Mathematical Techniques III
(PHY 317)

Problem Set 9

Due on Monday, December 7th

Problem 1. (*5 points*)

Compute the Fourier series for the following functions:

(a) $F(t) = (\sin t)^3$;

(b) $F(t) = \left|\left(\cos \frac{1}{3}t\right)^3\right|$;

(c) $F(t) = t^2$ for $-\pi < t < \pi$; and

(d) $F(t) = t|t|$ for $-\pi < t < \pi$.

(*Hint:* You might find the following trigonometric identities useful: $(\sin t)^3 = \frac{1}{4}\sin(3t) - \frac{3}{4}\sin t$, and $(\cos \frac{1}{3}t)^3 = \frac{1}{4}\cos t + \frac{3}{4}\cos \frac{1}{3}t$. Then again, you might not.)

Problem 2. (*5 points*)

Consider a uniform rod of length L, whose ends are kept at zero temperature. Let $\Theta(x,t)$ denote the temperature at time $t > 0$ at the position x along the rod. It satisfies the following equation:

$$\frac{\partial^2 \Theta(x,t)}{\partial x^2} = \frac{1}{\kappa}\frac{\partial \Theta(x,t)}{\partial t} , \tag{1}$$

where $\kappa > 0$ is a physical coefficient depending on the material of the rod. If at $t = 0$ the temperature distribution in the rod is given by a function $f(x)$, solve equation (1) to find the temperature distribution for later times $t > 0$. Compute the limit $\lim_{t\to\infty} \Theta(x,t)$ and interpret physically.

(*Hint*: Use separation of variables and Fourier sine series. You may assume that the series can be integrated and differentiated termwise.)

Problem 3. (*10 points*)

Compute the Fourier transforms of the following functions:

(a) $F(t) = e^{-|t|}$;

(b) $F(t) = e^{-t^2}$;

(c) $F(t) = t\, e^{-t^2}$;

(d) $F(t) = (\sin t)/t$; and

(e) $F(t) = (\sin \pi t)/(1 - t^2)$.

Problem 4. (*10 points*)

Find the Fourier transform and verify the inversion formula (using residue theory or some other method) for the following functions:

(a) $F(t) = 1/(t^4 + 1)$;

(b) $F(t) = t/(t^4 + 1)$; and

(c) $F(t) = \exp(-t^2)$.

Problem 5. (*10 points*)

Find the Fourier integral representation for solutions to the following differential equations:

$$\text{(a)} \qquad \frac{d^2 f(t)}{dt^2} + \frac{df(t)}{dt} + f(t) = e^{-t^2} \ ,$$

$$\text{(b)} \qquad \frac{d^2 f(t)}{dt^2} + 4\frac{df(t)}{dt} + f(t) = \begin{cases} 0 \ , & \text{for } t < 0; \\ e^{-t} \ , & \text{for } t \geq 0. \end{cases}$$

$$\text{(c)} \qquad \frac{d^2 f(t)}{dt^2} + 2\frac{df(t)}{dt} + 3f(t) = \begin{cases} 1 \ , & \text{for } |t| < 1; \\ 0 \ , & \text{otherwise.} \end{cases}$$

Problem 6. (*10 points*)

Consider the heat equation in Problem 2, but with an infinitely long rod. Let $\Theta(x,t)$ denote the temperature distribution at time t at the position $-\infty < x < \infty$ along the rod. Let the initial temperature distribution be given by $\Theta(x,0) = f(x)$, where $f(x)$ is a continuous function whose Fourier transform is $\hat{f}(k)$. Show that the function

$$\Theta(x,t) = \int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, e^{-\kappa k^2 t}\, dk$$

solves the heat flow equation (1) with the given initial condition.

Now insert the expression for the transform $\hat{f}(k)$, interchange the order of integration, and derive the formula

$$\Theta(x,t) = \frac{1}{2\sqrt{\pi\kappa t}} \int_{-\infty}^{\infty} f(y) e^{-(x-y)^2/4\kappa t}\, dy = \int_{-\infty}^{\infty} K(x-y,t)\, f(y)\, dy \ .$$

The expression $K(x-y,t) = e^{-(x-y)^2/4\kappa t}/(2\sqrt{\pi\kappa t})$, for $t > 0$, is known as the (one-dimensional) **heat kernel**. Prove that $K(x,t)$ obeys the heat equation with initial condition $K(x,0) = \delta(x)$.