QUANTISATION OF DERIVED POISSON STRUCTURES

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Abstract. We prove that every 0-shifted Poisson structure on a derived Artin n-stack admits a curved A_∞ deformation quantisation whenever the stack has perfect cotangent complex; in particular, this applies to LCI schemes, where it gives a DQ algebroid quantisation. Whereas the Kontsevich–Tamarkin approach to quantisation for smooth varieties hinges on invariance of the Hochschild complex under affine transformations, we instead exploit the observation that the Hochschild complex carries an anti-involution, and that such anti-involutive deformations of the complex of polyvectors are essentially unique. We also establish analogous statements for deformation quantisations in $\mathcal{C}^\infty$ and analytic settings.

Introduction

By [Kon4], every Poisson manifold admits a deformation quantisation. For smooth algebraic varieties in characteristic 0, Kontsevich and Yekutieli proved an analogous statement, showing in [Kon3, Yek1] that all Poisson structures admit DQ algebroid quantisations. Via local choices of connections, the question reduced to constructing quantisations of affine space. These could then be handled as in [Tam, Kon2, Yek2, VdB]: formality of the E_2 operad associates to the Hochschild complex a deformation of the P_2-algebra of multiderivations, and invariance under affine transformations ensures that it is the unique deformation.

We now consider generalisations of this question to singular schemes and more generally to derived stacks, considering quantisations of 0-shifted Poisson structures in the sense of [Pri3, CPT$^+$] and their analytic and $\mathcal{C}^\infty$ analogues [Pri9, Pri5]. For positively shifted structures, the analogous question is a formality, following from the equivalence $E_{n+1} \simeq P_{n+1}$ of operads. Quantisations for non-degenerate 0-shifted Poisson structures were established in [Pri4], and we now consider degenerate quantisations as well, addressing the remaining unsolved case of [Toë2, Conjecture 5.3][1], long regarded as the hardest[2].

The construction of non-degenerate quantisations in [Pri4, Pri10] only relied on the fact that the Hochschild complex is an anti-involutive deformation of the complex of multiderivations. Our strategy in this paper is closer to [Tam, Kon2] in that we establish an equivalence between the two complexes. As in [Pri4, Pri10], the key observation is still that the Hochschild complex of a differential graded-commutative algebra (CDGA) carries an anti-involution corresponding to the endofunctor on deformations sending

[1]Although the statement for derived DM stacks was specifically claimed in [Toë2, Theorem 5.4], it is not even stated in the reference provided, whose proof is only relevant to strictly positive shifts, and it lacks the constraints on the cotangent complex needed to rule out known affine counterexamples.

[2]See for instance the survey [PV, §3], but beware that quantisations for negative shifts, starting with the BV quantisations of [Pri6], take a far more subtle form than proposed there, and that the declared aim of [PV, §2] is somewhat moot given the uncited [Pri3].

an algebra to its opposite. Via a formality quasi-isomorphism for the E_2 operad corresponding to an even associator, the Hochschild complex becomes an anti-involutive deformation of the P_2-algebra of multiderivations.

We show (Corollary 1.19) that such deformations are essentially unique whenever the complexes of polyvectors and of multiderivations are quasi-isomorphic. This condition is satisfied when the CDGA has perfect cotangent complex, so gives an equivalence between polyvectors and the Hochschild complex (Theorem 2.20). For derived Deligne–Mumford stacks with perfect cotangent complex, this yields quantisations of 0-shifted Poisson structures (Corollaries 2.26 and 2.29), which take the form of DQ algebroid quantisations of the étale structure sheaf. In particular, the hypotheses of Corollary 2.29 are satisfied by schemes with LCI singularities, where hitherto the conclusion was only known for smooth varieties. Deformation quantisations of 1-shifted co-isotropic structures also follow as an immediate consequence (Remark 2.32).

In Section 3, we extend these results to derived Artin n-stacks. This follows by essentially the same argument, but is much more technically complicated because of the subtleties in formulating polyvectors and Hochschild complexes for Artin stacks. Locally, these are defined as Tate realisations $\widehat{\mathrm{Tot}}$ of double complexes; those total complexes do not satisfy the conditions of Corollary 1.19, so we introduce an intermediate category through which $\widehat{\mathrm{Tot}}$ factorises, and in which the P_2-algebra of polyvectors has no non-trivial involutive deformations.

For stacky thickenings of derived affine schemes, this leads to an equivalence between polyvectors and the Hochschild complex (Theorem 3.34). Via a form of étale functoriality, this yields quantisations of 0-shifted Poisson structures on derived Artin stacks $\mathfrak{X}$ with perfect cotangent complex (Corollary 3.37), which in turn give curved A_∞ deformations of the dg category of perfect $\mathscr{O}_{\mathfrak{X}}$-complexes.

Notation and terminology. We write CDGAs (commutative differential graded algebras) and DGAAs (differential graded associative algebras) as chain complexes (homological grading), and denote the differential on a chain complex by δ.

The graded vector space underlying a chain (resp. cochain) complex V is denoted by $V_\#$ (resp. $V^\#$). Since we often have to work with chain and cochain structures separately, we denote shifts as subscripts and superscripts, respectively, so $(V_{[i]})_n := V_{i+n}$ and $(V^{[i]})^n := V^{i+n}$.

Given a DGAA A, and A-modules M, N in chain complexes, we write $\underline{\mathrm{Hom}}_A(M,N)$ for the chain complex given by

$$\underline{\mathrm{Hom}}_A(M,N)_i = \mathrm{Hom}_{A_\#}(M_\#, N_{\#[i]}),$$

with differential $\delta f = \delta_N \circ f \pm f \circ \delta_M$, where $V_\#$ denotes the graded vector space underlying a chain complex V.

When we need to compare chain and cochain complexes, we make use of the equivalence u from chain complexes to cochain complexes given by $(uV)^i := V_{-i}$, and refer to this as rewriting the chain complex as a cochain complex (or vice versa). On suspensions, this has the effect that $u(V_{[n]}) = (uV)^{[-n]}$.

Contents

1. Involutively filtered deformations of Poisson algebras

We will assume that all filtrations are increasing and exhaustive, unless stated otherwise. An action of the algebraic group $\mathbb{G}_m$ on a vector space V is equivalent to a decomposition $V = \bigoplus_{i\le n} \mathcal{W}_i V$; given a $\mathbb{G}_m$-equivariant vector space V, we then define a weight filtration W on V by setting $W_n V := \bigoplus_{i\le n} \mathcal{W}_i V$. For $\mathbb{G}_m$-equivariant complexes U, V, we write $\mathcal{W}_i\underline{\mathrm{Hom}}(U,V)$ for the complex $\prod_j \underline{\mathrm{Hom}}(\mathcal{W}_j U, \mathcal{W}_{i+j} V)$ of homomorphisms of weight i, with similar conventions for complexes of derivations etc.; beware that the inclusion $\bigoplus_i \mathcal{W}_i\underline{\mathrm{Hom}}(U,V) \to \underline{\mathrm{Hom}}(U,V)$ is not surjective in general.

1.1. Involutively filtered deformations of $\mathcal{P}$-algebras.

We now adapt an idea developed in [Pri2, §14.3.2], using Rees constructions to interpret the problem of recovering a filtered algebra from its associated graded algebra as a deformation problem.

Definition 1.1. We say that a vector space V is involutively filtered if it is equipped with a filtration W and an involution e which preserves W and acts on $\mathrm{gr}^W_i V$ as multiplication by $(-1)^i$.

We say that a chain complex $V_\bullet$ is quasi-involutively filtered if it is equipped with a filtration W by subcomplexes and an involution e which preserves W and acts on $\mathrm{H}_*(\mathrm{gr}^W_i V)$ as multiplication by $(-1)^i$.

Observe that if V is involutively filtered, then the involution gives an eigenspace decomposition $V = V^{e=1} \oplus V^{e=-1}$. Because $(\mathrm{gr}^W_{i-1} V)^{e=(-1)^i} = 0$, we have $W_{2j+1}V^{e=1} = W_{2j}V^{e=1}$ and $W_{2j}V^{e=-1} = W_{2j-1}V^{e=-1}$. If $V_\bullet$ is quasi-involutively filtered, we similarly have that $W_{2j}V^{e=1} \subset W_{2j+1}V^{e=1}$ and $W_{2j-1}V^{e=-1} \subset W_{2j}V^{e=-1}$ are quasi-isomorphic subcomplexes.

Definition 1.2. Given a quasi-involutively filtered chain complex $(V_\bullet, W, e)$, define the involutively filtered chain complex $(V_\bullet, W^e, e)$ by setting

$$W^e_n V_\bullet := (W_n V_\bullet)^{e=(-1)^n} \oplus (W_{n-1}V_\bullet)^{e=(-1)^{n-1}}.$$

Lemma 1.3. *After localisation at filtered quasi-isomorphisms, the inclusion functor from involutively filtered chain complexes to quasi-involutively filtered chain complexes gives an equivalence of ∞-categories.*

Proof. We can identify the involutively filtered objects as those $(V_\bullet, W, e)$ for which the natural transformation $\varepsilon_V\colon (V_\bullet, W^e, e) \to (V_\bullet, W, e)$ is an isomorphism, so the functor $(V_\bullet, W, e) \mapsto (V_\bullet, W^e, e)$ is right adjoint to the inclusion functor, with ε the co-unit of the adjunction.

It remains to show that ε_V is a filtered quasi-isomorphism for all quasi-involutively filtered chain complexes. Observe that $W_{n-1}V_\bullet$ is contained in $W^e_n V_\bullet$, with quotient isomorphic to $\mathrm{gr}^W_n V_\bullet^{e=(-1)^n}$, which implies that the inclusion $W^e_n V_\bullet \hookrightarrow W_n V_\bullet$ is indeed a quasi-isomorphism by quasi-involutivity of W. □

Definition 1.4. Define the $\mathbb{G}_m$-equivariant algebra $\mathbb{Q}[\hbar]$ by setting $\hbar$ to have weight 1 for the $\mathbb{G}_m$-action (and implicitly homological degree 0).

Definition 1.5. For a quasi-involutively filtered chain complex $(V, W_i V, e)$ over $\mathbb{Q}$, we define

$$\mathrm{Rees}^e_W(V) := \bigoplus_n \hbar^n \{v \in W_n V \ : \ e(v) = (-1)^n v\}.$$

This is a $\mathbb{G}_m$-equivariant chain complex of $\mathbb{Q}[\hbar^2]$-modules, where the $\mathbb{G}_m$-action is induced by the action on $\hbar$.

Lemma 1.6. *The involutive Rees functor of Definition 1.5 gives an equivalence between the category of involutively filtered $\mathbb{Q}$-vector spaces and the category of flat $\mathbb{G}_m$-equivariant $\mathbb{Q}[\hbar^2]$-modules.*

For quasi-involutively filtered complexes (V, W, e), it also satisfies $\mathrm{Rees}^e_{W^e}(V) \cong \mathrm{Rees}^e_W(V)$*, so factorises the ∞-equivalence of Lemma 1.3.*

Proof. We adapt the equivalence between exhaustively filtered vector spaces and flat $\mathbb{G}_m$-equivariant $\mathbb{Q}[\hbar]$-modules from for instance [Pri2, Lemma 2.1].

The functor Rees^e has a left adjoint, which sends a $\mathbb{G}_m$-equivariant $\mathbb{Q}[\hbar^2]$-module M in complexes to the complex $M/(\hbar^2 - 1)$ equipped with involution $-1 \in \mathbb{G}_m$ and filtration

$$\begin{aligned} W_n(M/(\hbar^2-1)) &:= (\bigoplus_{i \le n} \mathcal{W}_i M)/(\hbar^2-1)(\bigoplus_{i\le n-2} \mathcal{W}_i M) \\ &\cong \mathcal{W}_n M \oplus \mathcal{W}_{n-1} M; \end{aligned}$$

flatness of M ensures that the map $W_n(M/(\hbar^2-1)) \to M/(\hbar^2-1)$ is indeed an inclusion.

Evaluation at $\hbar = 1$ gives an morphism $\mathrm{Rees}^e_W(V)/(\hbar^2-1) \to V$, the co-unit of the adjunction. When V is involutively filtered, this map is a filtered isomorphism. Even when V is not involutively filtered, it follows easily that $W_n(\mathrm{Rees}^e_W(V)/(\hbar^2-1)) \cong W^e_n V$. Thus the involutive Rees functor factorises the ∞-equivalence of Lemma 1.3, and the co-unit of the adjunction is a natural isomorphism when restricted to involutively filtered objects.

Meanwhile, for any $\mathbb{G}_m$-equivariant $\mathbb{Q}[\hbar^2]$-module M we have

$$\mathcal{W}_n \mathrm{Rees}^e_W(M/(\hbar^2-1)) \cong (\mathcal{W}_n \oplus \mathcal{W}_{n-1}M)^{e=(-1)^n} = \mathcal{W}_n M,$$

so the unit of the adjunction is a natural isomorphism. □

Remark 1.7. Although we are assuming that filtrations are exhaustive, beware that we are not assuming they are Hausdorff. For instance, the $\mathbb{G}_m$-equivariant $\mathbb{Q}[\hbar^2]$-module $V[\hbar^2, \hbar^{-2}]$ corresponds under Lemma 1.6 to the vector space V with trivial involution and filtration $W_i V = V$ for all i.

However, in applications we only ever work with filtrations W on algebras satisfying the stronger condition $W_{-1}V = 0$, and our filtrations on operads will always be complete.

For the remainder of this subsection, we fix a $\mathbb{G}_m$-equivariant operad $\mathcal{P}$ in chain complexes over $\mathbb{Q}$.

Definition 1.8. Define the category of quasi-involutively filtered $(\mathcal{P}, W, e)$-algebras to consist of quasi-involutively filtered chain complexes (V, W_iV, e) of $\mathbb{Q}$-vector spaces equipped with a $\mathcal{P}$-algebra structure which is compatible with the filtration in the sense that the structure maps $\mathcal{P}(n) \otimes V^{\otimes n} \to V$ restrict to maps

$$\mathcal{W}_i\mathcal{P}(n) \otimes W_r(V^{\otimes n}) \to W_{r+i}V$$

for all r and i, where $W_r(V^{\otimes n}) := \sum_{r_1+\ldots+r_n=r}(W_{r_1}V) \otimes \ldots \otimes (W_{r_n}V)$.

The involutive Rees functor is clearly lax monoidal, i.e. $\mathrm{Rees}^e_W(V)\otimes_{\mathbb{Q}[\hbar^2]}\mathrm{Rees}^e_W(V) \to \mathrm{Rees}^e_W(U \otimes_{\mathbb{Q}} V)$, as is its left adjoint $M \mapsto M/(\hbar^2 - 1)$, so Lemma 1.6 has the following immediate consequence:

Lemma 1.9. *The functor of Definition 1.5 gives an equivalence of ∞-categories from the category of quasi-involutively filtered $(\mathcal{P}, W, e)$-algebras localised at filtered quasi-isomorphisms to the ∞-category of $\mathbb{G}_m$-equivariant $\mathcal{P}$-algebras in chain complexes of flat $\mathbb{Q}[\hbar^2]$-modules, localised at quasi-isomorphisms.*

Note that there is a projective model structure on $\mathbb{G}_m$-equivariant $\mathcal{P}$-algebras in chain complexes, for which fibrations are surjections and weak equivalences are quasi-isomorphisms. Existence of this cofibrantly generated model structure follows from [Hir, Theorem 11.3.2] applied to the forgetful functor to $\mathbb{G}_m$-equivariant chain complexes of $\mathbb{Q}$-vector spaces.

Definition 1.10. Given a $\mathbb{G}_m$-equivariant $\mathcal{P}$-algebra A, we write $\underline{\mathrm{Der}}_{\mathcal{P},\mathbb{G}_m}(A, M)$ for the complex of $\mathbb{G}_m$-equivariant $\mathcal{P}$-derivations from A to M. We also write $\mathbf{R}\underline{\mathrm{Der}}_{\mathcal{P},\mathbb{G}_m}(A, M)$ for the complex of derived derivations, given as in [Qui] by $\underline{\mathrm{Der}}_{\mathcal{P},\mathbb{G}_m}(\tilde{A}, M)$ for any cofibrant replacement $\tilde{A}$ of A in the projective model structure.

Here, M is a Beck A-module, meaning that $A \oplus M$ is a $\mathbb{G}_m$-equivariant $\mathcal{P}$-algebra for which the projection map $A \oplus M \to A$ and the addition map $(A \oplus M) \times_A (A \oplus M) \to A \oplus M$ are both $\mathcal{P}$-algebra homomorphisms. Explicitly,

$$\underline{\mathrm{Der}}_{\mathcal{P},\mathbb{G}_m}(A, M)_n := \mathrm{Hom}(A, A \oplus \mathrm{cone}(M)_{[n+1]}) \times_{\mathrm{Hom}(A,A)} \{\mathrm{id}\},$$

where Homs on the right-hand side are in the category of $\mathcal{P}$-algebras in $\mathbb{G}_m$-equivariant chain complexes. The differential $\underline{\mathrm{Der}}_{\mathcal{P},\mathbb{G}_m}(A, M)_n \to \underline{\mathrm{Der}}_{\mathcal{P},\mathbb{G}_m}(A, M)_{n-1}$ is then induced by the obvious map $\mathrm{cone}(M)_{[n+1]} \to \mathrm{cone}(M)_{[n]}$.

Proposition 1.11. *Take quasi-involutively filtered $(\mathcal{P}, W, e)$-algebras A, B such that the filtration on B is complete (satisfied in particular if $W_{-1}B = 0$), together with a morphism $f\colon \mathrm{gr}^W A \to \mathrm{gr}^W B$ of the associated $\mathbb{G}_m$-equivariant $\mathcal{P}$-algebras.*

If

$$\mathrm{H}_i\mathbf{R}\underline{\mathrm{Der}}_{\mathcal{P},\mathbb{G}_m}(\mathrm{gr}^W A, \hbar^{2(n+1)}\mathrm{gr}^W B) \cong 0$$

for all $n \geq 0$ and all $i \geq -1$, then the homotopy fibre over f of the associated graded functor

$$\mathrm{gr}^W\colon\ \mathbf{R}\mathrm{map}_{(\mathcal{P},W,e)}(A, B) \to \mathbf{R}\mathrm{map}_{\mathcal{P},\mathbb{G}_m}(\mathrm{gr}^W A, \mathrm{gr}^W B)$$

on mapping spaces in the respective ∞-categories is contractible.

Proof. By Lemma 1.9, $\mathbf{R}\mathrm{map}_{(\mathcal{P},W,e)}(A,B) \simeq \mathbf{R}\mathrm{map}_{\mathbb{Q}[\hbar^2]\otimes\mathcal{P},\mathbb{G}_m}(\mathrm{Rees}^e_W A, \mathrm{Rees}^e_W B)$. Since the filtration on B is complete, we have $\mathrm{Rees}^e_W B \cong \varprojlim_n (\mathrm{Rees}^e_W B)/\hbar^{2n}$, the limit being taken in the $\mathbb{G}_m$-equivariant category (i.e. separately in each weight). Since the maps $(\mathrm{Rees}^e_W B)/\hbar^{2n+2} \to (\mathrm{Rees}^e_W B)/\hbar^{2n}$ are all surjective, they are fibrations in the projective model structure, and thus the limit is a homotopy limit, so

$$\mathbf{R}\mathrm{map}_{\mathcal{P}[\hbar^2],\mathbb{G}_m}(\mathrm{Rees}^e_W A, \mathrm{Rees}^e_W B) \simeq \mathrm{ho}\varprojlim_n \mathbf{R}\mathrm{map}_{\mathcal{P}[\hbar^2],\mathbb{G}_m}(\mathrm{Rees}^e_W A, (\mathrm{Rees}^e_W B)/\hbar^{2n}),$$

where we write $\mathcal{P}[\hbar^2]$ for the operad $\mathbb{Q}[\hbar^2]\otimes\mathcal{P}$

Since the natural map $(\mathrm{Rees}^e_W A)/\hbar^2 \to \mathrm{gr}^W A$ is a quasi-isomorphism, and similarly for B, we have

$$\mathbf{R}\mathrm{map}_{\mathcal{P}[\hbar^2],\mathbb{G}_m}(\mathrm{Rees}^e_W A, (\mathrm{Rees}^e_W B)/\hbar^2) \simeq \mathbf{R}\mathrm{map}_{\mathcal{P},\mathbb{G}_m}(\mathrm{gr}^W A, \mathrm{gr}^W B),$$

so it suffices to show that the maps

$$\mathbf{R}\mathrm{map}_{\mathcal{P}[\hbar^2],\mathbb{G}_m}(A, B/\hbar^{2n+2}) \to \mathbf{R}\mathrm{map}_{\mathcal{P}[\hbar^2],\mathbb{G}_m}(A, B/\hbar^{2n})$$

are all equivalences.

We now invoke a standard obstruction theory argument. Since $\hbar^{2n}\mathbb{Q} \subset \mathbb{Q}[\hbar^2]/\hbar^{2n+2}$ is an ideal, there is an obvious CDGA structure on the chain complex $C' := \mathrm{cone}(\hbar^{2n}\mathbb{Q} \to \mathbb{Q}[\hbar^2]/\hbar^{2n+2})$, with the quotient map $C' \to \mathbb{Q}[\hbar^2]/\hbar^{2n}$ being a quasi-isomorphism. Moreover, since $(\hbar^{2n})\cdot(\hbar^2) = 0$ in $\mathbb{Q}[\hbar^2]/\hbar^{2n+2}$, the natural surjection $C' \to \mathrm{cone}(\hbar^{2n}\mathbb{Q} \xrightarrow{0} \mathbb{Q})$ is also a CDGA map, with

$$\mathbb{Q}[\hbar^2]/\hbar^{2n+2} \cong C' \times_{\mathrm{cone}(\hbar^{2n}\mathbb{Q}\xrightarrow{0}\mathbb{Q})} \mathbb{Q}.$$

Tensoring with $\mathrm{Rees}^e_W(B)$ over $\mathbb{Q}[\hbar^2]$, this gives rise to a quasi-isomorphism

$$C := \mathrm{cone}(\hbar^{2n}\colon \mathrm{Rees}^e_W(B)/\hbar^2 \to \mathrm{Rees}^e_W(B)/\hbar^{2n+2}) \xrightarrow{\alpha} \mathrm{Rees}^e_W(B)/\hbar^{2n}$$

of $\mathbb{G}_m$-equivariant $\mathcal{P}[\hbar^2]$-algebras, together with an isomorphism

$$\mathrm{Rees}^e_W(B)/\hbar^{2n+2} \cong C \times_{\hbar^{2n}(\mathrm{Rees}^e_W(B)/\hbar^2)_{[-1]}\oplus \mathrm{Rees}^e_W(B)/\hbar^2} \mathrm{Rees}^e_W(B)/\hbar^2.$$

That fibre product is a homotopy fibre product because α is surjective, so we have a homotopy pullback square

$$\begin{tikzcd}
\mathbf{R}\mathrm{map}_{\mathcal{P}[\hbar^2],\mathbb{G}_m}(\mathrm{Rees}^e_W(A), \mathrm{Rees}^e_W(B)/\hbar^{2n+2}) \arrow[r] \arrow[d] & \mathbf{R}\mathrm{map}_{\mathcal{P},\mathbb{G}_m}(\mathrm{gr}^W A, \mathrm{gr}^W B) \arrow[d] \\
\mathbf{R}\mathrm{map}_{\mathcal{P}[\hbar^2],\mathbb{G}_m}(\mathrm{Rees}^e_W(A), \mathrm{Rees}^e_W(B)/\hbar^{2n}) \arrow[r] & \mathbf{R}\mathrm{map}_{\mathcal{P},\mathbb{G}_m}(\mathrm{gr}^W A, \mathrm{gr}^W B \oplus \hbar^{2n}\mathrm{gr}^W B_{[-1]}).
\end{tikzcd}$$

Taking homotopy fibres over $f \in \mathbf{R}\mathrm{map}_{\mathcal{P},\mathbb{G}_m}(\mathrm{gr}^W A, \mathrm{gr}^W B)$ gives a homotopy fibre sequence

$$\begin{aligned}
&\mathbf{R}\mathrm{map}_{\mathcal{P}[\hbar^2],\mathbb{G}_m}(\mathrm{Rees}^e_W(A), \mathrm{Rees}^e_W(B)/\hbar^{2n+2})_f \\
&\quad\to \mathbf{R}\mathrm{map}_{\mathcal{P}[\hbar^2],\mathbb{G}_m}(\mathrm{Rees}^e_W(A), \mathrm{Rees}^e_W(B)/\hbar^{2n})_f \\
&\qquad\to \mathbf{R}\mathrm{map}_{\mathcal{P},\mathbb{G}_m}(\mathrm{gr}^W A, \mathrm{gr}^W B \oplus \hbar^{2n}\mathrm{gr}^W B_{[-1]})_f,
\end{aligned}$$

but

$$\pi_j \mathbf{R}\mathrm{map}_{\mathcal{P},\mathbb{G}_m}(\mathrm{gr}^W A, \mathrm{gr}^W B \oplus \hbar^{2n}\mathrm{gr}^W B_{[-1]})_f \cong \mathrm{H}_{j-1}\mathbf{R}\underline{\mathrm{Der}}_{\mathcal{P},\mathbb{G}_m}(\mathrm{gr}^W A, \hbar^{2(n+1)}\mathrm{gr}^W B)$$

which is 0 by hypothesis for all $i > 0$. Thus the base of the fibration is contractible, which gives the desired equivalence. ☐

Corollary 1.12. *If A is a quasi-involutively filtered $(\mathcal{P}, W, e)$-algebra such that the filtration is complete and* $\mathrm{H}_i\mathbf{R}\underline{\mathrm{Der}}_{\mathcal{P},\mathbb{G}_m}(\mathrm{gr}^W A, \hbar^{2n}\mathrm{gr}^W A) \cong 0$ *for all $n > 0$ and all $i \geq -1$, then A is quasi-isomorphic to* $\mathrm{gr}^W A$ *as a $(\mathcal{P}, W, e)$-algebra.*

Proof. Proposition 1.11 implies that the ∞-category of quasi-involutively filtered $(\mathcal{P}, W, e)$-algebras B with fixed quasi-isomorphism $\mathrm{gr}^W B \simeq \mathrm{gr}^W A$ is contractible (by taking f to be the identity map on $\mathrm{gr}^W A$). Since A and $\mathrm{gr}^W A$ both lie in this category, it follows that they are quasi-isomorphic. □

Corollary 1.13. *The associated graded functor from the ∞-category of quasi-involutively filtered $(\mathcal{P}, W, e)$-algebras to the category of $\mathbb{G}_m$-equivariant $\mathcal{P}$-algebras becomes an equivalence when restricted to objects A satisfying the conditions:*

(1) $\mathrm{gr}^W_i A \simeq 0$ *for all $i < 0$, and*
(2) $\mathbf{R}\underline{\mathrm{Der}}_{\mathcal{P},\mathbb{G}_m}(\mathrm{gr}^W A, M) \simeq 0$ *for all Beck* $\mathrm{gr}^W A$*-modules M with $\mathcal{W}_i M \simeq 0$ for all $i < 2$.*

The quasi-inverse functor is given by sending a $\mathbb{G}_m$-equivariant $\mathcal{P}$-algebra to the underlying filtered algebra.

In particular, note that these conditions are satisfied if $\mathrm{gr}^W A$ has a cofibrant replacement generated in weights ≤ 1.

Proof. Given A satisfying condition (2), B satisfying condition (1), and a morphism $f\colon \mathrm{gr}^W A \to \mathrm{gr}^W B$, observe that the Beck A-module $\hbar^{2(n+1)}\mathrm{gr}^W B$ is acyclic in weights below $2n + 2$, so for all $n \geq 0$ we have

$$\mathbf{R}\underline{\mathrm{Der}}_{\mathcal{P},\mathbb{G}_m}(\mathrm{gr}^W A, \hbar^{2(n+1)}\mathrm{gr}^W B) \simeq 0$$

by hypothesis on A. Thus Proposition 1.11 ensures that the restricted functor is full and faithful. The functor sending a grading to its underlying increasing filtration is clearly a right inverse to gr^W, and hence a quasi-inverse. □

Remark 1.14. If we take a quasi-involutively filtered dg operad $\mathcal{Q}$ with an involutive equivalence $\mathrm{gr}^W \mathcal{Q} \simeq \mathcal{P}$, then the conditions of Proposition 1.11 give the same conclusion when A and B are quasi-involutively filtered $(\mathcal{Q}, W, e)$-algebras, with the same proof, because the associated graded pieces are the same. Since $\mathrm{gr}^W \mathcal{Q}$-algebras are not then canonically $\mathcal{Q}$-algebras, the analogous statement to Corollary 1.13 is just that the restricted functor is full and faithful.

Essential surjectivity also follows if we assume $\mathrm{H}_{<0}\mathcal{P} = 0$ and restrict to $\mathbb{G}_m$-equivariant $\mathcal{P}$-algebras B satisfying $\mathrm{H}_{<0}(B) = 0$, because if the condition $\mathrm{H}_{-2}\mathbf{R}\underline{\mathrm{Der}}_{\mathcal{P},\mathbb{G}_m}(B, \hbar^{2(n+1)}B) = 0$ holds for all $n \geq 0$, then there must exist a quasi-involutively filtered $(\mathcal{P}, W, e)$-algebra A with $\mathrm{gr}^W A \simeq B$. This follows by a similar argument to [KS, Hin2], by lifting quasi-free generators and looking at the condition $\delta^2 = 0$; it relies on the quasi-free objects being cofibrant when concentrated in non-negative chain degrees.

1.2. **Almost commutative Poisson algebras.** We now consider non-unital P_k-algebras (i.e. $(k-1)$-shifted Poisson algebras); these are non-unital CDGAs equipped with a Lie bracket of chain degree $k-1$ acting as a biderivation. They are governed by an operad P_k which can be written as $\mathrm{Com} \circ (s^{1-k}\mathrm{Lie})$ via a distributive law (cf. [LV, §8.6]), for the operads $\mathrm{Com}, \mathrm{Lie}$ governing non-unital commutative algebras and Lie algebras, where the shift $s\mathcal{P}$ of a dg operad $\mathcal{P}$ is given by $(s\mathcal{P})(n) := \mathcal{P}(n)_{[n-1]}$.

Definition 1.15. Define the $\mathbb{G}_m$-equivariant dg operad P_k^{ac} to be the dg operad $\mathrm{Com} \circ s^{1-k}\hbar^{-1}\mathrm{Lie}$, where $(\hbar^j\mathcal{P})(i) := \hbar^{j(i-1)}\mathcal{P}(i)$ for any operad $\mathcal{P}$, and as in Definition 1.4, $\hbar$ has degree 0 and weight 1 for the $\mathbb{G}_m$-action.

Define a quasi-involutive a.c. P_k-algebra over a CDGA R to be an $(R \otimes P_k^{ac}, W, e)$-algebra A in quasi-involutively filtered chain complexes. Here, $R \otimes \mathcal{P}$ is the operad $(R \circ \mathcal{P})(n) := R \otimes \mathcal{P}(n)$ with operad structure coming from the distributive law (as in [LV, §8.6]) $\mathcal{P}(n) \otimes R^{\otimes n} \to R \otimes \mathcal{P}(n)$ given by the multiplication on R.

Thus a $\mathbb{G}_m$-equivariant P_k^{ac}-algebra is a P_k-algebra equipped with a $\mathbb{G}_m$-action for which multiplication has weight 0 and the Lie bracket has weight -1. A quasi-involutive a.c. P_k-algebra over R is a P_k-algebra in R-modules, equipped with an increasing filtration W satisfying $W_i \cdot W_j \subset W_{i+j}$ and $[W_i, W_j] \subset W_{i+j-1}$, together with an involution $*$ preserving the filtration, satisfying $(a \cdot b)^* = a^* \cdot b^*$ and $[a, b]^* = -[a^*, b^*]$, and acting as $(-1)^i$ on $\mathrm{H}_*\mathrm{gr}^W_i$.

Note that the chain degree of a non-zero element of the operad P_k^{ac} is always $(1-k)$ times its weight.

Lemma 1.16. *The Koszul dual of the $\mathbb{G}_m$-equivariant dg operad P_k^{ac} is given by $s^{k-1}\hbar P_k^{ac}$.*

Proof. The proof of [LV, 8.6.11] adapts to this shifted and graded setting (cf. [Fre, Appendix C]) to give

$$
\begin{aligned}
(P_k^{ac})^! &= (\mathrm{Com} \circ s^{1-k}\hbar^{-1}\mathrm{Lie})^! \\
&= (s^{1-k}\hbar^{-1}\mathrm{Lie})^! \circ \mathrm{Com}^! \\
&\cong (s^{k-1}\hbar\mathrm{Com}) \circ \mathrm{Lie} \\
&= s^{k-1}\hbar(\mathrm{Com} \circ (s^{1-k}\hbar^{-1}\mathrm{Lie})) \\
&= s^{k-1}\hbar P_k^{ac}.
\end{aligned}
$$

□

We now fix a CDGA R over $\mathbb{Q}$. There is a model structure on the category of $\mathbb{G}_m$-equivariant R-chain complexes (the projective model structure), in which fibrations are surjections.

The following lemma runs on similar lines to [Mel]:

Lemma 1.17. *Given a $\mathbb{G}_m$-equivariant P_k^{ac}-algebra A in R-chain complexes which is cofibrant as a $\mathbb{G}_m$-equivariant R-CDGA, together with a Beck A-module M, there is a model for $\mathbf{R}\underline{\mathrm{Der}}_{P_k^{ac}\otimes R, \mathbb{G}_m}(A, M)$ which has a complete decreasing filtration $F^1 \supset F^2 \supset \ldots$ with associated graded complexes*

$$\underline{\mathrm{Hom}}_A(\mathrm{CoS}^p_A((\Omega^1_{A/R})_{[-k]}), \hbar^{p-1}M)^{\mathbb{G}_m}_{[-k]},$$

where Ω^1 denotes the complex of Kähler differentials of the underlying CDGA, and $\mathrm{CoS}^p_A(N) := \mathrm{CoSymm}^p_A(N) = (N^{\otimes_A p})^{\Sigma_p}$ the cosymmetric powers.

Proof. Adapting the formulae of [LV, §11.2] to the R-linear setting, the Koszul duality map $\alpha\colon (s(P_k^{ac})^!)^\vee \to P_k^{ac}$ gives rise to a cobar-bar adjunction $\Omega_\alpha \dashv \mathrm{B}_\alpha$ between P_k^{ac}-algebras in R-chain complexes and $s(P_k^{ac})^!$-coalgebras in R-chain complexes. Moreover, $\Omega_\alpha\mathrm{B}_\alpha A$ is a cofibrant resolution of A [LV, Theorem 11.4.7], so

$$\mathbf{R}\underline{\mathrm{Der}}_{P_k^{ac}\otimes R, \mathbb{G}_m}(A, M) \simeq \underline{\mathrm{Der}}_{P_k^{ac}\otimes R, \mathbb{G}_m}(\Omega_\alpha\mathrm{B}_\alpha A, M).$$

Since $\Omega_\alpha C$ is freely generated by $C_{[1]}$, the latter complex can be rewritten as

$$(\underline{\mathrm{Hom}}_R(\mathrm{B}_\alpha A, M)^{\mathbb{G}_m}_{[-1]}, \delta + [\omega_A \circ \alpha, -]),$$

where ω_A is the data defining the P_k^{ac}-algebra structure on A and the bracket is defined via the convolution product of [LV, 6.4.4] (which is of weight 0 for the $\mathbb{G}_m$-action).

Substituting for B_α, we can rewrite this complex as

$$(\prod_n (\hbar^{n-1} P_k^{ac}(n)_{[k(n-1)]} \otimes^{\Sigma_n} \underline{\mathrm{Hom}}_R(A^{\otimes n}, M))^{\mathbb{G}_m}, \delta + [\omega_A \circ \alpha, -]),$$
$$\cong (\underline{\mathrm{Hom}}_R(\mathrm{CoSymm}^+(\hbar^{-1}\mathrm{CoLie}(A_{[-1]})_{[1-k]})_{[k]}\hbar, M)^{\mathbb{G}_m}, \delta + [\omega_A \circ \alpha, -]).$$

Since the differential on $\mathrm{B}_\alpha A$ is non-increasing on cosymmetric powers, it follows that this dual complex has a complete decreasing filtration F, with F^p given by terms involving CoSymm^i for $i \geq p$. The associated graded pieces are then given by

$$\mathrm{gr}_F^p \cong (\underline{\mathrm{Hom}}_R(\mathrm{CoSymm}^p(\hbar^{-1}\mathrm{CoLie}(A_{[-1]})_{[1-k]})_{[k]}\hbar, M)^{\mathbb{G}_m}, \delta + [\omega_A \circ \alpha, -]).$$

In order to proceed further, we use the Koszul duality map $\beta\colon (s\mathrm{Lie})^\vee \to \mathrm{Com}$, which as in [LV, §11] gives us a cofibrant resolution $A' := \Omega_{\mathrm{Com}}\mathrm{B}_{\mathrm{Com}}A$ of the CDGA underlying A, on generators $(\mathrm{B}_{\mathrm{Com}}A)_{[1]}$. Repeating the argument above for this instance of Koszul duality gives

$$\begin{aligned}
&(\underline{\mathrm{Hom}}_R(\mathrm{CoSymm}^p(\hbar^{-1}\mathrm{CoLie}(A_{[-1]})_{[1-k]})_{[k]}\hbar, M)^{\mathbb{G}_m}, \delta + [\omega_A \circ \alpha, -])\\
&\cong \underline{\mathrm{Hom}}_{A'}(\mathrm{CoS}^p_{A'}((\hbar^{-1}\Omega^1_{A'/R})_{[-k]})\hbar, M)^{\mathbb{G}_m}_{[-k]},\\
&\simeq \underline{\mathrm{Hom}}_{A}(\mathrm{CoS}^p_{A}((\hbar^{-1}\Omega^1_{A/R})_{[-k]})\hbar, M)^{\mathbb{G}_m}_{[-k]},
\end{aligned}$$

where the final line follows because A is cofibrant and $A' \to A$ a quasi-isomorphism. □

Before we state the next proposition, recall that the cotangent complex $\mathbf{L}\Omega^1$ for CDGAs is defined as the left-derived functor of Kähler differentials Ω^1. For a morphism $f\colon A \to B$ of CDGAs, that means $\mathbf{L}\Omega^1_{B/A}$ (defined up to B-linear quasi-isomorphism) is given by the B-module $\Omega^1_{\tilde{B}/A} \otimes_{\tilde{B}} B$ for any factorisation $A \to \tilde{B} \to B$ of f as a cofibration followed by a weak equivalence.

Proposition 1.18. *If B is a non-negatively weighted $\mathbb{G}_m$-equivariant P_k^{ac}-algebra over a CDGA R for which the map $(\mathcal{W}_1\mathbf{L}\Omega^1_{B/\mathcal{W}_0 B}) \otimes^{\mathbf{L}}_{\mathcal{W}_0 B} B \to \mathbf{L}\Omega^1_{B/\mathcal{W}_0 B}$ is a quasi-isomorphism, then*

$$\mathbf{R}\underline{\mathrm{Der}}_{P_k^{ac}\otimes R, \mathbb{G}_m}(B, M) \simeq 0$$

for all Beck B-modules M with $\mathcal{W}_i M \simeq 0$ for all $i < 2$.

Proof. Without loss of generality, we may assume that B is cofibrant. We then have an exact triangle

$$\Omega^1_{\mathcal{W}_0 B} \otimes_{\mathcal{W}_0 B} B \to \Omega^1_B \to \Omega^1_{B/\mathcal{W}_0 B} \to \Omega^1_{\mathcal{W}_0 B} \otimes_{\mathcal{W}_0 B} B_{[-1]},$$

which by hypothesis simplifies to

$$\Omega^1_{\mathcal{W}_0 B} \otimes_{\mathcal{W}_0 B} B \to \Omega^1_B \to (\mathcal{W}_1\Omega^1_{B/\mathcal{W}_0 B}) \otimes_{\mathcal{W}_0 B} B \to \Omega^1_{\mathcal{W}_0 B} \otimes_{\mathcal{W}_0 B} B_{[-1]}.$$

Since Ω^1_B is quasi-isomorphic to a B-module freely generated by terms of weights $0, 1$, we thus have that $\mathbf{R}\underline{\mathrm{Hom}}_B(\mathrm{CoS}^p_B((\Omega^1_{B/R})_{[-k]}), N)^{\mathbb{G}_m}$ is acyclic for any Beck module N

with weights $> p$, or more generally with $\mathcal{W}_i N$ acyclic for $i \le p$. Setting $N := \hbar^{p-1}M$, the statement now follows from the convergent spectral sequence

$$\mathrm{Ext}^j_B(\mathrm{CoS}^p_B((\Omega^1_{B/R})_{[-k]}), \hbar^{p-1}M)^{\mathbb{G}_m} \implies \mathrm{H}_{k-j}\mathbf{R}\underline{\mathrm{Der}}_{P_k^{ac}\otimes R, \mathbb{G}_m}(B, M)$$

arising from Lemma 1.17. □

Corollary 1.19. *The associated graded functor from the ∞-category of quasi-involutive a.c. P_k-algebras to $\mathbb{G}_m$-equivariant P_k^{ac}-algebras becomes an equivalence when restricted to those objects B with $W_{-1}B = 0$ for which the map*

$$(\mathrm{gr}^W_1 \mathbf{L}\Omega^1_{B/W_0B}) \otimes^{\mathbf{L}}_{W_0B} \mathrm{gr}^W B \to \mathbf{L}\Omega^1_{\mathrm{gr}^W B/W_0B}$$

of commutative cotangent complexes is a quasi-isomorphism.

The quasi-inverse functor is given by sending a $\mathbb{G}_m$-equivariant P_k^{ac}-algebra to the underlying filtered algebra.

Proof. Substitute Proposition 1.18 into Corollary 1.13. □

Remark 1.20. If we were to consider non-involutive deformations instead, then the analogue of Corollary 1.19 would not hold. The non-involutive analogue of Corollary 1.13 involves homology of $\mathbf{R}\underline{\mathrm{Der}}_{\mathcal{P},\mathbb{G}_m}(\mathrm{gr}^W B, M)$ for M concentrated in weights ≥ 1, giving a first term which is seldom acyclic. This is similar to phenomena arising in [Pri6, Pri4, Pri10], where the only obstruction to quantisation is first-order, and can be eliminated by restricting to involutive quantisations.

Remark 1.21. Following Remark 1.14, if we take an involutively filtered dg operad $(\mathcal{P}, W)$ with an involutive equivalence $\mathrm{gr}^W\mathcal{P} \simeq P_k^{ac}$, then the conditions of Corollary 1.19 also ensure (via the argument of [KS, Hin2]) that for B a non-negatively weighted $\mathbb{G}_m$-equivariant P_k^{ac}-algebra concentrated in non-negative homological degrees, the space of quasi-involutive a.c. derived $\mathcal{P}$-algebras (B', W) with $\mathrm{gr}^W B' \simeq B$ is contractible.

In particular, when $k = 1$ we can take $\mathcal{P}$ to be the BD_1 operad, given by the PBW filtration on the associative operad as in [CPT$^+$, §3.5.1] (named by analogy with the Beilinson–Drinfeld (BD or BD_0) algebras of [CG, §2.2]). The argument above then gives an essentially unique filtered associative dg algebra (B', W) equipped with a filtered involution $(B')^{\mathrm{opp}} \cong B'$ and an equivalence $\mathrm{gr}^W B' \simeq B$. When B is an algebra of polyvectors, B' will thus be given by the ring of differential operators $\mathscr{D}(\omega^{\frac{1}{2}})$ on a square root $\omega^{\frac{1}{2}}$ of the dualising bundle whenever this exists, and B' gives a ring of twisted differential operators generalising $\mathscr{D}(\omega^{\frac{1}{2}})$ even when the dualising complex is not a line bundle, or has no square root.

2. Quantisations on derived Deligne–Mumford stacks

2.1. Hochschild complexes.

Definition 2.1. For a cofibrant CDGA A over R, we define the filtered chain complex

$$(\mathrm{CC}_{R,\oplus}(A)_\bullet, \tau^{\mathrm{HH}})$$

to be the direct sum total complex (written as a chain complex) of the double complex $\underline{\mathrm{CC}}^\bullet_R(A)$ given by

$$\underline{\mathrm{CC}}^n_R(A) = \underline{\mathrm{Hom}}_R(A^{\otimes_R n}, A),$$

with Hochschild differential $b\colon \underline{\mathrm{CC}}^{n-1} \to \underline{\mathrm{CC}}^{n}$ given by

$$\begin{aligned}(bf)(a_1,\ldots,a_n) =& a_1 f(a_2,\ldots,a_n)\\ &+\sum_{i=1}^{n-1}(-1)^i f(a_1,\ldots,a_{i-1},a_ia_{i+1},a_{i+2},\ldots,a_n)\\ &+(-1)^n f(a_1,\ldots,a_{n-1})a_n.\end{aligned}$$

The filtration τ^{HH} on $\underline{\mathrm{CC}}^{\bullet}_R(A)$ and $\mathrm{CC}_{R,\oplus}(A)_\bullet$ is given by good truncation in the Hochschild direction, so $\tau^{\mathrm{HH}}_p\mathrm{CC}_{R,\oplus}(A)_\bullet \subset \mathrm{CC}_{R,\oplus}(A)_\bullet$ is the subspace

$$\prod_{i=0}^{p-1}\underline{\mathrm{Hom}}(A^{\otimes_R i},A)_{[i]} \times \ker(b\colon \underline{\mathrm{Hom}}(A^{\otimes_R p},A) \to \underline{\mathrm{Hom}}(A^{\otimes_R (p+1)},A))_{[p]}.$$

Remarks 2.2. Beware that $\mathrm{CC}_{R,\oplus}(A)_\bullet$ is usually a proper subcomplex of the cohomological Hochschild complex $\mathrm{CC}_{R,\Pi}(A)_\bullet$ of A, which is defined by taking the product, rather than sum, total complex. Since $\mathrm{CC}_{R,\oplus}(A)_\bullet = \bigcup_p \tau^{\mathrm{HH}}_p\mathrm{CC}_{R,\Pi}(A)_\bullet$, our constructions will be consistent with those of [Pri4].

There is a filtration γ from [Pri10, Definition 1.18] which is quasi-isomorphic to τ^{HH} under the cofibrancy hypothesis of Definition 2.1, but which also behaves well whenever A is cofibrant as an R-module (rather than as an R-CDGA) and gives an involutive, rather than quasi-involutive, filtration analogous to Lemma 2.8. For simplicity, we will just use τ^{HH}.

Recall that a brace algebra B is a chain complex equipped with a cup product in the form of a chain map

$$B\otimes B \xrightarrow{\smile} B,$$

and braces in the form of maps

$$\{-\}\{-,\ldots,-\}_r\colon\ B\otimes B^{\otimes r} \to B_{[r]}$$

satisfying the conditions of [Vor, §3.2] (where brace algebras are called homotopy G-algebras) with respect to the differential. The commutator of the brace $\{-\}\{-\}_1$ is a Lie bracket, so for any brace algebra B, there is a natural DGLA structure on $B_{[-1]}$.

There is a natural brace algebra structure on $\mathrm{CC}_{R,\oplus}(A)$ by [Vor, Theorem 3.1].

The following are taken from [Pri10, §1.2.1]:

Definition 2.3. Given a brace algebra B, define the opposite brace algebra B^{opp} to have the same elements as B, but multiplication $b^{\mathrm{opp}} \smile c^{\mathrm{opp}} := (-1)^{\deg b\deg c}(c\smile b)^{\mathrm{opp}}$ and brace operations given by the multiplication $(\mathrm{B}B^{\mathrm{opp}})\otimes(\mathrm{B}B^{\mathrm{opp}}) \to \mathrm{B}B^{\mathrm{opp}}$ on the bar construction induced by the isomorphism $(\mathrm{B}B^{\mathrm{opp}}) \cong (\mathrm{B}B)^{\mathrm{opp}}$, the opposite coalgebra. Explicitly,

$$\{b^{\mathrm{opp}}\}\{c_1^{\mathrm{opp}},\ldots,c_m^{\mathrm{opp}}\} := \pm\{b\}\{c_m,\ldots,c_1\}^{\mathrm{opp}},$$

where $\pm = (-1)^{m(m+1)/2+(\deg b-m)(\sum_i \deg c_i - m)+\sum_{i<j}\deg c_i\deg c_j}$.

Observe that when a filtered brace algebra B is almost commutative, then so is B^{opp}.

Definition 2.4. Define a filtration τ on the brace operad Br of [Vor] by good truncation aritywise, so $(\tau_{-p}\mathrm{Br})(n) := \tau_{\geq p}\mathrm{Br}(n)$

We refer to (brace, τ)-algebras in filtered complexes as almost commutative brace algebras. We define a quasi-involutive a.c. brace algebra to be an almost commutative brace algebra (B, F) equipped with an involution $(B, F) \cong (B^{\mathrm{opp}}, F)$ of (brace, τ)-algebras which acts on $\mathrm{H}_*(\mathrm{gr}^F_i B)$ as multiplication by $(-1)^i$.

Lemma 2.5. *The filtered Hochschild complex* $(\mathrm{CC}_{R,\oplus}(A)_\bullet, \tau^{\mathrm{HH}})$ *naturally has the structure of an almost commutative brace algebra.*

Proof. The double complex $\underline{\mathrm{CC}}^\bullet_R(A)$ is a brace algebra in the cochain direction, as a consequence of [Vor, Theorem 3.1]. Compatibility of tensor products with good truncation then makes $(\underline{\mathrm{CC}}^\bullet_R(A), \tau^{\mathrm{HH}})$ a (Br, τ)-algebra in double complexes, and passing to total complexes gives the required result. □

Definition 2.6. Define $\mathrm{Pol}(A/R, 0)$ to be the $\mathbb{G}_m$-equivariant P_2^{ac}-algebra $\bigoplus_{p\geq 0} \underline{\mathrm{Hom}}_A(\Omega^p_{A/R}, A)_{[p]}$ with the obvious graded-commutative multiplication and the Schouten–Nijenhuis Lie bracket, given by interpreting $\underline{\mathrm{Hom}}_A(\Omega^p_{A/R}, A)$ as the complex of antisymmetric p-derivations $A^p \to A$.

Lemma 2.7. *For a cofibrant R-CDGA A, the HKR isomorphism gives a quasi-isomorphism* $\mathrm{HKR}\colon \mathrm{gr}^{\tau^{\mathrm{HH}}}\mathrm{CC}_{R,\oplus}(A)_\bullet \to \mathrm{Pol}(A/R, 0)$ *of $\mathbb{G}_m$-equivariant* $\mathrm{gr}^\tau\mathrm{Br}$*-algebras, where the* $\mathrm{gr}^\tau\mathrm{Br}$*-algebra structure on* $\mathrm{Pol}(A/R, 0)$ *comes from the quasi-isomorphism* $\mathrm{gr}^\tau\mathrm{Br} \to H_*\mathrm{Br} \cong P_2$ *of dg operads from* [MS1].

Proof. By turning the structural differential δ on A off and on again, the HKR isomorphism gives us a quasi-isomorphism $(\mathrm{HH}^p_R(A_\#), \delta) \simeq \underline{\mathrm{Hom}}_A(\Omega^p_A, A)$, since A is cofibrant[3]. Thus $\mathrm{gr}_p^{\tau^{\mathrm{HH}}}\underline{\mathrm{CC}}^\bullet_R(A)$ maps quasi-isomorphically to $\underline{\mathrm{Hom}}_A(\Omega^p_A, A)^{[-p]}$.

Now, the action of $\mathrm{gr}^\tau\mathrm{Br}$ on $\mathrm{HH}^*_R(A_\#)$ factors through its quotient $\mathrm{H}_*\mathrm{Br} \cong P_2$. This operad is generated by the commutative product and Lie bracket, and a simple check shows that the cup product and Gerstenhaber bracket on Hochschild cohomology act in the prescribed fashion under the HKR isomorphism. □

Lemma 2.8. *Given a cofibrant CDGA A over R, there is an anti-involution*

$$-i\colon \mathrm{CC}_{R,\oplus}(A)^{\mathrm{opp}} \to \mathrm{CC}_{R,\oplus}(A)$$

making $(\mathrm{CC}_{R,\oplus}(A), \tau^{\mathrm{HH}})$ *into a quasi-involutive a.c. brace algebra.*

Proof. The (brace, τ)-algebra structure is given by Lemma 2.5. The involution is given in [Bra, §2.1], and comes from the antipode on the cofree tensor coalgebra $\bigoplus(A_{[-1]})^{\otimes n}$ regarded as the universal enveloping coalgebra of a cofree graded Lie coalgebra. Quasi-involutivity is then a simple calculation as in [Pri4, Lemma 1.15], and compatibility with the brace structure follows as in [Pri4, Remark 2.22]. □

2.2. **Polydifferential operators.** There is a variant of the Hochschild complex defined using polydifferential operators instead of R-linear maps. In the algebraic setting it is quasi-isomorphic to the Hochschild complex, so leads to the same theory, but in smooth and analytic settings it gives a much better behaved object.

The following adapts [CR] as in [Pri9, Pri5]:

Definition 2.9. Define a $\mathcal{C}^\infty$-DGA (over a base $R = \mathbb{R}$), resp. an EFC-DGA (over a base $R = K$, a complete valued field), to be an R-CDGA for which:

[3]In fact, it suffices for A to be quasi-smooth in the sense of [Kon1] (not to be confused with the clashing sense of [Toë1]) or even ind-quasi-smooth, i.e. for A_0 to be ind-smooth and the graded algebra $A_\#$ to be freely, or even projectively, generated over it. The cofibrant hypothesis on A can be accordingly relaxed throughout this paper.

- A_0 is equipped with a $\mathcal{C}^\infty$-ring structure, resp. entire functional calculus, enhancing its commutative R-algebra structure, and
- the derivation $\delta\colon A_0 \to A_{-1}$ is a $\mathcal{C}^\infty$-derivation, resp. EFC-derivation.

A morphism $A \to B$ of $\mathcal{C}^\infty$-DGAs, resp. EFC-DGAs, is then a morphism of R-CDGAs preserving the $\mathcal{C}^\infty$ (resp. EFC) structure.

A $\mathcal{C}^\infty$ (resp. EFC) structure on A_0 is a product-preserving set-valued functor $\mathbb{R}^n \mapsto (A_0)^n$ (resp. $K^n \mapsto (A_0)^n$) on the category with objects $\{\mathbb{R}^n\}_{n\geq 0}$ (resp. $\{K^n\}_{n\geq 0}$) and morphisms consisting of $\mathcal{C}^\infty$ (resp. K-analytic) maps. Explicitly, the set A_0 is equipped, for every $\mathcal{C}^\infty$ (resp. analytic) function f in n variables with an operation $\Phi_f\colon (A_0)^n \to A_0$. The condition on $\delta\colon A_0 \to A_{-1}$ then says that the map $(\mathrm{id}, \delta)\colon A_0 \to A_0 \times A_{-1}$ is a $\mathcal{C}^\infty$ (resp. EFC) morphism.

As in [CR, Theorem 6.10], the corresponding categories of algebras carry cofibrantly generated model structures obtained by adjunction from the model category of chain complexes, so weak equivalences are quasi-isomorphisms and fibrations are surjections. For most applications, it suffices to know that examples of cofibrant $\mathcal{C}^\infty$-DGAs and EFC-DGAs A are given by CDGAs $A = A_{\geq 0}$ which are freely generated as graded-commutative A_0-algebras, with A_0 respectively a ring of smooth functions on $\mathbb{R}^n$ or a ring of analytic functions on K^n.[4] Morphisms are CDGA morphisms with additional restrictions in degree 0 corresponding to $\mathcal{C}^\infty$ (resp. holomorphic) morphisms between manifolds.

Both $\mathcal{C}^\infty$-DGAs and EFC-DGAs similarly have notions of derivations which are more restrictive than those of the underlying CDGAs, with resulting modules of differential forms; for rings of functions on manifolds, these correspond respectively to smooth and analytic forms on the manifold. This leads to a notion of differential operators, which have the expected form in terms of co-ordinates as in [Pri5, Definition 3.1]; for intrinsic definitions, see [Pri5, Remark 3.2].

Definition 2.10. For a cofibrant R-CDGA (resp. $\mathcal{C}^\infty$-DGA, resp. EFC-DGA) A, we define the filtered chain complex

$$(D^{\mathrm{poly}}_{\oplus}(A)_\bullet, \tau^{\mathrm{HH}})$$

as follows. First let

$$\underline{D}^{\mathrm{poly},n}(A) := \mathcal{D}\mathit{iff}(A^{\coprod n}, A),$$

the complex of R-linear algebraic (resp. $\mathcal{C}^\infty$, resp. EFC) differential operators from the n-fold coproduct $A^{\coprod n}$ (i.e. $A^{\otimes_R n}$ in the algebraic case) to the $A^{\coprod n}$-module A. This carries a Hochschild differential $b\colon \underline{D}^{\mathrm{poly},n-1} \to \underline{D}^{\mathrm{poly},n}$ given by the same formula as in Definition 2.1, leading to a double complex $\underline{D}^{\mathrm{poly},\bullet}(A)$. We then define $D^{\mathrm{poly}}_{\oplus}(A)_\bullet$ to be the direct sum total complex of $\underline{D}^{\mathrm{poly},\bullet}(A)$.

The filtration τ^{HH} on $\underline{D}^{\mathrm{poly},\bullet}(A)$ and $\mathcal{D}^{\mathrm{poly}}_{\oplus}(A)_\bullet$ is given by good truncation in the Hochschild direction.

The subcomplex $D^{\mathrm{poly}}_{\oplus}(A)_\bullet \subset \mathrm{CC}_{R,\oplus}(A)$ is clearly closed under the brace operations, where R is taken to be $\mathbb{R}$ (resp. K) in the $\mathcal{C}^\infty$ (resp. EFC) setting, so $D^{\mathrm{poly}}_{\oplus}(A)_\bullet$ is a brace algebra over R.

[4] In the remainder of the paper, cofibrancy is only used to give the correct cotangent complexes, so the condition on A can be relaxed by asking only that A_0 consist respectively of functions on a smooth manifold or holomorphic functions on a Stein manifold, and that $A = A_{\geq 0}$ be projectively generated as a graded-commutative A_0-algebra.

Definition 2.11. Given a $\mathcal{C}^\infty$-DGA A, resp. EFC-DGA A over K, define $\mathrm{Pol}(A,0)$ to be the $\mathbb{G}_m$-equivariant P_2^{ac}-algebra of $\mathcal{C}^\infty$, resp. EFC, polyvectors of A, defined by replacing the $\Omega^1_{A/R}$ in Definition 2.6 with the complexes $\Omega^1_{A,\mathcal{C}^\infty}$ and $\Omega^1_{A/K,\mathrm{EFC}}$ of $\mathcal{C}^\infty$ and EFC differentials, respectively.

Lemma 2.12. *For any cofibrant R-CDGA (resp. $\mathcal{C}^\infty$-DGA, resp. EFC-DGA) A, we have a natural quasi-isomorphism* $\mathrm{HKR}\colon \mathrm{gr}^{\tau^{\mathrm{HH}}} D^{\mathrm{poly}}_{\oplus}(A)_\bullet \to \mathrm{Pol}(A/R,0)$ *of $\mathbb{G}_m$-equivariant* $\mathrm{gr}^\tau\mathrm{Br}$*-algebras.*

Proof. Observe that for the order filtration F on differential operators, we have $\mathrm{gr}^F_j \underline{D}^{\mathrm{poly},n}(A) \cong \underline{\mathrm{Hom}}_A(\mathrm{Symm}^j_A((\Omega^1_A)^{\oplus n}), A)$. Thus $\mathrm{gr}^F \underline{D}^{\mathrm{poly},n}(A)$ is the A-linear dual of the free A-algebra generated by $(\Omega^1_A)^{\oplus n}$, and the Hochschild differential on these copies of Ω^1_A is easily seen to act as the total differential of the simplicial nerve of Ω^1_A. Thus the Eilenberg–Zilber theorem gives quasi-isomorphisms

$$\mathrm{gr}^F_p D^{\mathrm{poly}}_{\oplus}(A)_\bullet \to \underline{\mathrm{Hom}}_A(\mathrm{Symm}^p_A((\Omega^1_A)_{[-1]}), A) \cong \underline{\mathrm{Hom}}_A(\Omega^p_A), A)_{[p]}.$$

These are compatible with τ^{HH}, so the restriction to $D^{\mathrm{poly}}_{\oplus}(A)_\bullet$ of the HKR map is a quasi-isomorphism. The rest of the proof follows as in Lemma 2.7. $\square$

Lemma 2.13. *Given a cofibrant R-CDGA (resp. $\mathcal{C}^\infty$-DGA, resp. EFC-DGA) A, there is an anti-involution*

$$-i\colon D^{\mathrm{poly}}_{\oplus}(A)^{\mathrm{opp}}_\bullet \to D^{\mathrm{poly}}_{\oplus}(A)_\bullet$$

making $(D^{\mathrm{poly}}_{\oplus}(A)^{\mathrm{opp}}_\bullet(A), \tau^{\mathrm{HH}})$ into a quasi-involutive a.c. brace algebra.

Proof. Most of the properties follow immediately from Lemma 2.8, since $D^{\mathrm{poly}}_{\oplus}(A)_\bullet \subset \mathrm{CC}_{R,\oplus}(A)$ is a brace subalgebra. It only remains to show that the involution acts as $(-1)^j$ on $\mathrm{H}_*\mathrm{gr}^{\tau^{\mathrm{HH}}}_j D^{\mathrm{poly}}_{\oplus}(A)^{\mathrm{opp}}_\bullet$. By Lemma 2.12, these groups are $\mathrm{H}_{*+p}\underline{\mathrm{Hom}}_A(\Omega^p_A), A)$, on which the induced involution acts as $(-1)^j$ by [Pri4, Lemma 1.15], giving quasi-involutivity. $\square$

2.3. **Involutions from the Grothendieck–Teichmüller group.** The good truncation filtration τ of Definition 2.4 is defined similarly on any dg operad in place of Br, and in particular there is a good truncation on the quasi-isomorphic $\mathbb{Q}$-linear operad $\mathrm{C}_\bullet(E_2,\mathbb{Q})$ of chains on the topological operad E_2.[5]

As summarised in the preamble to the theorem in [Pet], the space of homotopy automorphisms of $\mathrm{C}_\bullet(E_2,\mathbb{Q})$ as an operad in dg coalgebras (equivalently, as a dg Hopf operad) is homotopy equivalent the Grothendieck–Teichmüller group $\mathrm{GT}(\mathbb{Q})$. By [MS1, Theorem 1.1], the brace operad Br is quasi-isomorphic to $\mathrm{C}_\bullet(E_2,\mathbb{Q})$, so it also admits an action by GT.

Lemma 2.14. *The involution of the brace operad* Br *which sends a brace algebra to its opposite (Definition 2.3) is induced under the Grothendieck–Teichmüller action above by an element $t \in \mathrm{GT}(\mathbb{Q})$ of order* 2.

Proof. In the notation of [Saf], [CW] gives an equivalence $\mathrm{Br} = \mathrm{Br}_{\mathrm{coAL}_\infty} \simeq \mathrm{C}_\bullet(E_1,\mathbb{Q}) \otimes_{BV} \mathrm{C}_\bullet(E_1,\mathbb{Q})$ of dg operads, where $\otimes_{BV}$ is the Boardman–Vogt tensor product. For the involution ι of E_1 which sends an associative algebra to its opposite, it follows from the description of Definition 2.3 that our involution is just $\mathrm{id} \otimes_{BV} \iota$.

[5]The latter is presumably what is meant by the filtration given by the Postnikov tower in [CPT+, §3.5.1], since the Postnikov tower itself consists of quotients rather than subspaces, with kernels given by good truncation.

In particular, this means that it comes from an involution of the topological operad $E_1 \otimes_{BV} E_1 \simeq E_2$, so *a fortiori* gives an involution of $\mathrm{C}_\bullet(E_2, \mathbb{Q})$ as a rational homotopy operad, hence an element $t \in \mathrm{GT}(\mathbb{Q})$ of order 2; it cannot be trivial because it acts non-trivially on $\mathrm{gr}^\tau \mathrm{Br} \simeq P_2$. In fact, since t is an involution of E_2 itself, the proof of [Hor, Theorem 7.1] implies that t must be the unique non-trivial object-preserving automorphism of the groupoid of parenthesised braids from [Dri, Proposition 4.1]. □

Definition 2.15. Denote the pro-unipotent radical of the pro-algebraic group GT (defined over $\mathbb{Q}$) by GT^1. Write $\mathrm{Levi}_{\mathrm{GT}}$ for the set of Levi decompositions $\mathrm{GT} \cong \mathbb{G}_m \ltimes \mathrm{GT}^1$; equivalently, this is the set of sections of the natural map $\mathrm{GT} \to \mathbb{G}_m$. We then define $\mathrm{Levi}^t_{\mathrm{GT}} \subset \mathrm{Levi}_{\mathrm{GT}}$ to be the set of sections w satisfying $w(-1) = t$.

Taking base change to arbitrary commutative $\mathbb{Q}$-algebras A, we can extend this definition to give sets $\mathrm{Levi}_{\mathrm{GT}}(A), \mathrm{Levi}^t_{\mathrm{GT}}(A)$ of decompositions of the pro-algebraic group $\mathrm{GT} \times \mathrm{Spec}\, A$ over A.

Lemma 2.16. *The functor* $\mathrm{Levi}^t_{\mathrm{GT}}$ *is an affine scheme over* $\mathbb{Q}$ *equipped with the structure of a trivial torsor for the subgroup scheme* $(\mathrm{GT}^1)^t \subset \mathrm{GT}^1$ *given by the centraliser of* t.

Proof. We expand the argument from [Pri4, Remark 2.22]. By the general theory [HM] of pro-algebraic groups in characteristic 0, the set $\mathrm{Levi}_{\mathrm{GT}}(\mathbb{Q})$ is non-empty, and for all commutative $\mathbb{Q}$-algebras A, the group $\mathrm{GT}^1(A)$ acts transitively on $\mathrm{Levi}_{\mathrm{GT}}(A)$ via the adjoint action. Because the graded quotients of the lower central series of GT^1 have non-zero weight for the adjoint $\mathbb{G}_m$-action, the centralisers of this action are trivial and $\mathrm{Levi}_{\mathrm{GT}}(A)$ is a torsor for $\mathrm{GT}^1(A)$.

Now, choose any Levi decomposition $w_0 \in \mathrm{Levi}_{\mathrm{GT}}(\mathbb{Q})$ and let $w_0(-1) = tu$ for $u \in \mathrm{GT}^1(\mathbb{Q})$. Since t and $w_0(-1)$ are both of order 2, we have $u = \mathrm{ad}_t(u^{-1})$. Writing $u = \exp(v)$ for v in the pro-nilpotent Lie algebra $\mathfrak{gt}^1$, and setting $u^{\frac{1}{2}} := \exp(\frac{1}{2}v)$, we have $u^{\frac{1}{2}} = \mathrm{ad}_t(u^{-\frac{1}{2}})$, giving $w := \mathrm{ad}_{u^{-\frac{1}{2}}} \circ w_0 \in \mathrm{Levi}^t_{\mathrm{GT}}(\mathbb{Q})$. Thus $\mathrm{Levi}^t_{\mathrm{GT}}$ is non-empty, so $\mathrm{Levi}^t_{\mathrm{GT}} \subset \mathrm{Levi}_{\mathrm{GT}}$ is a torsor for the subgroup $(\mathrm{GT}^1)^t \subset \mathrm{GT}^1$ fixing t under the adjoint action. □

As in [Pri4, §2.2], $\mathrm{Levi}_{\mathrm{GT}}$ is naturally isomorphic to the space of Drinfeld 1-associators, and the description of the automorphism t from the proof of Lemma 2.14 then implies that the subspace $\mathrm{Levi}^t_{\mathrm{GT}} \subset \mathrm{Levi}_{\mathrm{GT}}$ corresponds to 1-associators which are even.

Proposition 2.17. *Every* 1*-associator* $w \in \mathrm{Levi}_{\mathrm{GT}}$ *induces a zigzag* θ_w *of filtered quasi-isomorphisms between* (Br, τ) *and* (P_2, τ) *If the* 1*-associator is even (i.e.* $w \in \mathrm{Levi}^t_{\mathrm{GT}}$*), then the quasi-isomorphisms preserve the respective involutions, given by* $-1 \in \mathbb{G}_m$ *on* P_2^{ac} *and by Definition 2.3 on* Br.

The quasi-isomorphism θ_w *is compatible with the natural maps* $s^{-1}\mathrm{Lie} \to \mathscr{P}_2$ *and* $s^{-1}\mathrm{Lie} \to \mathrm{Br}$ *from the shifted Lie operad* $s^{-1}\mathrm{Lie}$.

Proof. By [MS1, Theorem 1.1], we have a zigzag of quasi-isomorphisms between Br and $\mathrm{C}_\bullet(E_2, \mathbb{Q})$, and hence between Br and $\mathrm{C}_\bullet(B\mathrm{PaB}, \mathbb{Q})$, the dg operad of chains on the operad of parenthesised braids. The proof of Lemma 2.14 shows that these quasi-isomorphisms can be chosen $\mathbb{Z}/2$-equivariantly, with Br carrying the involution of Definition 2.3 and PaB its unique non-trivial involution given by $t \in \mathrm{GT}(\mathbb{Q})$.

As explained succinctly in [Pet], formality of the operad $C_\bullet(B\mathrm{PaB},\mathbb{Q})$ is a consequence of its GT-action and the observation that the Grothendieck–Teichmüller group GT is a pro-unipotent extension of $\mathbb{G}_m$. The pro-unipotent radical $\mathrm{GT}^1 = \ker(\mathrm{GT} \to \mathbb{G}_m)$ acts trivially on homology $\mathrm{H}_*(B\mathrm{PaB},\mathbb{Q}) \cong P_2$, inducing a $\mathbb{G}_m$-action on P_2.

Since the $\mathbb{G}_m$-action has weight 0 on $\mathrm{H}_0\mathrm{Br}(2)$ and weight -1 on $\mathrm{H}_1\mathrm{Br}(2)$, and since these generate the whole operad, it follows that the $\mathbb{G}_m$-action on $\mathrm{H}_j\mathrm{Br}(k)$ has weight $-j$ for all j,k. Thus we have a $\mathbb{G}_m$-equivariant isomorphism

$$\mathrm{H}_*(B\mathrm{PaB},\mathbb{Q}) \cong P_2^{ac}$$

with the $\mathbb{G}_m$-equivariant operad $P_2^{ac} = \mathrm{Com}\circ s^{-1}\hbar^{-1}\mathrm{Lie}$ of Definition 1.15. The element $t \in \mathrm{GT}(\mathbb{Q})$ lies over $-1 \in \mathbb{G}_m(\mathbb{Q})$, so the involution t on Br induces the action of $-1 \in \mathbb{G}_m$ on P_2^{ac} under the isomorphism above.

Any Levi decomposition $w\colon \mathbb{G}_m \to \mathrm{GT}$ gives a $\mathbb{G}_m$-action on $C_\bullet(B\mathrm{PaB},\mathbb{Q})$, i.e. a weight decomposition. As in [Pet], the weight decomposition associated to w thus induces a zigzag of $\mathbb{G}_m$-equivariant quasi-isomorphisms between $C_\bullet(B\mathrm{PaB},\mathbb{Q})$ and P_2^{ac}. The involution $-1 \in \mathbb{G}_m$ of the latter then necessarily corresponds to the involution $w(-1) \in \mathrm{GT}$ of the former.

These quasi-isomorphisms of operads combine to give θ_w, and the quasi-isomorphisms respect the involutions when $w(-1) = t$. All quasi-isomorphisms automatically respect the good truncation filtrations, so θ_w is a filtered quasi-isomorphism.

Finally, the natural morphism from the Lie operad to Br is given in each arity by inclusion of the top weight term for the decreasing filtration, i.e.

$$\{\mathrm{Br}(n)\}_n \leftarrow \{\tau_{\geq n-1}\mathrm{Br}(n)\}_n \xrightarrow{\sim} \{\mathrm{H}_{n-1}\mathrm{Br}(n)_{[1-n]}\}_n \cong \{\mathrm{Lie}(n)_{[1-n]}\}_n = \{s^{-1}\mathrm{Lie}(n)\}_n.$$

The same construction applied to each dg operad in the zigzag ensures that our map $(P_2,\tau) \to (\mathrm{Br},\tau)$ in the homotopy category of filtered dg operads respects the natural maps from the shifted Lie operad on each side. □

Definition 2.18. Given a Levi decomposition $w \in \mathrm{Levi}_{\mathrm{GT}}(\mathbb{Q})$ of the pro-algebraic group GT over $\mathbb{Q}$, we denote by p_w the ∞-functor from almost commutative brace algebras to almost commutative P_2-algebras coming from the map $\theta_w\colon (P_2,\tau) \to (\mathrm{Br},\tau)$. This preserves the underlying filtered L_∞-algebras up to equivalence.

When $w \in \mathrm{Levi}^t_{\mathrm{GT}}(\mathbb{Q})$, this induces an ∞-functor from quasi-involutive a.c. brace algebras to quasi-involutive a.c. P_2-algebras, which we also denote by p_w.

Explicitly, Proposition 2.17 gives us a filtered quasi-isomorphism $\tilde{\theta}_w\colon \tilde{P}_2^{ac} \to (\mathrm{Br},\tau)$ from any $\mathbb{G}_m$-equivariant cofibrant replacement $\tilde{P}_2^{ac}$ of $\tilde{P}_2$, and $\tilde{\theta}_w$ is involutive when w is even. This automatically gives any involutive a.c. brace algebra the structure of an involutive a.c. $\tilde{P}_2^{ac}$-algebra, and we can then obtain p_w by applying the derived left adjoint of the forgetful functor from $\mathrm{Rees}_W^{-1\in\mathbb{G}_m}(P_2^{ac})$-algebras to $\mathrm{Rees}_W^{-1\in\mathbb{G}_m}(\tilde{P}_2^{ac})$-algebras induced by the morphism $\tilde{P}_2^{ac} \to P_2^{ac}$.

2.4. **Existence of deformation quantisations.**

2.4.1. *Formality.*

Lemma 2.19. *Take a cofibrant R-CDGA (resp. $\mathcal{C}^\infty$-DGA, resp. K-EFC-DGA) A with perfect cotangent complex $\Omega^1_{A/R}$ (resp. $\Omega^1_{A,\mathcal{C}^\infty}$, resp. $\Omega^1_{A/K,\mathrm{EFC}}$). For the non-negatively weighted $\mathbb{G}_m$-equivariant P_2^{ac}-algebra* $\mathrm{Pol}(A,0)$ *of polyvectors (of Definition 2.6, resp. 2.11), the map*

$$(\mathcal{W}_1\mathbf{L}\Omega^1_{\mathrm{Pol}(A,0)/A}) \otimes^{\mathbf{L}}_A \mathrm{Pol}(A,0) \to \mathbf{L}\Omega^1_{\mathrm{Pol}(A,0)/A}$$

of commutative cotangent complexes is a quasi-isomorphism.

Proof. Since $\Omega^1_{A/R}$ (resp. $\Omega^1_{A,\mathcal{C}^\infty}$, resp. $\Omega^1_{A/K,\mathrm{EFC}}$) is perfect, we have a $\mathbb{G}_m$-equivariant quasi-isomorphism

$$\bigoplus_{p\geq 0}\mathbf{L}\mathrm{Symm}^p_A(\underline{\mathrm{Hom}}_A(\Omega^1_{A/R},A)_{[1]}) \to \bigoplus_{p\geq 0}\underline{\mathrm{Hom}}_A(\Omega^p_{A/R},A)_{[p]}$$
$$\mathbf{L}\mathrm{Symm}_A(\mathcal{W}_1\mathrm{Pol}(A,0)) \to \mathrm{Pol}(A,0)$$

of CDGAs, from which the statement follows immediately. □

Theorem 2.20. *Given a cofibrant R-CDGA (resp. $\mathcal{C}^\infty$-DGA or K-EFC-DGA) A with perfect cotangent complex $\Omega^1_{A/R}$ (resp. $\Omega^1_{A,\mathcal{C}^\infty}$ or $\Omega^1_{A/K,\mathrm{EFC}}$), the quasi-involutively filtered DGLA underlying the complex of polydifferential operators $D^{\mathrm{poly}}_{\oplus}(A)_{[-1]}$ with the increasing filtration τ^{HH} of Definition 2.10 is filtered quasi-isomorphic to the graded DGLA $\mathrm{Pol}(A,0)_{[-1]}$ from Definition 2.6 (resp. Definition 2.11).*

This quasi-isomorphism depends only on a choice of even 1-associator $w \in \mathrm{Levi}^t_{\mathrm{GT}}$, and is natural with respect to all morphisms $(\mathcal{D}^{\mathrm{poly}}_{\oplus}(A),\tau^{\mathrm{HH}}) \to (\mathcal{D}^{\mathrm{poly}}_{\oplus}(A'),\tau^{\mathrm{HH}})$ in the ∞-category of quasi-involutive a.c. brace algebras.

When A is an R-CDGA with perfect cotangent complex, the same statements hold for the Hochschild complex $\mathrm{CC}_{R,\oplus}(A)$ in place of $D^{\mathrm{poly}}_{\oplus}(A)$.

Proof. Lemma 2.13 shows that $(D^{\mathrm{poly}}_{\oplus}(A),\tau^{\mathrm{HH}})$ is a quasi-involutive a.c. brace algebra. We have a quasi-isomorphism $\mathrm{HKR}\colon \mathrm{gr}^{\tau^{\mathrm{HH}}}D^{\mathrm{poly}}_{\oplus}(A) \xrightarrow{\sim} \mathrm{Pol}(A,0)$ of graded $\mathrm{gr}^{\tau}\mathrm{Br}$-algebras by Lemma 2.12.

Lemmas 2.8 and 2.7 give the corresponding statements for $\mathrm{CC}_{R,\oplus}(A)$ in the algebraic setting; since $D^{\mathrm{poly}}_{\oplus}(A) \to \mathrm{CC}_{R,\oplus}(A)$ is thus a filtered quasi-isomorphism, it suffices to prove the statements for polydifferential operators in place of the Hochschild complex.

Applying the ∞-functor p_w of Definition 2.18 for some even associator $w \in \mathrm{Levi}^t_{\mathrm{GT}}(\mathbb{Q})$ (or even a point in the space $\mathrm{Levi}^t_{\mathrm{GT}}(R)$) gives a quasi-involutive a.c. P_2-algebra $p_wD^{\mathrm{poly}}_{\oplus}(A)$ together with a zigzag of quasi-isomorphisms of $\mathbb{G}_m$-equivariant P^{ac}_2-algebras between its associated graded algebra and $\mathrm{Pol}(A,0)$.

By Lemma 2.19, $\mathbf{L}\Omega^1_{\mathrm{Pol}(A,0)/A}$ satisfies the conditions of Corollary 1.19, giving an essentially unique equivalence $\alpha_{w,A}\colon p_wD^{\mathrm{poly}}_{\oplus}(A) \simeq p_w\mathrm{gr}^{\tau^{\mathrm{HH}}}D^{\mathrm{poly}}_{\oplus}(A)$ of involutive a.c. P_2-algebras, natural with respect to quasi-involutive a.c. brace algebra morphisms of $D^{\mathrm{poly}}_{\oplus}(A)$. Composition with the HKR quasi-isomorphism then gives us the required equivalence

$$\mathrm{HKR}\circ\alpha_{w,A}\colon p_wD^{\mathrm{poly}}_{\oplus}(A) \simeq \mathrm{Pol}(A,0).$$ □

Remark 2.21. When applied to polynomial rings in the algebraic setting and to $\mathcal{C}^\infty(\mathbb{R}^n)$ in the smooth setting, the statement of Theorem 2.20 recovers [Kon2, Theorem 4] and [Tam, §3]. For more general smooth varieties it recovers [VdB, Theorem 1.1]. The preliminary steps are the same, but the arguments for eliminating the potential first-order deformation are very different, as we consider anti-involutive deformations while Tamarkin and Kontsevich looked at invariance under affine transformations, which do not exist even locally for our more general rings.

Complete intersection singularities have perfect cotangent complexes, so Theorem 2.20 also promotes Kontsevich's quasi-isomorphism from [Frø, Appendix, Proposition 1] to an L_∞ quasi-isomorphism.

2.4.2. *Affine quantisations.*

Definition 2.22. Given a differential graded Lie algebra (DGLA) L with homological grading, define the the Maurer–Cartan set by

$$\mathrm{MC}(L) := \{\omega \in L_{-1} \mid \delta\omega + \frac{1}{2}[\omega,\omega] = 0 \in L_{-2}\}.$$

Following [Hin1], define the Maurer–Cartan space $\underline{\mathrm{MC}}(L)$ (a simplicial set) of a nilpotent DGLA L by

$$\underline{\mathrm{MC}}(L)_n := \mathrm{MC}(L \otimes_{\mathbb{Q}} \Omega(\Delta^n)_\bullet),$$

where

$$\Omega(\Delta^n)_\bullet = \mathbb{Q}[t_0, t_1, \ldots, t_n, \delta t_0, \delta t_1, \ldots, \delta t_n]/(\sum t_i - 1, \sum \delta t_i)$$

is the commutative dg algebra of de Rham polynomial forms on the n-simplex, with the t_i of degree 0 and δt_i of chain degree -1, so $\Omega(\Delta^n)_{-m} = \Omega^m(\Delta^n)$, the space of m-forms.

Given an inverse system $L = \{L_\alpha\}_\alpha$ of nilpotent DGLAs, define

$$\mathrm{MC}(L) := \varprojlim_\alpha \mathrm{MC}(L_\alpha) \quad \underline{\mathrm{MC}}(L) := \varprojlim_\alpha \underline{\mathrm{MC}}(L_\alpha).$$

Note that $\mathrm{MC}(L) = \mathrm{MC}(\varprojlim_\alpha L_\alpha)$, but $\underline{\mathrm{MC}}(L) \neq \underline{\mathrm{MC}}(\varprojlim_\alpha L_\alpha)$.

Definition 2.23. Given a cofibrant R-CDGA A (resp. $\mathcal{C}^\infty$-DGA or EFC-DGA), as in [Pri3, Definition 1.1] we complete the P_2-algebra $\mathrm{Pol}(A,0)$ of Definition 2.6 (resp. Definition 2.11) to give a P_2-algebra

$$\widehat{\mathrm{Pol}}(A,0) := \prod_{p\geq 0} \mathcal{W}_p \mathrm{Pol}(A,0)$$

of polyvectors. This is $\prod_{p\geq 0} \underline{\mathrm{Hom}}_A(\Omega^p_{A/R}, A)_{[p]}$ in the algebraic setting and corresponding expressions using $\mathcal{C}^\infty$ or EFC differentials in the other settings.

This is not graded, but does have a decreasing filtration

$$F^i\widehat{\mathrm{Pol}}(A,0) := \prod_{p\geq i} \mathcal{W}_i \mathrm{Pol}(A,0)$$

The space $\mathcal{P}(A,0)$ of 0-shifted Poisson structures on A is then defined in [Pri3, Definition 1.5] to be

$$\underline{\mathrm{MC}}(F^2\widehat{\mathrm{Pol}}(A,n)_{[-1]});$$

this is the space of P_1-algebras with underlying CDGA quasi-isomorphic to A.

Definition 2.24. Given a cofibrant R-CDGA, $\mathcal{C}^\infty$-DGA or K-EFC-DGA A, adapting [Pri4, Definition 1.16] as in [Pri5, Pri9], we define the DGLA $Q\widehat{\mathrm{Pol}}(A,0)_{[-1]}$ of quantised polyvectors by setting

$$Q\widehat{\mathrm{Pol}}(A,0) := \prod_{p\geq 0} \tau_p^{\mathrm{HH}} D^{\mathrm{poly}}_{\oplus}(A)_\bullet \hbar^{p-1};$$

observe that because the Gerstenhaber bracket satisfies $[\tau_p^{\mathrm{HH}}, \tau_q^{\mathrm{HH}}] \subset \tau_{p+q-1}^{\mathrm{HH}}$, this is indeed a DGLA.

Define a decreasing filtration $\tilde{F}$ on $Q\widehat{\mathrm{Pol}}(A,0)$ by the subcomplexes

$$\tilde{F}^i Q\widehat{\mathrm{Pol}}(A,0) := \prod_{j\geq i} \tau_j^{\mathrm{HH}} D^{\mathrm{poly}}_{\oplus}(A)\hbar^{j-1}.$$

This filtration is complete and Hausdorff, with $[\tilde{F}^i, \tilde{F}^j] \subset \tilde{F}^{i+j-1}$. In particular, this makes $\tilde{F}^2 Q\widehat{\mathrm{Pol}}(A,0)_{[-1]}$ into a pro-nilpotent filtered DGLA.

The space $Q\mathcal{P}(A,0)$ of 0-shifted quantisations of A is then defined (adapting [Pri4, Definition 1.23]) to be

$$\underline{\mathrm{MC}}(\tilde{F}^2 Q\widehat{\mathrm{Pol}}(A,n)_{[-1]}).$$

The subspace $Q\mathcal{P}(A,0)^{sd} \subset Q\mathcal{P}(A,0)$ of self-dual quantisations then consists of fixed points for the involution $(-)^*$ given by $\Delta^*(\hbar) := i(\Delta)(-\hbar)$, for the involution i of Lemma 2.8.

Remark 2.25. Since for an R-CDGA A, the inclusion $D^{\mathrm{poly}}_{\oplus}(A)_\bullet \subset \mathrm{CC}_{R,\oplus}(A)_\bullet$ is a filtered quasi-isomorphism, replacing polyvectors with the Hochschild complex gives an equivalent construction in the algebraic setting. Similarly the filtration γ from [Pri10, Definition 1.18] is quasi-isomorphic to τ^{HH}, so the definitions are also equivalent to those of [Pri10].

As in [Pri10, Remark 2.15], $Q\mathcal{P}(A,0)$ is thus equivalent to the space of curved BD_1-algebras (almost commutative associative algebras over $R[\![\hbar]\!]$) deforming A; when A lies non-negative chain degrees no curvature is possible on objects, but curvature still leads to additional higher morphisms coming from inner automorphisms. The objects in the $\mathcal{C}^\infty$ and EFC settings admit a similar interpretation, but with additional constraints from the restriction to polydifferential operators.

As in [Pri10, Remark 2.15], $Q\mathcal{P}(A,0)^{sd}$ consists of curved BD_1-algebras $\tilde{A}$ deforming A and equipped with an anti-involution $\tilde{A}^{\mathrm{opp}} \cong \tilde{A}$ which is semilinear under the transformation $\hbar \mapsto -\hbar$.

Corollary 2.26. *Given a cofibrant R-CDGA (resp. $\mathcal{C}^\infty$-DGA or K-EFC-DGA) A with perfect cotangent complex $\Omega^1_{A/R}$ (resp. $\Omega^1_{A,\mathcal{C}^\infty}$ or $\Omega^1_{A/K,\mathrm{EFC}}$), the space $Q\mathcal{P}(A,0)$ of 0-shifted quantisations of A is equivalent to the Maurer–Cartan space*

$$\begin{aligned}&\underline{\mathrm{MC}}((A_{[-1]}\hbar \times \underline{\mathrm{Hom}}_A(\Omega^1_{A/R},A)\hbar \times \prod_{p\geq 2} \underline{\mathrm{Hom}}_A(\Omega^p_{A,\square},A)_{[p-1]}\hbar^{p-1})[\![\hbar]\!])\\ &\cong \underline{\mathrm{MC}}(F^2\widehat{\mathrm{Pol}}(A,0)_{[-1]} \times \hbar F^1\widehat{\mathrm{Pol}}(A,0)_{[-1]} \times \hbar^2\widehat{\mathrm{Pol}}(A,0)_{[-1]}[\![\hbar]\!])\end{aligned}$$

where the DGLA structure comes from the Schouten–Nijenhuis bracket, and $\Omega^p_{A,\square}$ is $\Omega^p_{A/R}$ (resp. $\Omega^p_{A,\mathcal{C}^\infty}$, resp. $\Omega^p_{A/K,\mathrm{EFC}}$).

The subspace $Q\mathcal{P}(A,0)^{sd} \subset Q\mathcal{P}(A,0)$ of self-dual quantisations is then equivalent to the Maurer–Cartan space

$$\begin{aligned}&\underline{\mathrm{MC}}((A_{[-1]}\hbar \times \underline{\mathrm{Hom}}_A(\Omega^1_{A/R},A)\hbar^2 \times \prod_{p\geq 2} \underline{\mathrm{Hom}}_A(\Omega^p_{A,\square},A)_{[p-1]}\hbar^{p-1})[\![\hbar^2]\!])\\ &\cong \underline{\mathrm{MC}}(F^2\widehat{\mathrm{Pol}}(A,0)_{[-1]} \times \hbar^2\widehat{\mathrm{Pol}}(\mathscr{O},0)_{[-1]}[\![\hbar^2]\!]).\end{aligned}$$

In particular, there exist self-dual associative quantisations for every Poisson structure on A.

Proof. The quasi-isomorphism of Theorem 2.20 gives us a quasi-isomorphism

$$\tilde{F}^2 Q\widehat{\mathrm{Pol}}(A,0)_{[-1]} \simeq \prod_{p\geq 2, i\leq p} \mathcal{W}_i\mathrm{Pol}(A,0)_{[-1]}\hbar^{p-1} = \prod_{i\geq 0} \hbar^{\max(i-1,1)}\mathcal{W}_i\mathrm{Pol}(A,0)_{[-1]}[\![\hbar]\!]$$

of pro-nilpotent filtered DGLAs, and hence an equivalence of the respective Maurer–Cartan spaces. We then use the isomorphism

$$F^2\widehat{\mathrm{Pol}}(A,0)\times \hbar F^1\widehat{\mathrm{Pol}}(A,0)\times \hbar^2\widehat{\mathrm{Pol}}(A,0)[\![\hbar]\!] \xrightarrow{\simeq} \prod_i \hbar^{\max(i-1,1)}\mathcal{W}_i\mathrm{Pol}(A,0)[\![\hbar]\!]$$

given by multiplying $\mathcal{W}_i\mathrm{Pol}(A,0)$ by $\hbar^{i-1}$; this gives a DGLA morphism because the Schouten–Nijenhuis bracket satisfies $[\mathcal{W}_i,\mathcal{W}_j]\subset \mathcal{W}_{i+j-1}$.

Now, the involution $*$ acts on $\mathrm{H}_*(\mathrm{gr}_i^{\tau^{\mathrm{HH}}}\mathrm{CC}_R(A))\hbar^j$ and on $\mathcal{W}_i\mathrm{Pol}(A,0)\hbar^j$ as $(-1)^{i+j-1}$. Under the isomorphism above given by division by $\hbar^{i-1}$, this corresponds to the involution $\hbar\mapsto -\hbar$ of $F^2\widehat{\mathrm{Pol}}(A,0)\times \hbar F^1\widehat{\mathrm{Pol}}(A,0)\times \hbar^2\widehat{\mathrm{Pol}}(A,0)[\![\hbar]\!]$, so the fixed points are $F^2\widehat{\mathrm{Pol}}(A,0)\times \hbar^2\widehat{\mathrm{Pol}}(A,0)[\![\hbar^2]\!]$.

Existence of quantisations then follows from the inclusion of the first term in $F^2\widehat{\mathrm{Pol}}(A,0)\times \hbar^2\widehat{\mathrm{Pol}}(\mathscr{O},0)[\![\hbar^2]\!]$, giving a morphism $\mathcal{P}(A,0)\to Q\mathcal{P}(A,0)^{sd}\subset Q\mathcal{P}(A,0)$. □

2.4.3. *Étale functoriality revisited.* Functoriality for Hochschild complexes and polyvectors is subtle, but exists with respect to homotopy étale morphisms as in [Pri4, §3.1] and [Pri3, §2.1.2]. We now revisit and generalise the results there.

Definition 2.27. Given a category $\mathcal{C}$, let $\int B\mathcal{C}$ be the Grothendieck construction of the nerve of $\mathcal{C}$. Explicitly, objects of $\int B\mathcal{C}$ are pairs $(m,\, A\colon [m]\to\mathcal{C}) = (m,\, A(0)\to A(1)\to \ldots\to A(m))$ for $m\geq 0$, and a morphism $u\colon (m,\, A\colon [m]\to\mathcal{C})\to(n,\, B\colon [n]\to\mathcal{C})$ is a morphism $u\colon \mathbf{n}\to\mathbf{m}$ in the simplex category such that $B=A\circ u$.

Lemma 2.28. *The category $\mathcal{C}$ is equivalent to the simplicial localisation of $\int B\mathcal{C}$ at the class $\mathcal{W}_0$ of morphisms $u\colon (m,\, A\colon [m]\to\mathcal{C})\to(n,\, A\circ u\colon [n]\to\mathcal{C})$ with $u(0)=0\in\mathbf{m}$.*

Proof. The functor $\rho\colon\ \int B\mathcal{C}\to\mathcal{C}$ is given by $\rho(m,A)=A(0)$ on objects and by $\rho(u)=A(\gamma)\colon\ A(0)\to A(u(0))$ on morphisms $u\colon (m,A\colon [m]\to\mathcal{C})\to(n,A\circ u\colon [n]\to\mathcal{C})$, for the unique morphism $\gamma\colon\ 0\to u(0)$ in $\mathbf{m}$.

In order to show that this induces an equivalence $L^{\mathcal{W}_0}(\int B\mathcal{C})\simeq\mathcal{C}$ of simplicial categories, by [DK, Theorem 2.2] it suffices to show that the functor $\rho^*\colon (s\mathrm{Set})^{\mathcal{C}}\to (s\mathrm{Set})^{\int B\mathcal{C},\mathcal{W}_0}$ from the model category of simplicial set-valued functors on $\mathcal{C}$ to the model category of $\mathcal{W}_0$-restricted $\int B\mathcal{C}$-diagrams is a right Quillen equivalence.

Now, giving a simplicial set-valued functor F on $\int B\mathcal{C}$ is equivalent to giving a bisimplicial set $\int F$ over the nerve $B\mathcal{C}$, with $(\int F)_n := \coprod_{x\in B_m\mathcal{C}} F(x)$. Thus the category $(s\mathrm{Set})^{\int B\mathcal{C},\mathcal{W}_0}$ is equivalent to $ss\mathrm{Set}{\downarrow}B\mathcal{C}$. Moreover, F sends morphisms in $\mathcal{W}_0$ to weak equivalences if and only if the maps $F(n,\, A(0)\to\ldots\to A(n)))\to F(0,\, A(0))$ are weak equivalences, which amounts to saying that the maps

$$(\int F)_n \xrightarrow{(\partial_1)^n} (\int F)_0 \times_{B_0\mathcal{C},(\partial_1)^n} B_n\mathcal{C}$$

are weak equivalences. When $\int F$ is Reedy fibrant, this is equivalent to saying that $\int F$ is a left fibration over $B\mathcal{C}$ in the sense of [dB].

The equivalence $\int^{-1}\colon\ s\mathrm{Set}{\downarrow}B\mathcal{C}\to (s\mathrm{Set})^{\int B\mathcal{C},\mathcal{W}_0}$ of categories is a right Quillen functor for the left fibration model structure on $s\mathrm{Set}{\downarrow}B\mathcal{C}$, and the observation above implies that it is a Quillen equivalence. Composing this with the right Quillen equivalence of [dB, Theorem A] (in its variant for left fibrations) then gives the required result. □

If $\mathcal{C}$ is the category of R-CDGAs, EFC-DGAs or $\mathcal{C}^\infty$-DGAs, then for any diagram $D\colon I \to \mathcal{C}$, we can define $D^{\mathrm{poly}}_{\oplus}(D)$ and $\mathrm{Pol}(D,0)$ by substituting $\underline{\mathrm{Hom}}_B(M,N)$ with the equaliser of the obvious diagram

$$\prod_{i\in I}\underline{\mathrm{Hom}}_{B(i)}(M(i),N(i)) \implies \prod_{f\colon\, i\to j \text{ in } I}\underline{\mathrm{Hom}}_{B(i)}(M(i),f_*N(j)).$$

throughout the corresponding definitions for algebras. These constructions behave well for diagrams $D\colon [m] \to \mathcal{C}_{c,\twoheadrightarrow}$ to the subcategory $\mathcal{C}_{c,\twoheadrightarrow} \subset \mathcal{C}$ of cofibrant objects and surjective morphisms. Since a morphism $u\colon [n] \to [m]$ naturally induces a morphism $u^\#\colon D^{\mathrm{poly}}_{\oplus}(D) \to D^{\mathrm{poly}}_{\oplus}(u^*D)$ by restriction, the constructions $D \mapsto (D^{\mathrm{poly}}_{\oplus}(D), \tau^{\mathrm{HH}})$ and $D \mapsto \mathrm{Pol}(D,0)$ define functors on the Grothendieck construction $\int B\mathcal{C}_{c,\twoheadrightarrow}$.

When $u(0) = 0$ and the morphisms in D are all homotopy étale, the morphism $u^\#$ is moreover a filtered quasi-isomorphism, so the restriction of $D^{\mathrm{poly}}_{\oplus}$ to the subcategory $\int B(\mathcal{C}_{c,\twoheadrightarrow,\text{ét}})$ of homotopy étale morphisms sends morphisms in $\mathcal{W}_0$ to filtered quasi-isomorphisms. It then follows from Lemma 2.28 and quasi-isomorphism invariance that $(D^{\mathrm{poly}}_{\oplus}, \tau^{\mathrm{HH}})$ and $\mathrm{Pol}(-,0)$ define ∞-functors on the ∞-category $\mathbf{L}\mathcal{C}^{\text{ét}}$ of R-CDGAs, EFC-DGAs or $\mathcal{C}^\infty$-DGAs, and homotopy étale morphisms; the HKR quasi-isomorphism is then a natural transformation between them. Taking Maurer–Cartan elements as in Definition 2.24 then gives an ∞-functor $Q\mathcal{P}(-,0)$ from $\mathbf{L}\mathcal{C}^{\text{ét}}$ to the ∞-category of simplicial sets.

2.4.4. *Global quantisations.* Incorporating the homotopy étale functoriality of §2.4.3 with the functoriality of Theorem 2.20 immediately leads to the following generalisation of Corollary 2.26 on taking derived global sections on the étale site:

Corollary 2.29. *Given a derived DM n-stack $\mathfrak{X}$ over R with perfect cotangent complex $\mathbf{L}\Omega^1_{\mathfrak{X}/R}$, the space $Q\mathcal{P}(\mathfrak{X},0) := \mathbf{R}\Gamma(\mathfrak{X}_{\text{ét}}, Q\mathcal{P}(\mathscr{O},0))$ of 0-shifted quantisations of $\mathfrak{X}$ from* [Pri4, Definitions 1.23, 3.9] *and its subspace $Q\mathcal{P}(\mathfrak{X},0)^{sd}$ of self-dual (or involutive) quantisations from* [Pri4, Definition 1.33] *are respectively equivalent to the Maurer–Cartan spaces*

$$\mathbf{R}\Gamma(\mathfrak{X}_{\text{ét}}, \underline{\mathrm{MC}}(F^2\widehat{\mathrm{Pol}}(\mathscr{O},0)_{[-1]} \times \hbar F^1\widehat{\mathrm{Pol}}(\mathscr{O},0)_{[-1]} \times \hbar^2\widehat{\mathrm{Pol}}(\mathscr{O},0)_{[-1]}[\![\hbar]\!])),$$
$$\mathbf{R}\Gamma(\mathfrak{X}_{\text{ét}}, \underline{\mathrm{MC}}(F^2\widehat{\mathrm{Pol}}(\mathscr{O},0)_{[-1]} \times \hbar^2\widehat{\mathrm{Pol}}(\mathscr{O},0)_{[-1]}[\![\hbar^2]\!])).$$

In particular, every Poisson structure $\pi \in \mathcal{P}(\mathfrak{X},0) = \mathbf{R}\Gamma(\mathfrak{X}_{\text{ét}}, \underline{\mathrm{MC}}(F^2\widehat{\mathrm{Pol}}(\mathscr{O},0)_{[-1]}))$ admits self-dual quantisations in the form of almost commutative curved A_∞-deformations $\mathscr{A}_\hbar$ of $\mathscr{O}_{\mathfrak{X}}$ with $\mathscr{A}_{-\hbar} \simeq \mathscr{A}_\hbar^{\mathrm{opp}}$.

The analogous statements for derived $\mathcal{C}^\infty$ and derived analytic DM n-stacks (in the sense of [Pri9]*) with perfect cotangent complexes also hold.*

Remarks 2.30. The hypotheses of Corollary 2.29 are satisfied by any derived Deligne–Mumford stack locally of finite presentation over the CDGA R. When $R = \mathrm{H}_0R$, this includes underived schemes X which are local complete intersections over R, in which case the cotangent complex $\mathbf{L}\Omega^1_{X/R}$ is concentrated in homological degrees $[0,1]$. For such underived schemes, a quantisation in the sense of the corollary reduces to the usual notion, namely a DQ algebroid deformation of $\mathscr{O}_X$ over $R[\![\hbar]\!]$. For more details, see [Pri10, Remarks 2.15, 4.10 and 4.17].

In analytic settings, Corollary 2.29 similarly gives DQ algebroid quantisations for local complete intersections. Since $\mathcal{C}^\infty$ spaces tend to embed in affine space under mild finiteness hypotheses, Corollary 2.26 also gives strict quantisations for LCI $\mathcal{C}^\infty$ spaces.

Our hypothesis that $\mathfrak{X}$ have perfect cotangent complex cannot be removed, since Mathieu's example [Mat] gives a non-quantisable Poisson structure on a non-LCI scheme.

Remark 2.31. The space $Q\mathcal{P}^+(\mathfrak{X},0)$ of strict deformation quantisations, i.e. in terms of BD_1-algebras (associative, almost commutative algebras) rather than algebroids, is given by replacing $Q\widehat{\mathrm{Pol}}(\mathscr{O}_{\mathfrak{X}},0)$ with the kernel $\tilde{F}^2Q\widehat{\mathrm{Pol}}^+(\mathscr{O}_{\mathfrak{X}},0)$ of the natural map $\tilde{F}^2Q\widehat{\mathrm{Pol}}(\mathscr{O}_{\mathfrak{X}},0)\to \hbar\mathscr{O}_{\mathfrak{X}}[\![\hbar]\!]$, and similarly by $\tilde{F}^2Q\widehat{\mathrm{Pol}}^+(\mathscr{O}_{\mathfrak{X}},0)^{sd} := \ker(\tilde{F}^2Q\widehat{\mathrm{Pol}}(\mathscr{O}_{\mathfrak{X}},0)\to \hbar\mathscr{O}_{\mathfrak{X}}[\![\hbar^2]\!])$ for $Q\mathcal{P}^+(\mathfrak{X},0)^{sd}$. These maps are not in general compatible with the formality quasi-isomorphism of Corollary 2.29, but they are pro-nilpotent surjections of DGLAs, so lead to homotopy fibre sequences

$$Q\mathcal{P}^+(\mathfrak{X},0)\to Q\mathcal{P}(\mathfrak{X},0)\to \mathbf{R}\Gamma(\mathfrak{X},\hbar\mathscr{O}_{\mathfrak{X}}[\![\hbar]\!])_{[-2]},$$
$$Q\mathcal{P}^+(\mathfrak{X},0)^{sd}\to Q\mathcal{P}(\mathfrak{X},0)^{sd}\to \mathbf{R}\Gamma(\mathfrak{X},\hbar\mathscr{O}_{\mathfrak{X}}[\![\hbar^2]\!])_{[-2]}.$$

In particular this means that if $\mathrm{H}^2(\mathfrak{X},\mathscr{O}_{\mathfrak{X}})=0$ then every (self-dual) quantisation in Corollary 2.29 comes from a strict (self-dual) quantisation.

Remark 2.32 (Quantisations of 1-shifted co-isotropic structures). In [MS2, §5.3], existence of quantisations for n-shifted co-isotropic structures is established as a direct consequence of the formality of the E_n operad, for $n>1$. Theorem 2.20 allows us to establish the corresponding result for 1-shifted co-isotropic structures as follows.

A 1-shifted quantised co-isotropic structure on a morphism $f\colon \mathfrak{X}\to\mathfrak{Y}$ of derived DM n-stacks consists of a quantised 1-shifted Poisson structure on $\mathfrak{Y}$ corresponding to a BD_2-deformation $\tilde{\mathscr{O}}_{\mathfrak{Y}}$ of $\mathscr{O}_{\mathfrak{Y}}$, a quantised 0-shifted Poisson structure Δ on $\mathfrak{X}$, and a morphism $f^{-1}\tilde{\mathscr{O}}_{\mathfrak{Y}}\to(\prod_{j\ge0}\tau_j^{\mathrm{HH}}\mathrm{CC}_{R,\oplus}(\mathscr{O}_{\mathfrak{X}})\hbar^j,\delta+\{\Delta,-\})$ of BD_2-algebras.

When $\mathfrak{X}$ has perfect cotangent complex, we may apply the ∞-equivalence p_w for any even associator w, and then composition with the filtered quasi-isomorphism $\mathrm{HKR}\circ\alpha_{w,B}$ from the proof of Theorem 2.20 makes the quantisation equivalent to a morphism

$$f^{-1}p_w\tilde{\mathscr{O}}_{\mathfrak{Y}}\to(\prod_{p\ge1}\mathbf{L}\Lambda^p_{\mathscr{O}_{\mathfrak{X}}}(T_{\mathfrak{X}/R})_{[p-1]}\hbar^p[\![\hbar]\!],\delta+\{\alpha_w\Delta,-\})$$

of strong homotopy P_2-algebras. Now, this is just the same as an $R[\![\hbar]\!]$-linear 1-shifted co-isotropic structure on an $R[\![\hbar]\!]$-linear deformation of the diagram $f\colon\mathfrak{X}\to\mathfrak{Y}$; in particular, taking the trivial deformation yields a quantisation for every 1-shifted co-isotropic structure on f.

The analogous statements hold in the $\mathcal{C}^\infty$ and EFC settings, with polydifferential operators in place of the Hochschild complex.

3. Quantisations on derived Artin stacks

When applied to (derived) Artin stacks, the definition of Poisson structures in [Pri3, §3] or [CPT$^+$] and their quantisations in [Pri4, Definitions 1.23, 3.9] is more subtle than for derived DM stacks, since polyvectors and the Hochschild complex are not functorial with respect to smooth morphisms.

This is resolved in [Pri3, §3.1] by observing that the formal completion of a derived Artin stack $\mathfrak{X}$ along an affine atlas $f\colon U\to\mathfrak{X}$ with f smooth can be recovered from a commutative algebra in double complexes.

These lead to sufficiently functorial constructions of polyvectors and Hochschild complexes, but only after passing to sum-product total complexes $\hat{\mathrm{Tot}}$ (Tate realisations in the terminology of [CPT$^+$]). Generalising quantisation results to derived Artin stacks is thus far from straightforward, and to establish them we introduce an intermediate category $\mathcal{U}_{P_k^{ac}}[\hbar^2]$ in §3.3 which delicately balances the requirements of functoriality and deformations.

Remark 3.1. While the results of §1 easily adapt to P_k^{ac}-algebras in double complexes, the P_{n+2}-algebra of n-shifted polyvectors only satisfies the analogue of Proposition 1.18 before applying $\hat{\mathrm{Tot}}$. In terms of our notation below, $\mathcal{P}ol(-,n)$ (roughly corresponding to $\mathrm{Pol}^{\mathrm{int}}(-,n)$ in [CPT$^+$]) is not sufficiently functorial, while the algebra $\hat{\mathrm{Tot}}_{\mathbb{G}_m}\mathcal{P}ol(-,n)$ (roughly corresponding to $\mathrm{Pol}^t(-,n)$ in [CPT$^+$]) cannot simply be described as derived symmetric powers of the tangent complex, so we cannot constrain its deformations.

3.1. **Double complexes and stacky Hochschild complexes.**

Definition 3.2. A stacky CDGA is a chain cochain complex (i.e. a double complex)

$$A^{\bullet}_{\bullet} = (A^0_{\bullet} \xrightarrow{\partial} A^1_{\bullet} \xrightarrow{\partial} A^2_{\bullet} \xrightarrow{\partial} \dots).$$

equipped with a commutative product $A \otimes A \to A$ and unit $\mathbb{Q} \to A$. Given a chain CDGA R, a stacky CDGA over R is then a morphism $R \to A$ of stacky CDGAs, where we silently regard R as a stacky CDGA concentrated in cochain degree 0.

A stacky $\mathcal{C}^\infty$-DGA (resp. EFC-DGA over K) is a stacky CDGA over $\mathbb{R}$ (resp. K) $A^{\bullet}_{\bullet}$ equipped with a $\mathcal{C}^\infty$-DGA (resp. EFC-DGA) structure on $A^0_{\bullet}$ and such that $\partial\colon A^0_0 \to A^1_0$ is a $\mathcal{C}^\infty$-derivation (resp. EFC-derivation) in the sense of [Pri9, Definition 2.9].

As explained in [Pri3, Remark 3.32], these correspond to the "graded mixed cdgas" of [CPT$^+$] (but beware that the latter are something of a misnomer, not having mixed differentials). The structure in the chain direction encodes derived information, while the cochain direction encodes stacky structure.

For general derived Artin n-stacks, these formal completions are constructed in [Pri3, §3.1] by forming affine hypercovers as in [Pri1], and then applying the functor D^* (left adjoint to denormalisation) to obtain a stacky CDGA. When $\mathfrak{X}$ is a derived Artin 1-stack, the formal completion of an affine atlas $U \to \mathfrak{X}$ is simply given by the relative de Rham complex

$$O(U) \xrightarrow{\partial} \Omega^1_{U/\mathfrak{X}} \xrightarrow{\partial} \Omega^2_{U/\mathfrak{X}} \xrightarrow{\partial} \dots,$$

which arises by applying the functor D^* to the Čech nerve of U over $\mathfrak{X}$.

Lemma 3.3. *There is a cofibrantly generated model structure on the category of cochain chain complexes in which fibrations are surjections and weak equivalences are levelwise quasi-isomorphisms in the chain direction. For any chain operad $\mathcal{P}$, this induces a cofibrantly generated model structure on $\mathcal{P}$-algebras in cochain chain complexes, in which fibrations and weak equivalences are those of the underlying cochain chain complexes.*

Similarly, for any $\mathbb{G}_m$-equivariant chain operad $\mathcal{P}$, there is a cofibrantly generated model structure on the category of $\mathbb{G}_m$-equivariant $\mathcal{P}$-algebras in cochain chain complexes with the same cofibrations, fibrations and weak equivalences.

Proof. This follows as in the proof of [Pri3, Lemma 3.4]. Each generating (trivial) cofibration in the non-$\mathbb{G}_m$-equivariant setting gives rise to a $\mathbb{Z}$-indexed family of generating (trivial) cofibrations in the $\mathbb{G}_m$-equivariant setting, corresponding to choices of weight for the generators. □

Note that since $\mathcal{P}$ is assumed to be a chain operad, the operations for the $\mathcal{P}$-algebras A in Lemma 3.3 are maps

$$\mathcal{P}(n)_i \otimes A^{m_1}_{j_1} \otimes \ldots \otimes A^{m_n}_{j_n} \to A^{m_1+\ldots+m_n}_{i+(j_1+\ldots+j_n)};$$

we refer to such $\mathcal{P}$-algebras in cochain chain complexes as stacky $\mathcal{P}$-algebras; beware that this conflicts slightly with the terminology of Definition 3.2 because we allow stacky $\mathcal{P}$-algebras to have terms of negative cochain degree, while our stacky CDGAs are concentrated in non-negative cochain degrees.

We also have a more subtle variant:

Lemma 3.4. *Given a chain CDGA R, there is a cofibrantly generated model structure on the category of $\mathbb{G}_m$-equivariant stacky P^{ac}_k-algebras A over R of non-negative weights (i.e. $\mathcal{W}_i A = 0$ for all $i < 0$), in which fibrations are surjections and weak equivalences are levelwise quasi-isomorphisms.*

Moreover, the forgetful functor from this category to the category of $\mathbb{G}_m$-equivariant stacky R-CDGAs preserves cofibrant objects.

Proof. We apply [Hir, Theorem 11.3.2] to the forgetful functor mapping to non-negatively weighted $\mathbb{G}_m$-equivariant double complexes of $\mathbb{Q}$-vector spaces. The only non-trivial condition to check for the first statement is that the left adjoint F sends pushouts of generating trivial cofibrations to levelwise quasi-isomorphisms. For the second statement it suffices to show that F sends pushouts of generating cofibrations to cofibrations of $\mathbb{G}_m$-equivariant stacky R-CDGAs.

Since the forgetful functor factors through the category of non-negatively weighted $\mathbb{G}_m$-equivariant stacky $s^{1-k}\hbar^{-1}$Lie-algebras (by forgetting the commutative R-algebra structure but keeping the Lie bracket), the left adjoint F factors as $(R \otimes_{\mathbb{Q}} \mathrm{Symm}_{\mathbb{Q}}) \circ L_+$, where L_+ is the left adjoint to the forgetful functor from that intermediate category. For the Lie bracket of chain degree $k-1$ and $\mathbb{G}_m$-weight -1, the functor L_+ sends a $\mathbb{G}_m$-equivariant double complex V to the quotient $L(V)/(\mathcal{W}_{<0}L(V))$ of the free graded Lie algebra $L(V)$ (with Koszul signs) by the Lie ideal generated by terms of negative weight.

Now, a cofibration of double complexes is an injective map $U \hookrightarrow V$ for which the quotient V/U is acyclic in the cochain direction (i.e. $\mathrm{H}^i(V_j/U_j) = 0$ for all i, j); it is trivial if it is moreover acyclic in the chain direction. On V/U there is thus a contracting cochain homotopy $h\colon (V/U)^{\#}_{\#} \to V/U^{\#[-1]}_{\#}$ such that the graded commutator $[\partial, h]$ is the identity, and if the cofibration is trivial, there is also a contracting chain homotopy $h'\colon (V/U)^{\#}_{\#} \to (V/U)^{\#}_{\#[1]}$ such that $[\delta, h']$ is the identity. If we give V an increasing filtration with $F_{-1}V = 0$, $F_0V = U$ and $F_1V = V$, then the Lie bracket induces an exhaustive increasing filtration F on $L_+(V)$, with $\mathrm{gr}^F L_+(V) \cong L_+(U \oplus (V/U))$. We can then define a derivation $H\colon \mathrm{gr}^F L_+(V)^{\#}_{\#} \to \mathrm{gr}^F L_+(V)^{\#[-1]}_{\#}$ (resp. $H'\colon \mathrm{gr}^F L_+(V)^{\#}_{\#} \to \mathrm{gr}^F L_+(V)^{\#}_{\#[1]}$), given on generators by 0 on U and h (resp. h') on V/U. It follows that the commutator $[\partial, H]$ (resp. $[\delta, H]$) is a derivation acting as 0 on generators U and as the identity on generators V/U, so it must act as multiplication by p on $\mathrm{gr}^F_p L_+(V)$.

The maps $F_pL_+V \to F_{p+1}L_+V$ are thus cofibrations (resp. trivial cofibrations) of double complexes for all $p \geq 0$, since $h/(p+1)$ and $H/(p+1)$ provide the necessary homotopies on the quotient, so $L_+U \to L_+V$ is a cofibration (resp. trivial cofibration) of double complexes. Passing to symmetric powers over R then gives that $F(U) \to F(V)$ is a cofibration (resp. trivial cofibration) of $\mathbb{G}_m$-equivariant CDGAs, as required. □

Hochschild complexes and multiderivations on stacky CDGAs are then defined in terms of the semi-infinite total complex:

Definition 3.5. The subcomplex $\hat{\mathrm{Tot}}\, V \subset \mathrm{Tot}^{\Pi} V$ is given by

$$(\hat{\mathrm{Tot}}\, V)_m := (\bigoplus_{i<0} V^i_{i+m}) \oplus (\prod_{i\geq 0} V^i_{i+m})$$

with differential $\partial \pm \delta$. This functor has the properties that it maps levelwise quasi-isomorphisms in the chain direction to quasi-isomorphisms, and that it is lax monoidal.

Applied to the internal Hom functor $\mathcal{H}om$, this construction gives chain complexes

$$\underline{\hat{\mathrm{Hom}}}_R(M,N) := \hat{\mathrm{Tot}}\, \mathcal{H}om_R(M,N)$$

for cochain complexes M, N of R-modules in chain complexes, and hence a dg enhancement of the monoidal category of R-cochain chain complexes.

Definition 3.6. For a stacky DGAA A (i.e. an associative algebra in double complexes) over a chain CDGA R, we define the internal cohomological Hochschild complex $\mathcal{CC}_{R,\oplus}(A)$ by replacing $\underline{\mathrm{Hom}}$ with $\mathcal{H}om$ in Definition 2.1 (as in [Pri10, Definition 2.7]) to give a chain cochain complex. Note that for the conventions we have chosen, this means that the Hochschild differential is acting in the chain direction.

We similarly define $\mathcal{D}^{\mathrm{poly}}_{\oplus}(A) \subset \mathcal{CC}_{R,\oplus}(A)$ to be the double subcomplex of polydifferential operators (including the $\mathcal{C}^\infty$ setting with $R = \mathbb{R}$ and the EFC setting with $R = K$, a complete valued field), defined exactly as in Definition 2.10, but with an additional grading coming from the cochain grading on A.

The filtration τ^{HH} on $\mathcal{CC}_{R,\oplus}(A)$ and $\mathcal{D}^{\mathrm{poly}}_{\oplus}(A)$ is then given by good truncation in the Hochschild direction.

Definition 3.7. For A a cofibrant stacky R-CDGA (resp. stacky $\mathcal{C}^\infty$-DGA, resp. stacky EFC-DGA), define the $\mathbb{G}_m$-equivariant stacky P^{ac}_{n+2}-algebra $\mathcal{P}ol(A,n)$ by

$$\mathcal{P}ol(A,n) := \bigoplus_{p\geq 0} \mathcal{H}om_A(\mathrm{CoS}^p_A((\Omega^1_{A,\square})_{[-n-1]}), A),$$

where p is the weight for the $\mathbb{G}_m$-action, the commutative multiplication and Lie bracket are defined in the usual way for polyvectors, and $\Omega^p_{A,\square}$ is $\Omega^p_{A/R}$ (resp. $\Omega^p_{A,\mathcal{C}^\infty}$, resp. $\Omega^p_{A/K,\mathrm{EFC}}$).

In particular, note that $\mathcal{P}ol(A,0) = \bigoplus_{p\geq 0} \mathcal{H}om_A(\Omega^p_{A,\square}, A)_{[p]}$.

3.2. **Étale functoriality and $\underline{\hat{\mathrm{Hom}}}$-equivalences.** We now consider conditions for a morphism $f\colon P \to Q$ of A-modules to be a $\underline{\hat{\mathrm{Hom}}}_A$-homotopy equivalence, i.e. for the homology class $[f] \in \mathrm{H}_0\, \underline{\hat{\mathrm{Hom}}}_A(P,Q)$ to have an inverse in $\mathrm{H}_0\, \underline{\hat{\mathrm{Hom}}}_A(Q,P)$. Writing $\sigma^{\leq n} M := M/\sigma^{>n} M$ for the brutal cotruncation in the cochain direction, we have:

Lemma 3.8. *If A is a stacky CDGA concentrated in non-negative cochain degrees, and P and Q are cofibrant A-modules in double complexes, with the chain complexes $(P \otimes_A A^0)^i, (Q \otimes_A A^0)^i$ zero for all $i < r$ and acyclic for all $i > s$, then a morphism $f\colon P \to Q$ is a $\underline{\hat{\mathrm{Hom}}}_A$-homotopy equivalence whenever the map $\mathrm{Tot}\, \sigma^{\leq s}(P \otimes_A A^0) \to \mathrm{Tot}\, \sigma^{\leq s}(Q \otimes_A A^0)$ is a quasi-isomorphism. Under this condition, the morphism $f^*\colon \mathcal{H}om_A(Q,A) \to \mathcal{H}om_A(P,A)$ is also a $\underline{\hat{\mathrm{Hom}}}_A$-homotopy equivalence.*

Proof. Since P is cofibrant, brutal truncation $\{\sigma^{\ge p}M\}_p$ of any A-module M in cochain chain complexes M induces surjections $\mathcal{H}om_A(P,\sigma^{\ge i}M) \to \mathcal{H}om_A(P,\sigma^{\ge i-1}M)$ with kernels $\mathcal{H}om_A(P,M^i)^{[-i]}$. The second hypothesis on P implies that $(\mathcal{H}om_A(P,M^i)^{[-i]})^m$ is acyclic for $i > m+s$, thus giving a quasi-isomorphism

$$\mathcal{H}om_A(P,M)^m \cong \varprojlim_n \mathcal{H}om_A(P,\sigma^{\le n}M)^m \xrightarrow{\sim} \mathcal{H}om_A(P,\sigma^{\le m+s}M)^m.$$

Moreover, if $M^i = 0$ for all $i < t$, then $\mathcal{H}om_A(P,\sigma^{\le m+s}M)^m = 0$ for all $m < t-s$, so $\mathcal{H}om_A(P,M)^m \simeq 0$ for all $m < t-s$. Applying $\hat{\mathrm{Tot}}$ then gives

$$\underline{\hat{\mathrm{Hom}}}(P,M) \simeq \mathrm{Tot}^{\Pi}\sigma^{\ge t-s}\mathcal{H}om(P,M).$$

Now, we have quasi-isomorphisms $\mathrm{Tot}\,\sigma^{\le s'}(P \otimes_A A^0) \to \mathrm{Tot}\,\sigma^{\le s'}(Q \otimes_A A^0)$ for all $s' \ge s$, giving quasi-isomorphisms $\mathrm{Tot}^{\Pi}\sigma^{\ge -s'}\mathcal{H}om_A(Q,M^i) \to \mathrm{Tot}^{\Pi}\sigma^{\ge -s'}\mathcal{H}om_A(P,M^i)$ for all i. Writing M as the limit of the tower $\ldots \to \sigma^{\le t+1}M \to \sigma^{\le t}M = (M^t)^{[-t]}$ and setting $s' = s+i-t$, these give a quasi-isomorphism

$$\mathrm{Tot}^{\Pi}\sigma^{\ge t-s}\mathcal{H}om_A(Q,M) \to \mathrm{Tot}^{\Pi}\sigma^{\ge t-s}\mathcal{H}om_A(P,M).$$

For all M bounded below in the cochain direction, we therefore have quasi-isomorphisms

$$f^*\colon\ \underline{\hat{\mathrm{Hom}}}_A(Q,M) \to \underline{\hat{\mathrm{Hom}}}_A(P,M).$$

The first hypothesis on P implies that $P \cong \sigma^{\ge r}P$, so it is bounded below. We may therefore take $M = P$, giving a class $[g] \in \mathrm{H}_0\,\underline{\hat{\mathrm{Hom}}}_A(Q,P)$ with $f^*[g] = [\mathrm{id}]$. Thus $[g]$ is inverse to $[f] \in \mathrm{H}_0\,\underline{\hat{\mathrm{Hom}}}_A(P,Q)$, so f is indeed a $\underline{\hat{\mathrm{Hom}}}_A$-homotopy equivalence.

Finally, observe that the contravariant functor $\mathcal{H}om_A(-,A)$ on A-modules is a $\mathcal{H}om_A$-enriched functor, having natural maps

$$\mathcal{H}om_A(M,N) \to \mathcal{H}om_A(\mathcal{H}om_A(N,A),\mathcal{H}om_A(M,A))$$

for all A-modules M,N in chain cochain complexes, compatible with composition. Applying $\hat{\mathrm{Tot}}$ then gives maps

$$\underline{\hat{\mathrm{Hom}}}_A(M,N) \to \underline{\hat{\mathrm{Hom}}}_A(\mathcal{H}om_A(N,A),\mathcal{H}om_A(M,A))$$

compatible with composition, so $[f]$ and $[g]$ give rise to mutually inverse elements of

$$\mathrm{H}_0\,\underline{\hat{\mathrm{Hom}}}_A(\mathcal{H}om_A(Q,A),\mathcal{H}om_A(P,A)) \quad \text{and} \quad \mathrm{H}_0\,\underline{\hat{\mathrm{Hom}}}_A(\mathcal{H}om_A(P,A),\mathcal{H}om_A(Q,A)).$$

□

If D denotes an $[m]$-diagram $(A(0) \xrightarrow{f_1} A(1) \xrightarrow{f_2} \ldots \xrightarrow{f_n} A(n))$ of cofibrant stacky CDGAs with all f_i surjective, then adapting [Pri4, §3.1] (cf. §2.4.3), gives a stacky involutive a.c. brace algebra $\mathcal{CC}_{R,\oplus}(D)$ (resp. $\mathcal{D}^{\mathrm{poly}}_{\oplus}(D)$) with restriction maps

$$\mathcal{CC}_{R,\oplus}(D) \to \mathcal{CC}_{R,\oplus}(u^*D)$$
$$\mathcal{D}^{\mathrm{poly}}_{R,\oplus}(D) \to \mathcal{D}^{\mathrm{poly}}_{R,\oplus}(u^*D)$$

for all maps $u\colon [m'] \to [m]$. For $\mathcal{C}$ the relevant category of stacky CDGAs, EFC-DGAs or $\mathcal{C}^{\infty}$-DGAs, these combine to give a functor on the Grothendieck construction $\int B\mathcal{C}_{c,\twoheadrightarrow}$ of the nerve (cf. §2.4.3).

Similarly, setting

$$\mathcal{P}ol(A,B;n) := \bigoplus_{p\ge 0} \mathcal{H}om_A(\mathrm{CoS}^p_A((\Omega^1_{A,\square})_{[-n-1]}),B),$$

where $\Omega^1_{A,\square}$ is $\Omega^1_{A/R}$ (resp. $\Omega^1_{A,\mathcal{C}^\infty}$, resp. $\Omega^1_{A/K,\mathrm{EFC}}$) and $\mathcal{P}ol(D,n)$ is

$$\mathcal{P}ol(A(0),n)\times_{\mathcal{P}ol(A(0),A(1);n)}\mathcal{P}ol(A(1),n)\ldots\times_{\mathcal{P}ol(A(m-1),A(m);n)}\mathcal{P}ol(A(m),n)$$

gives a stacky P^{ac}_{n+2}-algebra with restriction maps

$$\mathcal{P}ol(D,n)\to\mathcal{P}ol(u^*D,n)$$

for all maps $u\colon [m']\to[m]$.

When the maps f_i are all homotopy formally étale in the appropriate stacky sense, the map $\mathcal{P}ol(A(0)\xrightarrow{f_1}A(1)\xrightarrow{f_2}\ldots\xrightarrow{f_n}A(n),n)\to\mathcal{P}ol(A(0),n)$ induced by the inclusion $[0]\to[m]$ becomes a quasi-isomorphism on applying $\hat{\mathrm{Tot}}$, as shown in [Pri4, §3.1]. However, it will not usually be a levelwise filtered quasi-isomorphism. Crucially for us, a slightly stronger property than $\hat{\mathrm{Tot}}$-quasi-isomorphism holds, as we will see in Lemma 3.10 below.

Definition 3.9. Given a stacky CDGA A and A-modules P,Q in cochain chain complexes, we define $\mathbf{R}\,\underline{\hat{\mathrm{Hom}}}_A(P,Q)$ to be any of the quasi-isomorphic complexes $\underline{\hat{\mathrm{Hom}}}_A(\tilde{P},Q)$ given by replacing P with a cofibrant replacements $\tilde{P}\to P$ in the model structure of Lemma 3.3; all choices are quasi-isomorphic because all objects are fibrant, so quasi-isomorphisms between cofibrant objects are homotopy equivalences in the strict model structure, inducing quasi-isomorphisms on $\mathcal{H}om$ and hence on $\underline{\hat{\mathrm{Hom}}}$.

We say that a map $f\colon P\to Q$ is an $\mathbf{R}\,\underline{\hat{\mathrm{Hom}}}_A$-homotopy equivalence if the induced homology class $[f]\in \mathrm{H}_0\mathbf{R}\,\underline{\hat{\mathrm{Hom}}}_A(P,Q)$ has an inverse in $\mathrm{H}_0\mathbf{R}\,\underline{\hat{\mathrm{Hom}}}_A(Q,P)$.

Lemma 3.10. *Take a diagram $D=(A(0)\xrightarrow{f_1}A(1)\xrightarrow{f_2}\ldots\xrightarrow{f_m}A(m))$ of cofibrant stacky R-CDGAs (resp. stacky EFC-DGAs, resp. stacky $\mathcal{C}^\infty$-DGAs) concentrated in non-negative cochain degrees with all f_i surjective, such that*

1. *there exists $s\geq 0$ for which the chain complexes $(\Omega^1_{A(i),\square}\otimes_{A(i)}A(i)^0)^r$ are acyclic for all $r>s$, and*
2. *the maps f_i are all homotopy formally étale in the sense that the maps*

$$\mathrm{Tot}\,\sigma^{\leq s}(\Omega^1_{A(i-1),\square}\otimes_{A(i-1)}A(i)^0)\to\mathrm{Tot}\,\sigma^{\leq s}(\Omega^1_{A(i),\square}\otimes_{A(i)}A(i)^0)$$

are quasi-isomorphisms of $A(i)^0$-modules.

Then the natural morphisms

$$\mathcal{W}_i\mathcal{P}ol(D,n)\to\mathcal{W}_i\mathcal{P}ol(A(0),n)$$

are all $\mathbf{R}\,\underline{\hat{\mathrm{Hom}}}_{A(0)}$-homotopy equivalences.

Proof. Since each f_i is surjective and taking symmetric invariants is an exact functor, the question reduces to showing that for $A=A(i-1)$ and $B=A(i)$, the maps

$$\mathcal{H}om_B((\Omega^1_{B,\square})^{\otimes_B p},B)\to\mathcal{H}om_A((\Omega^1_{A,\square})^{\otimes_A p},B)\cong\mathcal{H}om_B((\Omega^1_{A,\square})^{\otimes_A p}\otimes_A B,B)$$

are $\mathbf{R}\,\underline{\hat{\mathrm{Hom}}}_{A(0)}$-homotopy equivalences for all $i>0$.

The boundedness hypotheses ensure that the chain complexes $((\Omega^1_{A,\square})^{\otimes p}\otimes_A A^0)^i$ and $((\Omega^1_{B,\square})^{\otimes p}\otimes_B B^0)^i$ are acyclic for $i>sp$. Combined with the homotopy formally étale hypothesis, this ensures that the maps $(\Omega^1_{A,\square})^{\otimes p}\otimes_A B\to(\Omega^1_{B,\square})^{\otimes p}$ satisfy the conditions of Lemma 3.8, so are $\underline{\hat{\mathrm{Hom}}}_B$-homotopy equivalences, and hence $\mathbf{R}\,\underline{\hat{\mathrm{Hom}}}_{A(0)}$-homotopy equivalences *a fortiori.* $\square$

Lemma 3.11. *If $A \to B$ is a morphism of cofibrant $\mathbb{G}_m$-equivariant stacky CDGAs with non-negative $\mathbb{G}_m$-weights, such that $\mathcal{W}_0 A \to \mathcal{W}_0 B$ is a levelwise quasi-isomorphism and the morphisms $\mathcal{W}_i A \to \mathcal{W}_i B$ are $\underline{\mathrm{H\hat{o}m}}_{\mathcal{W}_0 A}$-homotopy equivalences for all $i \geq 0$, then the morphisms*

$$\mathcal{W}_i(\Omega^1_A \otimes_A \mathcal{W}_0 A) \to \mathcal{W}_i(\Omega^1_B \otimes_B \mathcal{W}_0 B)$$

are $\underline{\mathrm{H\hat{o}m}}_{\mathcal{W}_0 A}$-homotopy equivalences for all i.

Proof. Since $\mathcal{W}_0 A \to \mathcal{W}_0 B$ is a levelwise quasi-isomorphism, so too is $\Omega^1_{\mathcal{W}_0 A} \to \Omega^1_{\mathcal{W}_0 B}$. Using the exact sequence $0 \to \Omega^1_{\mathcal{W}_0 A} \otimes_{\mathcal{W}_0 A} A \to \Omega^1_A \to \Omega^1_{A/\mathcal{W}_0 A} \to 0$, we can thus replace Ω^1_A and Ω^1_B with $\Omega^1_{A/\mathcal{W}_0 A}$ and $\Omega^1_{B/\mathcal{W}_0 B}$ to give an equivalent statement.

Because A is cofibrant and $\mathcal{W}_{>0} A$ is the augmentation ideal of the morphism $A \to \mathcal{W}_0 A$, we have $\Omega^1_{A/\mathcal{W}_0 A} \otimes_A \mathcal{W}_0 A \cong (\mathcal{W}_{>0} A)/((\mathcal{W}_{>0} A) \cdot (\mathcal{W}_{>0} A))$, so the Koszul resolution (equivalently, the commutative bar construction as in the proof of Lemma 1.17) gives a canonical $\mathcal{W}_0 A$-module resolution of $\Omega^1_{A/\mathcal{W}_0 A} \otimes_A \mathcal{W}_0 A$ equipped with an increasing filtration whose graded pieces are $\mathrm{CoLie}_n(\mathcal{W}_{>0} A_{[-1]})_{[1]}$. In weight i, this becomes a finite filtration only involving terms with $n \leq i$, so it suffices to observe that the maps $\mathcal{W}_j \mathrm{CoLie}^n_{\mathcal{W}_0 A}(\mathcal{W}_{>0} A_{[-1]}) \to \mathcal{W}_j \mathrm{CoLie}^n_{\mathcal{W}_0 B}(\mathcal{W}_{>0} B_{[-1]})$ are all $\underline{\mathrm{H\hat{o}m}}_{\mathcal{W}_0 A}$-homotopy equivalences because they are finite tensor expressions in the terms $\mathcal{W}_k A, \mathcal{W}_k B$. □

3.3. **Poisson L_∞-morphisms in the Tate category.** For this section, we fix a chain CDGA R over $\mathbb{Q}$ to act as our base ring.

We now introduce a variant of Definition 3.5 incorporating non-negatively weighted $\mathbb{G}_m$-actions.

Definition 3.12. Define the non-negatively weighted Tate dg category $\mathcal{T}^+_{R,dg}$ as follows. Objects are $\mathbb{G}_m$-equivariant R-modules in chain cochain complexes for which the $\mathbb{G}_m$-weights are non-negative. Morphisms are given by the complexes

$$\mathcal{T}^+_{R,dg}(M,N) := \prod_{i \geq 0} \underline{\mathrm{H\hat{o}m}}_R(\mathcal{W}_i M, \mathcal{W}_i N)$$

with the obvious composition rule.

The non-negatively weighted Tate category $\mathcal{T}^+_R$ is defined to have the same objects, but with morphisms $\mathrm{Z}_0 \mathcal{T}^+_{R,dg}(M,N)$.

Note that the tensor product defines a bifunctor on these categories, since $\hat{\mathrm{Tot}}$ commutes with finite direct sums, so

$$\mathcal{T}^+_{R,dg}(M \otimes_R M', P) = \prod_{i \geq 0, j \geq 0} \underline{\mathrm{H\hat{o}m}}_R(\mathcal{W}_i M \otimes_R \mathcal{W}_j M', \mathcal{W}_{i+j} P),$$

meaning that the maps

$$\underline{\mathrm{H\hat{o}m}}_R(\mathcal{W}_i M, \mathcal{W}_i N) \otimes_R \underline{\mathrm{H\hat{o}m}}_R(\mathcal{W}_j M', \mathcal{W}_j N') \to \underline{\mathrm{H\hat{o}m}}_R(\mathcal{W}_i M \otimes_R \mathcal{W}_j M', \mathcal{W}_{i+j}(N \otimes N'))$$

coming from lax monoidality of $\hat{\mathrm{Tot}}$ induce natural maps

$$\mathcal{T}^+_{R,dg}(M,N) \otimes_R \mathcal{T}^+_{R,dg}(M',N') \to \mathcal{T}^+_{R,dg}(M \otimes_R M', N \otimes_R N').$$

Note that our hypothesis that the $\mathbb{G}_m$-weights be non-negative is essential for this to hold, since otherwise the expression $\mathcal{W}_n(M \otimes_R M') = \bigoplus_{i+j=n} \mathcal{W}_i M \otimes_R \mathcal{W}_j M'$ would not be finite.

Also observe that $\mathcal{T}^+_{R,dg}$ has a dg-subcategory consisting of $\mathbb{G}_m$-equivariant R-linear morphisms of double complexes, which we refer to as strict morphisms, given by the subcomplex $\prod_{i\ge 0} \mathrm{Z}^0\mathcal{H}om_R(\mathcal{W}_iM, \mathcal{W}_iN) \subset \mathcal{T}^+_{R,dg}(M,N)$.

Definition 3.13. Given a stacky CDGA A equipped with a $\mathbb{G}_m$-action of non-negative weights, we define $\mathcal{T}^+_{A,dg}$ to be the dg category whose objects are $\mathbb{G}_m$-equivariant A-modules in chain cochain complexes for which the $\mathbb{G}_m$-weights are non-negative and whose complexes $\mathcal{T}^+_{A,dg}(M,N)$ of morphisms are given by the natural equaliser

$$\mathcal{T}^+_{R,dg}(M,N) \implies \mathcal{T}^+_{R,dg}(A\otimes_R M, N)$$

coming from the observations above.

The category $\mathcal{T}^+_A$ is then defined to have the same objects, but with morphisms $\mathrm{Z}_0\mathcal{T}^+_{A,dg}(M,N)$.

Since the bifunctor $\underline{\hat{\mathrm{Hom}}}_R$ respects finite colimits in the first argument, for any morphism $A\to B$ of non-negatively weighted $\mathbb{G}_m$-equivariant CDGAs, the forgetful functor $\mathcal{T}^+_{B,dg}\to\mathcal{T}^+_{A,dg}$ then has a left adjoint given by $M\mapsto M\otimes_A B$.

Since $\mathcal{T}^+_R$ does not contain arbitrary coproducts, and in particular since infinite direct sums are not coproducts there, we cannot directly adapt §1.2. Instead, in order to express our obstructions in terms of polyvectors we have to enlarge the class of morphisms as follows.

For now, fix a non-negatively weighted $\mathbb{G}_m$-equivariant chain CDGA S, which in applications will be either $R[\hbar^2]$ or $R[\hbar^2]/\hbar^{2m}$.

Definition 3.14. Take a symmetric monoidal pre-triangulated dg category $\mathcal{C}$ equipped with a tensor functor from the dg category of $\mathbb{G}_m$-representations, and take P^{ac}_k-algebras A,B in $\mathcal{C}$; write $\hbar^p\colon \mathcal{C}\to\mathcal{C}$ for the dg functor given by tensoring with the $\mathbb{G}_m$-representation of weight p.

We then define $\mathbb{G}_m$-equivariant Poisson L_∞-morphisms from A to B to consist of sequences $f=(f_1,f_2,\ldots)$ with $f_p\in\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathrm{Symm}^p(A_{[1-k]}),\hbar^{p-1}B)_{p-k}$, such that

- f is an L_∞-morphism (i.e. satisfies the formulae of [LV, Proposition 10.2.13]) and
- f satisfies the multiplicative property

$$f_{n+1}(ab,x_1,\ldots,x_n)=\sum_{p=0}^{n}(-1)^{(\deg b)\sum_{i=1}^{p}(\deg x_i)}f_{p+1}(a,x_1,\ldots,x_p)f_{n-p}(b,x_{p+1},\ldots,x_n).$$

When A,B are weighted $\mathbb{G}_m$-equivariant stacky P^{ac}_k-algebras over S, we write $\mathrm{Tate}_{P^{ac}_k,S,L_\infty}(A,B)$ for the set of Poisson L_∞-morphisms from A to B in $\mathcal{T}^+_{S,dg}$.

Again, observe that the category from Lemma 3.4 arises as a subcategory of $\mathrm{Tate}_{P^{ac}_k,R,L_\infty}(A,B)$, consisting of morphisms f with $f_1\in\mathrm{Z}_0\mathrm{Z}^0\mathcal{H}om_R(A,B)\subset\mathcal{T}^+_{R,dg}(A,B)$ and $f_p=0$ for all $p>1$; we refer to these as strict morphisms.

Lemma 3.15. *Poisson L_∞-morphisms are closed under composition of L_∞-morphisms.*

Proof. This can be checked by direct substitution.

Alternatively, we can form bar constructions with respect to the P_k and Lie operads (cf. Lemma 1.17) in the ind-category $\mathrm{ind}(\mathcal{C})$, which does have infinite coproducts. An L_∞-morphism f corresponds to a cocommutative coalgebra map

$\mathrm{B}_{\mathrm{Lie}}(f)\colon \mathrm{B}_{s^{1-k}\hbar^{-1}\mathrm{Lie}}A \to \mathrm{B}_{s^{1-k}\hbar^{-1}\mathrm{Lie}}B$ in the ind-category, with f being given by restriction to cogenerators B. Since $\mathrm{B}_{s^{1-k}\hbar^{-1}\mathrm{Lie}}A$ is a quotient of the P_k^{ac}-coalgebra $\mathrm{B}_{P_k^{ac}}A$, the same procedure defines a map $\mathrm{B}_{P_k^{ac}}(f)\colon \mathrm{B}_{P_k^{ac}}A \to \mathrm{B}_{P_k^{ac}}B$ of $\mathbb{G}_m$-equivariant graded P_k^{ac}-coalgebras (i.e. a degree 0 morphism in the dg category but not necessarily closed). The conditions of Definition 3.14 are then precisely the evaluation on cogenerators B of the condition that $\mathrm{B}_{P_k^{ac}}(f)$ be closed under the differential.

Thus a Poisson L_∞-morphism from A to B in $\mathcal{C}$ corresponds to a P_k^{ac}-coalgebra map $\mathrm{B}_{P_k^{ac}}A \to \mathrm{B}_{P_k^{ac}}B$ in $\mathrm{ind}(\mathcal{C})$ whose co-restriction to the natural coalgebra quotient $\mathrm{B}_{s^{1-k}\hbar^{-1}\mathrm{Lie}}B$ factors through the corresponding quotient $\mathrm{B}_{s^{1-k}\hbar^{-1}\mathrm{Lie}}A$ of $\mathrm{B}_{P_k^{ac}}A$. This property is manifestly closed under composition. □

Definition 3.16. Given a $\mathbb{G}_m$-equivariant stacky P_k^{ac}-algebra A over S and a Beck A-module M in the strict category of $\mathbb{G}_m$-equivariant S-modules in double complexes, define the chain complex $\underline{\widehat{\mathbb{D}\mathrm{er}}}_{P_k^{ac},S,L_\infty}(A,M)$ to consist in degree n of $\mathbb{G}_m$-equivariant Poisson L_∞-derivations $A \to \mathrm{cone}(M)_{[n]}$ in $\mathcal{T}_S^+$, with differential $\delta\colon \underline{\widehat{\mathbb{D}\mathrm{er}}}_{P_k^{ac},S,L_\infty}(A,M)_n \to \underline{\widehat{\mathbb{D}\mathrm{er}}}_{P_k^{ac},S,L_\infty}(A,M)_{n-1}$ induced by the obvious map $\mathrm{cone}(M)_{[n]} \to \mathrm{cone}(M)_{[n-1]}$.

The following acts as our substitute for Lemma 1.17 in the stacky setting:

Lemma 3.17. *Given a non-negatively weighted $\mathbb{G}_m$-equivariant stacky P_k^{ac}-algebra A over R which is cofibrant as a $\mathbb{G}_m$-equivariant R-CDGA, together with a Beck A-module M, there is a complete decreasing filtration $F^1 \supset F^2 \supset \ldots$ on the complex $\underline{\widehat{\mathbb{D}\mathrm{er}}}_{P_k^{ac},R,L_\infty}(A,M)$ with associated graded complexes*

$$\mathcal{T}_A^+(\mathrm{Symm}_A^p((\Omega^1_{A/R})_{[-k]}), \hbar^{p-1}M)_{[-k]},$$

where Ω^1 denotes the double complex of Kähler differentials of the underlying stacky CDGA.

Proof. Unwinding Definition 3.14, a $\mathbb{G}_m$-equivariant Poisson L_∞-derivation in $\mathcal{T}_R^+$ is an L_∞-derivation θ for which the elements $\theta_p \in \mathcal{T}_{R,dg}^+(\mathrm{Symm}^p(A_{[1-k]}), \hbar^{p-1}M)_{p-k}$ satisfy $\theta_p(ab, y_2, \ldots, y_p) = a\theta_p(b, y_2, \ldots, y_p) \pm \theta_p(a, y_2, \ldots, y_p)b$, which, since the operations are symmetric, says precisely that they are multiderivations.

Thus if we forget differentials we have an isomorphism of graded R-modules

$$\underline{\widehat{\mathbb{D}\mathrm{er}}}_{P_k^{ac},R,L_\infty}(A,M)_\# \cong \prod_{p\geq 1} \mathcal{T}_{A,dg}^+(\mathrm{Symm}_A^p((\Omega^1_{A/R})_{[-k]}), \hbar^{p-1}M)_{[\#-k]},$$

and we can then set F^j to be the product of the terms with $p \geq j$. Since $\underline{\widehat{\mathbb{D}\mathrm{er}}}_{P_k^{ac},R,L_\infty}(A,M)$ is a subcomplex of the complex of L_∞-derivations, the pth component $(\delta\theta)_p$ is an expression in terms of θ_j for $j \leq p$. If $\theta_p = 0$ for all $p < j$, this implies that $(\delta\theta)_p = 0$ for all $p < j$, so $\delta(F^j) \subset F^j$ and we have a filtration by subcomplexes. Moreover, $(\delta\theta)_j = \delta(\theta_j)$ under those conditions, so $\delta\theta \in \delta(\theta_j) + F^{j+1}$, giving the required description of the associated graded complex. □

Definition 3.18. Say that a morphism $A \to B$ of stacky $\mathcal{P}$-algebras is an abelian extension if it is surjective and if whenever a $\mathcal{P}$-algebra operation on A has more than one input in $\ker(A \to B)$, the output is zero. Say that a morphism $A \to B$ is a nilpotent extension if it is a finite composition of abelian extensions.

Note that the condition implies that $\ker(A \to B)$ is naturally a Beck A-module, and that this structure is induced by a Beck B-module structure. It also gives us a stacky $\mathcal{P}$-algebra isomorphism $A \times_B A \cong A \times_B (B \oplus \ker(A \to B))$.

The following proposition is the technical key to this section:

Proposition 3.19. *Take strict morphisms $B' \xrightarrow{g} B \xrightarrow{h} \bar{B} \xleftarrow{f} A$ of non-negatively weighted $\mathbb{G}_m$-equivariant stacky P_k^{ac}-algebras over S, such that g is an abelian extension with kernel I, h is a nilpotent extension, and the natural Beck B-module structure on I is induced by a Beck $\bar{B}$-module structure.*

We then have a short exact sequence

$$0 \to \mathrm{Z}_0 \,\underline{\widehat{\mathrm{Der}}}_{P_k^{ac},S,L_\infty}(A, f_*I) \to \mathrm{Tate}_{P_k^{ac},S,L_\infty}(A,B')_f \xrightarrow{g_*} \mathrm{Tate}_{P_k^{ac}.S,L_\infty}(A,B)_f$$

of groups and sets, where $(-)_f$ denotes the fibre over $f \in \mathrm{Tate}_{P_k^{ac},S,L_\infty}(A,\bar{B})$.

Moreover, if A is cofibrant as a stacky CDGA in the model structure of Lemma 3.3, then the short exact sequence extends to a further term $o_g\colon \mathrm{Tate}_{P_k^{ac},S,L_\infty}(A,B)_f \to \mathrm{H}_{-1}\,\underline{\widehat{\mathrm{Der}}}_{P_k^{ac},S,L_\infty}(A, f_*I)$, so $o_g(\phi) = 0$ if and only if ϕ lies in the image of g_*.

Proof. Given two elements $\phi, \psi \in \mathrm{Tate}_{P_k^{ac},S,L_\infty}(A,B')_f$ with $g \circ \phi = g \circ \psi$, consider the element $\phi - \psi \in \prod_{p\geq 1} \mathcal{T}^+_{S,dg}(\mathrm{Symm}^p_S(A_{[1-k]}), \hbar^{p-1}B)_{p-k}$. Since g is an abelian extension with I a Beck $\bar{B}$-module, we have an isomorphism

$$\begin{aligned} B' \times_{\bar{B}} (\bar{B} \oplus I) &\xrightarrow{\cong} B' \times_B B' \\ (b, (\bar{b}, x)) &\mapsto (b, b + x) \end{aligned}$$

of P_k^{ac}-algebras. Noting that the inclusion functor from the strict dg category of double complexes to $\mathcal{T}^+_{S,dg}$ preserves finite limits, it follows that $(f, \phi - \psi)\colon A \to \bar{B} \oplus I$ defines a Poisson L_∞-morphism of $\mathbb{G}_m$-equivariant P_k^{ac}-algebras $\mathcal{T}^+_S$. Expanding this out in terms of Definition 3.14, this says precisely that $\phi - \psi \in \mathrm{Z}_0\,\underline{\widehat{\mathrm{Der}}}_{P_k^{ac},S,L_\infty}(A, f_*I)$, which gives the first statement.

For the second statement, we start with an intermediate lemma.

Lemma. *When A is cofibrant as a stacky CDGA, every element $\rho \in \mathrm{Tate}_{P_k^{ac},S,L_\infty}(A,B)_f$ admits a lift $\tilde{\rho}$ in $\prod_{p\geq 1} \mathcal{T}^+_{S,dg}(\mathrm{Symm}^p_S(A_{[1-k]}), \hbar^{p-1}B')_{p-k}$ satisfying the multiplicativity property of Definition 3.14.*

Proof of lemma. Since this lifting property is preserved on passage to a retract, we may assume that A is freely generated as a commutative algebra by a trigraded (i.e. $\mathbb{G}_m$-equivariant bigraded) $\mathbb{Q}$-vector space $V = \bigoplus_i \mathcal{W}_i V^{\#}_{\#}$; forgetting the differentials gives $A^{\#}_{\#} \cong \bigoplus_n S \otimes_{\mathbb{Q}} \mathrm{Symm}^n_{\mathbb{Q}}(V^{\#}_{\#})$.

If we let α be the restriction of ρ to V, we can then choose a lift $\tilde{\alpha} \in \prod_{p\geq 1} \mathcal{T}^+_{\mathbb{Q}}(\mathrm{Symm}^p_{\mathbb{Q}}(V_{[1-k]}), \hbar^{p-1}B')_{p-k}$, since $B' \to B$ is surjective. Repeated application of the multiplicativity property then gives us elements $\tilde{\rho}_p$ of $\mathrm{Tot}^{\Pi}(\mathcal{H}om_S(\mathrm{Symm}^p_S(A_{[1-k]}), \hbar^{p-1}B'))_{p-k}$ agreeing with $\tilde{\alpha}_p$ on generators $V \subset A$, but we need to check that each term lies in $\mathcal{T}^+_{dg} \subset \mathrm{Tot}^{\Pi}\mathcal{H}om$. The iterated multiplicativity property gives, for $r := \sum_{j=1}^p r_j$ and $L = r + 1 - p$,

$$\begin{aligned} &\tilde{\rho}_p(\mathrm{Symm}^{r_1} V \otimes \ldots \otimes \mathrm{Symm}^{r_p} V) \\ &\subset \sum_{\sum_{l=1}^L m_l = r} \tilde{\alpha}_{m_1}(V, \ldots, V) \cdots \tilde{\alpha}_{m_L}(V, \ldots, V), \end{aligned}$$

which in particular means that $m_l \leq p$ for all l, so $\tilde{\rho}_p$ depends only on $\{\tilde{\alpha}_m\}_{m\leq p}$.

Writing $\tilde{\alpha}_{p,i}$ for the component of $\tilde{\alpha}$ in $\underline{\mathrm{H\hat{o}m}}_{\mathbb{Q}}(\mathcal{W}_i\mathrm{Symm}^p_{\mathbb{Q}}(V_{[1-k]}), \mathcal{W}_{i-p+1}B')_{p-k}$, and $\tilde{\rho}_{p,i}$ for the restriction of $\tilde{\alpha}_p$ to $\mathcal{W}_i$, we moreover have that $\tilde{\rho}_{p,i}$ depends only on the finite set $\{\tilde{\alpha}_{m,j}\}_{m\le p, j\le i}$. Since the set is finite, by definition of $\underline{\mathrm{H\hat{o}m}}$ there exists an integer N such that $\tilde{\alpha}_{m,j}$ lies in cochain degrees $[-N, \infty)$ for all $m \le p, j \le i$.

We can decompose $\tilde{\alpha}_{m,j}$ as $\tilde{\alpha}^+_{m,j} + \tilde{\alpha}^-_{m,j}$ into terms of non-negative and negative cochain degrees. Since f is a strict morphism, the image of $\tilde{\alpha}^-_{m,j}$ must lie in the kernel J' of the nilpotent surjection $B' \to \bar{B}$. If n is the index of nilpotence of J', then at most $n-1$ of the terms $\tilde{\alpha}^-_{m,j}$ can feature in a product to give a non-zero contribution. Thus the cochain degree of $\tilde{\rho}_{p,i}$ is bounded below by $-(n-1)N$ (crucially independent of r), so $\tilde{\rho}_{p,i} \in \underline{\mathrm{H\hat{o}m}}_S(\mathcal{W}_i Symm^p_S(A_{[1-k]}), \mathcal{W}_{i+1-p}B')_{p-k}$, proving the lemma. □

To complete the proof of the proposition, we can now proceed by a standard obstruction argument.

Given a morphism $\rho \in \mathrm{Tate}_{P^{ac}_k,S,L_\infty}(A,B)_f$, we choose a lift $\tilde{\rho}$ as in the lemma. This will be an L_∞ morphism if and only if the induced coalgebra morphism $\mathrm{B}_{\mathrm{Lie}}(\tilde{\rho})\colon \mathrm{B}_{s^{1-k}\mathrm{Lie}}A \to \mathrm{B}_{s^{1-k}\mathrm{Lie}}B'$ as in the proof of Lemma 3.15 commutes with $\partial \pm \delta$. The resulting commutator $\kappa(\tilde{\rho}) \in \prod_p \mathcal{T}^+_{S,dg}(\mathrm{Symm}^p_S(A_{[1-k]}), \hbar^{p-1}B')_{p-k-1}$ is thus the obstruction to $\tilde{\rho}$ being an L_∞ morphism. Since its projection $\kappa(\rho)$ to B is 0, it follows that $\kappa(\tilde{\rho})$ is an L_∞ derivation from A to I. Because $B' \to B$ is an abelian extension, it moreover follows that $\kappa(\tilde{\rho})$ is a Poisson L_∞ derivation and that the Poisson L_∞ A-module structure on I is that induced from $\rho\colon A \to B$, so

$$\kappa(\tilde{\rho}) \in \mathrm{Z}_{-1}\,\underline{\mathbb{D}\widehat{\mathrm{er}}}_{P^{ac}_k,S,L_\infty}(A, f_*I).$$

Any other lift of ρ is of the form $\tilde{\rho} + \theta$ for $\theta \in \underline{\mathbb{D}\widehat{\mathrm{er}}}_{P^{ac}_k,S,L_\infty}(A, f_*I)_0$, since $B' \to B$ is an abelian extension. Then $\kappa(\tilde{\rho}+\theta) = \kappa(\tilde{\rho}) + [\partial \pm \delta, \theta]$, the abelian property killing all other terms, so there exists a lift of ρ to $\mathrm{Tate}_{P^{ac}_k,S,L_\infty}(A, B')$ if and only if $\kappa(\tilde{\rho})$ is a boundary, i.e. whenever $o_g(\rho) := [\kappa(\tilde{\rho})]$ is 0 in $\mathrm{H}_{-1}\,\underline{\mathbb{D}\widehat{\mathrm{er}}}_{P^{ac}_k,S,L_\infty}(A, f_*I)$. □

Remark 3.20. The cofibrancy condition in Proposition 3.19 is stronger than the proof uses, since it suffices for A to be a retract of a quasi-free object. That is an instance of a general phenomenon that deformation theory works most naturally with derived categories of the second kind.

Definition 3.21. Define the category $\mathcal{U}_{P^{ac}_k[\hbar^2]/\hbar^{2m},R}$ (resp. $\mathcal{U}_{P^{ac}_k[\hbar^2],R}$) as follows. Objects are non-negatively weighted $\mathbb{G}_m$-equivariant stacky $P^{ac}_k \otimes_{\mathbb{Q}} R[\hbar^2]/\hbar^{2m}$-algebras (resp. $P^{ac}_k \otimes_{\mathbb{Q}} R[\hbar^2]$-algebras) which are levelwise flat over $R[\hbar^2]/\hbar^{2m}$ (resp. $R[\hbar^2]$).

Morphisms from A to B then consist of those elements of $\mathrm{Tate}_{P^{ac}_k,R[\hbar^2]/\hbar^{2m},L_\infty}(A,B)$ (resp. $\mathrm{Tate}_{P^{ac}_k,R[\hbar^2],L_\infty}(A,B)$) which are strict morphisms $\mod \hbar^2$, i.e. morphisms of $\mathbb{G}_m$-equivariant $P^{ac}_k \otimes_{\mathbb{Q}} R$-algebras in the strict category of double complexes.

We then say that a morphism $A \to B$ is a weak equivalence if the induced (strict) morphism $A/\hbar^2 \to B/\hbar^2$ is a levelwise quasi-isomorphism, i,e. induces isomorphisms $\mathrm{H}_i(A^j/\hbar^2) \to \mathrm{H}^i(B^j/\hbar^2)$ for all i,j.

In particular, note that $\mathcal{U}_{P^{ac}_k,R} = \mathcal{U}_{P^{ac}_k[\hbar^2]/\hbar^2,R}$ is the strict category of non-negatively weighted $\mathbb{G}_m$-equivariant stacky $P^{ac}_k \otimes_{\mathbb{Q}} R$-algebras, i.e. the category with morphisms of cochain degree 0 and with no higher L_∞ terms.

Also observe that since $\mathcal{W}_i(\hbar^{2m}B) = 0$ for all $i < 2m$, we have $\underline{\mathrm{H\hat{o}m}}_{R[\hbar^2]}(\mathcal{W}_iA, \mathcal{W}_iB) = \varprojlim_m \underline{\mathrm{H\hat{o}m}}_{R[\hbar^2]}(\mathcal{W}_iA, \mathcal{W}_i(B/\hbar^{2m}))$, so substitution in Definition

3.12 gives $\mathcal{T}^+_{R[\hbar^2],dg}(A,B) \cong \varprojlim_m \mathcal{T}^+_{R[\hbar^2]/\hbar^{2m},dg}(A/\hbar^{2m}, B/\hbar^{2m})$ and thus

$$\mathcal{U}_{P_k^{ac}[\hbar^2],R}(A,B) \cong \varprojlim_m \mathcal{U}_{P_k^{ac}[\hbar^2]/\hbar^{2m},R}(A/\hbar^{2m}, B/\hbar^{2m}).$$

Definition 3.22. Letting $\mathcal{U}$ be either of the categories $\mathcal{U}_{P_k^{ac}[\hbar^2]/\hbar^{2m},R}$ or $\mathcal{U}_{P_k^{ac}[\hbar^2],R}$, and taking objects $A, B \in \mathcal{U}$, define the simplicial set $\underline{\mathcal{U}}(A,B)$ to be given by

$$n \mapsto \mathcal{U}(A, B \otimes \Omega(\Delta^n)_\bullet),$$

for the CDGA $\Omega(\Delta^n)_\bullet$ of polynomial de Rham forms as in Definition 2.22.

Corollary 3.23. *Given $A, B \in \mathcal{U}_{P_k^{ac}[\hbar^2]/\hbar^{2m+2},R}$, with A cofibrant in the model structure on $\mathbb{G}_m$-equivariant $P_k^{ac} \otimes R[\hbar^2]/\hbar^{2m+2}$-algebras from Lemma 3.3, the map*

$$\theta \colon \underline{\mathcal{U}}_{P_k^{ac}[\hbar^2]/\hbar^{2m+2},R}(A,B) \to \underline{\mathcal{U}}_{P_k^{ac}[\hbar^2]/\hbar^{2m},R}(A/\hbar^{2m}, B/\hbar^{2m})$$

is a Kan fibration of simplicial sets.

Moreover, for each element $f \in \underline{\mathcal{U}}_{P_k^{ac}[\hbar^2]/\hbar^{2m},R}(A/\hbar^{2m}, B/\hbar^{2m})$, there is a functorial obstruction $o(f) \in \mathrm{H}_{-1}\,\underline{\mathbb{D}\widehat{\mathrm{er}}}_{P_k^{ac},R,L_\infty}(A/\hbar^2, f_*\hbar^{2m}(B/\hbar^2))$ which vanishes if and only if the fibre $\theta^{-1}(f)$ is non-empty. If $o(f)$ vanishes, then the homotopy group $\pi_i(\theta^{-1}(f))$ is naturally a torsor for $\mathrm{H}_i\,\underline{\mathbb{D}\widehat{\mathrm{er}}}_{P_k^{ac},\mathcal{T}_R^+,L_\infty}(A/\hbar^2, f_*\hbar^{2m}(B/\hbar^2))$.

Proof. These are fairly standard consequences of Proposition 3.19. Since finite limits behave well in $\mathrm{Tate}_{P_k^{ac}[\hbar^2]/\hbar^{2m},R,L_\infty}$, for any finite simplicial set K we have

$$\mathrm{Hom}_{s\mathrm{Set}}(K, \underline{\mathcal{U}}_{P_k^{ac}[\hbar^2]/\hbar^{2m},R}(A,B)) \cong \mathcal{U}_{P_k^{ac}[\hbar^2]/\hbar^{2m},R}(A, B \otimes \Omega(K)_\bullet),$$

where $\Omega^n(K) := \mathrm{Hom}_{s\mathrm{Set}}(K, \Omega^n(\Delta^\bullet))$.

Letting $S := R[\hbar^2]/\hbar^{2m+2}$, the relative horn-filling condition for θ then amounts to surjectivity of

$$\begin{aligned}&\mathrm{Tate}_{P_k^{ac},S,L_\infty}(A, B \otimes \Omega(\Delta^n)_\bullet)_F \to \\ &\mathrm{Tate}_{P_k^{ac},S,L_\infty}(A, (B \otimes \Omega(\Delta^n)_\bullet/\hbar^{2m}) \times_{B\otimes\Omega(\Lambda^{n,k})_\bullet/\hbar^{2m}} B \otimes \Omega(\Lambda^{n,k})_\bullet)_F,\end{aligned}$$

for all $F \in \underline{\mathcal{U}}_{P_k^{ac},R}(A/\hbar^2, B/\hbar^2)_n \subset \mathrm{Tate}_{P_k^{ac},S,L_\infty}(A, B \otimes \Omega(\Delta^n)_\bullet/\hbar^2)$, where the subscript denotes the fibre over F.

Now, the map

$$B \otimes \Omega(\Delta^n)_\bullet \to (B \otimes \Omega(\Delta^n)_\bullet/\hbar^{2m}) \times_{B\otimes\Omega(\Lambda^{n,k})_\bullet/\hbar^{2m}} B \otimes \Omega(\Lambda^{n,k})_\bullet$$

is an abelian extension with kernel $\hbar^{2m}(B/\hbar^2) \otimes \ker(\Omega(\Delta^n)_\bullet \to \Omega(\Lambda^{n,k})_\bullet)$. Since the latter complex is acyclic, the obstruction of Proposition 3.19 vanishes, giving the required surjectivity for θ to be a Kan fibration.

The obstruction $o(f)$ is then given by applying Proposition 3.19 to the abelian extension $B \to B/\hbar^{2m}$ of $P_k^{ac} \otimes R[\hbar^2]/\hbar^{2m+2}$-algebras. Applying Proposition 3.19 to the extensions $B \otimes \Omega(\Delta^n)_\bullet \to B \otimes \Omega(\Delta^n)_\bullet/\hbar^{2m}$ and writing $V(n) := \underline{\mathbb{D}\widehat{\mathrm{er}}}_{P_k^{ac},R,L_\infty}(A/\hbar^2, f_*\hbar^{2m}(B/\hbar^2) \otimes \Omega(\Delta^n)_\bullet)$ gives us a faithful transitive action of $\mathrm{Z}_0V(n)$ on $\theta^{-1}(f)_n$ for all n, and hence of $\pi_i\mathrm{Z}_0V(-)$ on $\pi_i\theta^{-1}(f)$. Since A is cofibrant, the simplicial chain complex $V(-)$ is a Reedy fibrant replacement of the chain complex $V(0)$, so the simplicial abelian group $\mathrm{Z}_0V(-)$ has homotopy groups $\pi_i\mathrm{Z}_0V(-)$ isomorphic to $\mathrm{H}_iV(0)$, as required. □

Corollary 3.24. *For $\mathcal{U} := \mathcal{U}_{P^{ac}_k[\hbar^2]/\hbar^{2m},R}$ (resp. $\mathcal{U} := \mathcal{U}_{P^{ac}_k[\hbar^2],R}$), with $A, B \in \mathcal{U}$ such that A is cofibrant in the model structure on $\mathbb{G}_m$-equivariant stacky $P^{ac}_k \otimes R[\hbar^2]/\hbar^{2m}$-algebras (resp. $P^{ac}_k \otimes R[\hbar^2]$-algebras) from Lemma 3.4, the simplicial set $\underline{\mathcal{U}}(A,B)$ is a model for the derived mapping space*

$$\mathbf{R}\mathrm{map}_{\mathcal{U}}(A,B)$$

of morphisms from A to B in the simplicial localisation $L^{\mathcal{W}}\mathcal{U}$ of $\mathcal{U}$ at weak equivalences.

Proof. In the model category of restricted diagrams from [TV, §2.3.2], the natural morphisms $\mathcal{U}(A,-) \to \mathcal{U}(A, - \otimes \Omega^\bullet(\Delta^n))$ are all weak equivalences, since the maps $B \to B \otimes \Omega^\bullet(\Delta^n)$ are weak equivalences in $\mathcal{U}$. Because $\underline{\mathcal{U}}(A,B)$ is a model for $\mathrm{ho}\varinjlim_{n \in \Delta^{\mathrm{opp}}} \mathcal{U}(A, - \otimes \Omega^\bullet(\Delta^n))$, the natural map $\mathcal{U}(A,-) \to \underline{\mathcal{U}}(A,-)$ is also a weak equivalence in the category of restricted diagrams.

By [DK], as interpreted in [TV, Theorem 2.3.5], it then suffices to show that the functor $\underline{\mathcal{U}}(A,-)$ preserves weak equivalences, which we prove by induction on m. For $m = 1$, this follows from [Hov, §5], since it is a right function complex with respect to the model structure of Lemma 3.4. For the natural maps

$$\underline{\mathcal{U}}_{P^{ac}_k[\hbar^2]/\hbar^{2m+2},R}(A,B) \to \underline{\mathcal{U}}_{P^{ac}_k[\hbar^2]/\hbar^{2m},R}(A/\hbar^{2m}, B/\hbar^{2m}),$$

the description of Corollary 3.23 implies that the homotopy fibres are invariant under weak equivalences in B, which gives the inductive step. The case $\mathcal{U} := \mathcal{U}_{P^{ac}_k[\hbar^2],R}$ then follows by taking the limit over m; since all maps are fibrations, the limit is a homotopy limit. □

3.4. **Uniqueness results for deformations.**

3.4.1. *Uniqueness for stacky $P^{ac}_k[\hbar^2]$-algebras.*

Corollary 3.25. *Take $A, B \in \mathcal{U}_{P^{ac}_k[\hbar^2],R}$ such that A is cofibrant in the model structure on $\mathbb{G}_m$-equivariant stacky $P^{ac}_k \otimes_{\mathbb{Q}} R[\hbar^2]$-algebras from Lemma 3.4. If*

$$\mathrm{H}_i\, \widehat{\mathbb{D}\mathrm{er}}_{P^{ac}_k,R,L_\infty}(A/\hbar^2 A, \hbar^{2n} f_*(B/\hbar^2 B)) \cong 0$$

for all pairs (i,n) with $i \geq -1$ and $1 \leq n \leq m$, and all morphisms $f\colon A/\hbar^2 A \to B/\hbar^2 B$ in $\mathcal{U}_{P^{ac}_k,R}$, then the natural map

$$\underline{\mathcal{U}}_{P^{ac}_k[\hbar^2],R}(A,B) \to \underline{\mathcal{U}}_{P^{ac}_k,R}(A/\hbar^2 A, B/\hbar^2 B)$$

is a trivial fibration.

Proof. Under these hypotheses, Corollary 3.23 gives fibrations

$$\underline{\mathcal{U}}_{P^{ac}_k[\hbar^2]/\hbar^{2m+2},R}(A/\hbar^{2m+2}, B/\hbar^{2m+2}) \to \underline{\mathcal{U}}_{P^{ac}_k[\hbar^2]/\hbar^{2m},R}(A/\hbar^{2m}, B/\hbar^{2m})$$

for all $m \geq 1$, and shows that their fibres are contractible, making them trivial fibrations. The result then follows by taking the limit over all m. □

Definition 3.26. Define the lax monoidal dg functor $\hat{\mathrm{Tot}}_{\mathbb{G}_m}$ from $\mathbb{G}_m$-equivariant double complexes to $\mathbb{G}_m$-equivariant chain complexes by $\hat{\mathrm{Tot}}_{bG_m} A := \bigoplus_i \hat{\mathrm{Tot}}\, \mathcal{W}_i A$.

Lemma 3.27. *There is an ∞-functor $\hat{\mathrm{Tot}}_{\mathbb{G}_m}$ from the ∞-category of non-negatively weighted $\mathbb{G}_m$-equivariant stacky $P^{ac}_k \otimes_{\mathbb{Q}} R[\hbar^2]$-algebras to the ∞-category of $\mathbb{G}_m$-equivariant $P^{ac}_k \otimes_{\mathbb{Q}} R[\hbar^2]$-algebras, and this extends to an ∞-functor on $L^{\mathcal{W}}\mathcal{U}_{P^{ac}_k[\hbar^2],R}$.*

Proof. Since $\hat{\mathrm{Tot}}$ is lax monoidal, $\hat{\mathrm{Tot}}_{\mathbb{G}_m}$ defines a functor from $\mathbb{G}_m$-equivariant stacky $\mathcal{P}$-algebras to $\mathbb{G}_m$-equivariant $\mathcal{P}$-algebras for all chain operads $\mathcal{P}$, and this automatically yields an ∞-functor because it preserves weak equivalences, sending levelwise quasi-isomorphisms to quasi-isomorphisms.

Similarly, $\hat{\mathrm{Tot}}_{\mathbb{G}_m}$ defines a functor from $\mathcal{U}_{P_k^{ac}[\hbar^2],R}$ to the category of $\mathbb{G}_m$-equivariant Poisson L_∞-morphisms between $\mathbb{G}_m$-equivariant $P_k^{ac} \otimes_{\mathbb{Q}} R[\hbar^2]$-algebras. As in the proof of Lemma 3.15, these are *a fortiori* $R[\hbar^2]$-linear $(P_k)_\infty$-morphisms, i.e. ∞-morphisms in the sense of [LV, §10.2.2].

Composing with the $R[\hbar^2]$-linear analogue of the rectification functor $\Omega_{P_k^{ac}}\mathrm{B}_{P_k^{ac}}$ of [LV, Theorem 11.4.7] thus gives us a functor $F := \Omega_{P_k^{ac}}\mathrm{B}_{P_k^{ac}}\hat{\mathrm{Tot}}_{\mathbb{G}_m}$ on $\mathcal{U}_{P_k^{ac}[\hbar^2],R}$, with the restriction of F to the category of strict morphisms admitting a natural quasi-isomorphism from $\hat{\mathrm{Tot}}_{\mathbb{G}_m}$. In order to show that F induces the required ∞-functor, it suffices to show that it preserves weak equivalences. By the $R[\hbar^2]$-linear analogue of [LV, Proposition 11.4.11], that amounts to showing that the functor $\hat{\mathrm{Tot}}_{\mathbb{G}_m}$ sends weak equivalences to ∞-quasi-isomorphisms (i.e. $(P_k)_\infty$-morphisms whose first component is a quasi-isomorphism).

If $f\colon A \to B$ is a weak equivalence in $\mathcal{U}$, then $f_1\colon A/\hbar^2 \to B/\hbar^2$ is a levelwise quasi-isomorphism, so $f_1\colon \hat{\mathrm{Tot}}\,\mathcal{W}_i(A/\hbar^2) \to \hat{\mathrm{Tot}}\,\mathcal{W}_i(B/\hbar^2)$ is a quasi-isomorphism. Now, the flatness hypotheses on objects of $\mathcal{U}_{P_k^{ac}[\hbar^2],R}$ ensure that the maps $\hbar^{2k}\colon (A/\hbar^2) \to (\hbar^{2k}A/\hbar^{2k+2}A)$ are isomorphisms and similarly for B, so by induction we see that $f_1\colon \hat{\mathrm{Tot}}\,\mathcal{W}_i(A/\hbar^{2k}) \to \hat{\mathrm{Tot}}\,\mathcal{W}_i(B/\hbar^{2k})$ is a quasi-isomorphism for all k. Taking $k > i/2$ then shows that the map $f_1\colon \hat{\mathrm{Tot}}\,\mathcal{W}_iA \to \hat{\mathrm{Tot}}\,\mathcal{W}_iB$ is a quasi-isomorphism for all i. ☐

Definition 3.28. Given a non-negatively weighted $\mathbb{G}_m$-equivariant stacky P_k^{ac}-algebra A over R and a Beck A-module M in the strict category of $\mathbb{G}_m$-equivariant double complexes, define $\mathbf{R}\,\underline{\hat{\mathbb{D}\mathrm{er}}}_{P_k^{ac},R,L_\infty}(A, M) := \underline{\hat{\mathbb{D}\mathrm{er}}}_{P_k^{ac},R,L_\infty}(\tilde{A}, M)$ for any cofibrant replacement $\tilde{A}$ of A as a $\mathbb{G}_m$-equivariant stacky P_k^{ac}-algebra; note that by Lemma 3.17 this is well-defined up to quasi-isomorphism because Lemma 3.4 implies that $\tilde{A}$ is cofibrant as a stacky CDGA.

Theorem 3.29. *For $\bar{A} := (A/\hbar^2A)$, the functors*

$$A \mapsto \bigoplus_i \hat{\mathrm{Tot}}\,\mathcal{W}_iA \quad \text{and} \quad A \mapsto \bigoplus_i \hat{\mathrm{Tot}}\,\mathcal{W}_i(\bar{A}[\hbar^2])$$

from the ∞-category of non-negatively weighted $\mathbb{G}_m$-equivariant stacky $P_k^{ac} \otimes_{\mathbb{Q}} R[\hbar^2]$-algebras to the ∞-category of $\mathbb{G}_m$-equivariant $P_k^{ac} \otimes_{\mathbb{Q}} R[\hbar^2]$-algebras become naturally equivalent when restricted to objects A which are flat over $R[\hbar^2]$ and satisfy

(†) $\mathbf{R}\,\underline{\hat{\mathbb{D}\mathrm{er}}}_{P_k^{ac},R,L_\infty}(\bar{A}, M) \simeq 0$ *for all Beck $\bar{A}$-modules M of pure $\mathbb{G}_m$-weight ≥ 2.*

Proof. We adapt the proof of Corollary 1.13. As a preliminary, note that for all $P, N \in \mathcal{T}_A^+$, we have $\mathcal{T}_A^+(P, N) \cong \varprojlim_r \mathcal{T}_A^+(P, N/\mathcal{W}_{>r}N)$ essentially by construction. Since $\bar{A}$ is cofibrant, we can therefore deduce by a filtration argument that $\mathbf{R}\,\underline{\hat{\mathbb{D}\mathrm{er}}}_{P_k^{ac},R,L_\infty}(\bar{A}, N) \simeq 0$ whenever $\mathbf{R}\,\underline{\hat{\mathbb{D}\mathrm{er}}}_{P_k^{ac},R,L_\infty}(\bar{A}, \mathcal{W}_rN) \simeq 0$ for all r, which the condition (†) above ensures for all Beck $\bar{A}$-modules N with $\mathcal{W}_iN = 0$ for all $i < 2$.

In particular, whenever A satisfies the condition (†) and B is non-negatively weighted, we have $\mathbf{R}\,\underline{\hat{\mathbb{D}\mathrm{er}}}_{P_k^{ac},R,L_\infty}(\bar{A}, \hbar^{2i}\bar{B}) \simeq 0$ for all $i \geq 1$, so Corollary 3.25 implies that the map $\underline{\mathcal{U}}_{P_k^{ac}[\hbar^2],R}(A, B) \to \underline{\mathcal{U}}_{P_k^{ac},R}(A/\hbar^2A, B/\hbar^2B)$ is a weak equivalence. By Corollary 3.24, this means that the ∞-functor $A \mapsto A/\hbar^2A$ from $L^{\mathcal{W}}\mathcal{U}_{P_k^{ac}[\hbar^2],R}$ to $L^{\mathcal{W}}\mathcal{U}_{P_k^{ac},R}$ becomes full and faithful when restricted to objects satisfying (†) above. Since the functor $C \mapsto C[\hbar^2]$ is right inverse to reduction mod $\hbar^2$, it follows that the identity functor is naturally equivalent to the functor $A \mapsto (A/\hbar^2 A)[\hbar^2]$ on this ∞-subcategory $(L^{\mathcal{W}}\mathcal{U}_{P_k^{ac}[\hbar^2],R})^{\%}$.

Since the ∞-functor $\widehat{\mathrm{Tot}}_{\mathbb{G}_m}$ factors through $(L^{\mathcal{W}}\mathcal{U}_{P_k^{ac}[\hbar^2],R})$ by Lemma 3.27, composition gives us the required equivalence $\widehat{\mathrm{Tot}}_{\mathbb{G}_m} A \simeq \widehat{\mathrm{Tot}}_{\mathbb{G}_m}((A/\hbar^2 A)[\hbar^2])$, natural in objects A satisfying (†). □

3.4.2. *Uniqueness for involutive a.c. stacky P_k-algebras.* Adapting Definitions 1.1 and 1.15, we have:

Definition 3.30. We say that a cochain chain complex $V_\bullet^\bullet$ is quasi-involutively filtered if it is equipped with a filtration W by double subcomplexes and an involution e which preserves W and acts on $\mathrm{H}_*(\mathrm{gr}_i^W V^j)$ as multiplication by $(-1)^i$ for all j.

Define a quasi-involutive a.c. stacky P_k-algebra over R to be an $(R \otimes P_k^{ac}, W, e)$-algebra A in quasi-involutively filtered cochain chain complexes.

We have the following immediate analogue of Lemma 1.9:

Lemma 3.31. *The Rees functor of Definition 1.5 gives an equivalence of ∞-categories from the category of quasi-involutive a.c. stacky P_k-algebras over a chain CDGA R, localised at filtered levelwise quasi-isomorphisms, to the ∞-category of $\mathbb{G}_m$-equivariant $R \otimes_{\mathbb{Q}} P_k^{ac}$-algebras in cochain chain complexes of flat $\mathbb{Q}[\hbar^2]$-modules, localised at levelwise quasi-isomorphisms.*

Applying Lemmas 1.9 and 3.31 and taking functorial cofibrant replacement, Theorem 3.29 immediately gives the following:

Corollary 3.32. *The functors $\widehat{\mathrm{Tot}}$ and $\widehat{\mathrm{Tot}}\,\mathrm{gr}^W$ from the ∞-category of quasi-involutive a.c. stacky P_k-algebras in double complexes (localised at filtered levelwise quasi-isomorphisms) to the ∞-category of quasi-involutive a.c. P_k-algebras become naturally equivalent when restricted to objects A satisfying the conditions:*

1. $W_{-1}A^j = 0$ *for all j, and*
2. $\mathbf{R}\,\underline{\widehat{\mathbb{D}\mathrm{er}}}_{P_k^{ac},\mathcal{T}_R^+,L_\infty}(\mathrm{gr}^W A, M) \simeq 0$ *for all Beck W_0A-modules M of pure $\mathbb{G}_m$-weight ≥ 2.*

The following analogue of Proposition 1.18 is now an immediate consequence of Lemmas 3.17 and 3.11:

Proposition 3.33. *If B is a non-negatively weighted $\mathbb{G}_m$-equivariant stacky P_k^{ac}-algebra over a CDGA R for which the map $(\mathcal{W}_1\mathbf{L}\Omega^1_{B/\mathcal{W}_0B})\otimes^{\mathbf{L}}_{\mathcal{W}_0B} B \to \mathbf{L}\Omega^1_{B/\mathcal{W}_0B}$ of commutative cotangent complexes is a quasi-isomorphism, then*

$$\mathbf{R}\,\underline{\widehat{\mathbb{D}\mathrm{er}}}_{P_k^{ac},\mathcal{T}_R^+,L_\infty}(B, M) \simeq 0$$

for all Beck $\mathcal{W}_0B$-modules M pure of $\mathbb{G}_m$-weight ≥ 2.

Moreover, if A is another such algebra, equipped with a morphism $A \to B$ such that $\mathcal{W}_0A \to \mathcal{W}_0B$ is a levelwise quasi-isomorphism and if the morphisms $\mathcal{W}_iA \to \mathcal{W}_iB$ are $\mathbf{R}\,\underline{\widehat{\mathrm{Hom}}}_{\mathcal{W}_0A}$-homotopy equivalences for all $i \geq 0$, then we also have $\mathbf{R}\,\underline{\widehat{\mathbb{D}\mathrm{er}}}_{P_k^{ac},\mathcal{T}_R^+,L_\infty}(B, M) \simeq 0$.

3.5. **Quantisations on derived Artin stacks.** We are now in a position to generalise Theorem 2.20 to stacky CDGAs, and hence Corollary 2.29 to derived Artin n-stacks.

By [Pri4, Lemma 1.14], based on [Vor, §3], $\mathcal{CC}_R(A)$ is equipped with a stacky brace algebra structure. Since A is commutative, Lemma 2.8 moreover adapts to make $(\mathcal{CC}_{R,\oplus}(A), \tau^{\mathrm{HH}})$ (resp. $\mathcal{D}^{\mathrm{poly}}_{\oplus}(A)$) into a quasi-involutive stacky a.c. brace algebra.

For $w \in \mathrm{Levi}^t_{\mathrm{GT}}(\mathbb{Q})$, Definition 2.18 gives an equivalence p_w between stacky quasi-involutive a.c. brace algebras and stacky quasi-involutive a.c. P_2-algebras, by considering the respective algebras in the dg category of cochain complexes of R-modules in chain complexes. However, the proof of Theorem 2.20 does not immediately adapt to this setting, because functoriality for stacky Hochschild complexes and stacky polyvectors is much more subtle, which is why we have had to involve $\underline{\hat{\mathrm{Hom}}}$.

Writing $\Omega^1_{A,\Box}$ for the cotangent module associated to the relevant theory (commutative, $\mathcal{C}^\infty$ or EFC) as in Corollary 2.26, we have:

Theorem 3.34. *Take a cofibrant stacky R-CDGA (resp. stacky $\mathcal{C}^\infty$-DGA or stacky K-EFC-DGA) A with*

(‡) *cotangent complex $(\Omega^1_{A,\Box})^\#$ perfect as an $A^\#$-module.*

Then the quasi-involutively filtered DGLA underlying the complex of polydifferential operators

$$(\hat{\mathrm{Tot}}\, \mathcal{D}^{\mathrm{poly}}_{\oplus}(A)_{[-1]}, \tau^{\mathrm{HH}})$$

is filtered quasi-isomorphic to the graded DGLA

$$\mathrm{Pol}_R(A,0)_{[-1]} := \bigoplus_{p\geq 0} \underline{\hat{\mathrm{Hom}}}_A(\Omega^p_{A,\Box}, A)_{[p-1]}$$

of derived polyvectors on A, where the Lie algebra structure is given by the Schouten–Nijenhuis bracket.

This quasi-isomorphism depends only on a choice of even 1-associator $w \in \mathrm{Levi}^t_{\mathrm{GT}}$, and is natural with respect to homotopy étale functoriality induced by [Pri4, §3.1], [Pri3, §3.4.2] *and its $\mathcal{C}^\infty$ and EFC analogues.*

When A is a cofibrant stacky R-CDGA with the condition above on the cotangent complex, the same statements hold for the Hochschild complex $\mathcal{CC}_{R,\oplus}(A)$ in place of $\mathcal{D}^{\mathrm{poly}}_{\oplus}(A)$.

Proof. We adapt the proof of Theorem 2.20. As explained in §3.1, the cofibrant hypothesis ensures that Lemma 2.8 adapts to show that $(D^{\mathrm{poly}}_{\oplus}(A), \tau^{\mathrm{HH}})$ is a quasi-involutive a.c. stacky brace algebra, with a levelwise graded quasi-isomorphism $\mathrm{HKR}\colon \mathrm{gr}^{\tau^{\mathrm{HH}}}\mathcal{D}^{\mathrm{poly}}_{\oplus}(A) \xrightarrow{\sim} \mathcal{P}ol(A,0)$. Since $\mathcal{D}^{\mathrm{poly}}_{\oplus}(A) \to \mathcal{CC}_{R,\oplus}(A)$ is a filtered levelwise quasi-isomorphism in the stacky R-CDGA setting, it suffices to focus on $\mathcal{D}^{\mathrm{poly}}_{\oplus}(A)$ in all settings.

For any even associator w, the ∞-functor p_w of Definition 2.18 then gives an involutive a.c. stacky P_2-algebra $p_w(\mathcal{D}^{\mathrm{poly}}_{R,\oplus}, \tau^{\mathrm{HH}})$, with its associated graded algebra having a zigzag of levelwise quasi-isomorphisms of $\mathbb{G}_m$-equivariant stacky P_2^{ac}-algebras to $\mathrm{Pol}(A,0)$.

Since $(\Omega^1_{A,\Box})^\#$ is assumed to be perfect, Lemma 2.19 adapts verbatim to give a levelwise quasi-isomorphism $\mathbf{L}\Omega^1_{\mathcal{P}ol(A,0)/A} \simeq \mathcal{P}ol(A,0) \otimes^{\mathbf{L}}_A \mathcal{W}_1\mathcal{P}ol(A,0)$ of A-modules. Thus $\mathcal{P}ol(A,0)$ satisfies the conditions of Proposition 3.33.

By base change, we know that $\Omega^1_{A,\Box} \otimes_A A^0$ must be a perfect A^0-module in double complexes, so all but finitely many of the A^0-modules $(\Omega^1_{A,\Box} \otimes_A A^0)^r$ must be acyclic.

Thus if we take a diagram $D = (A(0) \to A(1) \to \ldots A(n))$ of homotopy formally étale surjections between $m+1$ such stacky CDGAs, the hypotheses of Lemma 3.10 are satisfied, so the maps $\mathcal{W}_i\mathcal{P}ol(D,0) \to \mathcal{W}_i\mathcal{P}ol(A(0),0)$ are $\mathbf{R}\,\underline{\widehat{\mathrm{Hom}}}_{A(0)}$-homotopy equivalences.

Thus $\mathcal{P}ol(D,0)$ also satisfies the conditions of Proposition 3.33, so $p_w(\mathcal{D}^{\mathrm{poly}}_{R,\oplus}(D), \tau^{\mathrm{HH}})$ satisfies the conditions of Corollary 3.32, giving a zigzag of filtered involutive quasi-isomorphisms

$$\alpha_{w,D} \colon\ p_w \bigcup_i \hat{\mathrm{Tot}}(\tau_i^{\mathrm{HH}}\mathcal{D}^{\mathrm{poly}}_{\oplus}(D)) \simeq \bigoplus_i \hat{\mathrm{Tot}}\,\mathcal{W}_i\mathcal{P}ol(D,0),$$

natural with respect to all morphisms $(\mathcal{D}^{\mathrm{poly}}_{\oplus}(D), \tau^{\mathrm{HH}}) \to (\mathcal{D}^{\mathrm{poly}}_{\oplus}(D'), \tau^{\mathrm{HH}})$ in the ∞-category of quasi-involutive a.c. stacky brace algebras, for all such diagrams D, D'.

If we write $\mathcal{C}$ for the category of stacky R-CDGAs (resp. stacky $\mathcal{C}^\infty$-DGAs or stacky K-EFC-DGAs), with $\mathcal{C}_{c,\twoheadrightarrow}$ the subcategory of cofibrant objects and surjective morphisms, then as in §3.2 we have functors $D \mapsto (\mathcal{D}^{\mathrm{poly}}_{\oplus}(D), \tau^{\mathrm{HH}})$ and $D \mapsto \mathcal{P}ol(D,0)$ on the Grothendieck construction $\int B\mathcal{C}_{c,\twoheadrightarrow}$ of the nerve. If we further restrict to the full subcategory $\mathcal{C}^{lfp}_{c,\twoheadrightarrow} \subset \mathcal{C}_{c,\twoheadrightarrow}$ on objects satisfying (‡), then the natural transformation α_w above gives a natural equivalence between the respective functors from $\int B\mathcal{C}^{lfp}_{c,\twoheadrightarrow}$ to the category of quasi-involutive a.c. P_2-algebras.

By Lemma 2.28, Consider the ∞-category $\mathbf{L}\mathcal{C}^{lfp,\text{ét}}$ given by the localisation at levelwise quasi-isomorphisms of the category of homotopy formally étale morphisms between stacky R-CDGAs (resp. stacky $\mathcal{C}^\infty$-DGAs or stacky K-EFC-DGAs) satisfying (‡). By Lemma 2.28, $\mathbf{L}\mathcal{C}^{lfp,\text{ét}}$ arises as a simplicial localisation of $\int B(\mathcal{C}^{lfp,\text{ét}}_{c,\twoheadrightarrow})$, for the subcategory $\mathcal{C}^{lfp,\text{ét}}_{c,\twoheadrightarrow} \subset \mathcal{C}^{lfp}_{c,\twoheadrightarrow}$ of homotopy formally étale morphisms. By [Pri4, §3.1], [Pri3, §3.4.2], our totalised functors descend to that localisation, so we also have an equivalence

$$\alpha_w \colon\ \bigcup_i \hat{\mathrm{Tot}}(\tau_i^{\mathrm{HH}}\mathcal{D}^{\mathrm{poly}}_{\oplus}(-)) \simeq \bigoplus_i \hat{\mathrm{Tot}}\,\mathcal{W}_i\mathcal{P}ol(-,0)$$

of ∞-functors from $\mathbf{L}\mathcal{C}^{lfp,\text{ét}}$ to the ∞-category of quasi-involutive a.c. P_2-algebras. □

Definition 3.35. Given a cofibrant stacky R-CDGA, stacky $\mathcal{C}^\infty$-DGA or stacky K-EFC-DGA A, adapting [Pri4, Definition 1.16] as in [Pri5, Pri9], we define the filtered DGLA $(Q\widehat{\mathrm{Pol}}(A,0)_{[-1]}, \tilde{F})$ of quantised polyvectors by setting

$$\tilde{F}^i Q\widehat{\mathrm{Pol}}(A,0) := \prod_{p \geq i} \hat{\mathrm{Tot}}\,\tau_p^{\mathrm{HH}}\mathcal{D}^{\mathrm{poly}}_{\oplus}(A)\hbar^{p-1};$$

observe the Gerstenhaber bracket satisfies $[\tau_p^{\mathrm{HH}}, \tau_q^{\mathrm{HH}}] \subset \tau_{p+q-1}^{\mathrm{HH}}$, so $[\tilde{F}^i, \tilde{F}^j] \subset \tilde{F}^{i+j-1}$, making $\tilde{F}^2 Q\widehat{\mathrm{Pol}}(A,0)_{[-1]}$ into a pro-nilpotent filtered DGLA.

The space $Q\mathcal{P}(A,0)$ of 0-shifted quantisations of A is then defined (adapting [Pri4, Definition 1.23]) to be

$$\varprojlim_i \underline{\mathrm{MC}}(\tilde{F}^2 Q\widehat{\mathrm{Pol}}(A,0)_{[-1]}/\tilde{F}^i).$$

The subspace $Q\mathcal{P}(A,0)^{sd} \subset Q\mathcal{P}(A,0)$ of self-dual quantisations then consists of fixed points for the involution $(-)^*$ given by $\Delta^*(\hbar) := i(\Delta)(-\hbar)$, for the involution i of Lemma 2.8.

These definitions all extend to diagrams $(A(0) \to \ldots \to A(m))$ in place of A, giving homotopy étale functoriality as in the proof of Theorem 3.34.

For a strongly quasi-compact derived Artin n-stack $\mathfrak{X}$, the space $Q\mathcal{P}(\mathfrak{X},0)$ and its variants are defined in [Pri4, §3.1] by first taking an Artin $(n+2)$-hypergroupoid resolution $X_\bullet$ of $\mathfrak{X}$, then applying the left adjoint D^* of the denormalisation functor D, forming a cosimplicial stacky CDGA $j \mapsto D^*O(X^{\Delta^j})$ with homotopy formally étale structure morphisms. This can be thought of as giving a formally étale simplicial resolution of $\mathfrak{X}$ by derived Lie algebroids. We then set

$$\mathcal{P}(\mathfrak{X},0) := \operatorname{holim}_{\substack{\longleftarrow \\ j\in\Delta}} \mathcal{P}(D^*O(X^{\Delta^j}),0), \quad Q\mathcal{P}(\mathfrak{X},0) := \operatorname{holim}_{\substack{\longleftarrow \\ j\in\Delta}} Q\mathcal{P}(D^*O(X^{\Delta^j}),0).$$

Similar constructions work verbatim for EFC and $\mathcal{C}^\infty$ analogues.

These constructions extend beyond the strongly quasi-compact setting, either by allowing the hypercover to involve disjoint unions of derived affine schemes, or as in [Pri7] by working directly with a functor on $D_*\mathfrak{X}$ on stacky CDGAs which admits formally étale affine hypercovers. Indeed, analogously to [Pri8, §2.3.4], this approach allows the definitions to extend to any homogeneous derived stack with a bounded below cotangent complex.

Lemma 3.36. *If Y is a derived Artin m-hypergroupoid, then the stacky CDGA $A := D^*O(Y)$ satisfies $(\Omega^1_{A,\Box}\otimes_A A^0)^r \simeq 0$ for all $r > m$. If the associated derived Artin m-stack $\mathfrak{Y} := Y^\sharp$ has perfect cotangent complex, then $(\Omega^1_A)^\#$ is perfect as an $A^\#$-module.*

Proof. First, note that the double complex $\Omega^1_{A,\Box}\otimes_A A^0$ is just the Dold–Kan normalisation N_c of the cosimplicial chain complex $\Omega^1_{Y,\Box}\otimes_{O(Y)} A^0$, since both constructions have the same right adjoint. The term N^r_c in cochain degree r is thus isomorphic to the cokernel $\Omega^1_{Y_r/M_{\Lambda^{r,0}}Y}\otimes_{O(Y_r)} O(Y_0)$ of the $(r,0)$th partial matching map, which by hypothesis is a trivial cofibration in degrees $r > m$, so $(\Omega^1_{A,\Box}\otimes_A A^0)^r \simeq 0$.

The map $\mathcal{H}om_A(\Omega^1_A, N)^0 \to \mathcal{H}om_A(\Omega^1_A, \sigma^{\le m}N)^0$ to the brutal cotruncation is thus a quasi-isomorphism for all A^0-modules N in double complexes. That quasi-isomorphism extends to all A-modules N in double complexes which are concentrated in non-negative cochain degrees, since $N \cong \varprojlim_r \sigma^{\ge r}N$ with the quotients being A^0-modules.

All the partial matching maps are smooth, so the argument of the first paragraph also shows that the chain complexes $(\Omega^1_{A,\Box}\otimes_A A^0)^r$ are projective A^0-modules in chain complexes, and in particular perfect complexes, for all $r > 0$. Since $Y_0 \to \mathfrak{Y}$ is an Artin m-atlas and $\mathfrak{Y}$ has perfect cotangent complex, so does Y_0, meaning that the A^0-module $\Omega^1_{A^0,\Box} = (\Omega^1_{A,\Box}\otimes_A A^0)^0$ is also a perfect complex. Since the vanishing result above implies that $(\Omega^1_A\otimes_A A^0)^\# \simeq \bigoplus_{r=0}^m ((\Omega^1_{A,\Box}\otimes_A A^0)^r)^{[-r]}$, we deduce that it is perfect as an A^0-module.

The functor $\mathcal{H}om_A(\Omega^1_A,-)^0$ thus commutes with filtered homotopy colimits of A^0-modules in double complexes, and hence (via quotients of the brutal truncation filtration) with filtered colimits of A-modules concentrated in cochain degrees $[0,m]$. Since cotruncation commutes with filtered colimits, it follows from the cotruncation property above that $\mathcal{H}om_A(\Omega^1_A,-)^0$ commutes with all filtered homotopy colimits of A-modules, which is equivalent to saying that $(\Omega^1_A)^\#$ is perfect as an $A^\#$-module. $\Box$

Corollary 3.37. *Given a derived Artin n-stack $\mathfrak{X}$ over R with perfect cotangent complex, any even associator $w \in \mathrm{Levi}^t_{\mathrm{GT}}$ gives rise to a map*

$$\mathcal{P}(\mathfrak{X},0) \to Q\mathcal{P}^{sd}(\mathfrak{X},0)$$

from the space of 0*-shifted Poisson structures on* $\mathfrak{X}$ *to the space of self-dual* E_1 *quantisations of* $\mathfrak{X}$ *in the sense of* [Pri4, Definitions 1.23, 3.9]*. These quantisations give rise to curved* A_∞ *deformations* $(\mathrm{per}_{dg}(\mathfrak{X})[\![\hbar]\!], \{m^{(i)}\}_{i\geq 0})$ *of the dg category* $\mathrm{per}_{dg}(\mathfrak{X})$ *of perfect* $\mathscr{O}_{\mathfrak{X}}$*-complexes,* $\hbar$*-semilinearly anti-involutive with respect to the dg endofunctor* $\mathscr{H}om_{\mathscr{O}_{\mathfrak{X}}}(-, \mathscr{O}_{\mathfrak{X}})$ *on* $\mathrm{per}_{dg}(\mathfrak{X})$*.*

The analogous statements for derived $\mathcal{C}^\infty$ *and derived analytic Artin* n*-stacks (in the sense of* [Pri9]*) with perfect cotangent complexes also hold.*

Proof. The first statement and its $\mathcal{C}^\infty$ and analytic analogues follow immediately by substituting Theorem 3.34 in the definitions above, via Lemma 3.36, and passing to homotopy limits.

By [Pri4, Proposition 3.11], E_1 quantisations of $\mathfrak{X}$ give rise to curved A_∞ deformations of $\mathrm{per}_{dg}(\mathfrak{X})$. Since the identity $\mathscr{O}_{\mathfrak{X}} \cong \mathscr{O}_{\mathfrak{X}}^{\mathrm{opp}}$ extends uniquely to the involution $(-)^\vee$ on $\mathrm{per}_{dg}(\mathfrak{X})^{\mathrm{opp}}$, the self-dual condition on a curved A_∞ deformation is that the following diagram commutes for each of the A_∞ operations $m^{(i)}(\hbar)$:

$$\begin{tikzcd} \mathcal{H}om(P_0,P_1)\otimes\ldots\otimes\mathcal{H}om(P_{i-1},P_i) \arrow[r, "m^{(i)}_{P_0,\ldots,P_i}(-\hbar)"] \arrow[d, "\simeq"'] & \mathcal{H}om(P_0,P_i)[\![\hbar]\!] \arrow[d, "\simeq"] \\ \mathcal{H}om(P_i^\vee,P_{i-1}^\vee)\otimes\ldots\otimes\mathcal{H}om(P_1^\vee,P_0^\vee) \arrow[r, "\pm m^{(i)}_{P_i^\vee,\ldots,P_0^\vee}"] & \mathcal{H}om(P_i^\vee,P_0^\vee)[\![\hbar]\!]. \end{tikzcd}$$ □

Remark 3.38. The hypotheses of Corollary 3.37 are satisfied by any derived Artin stack locally of finite presentation over the CDGA R. When $R = \mathrm{H}_0R$, this includes those underived Artin n-stacks $\mathfrak{X}$ which admit Artin n-atlases by schemes which are local complete intersections over R, in which case the cotangent complex $\mathbf{L}\Omega^1_{\mathfrak{X}/R}$ is concentrated in homological degrees $[-n, 1]$.

Remark 3.39 (Quantisations of 1-shifted co-isotropic structures)*.* Generalising Remark 2.32 to derived Artin stacks, we can use the proof of Theorem 3.34 to deduce that every 1-shifted co-isotropic structure on a morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ of derived Artin n-stacks admits a quantisation, provided $\mathfrak{X}$ has perfect cotangent complex. The reasoning is exactly the same as in Remark 2.32, with the quantisation functorially giving, for all morphisms $A \to B$ of stacky CDGAs with A and B homotopy formally étale over $\mathfrak{Y}$ and $\mathfrak{X}$ respectively, a 0-shifted quantisation Δ of B, a 1-shifted quantisation $\tilde{A}$ of A and a morphism $\tilde{A} \to (\prod_{j\geq 0} \tau_j^{\mathrm{HH}} \mathcal{C}\mathcal{C}_R(B)\hbar^j, \delta + \{\Delta, -\})$ of BD_2-algebras in an unweighted Tate category over $R[\![\hbar]\!]$.

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