathrm{dg},\mathbb{A}^1}$.

The mixed Weil cohomology theory E when restricted to \mathcal{B} thus admits a good truncation filtration, whose associated graded is quasi-isomorphic to

$$\mathrm{H}^* E: \mathrm{H}^0 \mathcal{B}^{\mathrm{opp}} \rightarrow \mathcal{C}_{\mathrm{dg}}(k);$$

think of this as a formal Weil cohomology theory. Note that this is a strong monoidal functor determined by the Chern character $\mathrm{CH}^s(Y) \rightarrow \mathrm{H}^{2s}(Y, E(s))$.

Since $\mathrm{H}^* E$ is finite-dimensional, we can then form the dg coalgebra $C_{\mathrm{H}^* E}(\mathcal{B})$, without needing to take a cofibrant replacement of \mathcal{B} . Explicitly, this is given by the total complex of

$$\begin{aligned} n \mapsto & \bigoplus_{X_0, \dots, X_n, r_0, \dots, r_n} \mathrm{H}^{*+2r_0}(X_0, E(r_0)) \otimes_k \mathcal{B}(M_{k,r_0}(X_0), M_{k,r_1}(X_1)) \otimes_k \dots \\ & \dots \otimes_k \mathcal{B}(M_{k,r_{n-1}}(X_{n-1}), M_{k,r_n}(X_n)) \otimes_k \mathrm{H}^{*+2r_n}(X_n, E(r_n))^\vee, \end{aligned}$$

with the dg coalgebra structure of Proposition 1.10. We then have

$$\mathcal{D}_{\mathrm{dg}}(C_{\mathrm{H}^* E}(\mathcal{B})) \simeq \mathcal{D}_{\mathrm{dg}}(\mathcal{B}) / \ker \mathrm{H}^* E \simeq \mathcal{D}_{\mathrm{dg}}(\mathcal{B}) / \ker E \simeq \mathcal{M}_{\mathrm{dg},\mathbb{A}^1} / \ker E.$$

Remark 2.19. If we write $\mathcal{Z}(X, \bullet)$ for the k -linearisation of Bloch's cycle complex as in [Blo, Proposition 1.3], then we can follow [Han] in defining $\mathcal{Z}(X \times Y, \bullet) \hat{\otimes} \mathcal{Z}(Y \times Z, \bullet) \subset \mathcal{Z}(X \times Y, \bullet) \otimes_k \mathcal{Z}(Y \times Z, \bullet)$ to be the quasi-isomorphic subcomplex of cycles intersecting transversely. We then have a bicomplex

$$\begin{aligned} n \mapsto & \bigoplus_{X_0, \dots, X_n, r_0, \dots, r_n} \mathrm{H}^{*+2r_0}(X_0, E(r_0)) \otimes_k \mathcal{Z}^{d_0+r_1-r_0}(X_0 \times X_1, -*) \hat{\otimes} \dots \\ & \dots \hat{\otimes} \mathcal{Z}^{d_{n-1}+r_n-r_{n-1}}(X_{n-1} \times X_n, -*) \otimes_k \mathrm{H}^{*+2r_n}(X_n, E(r_n))^\vee, \end{aligned}$$

for X_i of dimension d_i , and with differentials as in Definition 1.9.

The formulae of Proposition 1.10 make this into a dg coalgebra, which should be Morita equivalent to the dg coalgebra $C_{\mathrm{H}^* E}(\mathcal{B})$ of Remark 2.18.

2.6. Monoidal comparisons. We now consider the case where (\mathcal{A}, \boxtimes) is a monoidal dg category and $\omega: \mathcal{A} \rightarrow \mathrm{per}_{\mathrm{dg}}(k)$ a strong monoidal dg functor. We also assume that D is a universal bialgebra in the sense of §1.3.3.

Note that since C is a dg bialgebra by Lemma 1.36, the dg category $\mathcal{C}_{\mathrm{dg}}(C)$ has a monoidal structure \otimes_k , where the coaction on $N \otimes N'$ is the composition

$$N \otimes N' \rightarrow (N \otimes C) \otimes (N' \otimes C) \cong (N \otimes N') \otimes (C \otimes C) \rightarrow (N \otimes N') \otimes C$$

of the co-actions with the multiplication on C .

Lemma 2.20. *For $M, M' \in \mathcal{D}_{\text{dg}}(\mathcal{A})$ and $N, N' \in \mathcal{D}_{\text{dg}}(\mathcal{A}^{\text{opp}})$, there is a natural transformation*

$$(M \otimes_{\mathcal{A}} N) \otimes_k (M' \otimes_{\mathcal{A}} N') \rightarrow (M \boxtimes M') \otimes_{\mathcal{A}} (N \boxtimes N').$$

Proof. When $M, M' = h_X, h_Y$ and $N, N' = h_Y, h'_Y$ for $X, X', Y, Y' \in \mathcal{A}$, this is just the map

$$\mathcal{A}(X, Y) \otimes_k \mathcal{A}(X', Y') \rightarrow \mathcal{A}(X \boxtimes X', Y \boxtimes Y')$$

given by the bilinearity of \boxtimes . This extends uniquely to complexes and direct summands. \square

Proposition 2.21. *The dg functor $(- \otimes_{\mathcal{A}} P): \mathcal{D}_{\text{dg}}(\mathcal{A}) \rightarrow \mathcal{C}_{\text{dg}}(C)$ is lax monoidal, with the transformations*

$$(M \otimes_{\mathcal{A}} P) \otimes_k (M' \otimes_{\mathcal{A}} P) \rightarrow (M \boxtimes M') \otimes_{\mathcal{A}} P$$

being quasi-isomorphisms.

Proof. Lemma 2.20 gives the required transformations

$$(M \otimes_{\mathcal{A}} P) \otimes_k (M' \otimes_{\mathcal{A}} P) \rightarrow (M \boxtimes M') \otimes_{\mathcal{A}} (P \boxtimes P) \rightarrow (M \boxtimes M') \otimes_{\mathcal{A}} P.$$

The quasi-isomorphism $P \rightarrow \omega$ then maps these transformations quasi-isomorphically to

$$(M \otimes_{\mathcal{A}} \omega) \otimes_k (M' \otimes_{\mathcal{A}} \omega) \rightarrow (M \boxtimes M') \otimes_{\mathcal{A}} \omega,$$

i.e.

$$\omega(M) \otimes_k \omega(M') \rightarrow \omega(M \boxtimes M'),$$

which is an isomorphism because ω is required to be a strong monoidal dg functor. \square

Example 2.22 (Motives). The model category \mathcal{M}^{eff} of k -linear presheaves from §2.5 is monoidal, as is its localisation $\mathcal{M}_{\mathbb{A}^1}^{\text{eff}}$. However, the tensor product does not preserve fibrant objects, so the dg categories $\mathcal{M}_{\text{dg}}^{\text{eff}}, \mathcal{M}_{\text{dg}, \mathbb{A}^1}^{\text{eff}}$ of fibrant cofibrant objects are not monoidal. [At best, they are multicategories (a.k.a. coloured operads), with $\underline{\text{Hom}}_{\mathcal{M}_{\text{dg}}^{\text{eff}}}(X_1, \dots, X_n; Y) := \underline{\text{Hom}}_{\mathcal{M}^{\text{eff}}}(X_1 \otimes \dots \otimes X_n; Y)$.]

However, if we take the dg category $\mathcal{M}_{\text{dg}}^{\text{eff}'}$ of cofibrant objects in \mathcal{M} , with $\mathcal{M}_{\text{dg}, c}^{\text{eff}'}$ the full subcategory of compact objects, then $\mathcal{M}_{\text{dg}}^{\text{eff}'}$ and $\mathcal{M}_{\text{dg}, c}^{\text{eff}'}$ are monoidal dg categories. The dg categories $\mathcal{M}_{\text{dg}}^{\text{eff}}, \mathcal{M}_{\text{dg}, c}^{\text{eff}}$ are quasi-equivalent to the dg quotients of $\mathcal{M}_{\text{dg}}^{\text{eff}'}, \mathcal{M}_{\text{dg}, c}^{\text{eff}'}$ by the class of weak equivalences in \mathcal{M}^{eff} . By [CD1, Theorem 1], a mixed Weil cohomology theory then gives rise to a contravariant monoidal dg functor E from $\mathcal{M}_{\text{dg}}^{\text{eff}'}$ to cohomologically finite complexes, by setting $E := \underline{\text{Hom}}_{\mathcal{M}^{\text{eff}}}(-, \mathcal{E})$ for the associated presheaf \mathcal{E} of DGAs.

Since symmetric monoidal dg categories do not form a model category, we cannot then mimic the construction of §2.5 and replace E with a monoidal dg functor from a cofibrant replacement of $\mathcal{M}_{\text{dg}}^{\text{eff}'}$ to finite-dimensional complexes. However, we can apply Proposition 2.21 if we can find a Weil cohomology theory taking values in finite-dimensional complexes.

Objects of $\mathcal{M}_{\text{dg}}^{\text{eff}'}$ are formal k -linear complexes of smooth varieties over S . When S is a field admitting resolution of singularities, we can instead consider the model category \mathcal{N}^{eff} of presheaves on the category of pairs $j: U \rightarrow X$, where X is smooth and projective

over S , with U the complement of a normal crossings divisor. Then for any mixed Weil cohomology theory E , there is an associated formal theory

$$E_f(j: U \rightarrow X) := \left(\bigoplus_{a,b} H^a(X, \mathbf{R}^b j_* \mathcal{E}_U, d_2), \right)$$

where d_2 is the differential on the second page of the Leray spectral sequence. Alternatively, this can be rewritten (as in [Del2, 3.2.4]) in terms of $H^a(\tilde{D}^b, E(-b))$ and Gysin maps, where \tilde{D}^n consists of local disjoint unions of n -fold intersections in D .

The constructions of §2.5 all adapt from \mathcal{M} to \mathcal{N} , and the restriction of E_f to $\mathcal{N}_{\text{dg},c}^{\text{eff}'}$ takes values in finite-dimensional complexes, so we have a dg bialgebra $C := C_{E_f}(\mathcal{N}_{\text{dg},c}^{\text{eff}'})$, and Proposition 2.21 gives a monoidal dg functor

$$\mathcal{N}_{\text{dg}}^{\text{eff}'} \rightarrow \mathcal{C}_{\text{dg}}(C),$$

inducing an equivalence $\mathcal{N}_{\text{dg}}^{\text{eff}'}/\ker E \simeq \mathcal{D}_{\text{dg}}(C)$. With some work (see Appendix A.1.3) it follows that $\mathcal{N}_{\text{dg}}^{\text{eff}'}/\ker E \simeq \mathcal{M}_{\text{dg}}^{\text{eff}'}/\ker E$ for all known Weil cohomology theories, and (for Betti cohomology) that E_f is quasi-isomorphic to E .

Corollaries 2.15 and 2.16 then ensure that C is essentially equivalent to the dg coalgebra of §2.5, so the constructions above give a strong compatibility result for the comparisons of §2.5 with respect to the monoidal structures.

By Proposition 1.23, the multiplication on C comes from the fibred dg functor $\boxtimes: (\mathcal{N}^{\text{eff}'} \otimes \mathcal{N}^{\text{eff}'}, \omega \otimes \omega) \rightarrow (\mathcal{N}^{\text{eff}'}, \omega)$. Applying [Ayo3, Corollary 1.14] to these fibred dg categories gives bialgebra structures on the coalgebras $[C \otimes C], [C] \in \mathcal{D}(k)$. By Ayoub's weak universal property [Ayo3, Proposition 1.55], the functors \boxtimes and $\mathbb{1}$ on derived categories induce morphisms $[C \otimes C] \rightarrow [C]$ and $[k] \rightarrow [C]$ of commutative bialgebras; the relations between these morphisms force them to be Ayoub's multiplication and unit maps. Functoriality of the comparison of universal constructions in Remark 2.11 then ensures that this weak bialgebra structure must come from our dg bialgebra structure.

APPENDIX A. FORMAL WEIL COHOMOLOGY THEORIES

A.1. Quasi-projective pairs and localisation.

A.1.1. Quasi-projective pairs.

Definition A.1. Given a field F admitting resolution of singularities, we let SmQP/F be the category of pairs $j: U \rightarrow X$, where X is smooth and projective over F , with U the complement of a normal crossings divisor. We say that a morphism $(U, X) \rightarrow (U', X')$ in SmQP/F is an equivalence (or in \mathcal{E}) if it induces an isomorphism $U \rightarrow U'$.

Lemma A.2. *The pairs $(\text{SmQP}/F, \mathcal{E})$ and $(\mathcal{E}, \mathcal{E})$ admit right calculi of fractions in the sense of [DK1, §7].*

Proof. We begin with the case $(\text{SmQP}/F, \mathcal{E})$, noting that \mathcal{E} contains all identities and is closed under composition. For any diagram $(V, Y) \xrightarrow{f} (U, X) \xleftarrow{a} (U, X')$ in SmQP/F

(so a is an equivalence), we first need to find a commutative diagram

$$\begin{array}{ccc} (V, Y') & \xrightarrow{f'} & (U, X') \\ b \downarrow & & \downarrow a \\ (V, Y) & \xrightarrow{f} & (U, X), \end{array}$$

with b an equivalence. To do this, we first form the fibre product $X' \times_X Y$, and observe that the isomorphism $V = U \times_U V$ gives us a map $V \hookrightarrow X' \times_X Y$. Taking Y' to be a resolution of singularities of the closure of V in $X' \times_X Y$ gives the required diagram.

Secondly, we need to show that if any parallel arrows $f, g: (V, Y) \rightarrow (U, X')$ in SmQP/F satisfy $af = ag$ for some equivalence $a: (U, X') \rightarrow (U, X)$, then there exists an equivalence $b: (V, Y') \rightarrow (V, Y)$ with $fb = gb$. The condition $af = ag$ implies that the maps $f, g: V \rightarrow U$ are equal. There is therefore a diagonal map

$$V \hookrightarrow Y \times_{f, X', g} Y.$$

Taking Y' to be a resolution of singularities of the closure of V in $Y \times_{f, X', g} Y$ then gives the construction required. Thus $(\text{SmQP}/F, \mathcal{E})$ admits a right calculus of fractions.

Finally, note that \mathcal{E} satisfies the two out of three property, so as observed in [DK1, 7.1], it follows that $(\mathcal{E}, \mathcal{E})$ admits a right calculus of fractions. \square

A.1.2. Localisation and DG quotients.

Definition A.3. Given a category \mathcal{C} and a subcategory \mathcal{W} , we follow [DK1, §7] in writing $\mathcal{C}[\mathcal{W}^{-1}]$ for the localised category given by formally inverting all morphisms in \mathcal{W} .

Definition A.4. Given a category \mathcal{C} and a subcategory \mathcal{W} , and an object $Y \in \mathcal{C}$, we write

$$\mathcal{C}\mathcal{W}^{-1}(X, Y)$$

for the category whose objects are spans

$$Y \xleftarrow{u} Y' \xrightarrow{f} X$$

with u in \mathcal{W} , and whose morphisms are commutative diagrams

$$\begin{array}{ccccc} Y & \xleftarrow{u_1} & Y_1 & \xrightarrow{f_1} & X \\ \parallel & & \downarrow v & & \parallel \\ Y & \xleftarrow{u_2} & Y_2 & \xrightarrow{f_2} & X, \end{array}$$

with v in \mathcal{W} .

Note that this category is denoted in [DK2, 5.1] by $N^{-1}\mathcal{C}\mathcal{W}^{-1}(Y, X)$.

Definition A.5. Given a category \mathcal{C} , write $k\mathcal{C}$ for the k -linear category with the same objects as \mathcal{C} , but with morphisms given by the free k -modules

$$(k\mathcal{C})(X, Y) := k(\mathcal{C}(X, Y)).$$

Definition A.6. Given $\mathcal{F} \in \mathcal{C}_{\text{dg}}(k\mathcal{C})$ (i.e. a contravariant dg functor from \mathcal{C} to cochain complexes over k) and $Y \in \mathcal{C}$, define the cochain complex $\mathcal{F}\mathcal{W}^{-1}(Y)$ by

$$\mathcal{F}\mathcal{W}^{-1}(Y) := \text{holim}_{Y' \in \mathcal{W}Y} \mathcal{F}(Y').$$

Explicitly, this can be realised as the direct sum total complex of the simplicial cochain complex

$$\bigoplus_{Y'_0 \rightarrow Y} \mathcal{F}(Y'_0) \leftarrow \bigoplus_{Y'_1 \rightarrow Y'_0 \rightarrow Y} \mathcal{F}(Y'_0) \leftarrow \dots$$

Beware that this construction is not functorial in Y .

Proposition A.7. *Take a small category \mathcal{C} and a subcategory \mathcal{W} such that $(\mathcal{C}, \mathcal{W})$ and $(\mathcal{W}, \mathcal{W})$ admit right calculi of fractions. Let \mathcal{D} be the localised category $\mathcal{C}[\mathcal{W}^{-1}]$ given by formally inverting all morphisms in \mathcal{W} . Then the functor $\lambda: \mathcal{C} \rightarrow \mathcal{D}$ gives a left Quillen functor*

$$\lambda_!: \mathcal{C}_{\text{dg}}(k\mathcal{C}) \rightarrow \mathcal{C}_{\text{dg}}(k\mathcal{D}),$$

left adjoint to λ^{-1} , making $\mathcal{C}_{\text{dg}}(k\mathcal{D})$ Quillen-equivalent to the left Bousfield localisation of $\mathcal{C}_{\text{dg}}(k\mathcal{C})$ at the image $k\mathcal{W}$ of \mathcal{W} under the Yoneda embedding $k: \mathcal{C} \rightarrow \mathcal{C}_{\text{dg}}(k\mathcal{C})$.

Proof. The functor $\lambda_!$ satisfies $\lambda_!(kC) = k\lambda(C)$, which then determines $\lambda_!$ by right Kan extension. We begin by computing this for cofibrant $k\mathcal{C}$ -modules.

Combining [DK1, Propositions 7.2 and 7.3], the morphism

$$BC\mathcal{W}^{-1}(X, Y) \rightarrow \mathcal{D}(\lambda X, \lambda Y)$$

is a weak equivalence of simplicial sets for all $X, Y \in \mathcal{C}$. Since $k\mathcal{D}(\lambda X, \lambda Y) = (\lambda_! kX)(\lambda Y)$, and $BC\mathcal{W}^{-1}(X, Y) = (kX)\mathcal{W}^{-1}(Y)$, this gives a quasi-isomorphism

$$(kX)\mathcal{W}^{-1}(Y) \rightarrow (\lambda_! kX)(\lambda Y),$$

functorial in X (but not in Y). Since any cofibrant $k\mathcal{C}$ -module \mathcal{F} is a retraction of a filtered colimit of finite complexes of kX 's, this gives quasi-isomorphisms

$$\mathcal{F}\mathcal{W}^{-1}(Y) \xrightarrow{\sim} (\lambda_! \mathcal{F})(\lambda Y)$$

for all $Y \in \mathcal{C}$.

Now, the unit $\mathcal{F} \rightarrow \lambda^{-1}\lambda_!\mathcal{F}$ of the adjunction gives maps

$$\mathcal{F}(Y) \rightarrow (\lambda_! \mathcal{F})(\lambda Y)$$

for all $Y \in \mathcal{C}$, and these factor through the maps above, giving

$$\mathcal{F}(Y) \rightarrow \mathcal{F}\mathcal{W}^{-1}(Y) \xrightarrow{\sim} (\lambda_! \mathcal{F})(\lambda Y).$$

The $k\mathcal{C}$ -module \mathcal{F} will be $k\mathcal{W}$ -local if and only if \mathcal{F} maps morphisms in \mathcal{W} to quasi-isomorphisms. If this is the case, then the map $\mathcal{F}(Y) \rightarrow \mathcal{F}\mathcal{W}^{-1}(Y)$ is a quasi-isomorphism, so the unit

$$\mathcal{F}(Y) \rightarrow (\lambda_! \mathcal{F})(\lambda Y)$$

is also a quasi-isomorphism.

Because λ is essentially surjective on objects, the functor λ^{-1} reflects quasi-isomorphisms. Thus the co-unit $\mathbf{L}\lambda_!\lambda^{-1}\mathcal{G} \rightarrow \mathcal{G}$ of the derived adjunction is a quasi-isomorphism for all \mathcal{G} . Since λ maps \mathcal{W} to isomorphisms, any object in the image of λ^{-1} is $k\mathcal{W}$ -local. It therefore suffices to show that for any cofibrant $\mathcal{F} \in DG(k\mathcal{C})$, the unit $\mathcal{F}(Y) \rightarrow (\lambda_! \mathcal{F})(\lambda Y)$ of the adjunction is a $k\mathcal{W}$ -local equivalence. Now, for any

$k\mathcal{W}$ -local object \mathcal{G}

$$\begin{aligned}
\mathbf{R}\underline{\mathrm{Hom}}_{k\mathcal{C}}(\mathcal{F}, \mathcal{G}) &\simeq \mathbf{R}\underline{\mathrm{Hom}}_{k\mathcal{C}}(\mathcal{F}, \lambda^{-1}\mathbf{L}\lambda_!\mathcal{G}) \\
&\simeq \mathbf{R}\underline{\mathrm{Hom}}_{k\mathcal{D}}(\lambda_!\mathcal{F}, \mathbf{L}\lambda_!\mathcal{G}) \\
&\simeq \mathbf{R}\underline{\mathrm{Hom}}_{k\mathcal{D}}(\mathbf{L}\lambda_!\lambda^{-1}\lambda_!\mathcal{F}, \mathbf{L}\lambda_!\mathcal{G}) \\
&\simeq \mathbf{R}\underline{\mathrm{Hom}}_{k\mathcal{C}}(\lambda^{-1}\lambda_!\mathcal{F}, \lambda^{-1}\mathbf{L}\lambda_!\mathcal{G}) \\
&\simeq \mathbf{R}\underline{\mathrm{Hom}}_{k\mathcal{C}}(\lambda^{-1}\lambda_!\mathcal{F}, \mathcal{G}),
\end{aligned}$$

as required. \square

Corollary A.8. *In the setting of Proposition A.7, the functor $\lambda_!$ gives a quasi-equivalence $(k\mathcal{W})^\perp \rightarrow \mathcal{D}_{\mathrm{dg}}(k\mathcal{D})$ of dg categories. Moreover, the map $(\mathcal{W})^\perp \rightarrow \mathcal{D}_{\mathrm{dg}}(k\mathcal{C})/\mathcal{D}_{\mathrm{dg}}(k\mathcal{W})$ to the dg quotient is a quasi-equivalence.*

Proof. First observe that $(k\mathcal{W})^\perp \subset \mathcal{D}_{\mathrm{dg}}(k\mathcal{C})$ consists of the fibrant cofibrant objects in the Bousfield model structure, automatically giving the quasi-equivalence $(k\mathcal{W})^\perp \rightarrow \mathcal{D}_{\mathrm{dg}}(k\mathcal{D})$.

Fibrant replacement in the Bousfield model structure gives us morphisms $r: M \rightarrow \hat{M}$ in for each $M \in \mathcal{D}_{\mathrm{dg}}(k\mathcal{C})$, with $\hat{M} \in (k\mathcal{W})^\perp = \mathcal{D}_{\mathrm{dg}}(k\mathcal{W})^\perp$ and $\mathrm{cone}(r) \in \mathcal{D}_{\mathrm{dg}}(k\mathcal{W})$. Thus $\mathcal{D}_{\mathrm{dg}}(k\mathcal{W})$ is right admissible in the sense of [Dri, §12.6], giving the quasi-equivalence $(\mathcal{W})^\perp \rightarrow \mathcal{D}_{\mathrm{dg}}(k\mathcal{C})/\mathcal{D}_{\mathrm{dg}}(k\mathcal{W})$. \square

Now write Sm/F for the category of smooth schemes over F .

Corollary A.9. *The excision functor $(X, D) \mapsto X \setminus D$ induces quasi-equivalences $\mathcal{D}_{\mathrm{dg}}(\mathrm{SmQP}/F, k)/\mathcal{D}_{\mathrm{dg}}(k\mathcal{E}) \leftarrow (k\mathcal{E})^\perp \rightarrow \mathcal{D}_{\mathrm{dg}}(\mathrm{Sm}/F, k)$.*

Proof. This comes from applying Lemma A.2 to Corollary A.8, noting that the excision functor is essentially surjective, so gives an equivalence

$$(\mathrm{SmQP}/F)[\mathcal{E}^{-1}] \simeq \mathrm{Sm}/F.$$

\square

A.1.3. *Formal Weil cohomology theories.* As in Example 2.22, for any k -linear mixed Weil cohomology theory E over the field F , we can now define the formal Weil cohomology theory

$$E_f: (\mathrm{SmQP}/F)^{\mathrm{opp}} \rightarrow \mathcal{C}_{\mathrm{dg}}(k)$$

by

$$E_f(U \xrightarrow{j} X) := \left(\bigoplus_{a,b} \mathrm{H}^a(X, \mathbf{R}^b j_* E_U, d_2), \right)$$

where d_2 is the differential on the second page of the Leray spectral sequence.

Weight considerations or standard results on Gysin maps imply that the Leray spectral sequence degenerates at E_2 (at least for all known Weil theories), so any equivalence $(U, X) \rightarrow (U, X')$ in SmQP/F induces a quasi-isomorphism on E_f .

Writing $\mathcal{N}_{\mathrm{dg}}^{\mathrm{eff}'} := \mathcal{D}_{\mathrm{dg}}(\mathrm{SmQP}/K, k)$ as in Example 2.22, the functor E_f extends k -linearly, giving

$$E_f^\vee: \mathcal{N}_{\mathrm{dg}}^{\mathrm{eff}'} / \mathcal{D}_{\mathrm{dg}}(k\mathcal{E}) \rightarrow \mathcal{C}_{\mathrm{dg}}(k),$$

since \mathcal{E} lies in the kernel of E_f . By Corollary A.9, $\mathcal{N}_{\mathrm{dg}}^{\mathrm{eff}'} / \mathcal{D}_{\mathrm{dg}}(k\mathcal{E})$ is quasi-equivalent to $\mathcal{M}_{\mathrm{dg}}^{\mathrm{eff}'}$.

Moreover, since the Leray spectral sequence degenerates, we have $\ker E_f = \ker E$ on $\mathcal{N}_{\text{dg}}^{\text{eff}'}$, so

$$\mathcal{N}_{\text{dg}}^{\text{eff}'}/\ker E_f = \mathcal{N}_{\text{dg}}^{\text{eff}'}/\ker E \simeq \mathcal{M}_{\text{dg}}^{\text{eff}'}/\ker E.$$

A.2. Mixed Hodge structures on Betti cohomology.

Definition A.10. For $\Lambda \subset \mathbb{R}$ a subfield, define MHS_Λ to be the tensor category of mixed Hodge structures in finite-dimensional vector spaces over Λ . Explicitly, an object of MHS_Λ consists of a finite-dimensional vector space V over Λ equipped with an increasing (weight) filtration W , and a decreasing (Hodge) filtration F on $V \otimes_\Lambda \mathbb{C}$ (both exhaustive and Hausdorff), such that

$$\text{gr}_F^p \text{gr}_F^q \text{gr}_n^W V = 0$$

for $p + q \neq n$.

The functor forgetting the filtrations is faithful, so by Tannakian duality there is a corresponding affine group scheme which (following [Ara]) we refer to as the universal Mumford–Tate group MT_Λ ; this allows us to identify MHS_Λ with the category of finite-dimensional MT_Λ -representations.

Denote the pro-reductive quotient of MT_Λ by PMT_Λ — representations of this correspond to Hodge structures (i.e. direct sums of pure Hodge structures) over Λ . The assignment of weights to pure Hodge structures defines a homomorphism $\mathbb{G}_{m,\Lambda} \rightarrow \text{PMT}_\Lambda$.

Definition A.11. For $\Lambda \subset \mathbb{R}$ a subfield, a Λ -Hodge complex in the sense of [Bei, Definition 3.2] is a tuple $(V_F, V_\Lambda, V_{\mathbb{C}}, \phi, \psi)$, where (V_Λ, W) is a filtered complex of Λ -modules, $(V_{\mathbb{C}}, W)$ is a filtered complex of complex vector spaces, (V_F, W, F) a bifiltered complex of complex vector spaces, and

$$\phi: V_\Lambda \otimes_\Lambda \mathbb{C} \rightarrow V_{\mathbb{C}} \quad \psi: V_F \rightarrow V_{\mathbb{C}}$$

are W -filtered quasi-isomorphisms; these must also satisfy the conditions that

- (1) the cohomology $\bigoplus_i H^i(V_\Lambda)$ is finite-dimensional over Λ ;
- (2) for any $n \in \mathbb{Z}$, the differential in the filtered complex $(\text{gr}_n^W V_F, \text{gr}_n^W F)$ is strictly compatible with the filtration, or equivalently the map $H^*(F^p \text{gr}_n^W V_F) \rightarrow H^*(\text{gr}_n^W V_F)$ is injective;
- (3) the induced Hodge filtration together with the isomorphism $H^i(\text{gr}_n^W V_\Lambda) \otimes_\Lambda \mathbb{C} \rightarrow H^i(\text{gr}_n^W V_F)$ defines a pure Λ -Hodge structure of weight n on $H^i(\text{gr}_n^W V_\Lambda)$.

Example A.12. For a sheaf \mathcal{F} on $Y(\mathbb{C})$, write $\mathcal{C}_Y^\bullet(\mathcal{F})$ for the Godement resolution of \mathcal{F} — this is a cosimplicial diagram of flabby sheaves. Write $\mathbf{C}^\bullet(X, \mathcal{F})$ for the global sections of $\mathcal{C}_X^\bullet(\mathcal{F})$.

Take a smooth projective complex variety X and a complement $j: Y \hookrightarrow X$ of a normal crossings divisor D . Then set $A_\Lambda^\bullet := \mathbf{C}^\bullet(X, j_* \mathcal{C}_Y^\bullet(\Lambda))$, $A_{\mathbb{C}}^\bullet := \mathbf{C}^\bullet(X, j_* \mathcal{C}_Y^\bullet(N_c^{-1} \Omega_Y^\bullet))$, and $A_F^\bullet := \mathbf{C}^\bullet(X, N_c^{-1} \Omega_X^\bullet(D))$, where N_c^{-1} is the Dold–Kan denormalisation functor from cochain complexes to cosimplicial modules. The filtration W is given by décalage of the good truncation filtration on j_* in each case. Then $N_c A$ is mixed Hodge complex, and on applying the Thom–Sullivan functor Th from cosimplicial DG algebras to DG algebra, we obtain a commutative algebra $\text{Th}(A)$ in mixed Hodge complexes.

Definition A.13. For $\Lambda \subset \mathbb{R}$ a subfield, define MHS_Λ to be the category of mixed Hodge structures in finite-dimensional vector spaces over Λ , and write $\Pi(\text{MHS}_\Lambda)$ for the group scheme over Λ corresponding to the forgetful functor from MHS_Λ to Λ -vector spaces.

Definition A.14. Given a cosimplicial vector space V^\bullet and a simplicial set K , define $(V^\bullet)^K$ to be the cosimplicial vector space given by $((V^\bullet)^K)^n = (V^n)^{K_n}$, with operations $\partial^i: ((V^\bullet)^K)^n \rightarrow ((V^\bullet)^K)^{n+1}$ defined by composing

$$(V^n)^{K_n} \xrightarrow{(\partial^i)^{K_n}} (V^{n+1})^{K_n} \xrightarrow{(V^{n+1})^{\partial_i}} (V^{n+1})^{K_{n+1}},$$

and operations $\sigma^i: ((V^\bullet)^K)^n \rightarrow ((V^\bullet)^K)^{n-1}$ defined similarly.

In particular, $(V^\bullet)^{\Delta^1}$ is a path object over V^\bullet , with the two vertices $\Delta^0 \rightarrow \Delta^1$ inducing two maps $(V^\bullet)^{\Delta^1} \rightarrow V^\bullet$.

Definition A.15. The Λ -algebra $O(\text{MT}_\Lambda)$ admits both left and right multiplication by MT_Λ . These induce two different ind-mixed Hodge structures on $O(\text{MT}_\Lambda)$, which we refer to as the left and right mixed Hodge structures $(O(\text{MT}_\Lambda), W^l, F_l), (O(\text{MT}_\Lambda), W^r, F_r)$.

Example A.16. Given $A_{\Lambda, \mathcal{H}}^\bullet(X, D) := (A_\Lambda^\bullet, \phi, A_{\mathbb{C}}^\bullet, \psi, A_F^\bullet)$ as in Example A.12, we can define $A_{\text{MHS}}^\bullet(X, D; \Lambda)$ to be the limit of the diagram

$$\begin{array}{ccc} (W \otimes W^l)_0(A_\Lambda^\bullet \otimes_\Lambda O(\text{MT}_\Lambda)) & \longrightarrow & (W \otimes W^l)_0(A_{\mathbb{C}}^\bullet \otimes_\Lambda O(\text{MT}_\Lambda)) \\ (W \otimes W^l)_0((A_{\mathbb{C}}^\bullet)^{\Delta^1} \otimes_\Lambda O(\text{MT}_\Lambda)) & \longrightarrow & (W \otimes W^l)_0(A_{\mathbb{C}}^\bullet \otimes_\Lambda O(\text{MT}_\Lambda)) \\ (W \otimes W^l)_0(F \otimes F_l)^0(A_F^\bullet \otimes_\Lambda O(\text{MT}_\Lambda)) & & \end{array}$$

giving a cosimplicial algebra. The right Hodge structure on $O(\text{MT}_\Lambda)$ then gives us a cosimplicial algebra

$$(A_{\text{MHS}}^\bullet(X, D; \Lambda), W^r, F_r)$$

in $\text{ind}(\text{MHS}_\Lambda)$.

Proposition A.17. *The Λ -Hodge complex associated to the cosimplicial algebra $A_{\text{MHS}}^\bullet(X, D; \Lambda)$ in $\text{ind}(\text{MHS}_\Lambda)$ is canonically quasi-isomorphic to $A_{\Lambda, \mathcal{H}}^\bullet(X, D)$ as a commutative algebra in cosimplicial Λ -Hodge complexes.*

Proof. If we set

$$\begin{aligned} B_\Lambda &:= A_\Lambda^\bullet \times_{A_{\mathbb{C}}^\bullet} (A_{\mathbb{C}}^\bullet)^{\Delta^1} \\ B_{\mathbb{C}} &:= (A_{\mathbb{C}}^\bullet)^{\Delta^1} \\ B_F &:= (A_{\mathbb{C}}^\bullet)^{\Delta^1} \times_{A_{\mathbb{C}}^\bullet} A_F^\bullet, \end{aligned}$$

then $B := (B_\Lambda, B_{\mathbb{C}}, B_F)$ has the natural structure of a Λ -Hodge complex, and the morphism $\sigma^0: \Delta^1 \rightarrow \Delta^0$ induces a quasi-isomorphism $A_{\Lambda, \mathcal{H}}^\bullet(X, D) \rightarrow B$.

There is a map from the Λ -Hodge complex associated to $A_{\text{MHS}}^\bullet(X, D; \Lambda)$ to B given by projections. We need to show that these projections preserve the Hodge and weight filtrations, and are (bi)filtered quasi-isomorphisms.

Now, for any mixed Hodge structure V there is a canonical isomorphism $V \cong V \otimes^{\text{MT}_\Lambda} O(\text{MT}_\Lambda) := (V \otimes O(\text{MT}_\Lambda))^{\text{MT}_\Lambda}$, where the Mumford–Tate action combines the action on V with the left action on $O(\text{MT}_\Lambda)$. The mixed Hodge structure on V then corresponds to the Mumford–Tate action on $(V \otimes O(\text{MT}_\Lambda))^{\text{MT}_\Lambda}$ induced by the right action on $O(\text{MT}_\Lambda)$.

Since $W_n V$ is a sub-MHS, it follows that $W_n V \cong V \otimes^{\text{MT}_\Lambda} W_n^r O(\text{MT}_\Lambda)$, and since W_n is an idempotent functor, this is also isomorphic to $(W_n V) \otimes^{\text{MT}_\Lambda} W_n^r O(\text{MT}_\Lambda)$. In

particular,

$$V \otimes^{\mathrm{MT}_\Lambda} W_m^l W_n^r O(\mathrm{MT}_\Lambda) \subset (W_n V) \otimes^{\mathrm{MT}_\Lambda} W_m^l O(\mathrm{MT}_\Lambda) \subset \sum_{\substack{i \leq n, j \leq m, \\ i+j=0}} (W_i V) \otimes (W_j^l O(\mathrm{MT}_\Lambda)),$$

which is 0 for $m+n < 0$. Thus $W_{-n-1}^l W_n^r O(\mathrm{MT}_\Lambda) = 0$. A similar argument shows that $F_l^{1-p} F_r^p(\mathcal{O}(\mathrm{MT}_\Lambda) \otimes_\Lambda \mathbb{C}) = 0$.

Now, the weight filtration $W_n A_{\mathrm{MHS}}^\bullet(X, D; \Lambda)$ is given by replacing $O(\mathrm{MT}_\Lambda)$ with $W_n^r O(\mathrm{MT}_\Lambda)$ in the definition of $A_{\mathrm{MHS}}^\bullet(X, D; \Lambda)$, and the Hodge filtration $F^p A_{\mathrm{MHS}}^\bullet(X, D; \Lambda)$ by replacing $O(\mathrm{MT}_\Lambda) \otimes_\Lambda \mathbb{C}$ with $F_r^p(O(\mathrm{MT}_\Lambda) \otimes_\Lambda \mathbb{C})$. Projection onto the first factor gives a map from $W_n A_{\mathrm{MHS}}^\bullet(X, D; \Lambda)$ to $\sum_i (W_i A_\Lambda^\bullet) \otimes W_{-i}^l W_n^r O(\mathrm{MT}_\Lambda)$, which by the vanishing above is contained in $(W_n A_\Lambda^\bullet) \otimes O(\mathrm{MT}_\Lambda)$. Using similar arguments for the other factors and composing with the co-unit $O(\mathrm{MT}_\Lambda) \rightarrow \Lambda$ gives compatible (bi)filtered morphisms

$$\begin{aligned} A_{\mathrm{MHS}}^\bullet(X, D; \Lambda) &\rightarrow B_\Lambda \\ A_{\mathrm{MHS}}^\bullet(X, D; \Lambda) \otimes_\Lambda \mathbb{C} &\rightarrow B_{\mathbb{C}} \\ A_{\mathrm{MHS}}^\bullet(X, D; \Lambda) \otimes_\Lambda \mathbb{C} &\rightarrow B_F, \end{aligned}$$

and it only remains to establish quasi-isomorphism.

The data $N_c A_{\Lambda, \mathcal{H}}^\bullet(X, D)$ of Example A.12 define a Λ -Hodge complex, so by [Bei, Theorem 3.4], there exists a complex V^\bullet of mixed Hodge structures whose associated Hodge complex is quasi-isomorphic to $N_c A_{\Lambda, \mathcal{H}}^\bullet(X, D)$.

Now, observe that $N_c A_{\mathrm{MHS}}^\bullet(X, D; \Lambda)$ is a cocone calculating absolute Hodge cohomology, so

$$\begin{aligned} N_c A_{\mathrm{MHS}}^\bullet(X, D; \Lambda) &\simeq \mathbf{R}\Gamma_{\mathcal{H}}(A_{\Lambda, \mathcal{H}}^\bullet(X, D) \otimes (O(\mathrm{MT}_\Lambda), W^l, F_l)) \\ &\simeq \mathbf{R}\Gamma_{\mathcal{H}}(V^\bullet \otimes (O(\mathrm{MT}_\Lambda), W^l, F_l)) \\ &\simeq \mathbf{R}\underline{\mathrm{Hom}}_{\mathrm{MHS}, \Lambda}(\Lambda, V^\bullet \otimes (O(\mathrm{MT}_\Lambda), W^l, F_l)) \\ &\simeq \underline{\mathrm{Hom}}_{\mathrm{MHS}, \Lambda}(\Lambda, V^\bullet \otimes (O(\mathrm{MT}_\Lambda), W^l, F_l)) \\ &\cong V^\bullet \otimes^{\mathrm{MT}_\Lambda} O(\mathrm{MT}_\Lambda) \\ &\cong V^\bullet, \end{aligned}$$

with the last two properties following because $V^\bullet \otimes O(\mathrm{MT}_\Lambda)$ is an injective MT_Λ -representation and because $\mathrm{ind}(\mathrm{MHS}_\Lambda)$ is equivalent to the category of $O(\mathrm{MT}_\Lambda)$ -comodules in Λ -vector spaces. The quasi-isomorphisms above all respect mixed Hodge structures (via the right action on $O(\mathrm{MT}_\Lambda)$), completing the proof. \square

A.3. Splittings for Betti cohomology. Since MT_Λ is an affine group scheme, it is an inverse limit of linear algebraic groups, so by [HM], there exists a Levi decomposition $\mathrm{MT}_\Lambda \cong \mathrm{PMT}_\Lambda \ltimes \mathrm{R}_u(\mathrm{MT}_\Lambda)$ of the universal Mumford–Tate group as the semidirect product of its pro-reductive quotient and its pro-unipotent radical. Beware that this decomposition is not canonical; it might be tempting to think that the functor $V \mapsto \mathrm{gr}^W V$ yields the required section by Tannaka duality, but it is not compatible with the fibre functors.

Moreover, Levi decompositions are conjugate under the action of the radical $\mathrm{R}_u(\mathrm{MT}_\Lambda)$, so the set of decompositions is isomorphic to the quotient

$$\mathrm{R}_u(\mathrm{MT}_\Lambda) / \mathrm{R}_u(\mathrm{MT}_\Lambda)^{\mathrm{PMT}_\Lambda}$$

by the centraliser of PMT_Λ . For any element u of $\mathrm{R}_u(\mathrm{MT}_\Lambda)$, we must have $(u - \mathrm{id})W_n V \subset W_{n-1}V$ for all mixed Hodge structures V . However, any element in the centraliser necessarily has weight 0 for the \mathbb{G}_m -action, so must be 1. Thus the set of Levi decompositions is a torsor under

$$\mathrm{R}_u(\mathrm{MT}_\Lambda).$$

Proposition A.18. *Each choice of Levi decomposition for the universal Mumford–Tate group MT_Λ gives rise to a zigzag of W -filtered quasi-isomorphisms between the cosimplicial algebra-valued functors*

$$\begin{aligned} (X, D) &\mapsto A_\Lambda^\bullet(X, D) \\ (X, D) &\mapsto N_c^{-1}(\mathrm{H}^*(X, \mathbf{R}^*j_*\Lambda), d_2), \end{aligned}$$

where d_2 is the differential on the E_2 page of the Leray spectral sequence and $j: X \setminus D \rightarrow X$.

Proof. A choice of Levi decomposition is equivalent to a retraction of MHS_Λ onto HS_Λ , and $V \in \mathrm{MHS}_\Lambda$ is canonically isomorphic to $\mathrm{gr}^W V$. Since the weight filtration is a functorial filtration by mixed Hodge substructures, it is necessarily preserved by any such retraction, which thus amounts to giving a functorial W -filtered isomorphism $V \cong \mathrm{gr}^W V$ for all mixed Hodge structures V .

Proposition A.17 gives a zigzag of functorial W -filtered quasi-isomorphisms between the cosimplicial algebra $A_{\mathrm{MHS}}^\bullet(X, D; \Lambda)$ and the Betti complex $A_\Lambda^\bullet(X, D)$. A choice of Levi decomposition then gives a W -filtered isomorphism $A_{\mathrm{MHS}}^\bullet(X, D; \Lambda) \cong \mathrm{gr}^W A_{\mathrm{MHS}}^\bullet(X, D; \Lambda)$. Applying Proposition A.17 to the associated gradeds then gives a zigzag of filtered quasi-isomorphisms between $\mathrm{gr}^W A_{\mathrm{MHS}}^\bullet(X, D; \Lambda)$ and $\mathrm{gr}^W A_{\mathrm{MHS}}^\bullet(X, D; \Lambda)$, which maps quasi-isomorphically to $N_c^{-1}(\mathrm{H}^*(X, \mathbf{R}^*j_*\Lambda), d_2)$. \square

Corollary A.19. *If E_B denotes the mixed Weil cohomology theory associated to Betti cohomology, and $E_{B,f}$ its formal analogue as in Examples 2.22 and §A.1.3, then each choice of Levi decomposition for the universal Mumford–Tate group $\mathrm{MT}_\mathbb{Q}$ gives a zigzag of quasi-isomorphisms between E_B and $E_{B,f}$.*

Proof. The functor E_B is given by $X \mapsto \mathrm{Th}(C^\bullet(X(\mathbb{C}), \mathbb{Q}))$, so there is a canonical quasi-isomorphism from $\mathrm{Th}(A_\Lambda^\bullet(X(\mathbb{C}), D(\mathbb{C})))$ to $E_B(X)$. Proposition A.18 thus gives a zigzag of quasi-isomorphisms from $E_B(X)$ to $\mathrm{Th} N_c^{-1}(\mathrm{H}^*(X(\mathbb{C}), \mathbf{R}^*j_*\Lambda), d_2)$, functorial in $(X \setminus D \xrightarrow{j} X)$ in SmQP/F . The functors Th and N_c^{-1} are homotopy inverses, so this is quasi-isomorphic to $(\mathrm{H}^*(X(\mathbb{C}), \mathbf{R}^*j_*\Lambda), d_2)$, which is just $E_{B,f}(X \setminus D \xrightarrow{j} X)$. \square

REFERENCES

- [Ara] Donu Arapura. The Hodge theoretic fundamental group and its cohomology. In *The geometry of algebraic cycles*, volume 9 of *Clay Math. Proc.*, pages 3–22. Amer. Math. Soc., Providence, RI, 2010. arXiv:0902.4252v2 [math.AG].
- [Ayo1] Joseph Ayoub. Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. I. *Astérisque*, (314):x+466 pp. (2008), 2007.
- [Ayo2] Joseph Ayoub. La réalisation étale et les opérations de Grothendieck. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(1):1–145, 2014.
- [Ayo3] Joseph Ayoub. L’algèbre de Hopf et le groupe de Galois motiviques d’un corps de caractéristique nulle, I. *J. Reine Angew. Math.*, 693:1–149, 2014.
- [Bei] A. A. Beilinson. Notes on absolute Hodge cohomology. In *Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983)*, volume 55 of *Contemp. Math.*, pages 35–68. Amer. Math. Soc., Providence, RI, 1986.

- [BK1] C. Barwick and D. M. Kan. Relative categories: another model for the homotopy theory of homotopy theories. *Indag. Math. (N.S.)*, 23(1-2):42–68, 2012.
- [BK2] A. I. Bondal and M. M. Kapranov. Representable functors, Serre functors, and reconstructions. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53(6):1183–1205, 1337, 1989.
- [BK3] A. I. Bondal and M. M. Kapranov. Framed triangulated categories. *Mat. Sb.*, 181(5):669–683, 1990.
- [Blo] Spencer Bloch. Algebraic cycles and higher K -theory. *Adv. in Math.*, 61(3):267–304, 1986.
- [BV] Alexander Beilinson and Vadim Vologodsky. A DG guide to Voevodsky’s motives. *Geom. Funct. Anal.*, 17(6):1709–1787, 2008.
- [CD1] Denis-Charles Cisinski and Frédéric Déglise. Mixed Weil cohomologies. *Adv. Math.*, 230(1):55–130, 2012.
- [CD2] Denis-Charles Cisinski and Frédéric Déglise. Triangulated categories of mixed motives. arXiv:0912.2110v3 [math.AG], 2012.
- [CLM] Joseph Chuang, Andrey Lazarev, and Wajid Mannan. Koszul-Morita duality. *J. Noncommut. Geom.*, 10(4):1541–1557, 2016.
- [Del1] P. Deligne. Le groupe fondamental de la droite projective moins trois points. In *Galois groups over \mathbf{Q} (Berkeley, CA, 1987)*, volume 16 of *Math. Sci. Res. Inst. Publ.*, pages 79–297. Springer, New York, 1989.
- [Del2] Pierre Deligne. Théorie de Hodge. II. *Inst. Hautes Études Sci. Publ. Math.*, (40):5–57, 1971.
- [DK1] W. G. Dwyer and D. M. Kan. Calculating simplicial localizations. *J. Pure Appl. Algebra*, 18(1):17–35, 1980.
- [DK2] W. G. Dwyer and D. M. Kan. Simplicial localizations of categories. *J. Pure Appl. Algebra*, 17(3):267–284, 1980.
- [DMOS] Pierre Deligne, James S. Milne, Arthur Ogus, and Kuang-yen Shih. *Hodge cycles, motives, and Shimura varieties*, volume 900 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1982.
- [Dri] Vladimir Drinfeld. DG quotients of DG categories. *J. Algebra*, 272(2):643–691, 2004.
- [FI] Hiroshi Fukuyama and Isamu Iwanari. Monoidal infinity category of complexes from Tannakian viewpoint. *Math. Ann.*, 356(2):519–553, 2013.
- [Han] Masaki Hanamura. Mixed motives and algebraic cycles. I. *Math. Res. Lett.*, 2(6):811–821, 1995.
- [HM] G. Hochschild and G. D. Mostow. Pro-affine algebraic groups. *Amer. J. Math.*, 91:1127–1140, 1969.
- [HMS] Annette Huber and Stefan Müller-Stach. On the relation between Nori motives and Kontsevich periods. arXiv:1105.0865v4, 2011.
- [Iwa] Iwasamu Iwanari. Tannakization in derived algebraic geometry. *J. K-Theory*, 14(3):642–700, 2014. arXiv:1112.1761v3 [math.AG].
- [JS] André Joyal and Ross Street. An introduction to Tannaka duality and quantum groups. In *Category theory (Como, 1990)*, volume 1488 of *Lecture Notes in Math.*, pages 413–492. Springer, Berlin, 1991.
- [Kel1] Bernhard Keller. On the cyclic homology of exact categories. *J. Pure Appl. Algebra*, 136(1):1–56, 1999.
- [Kel2] Bernhard Keller. On differential graded categories. In *International Congress of Mathematicians. Vol. II*, pages 151–190. Eur. Math. Soc., Zürich, 2006.
- [Lei] Tom Leinster. Up-to-homotopy monoids. arXiv:math.QA/9912084v2, 2000.
- [Lur] Jacob Lurie. Derived algebraic geometry VIII: Quasi-coherent sheaves and Tannaka duality theorems. available at www.math.harvard.edu/~lurie/papers/DAG-VIII.pdf, 2011.
- [Mit] Barry Mitchell. Rings with several objects. *Advances in Math.*, 8:1–161, 1972.
- [Pos] Leonid Positselski. Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence. *Mem. Amer. Math. Soc.*, 212(996):vi+133, 2011.
- [Pri1] J. P. Pridham. The homotopy theory of strong homotopy algebras and bialgebras. *Homology, Homotopy Appl.*, 12(2):39–108, 2010. arXiv:0908.0116v2 [math.AG].
- [Pri2] J. P. Pridham. Tannaka duality for enhanced triangulated categories II: t -structures and homotopy types. <http://www.maths.ed.ac.uk/~jpridham/HHtannaka2.pdf> or §§3–4 of arXiv:1309.0637v5 [math.KT], 2014.
- [Qui] Daniel Quillen. Rational homotopy theory. *Ann. of Math. (2)*, 90:205–295, 1969.

- [Tab1] Gonçalo Tabuada. Invariants additifs de DG-catégories. *Int. Math. Res. Not.*, (53):3309–3339, 2005.
- [Tab2] Gonçalo Tabuada. Une structure de catégorie de modèles de Quillen sur la catégorie des dg-catégories. *C. R. Math. Acad. Sci. Paris*, 340(1):15–19, 2005.
- [Toë1] Bertrand Toën. Dualité de Tannaka supérieure I: Structures monoidales. MPI preprint, 2000.
- [Toë2] Bertrand Toën. The homotopy theory of dg-categories and derived Morita theory. *Invent. Math.*, 167(3):615–667, 2007.
- [TV] Bertrand Toën and Michel Vaquié. Moduli of objects in dg-categories. *Ann. Sci. École Norm. Sup. (4)*, 40(3):387–444, 2007.
- [Ver] Jean-Louis Verdier. Des catégories dérivées des catégories abéliennes. *Astérisque*, (239):xii+253 pp. (1997), 1996. With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis.
- [VSF] Vladimir Voevodsky, Andrei Suslin, and Eric M. Friedlander. *Cycles, transfers, and motivic homology theories*, volume 143 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2000.
- [Wal] James Wallbridge. Tannaka duality over ring spectra. arXiv:1204.5787v1 [math.AG], 2012.