Mathematical Techniques III
(PHY 317)

Problem Set 1

Due on Monday, October 5th by 6pm

Problem 1. (*10 points*)

Let $\mathbb{R}^N = \{(v_1, v_2, \ldots, v_N) \mid v_i \in \mathbb{R}\}$ be the set of N-tuples of real numbers. Show that $\mathbb{R}^N$ is a vector space with respect to the following operations:

(addition)

$$(v_1, v_2, \ldots, v_N) + (w_1, w_2, \ldots, w_N) \\ = (v_1 + w_1, v_2 + w_2, \ldots, v_N + w_N) \ ,$$

(multiplication by scalars)

$$\lambda\,(v_1, v_2, \ldots, v_N) = (\lambda\, v_1, \lambda\, v_2, \ldots, \lambda\, v_N) \quad \text{for } \lambda \in \mathbb{R}.$$

Determine which of the following subsets of $\mathbb{R}^N$ are vector subspaces:

1. The subset $\{(v_1, v_2, \ldots, v_N) \mid v_1 + v_2 + \cdots + v_N = 0\}$,
2. The subset $\{(v_1, v_2, \ldots, v_N) \mid v_1 + v_2 + \cdots + v_N = 1\}$, and
3. The subset $\{(v_1, v_2, \ldots, v_N) \mid v_i \in \mathbb{Q}\}$, where $\mathbb{Q}$ denotes the rational numbers.

Problem 2. (*10 points*)

Show that the set $\mathcal{F}$ consisting of real-valued functions on the interval $[-1, 1]$,

$$f : [-1, 1] \to \mathbb{R} \\ t \mapsto f(t) \ ,$$

is a vector space under the following operations:

$$(f + g)(t) = f(t) + g(t) \qquad \text{(addition)}$$
$$(\lambda\, f)(t) = \lambda\, f(t) \qquad \text{(multiplication by scalars)}$$

for $f, g \in \mathcal{F}$ and $\lambda \in \mathbb{R}$.

Determine which of the following subsets of $\mathcal{F}$ are vector subspaces:

1. The subset $\{f \in \mathfrak{F} \mid f(0) = 0\}$,
2. The subset $\{f \in \mathfrak{F} \mid f(0) = 1\}$,
3. The subset $\{f \in \mathfrak{F} \mid f(-1) = f(1)\}$, and
4. The subset consisting of functions which satisfy the differential equation
$$f'' + 2f' - 5f = 0 \ ,$$
where the prime indicates derivative with respect to t.

Problem 3. (*15 points*)

Show that the rotation R_θ in the plane through an angle θ about the origin is a linear transformation. Show that the matrix R_θ representing R_θ with respect to the canonical basis is given by

$$\mathsf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \ ,$$

and compute its determinant. Prove in addition that $\mathsf{R}_\theta \, \mathsf{R}_\phi = \mathsf{R}_{\theta+\phi}$ and $\mathsf{R}_\theta^{-1} = \mathsf{R}_{-\theta}$.

(These properties make the matrices $\{\mathsf{R}_\theta \mid 0 \leq \theta < 2\pi\}$ into a group, called SO(2).)

Show that the reflection F of the plane in the x-axis is linear. Show that the matrix F which represents F with respect to the canonical basis is given by

$$\mathsf{F} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ ,$$

and compute its inverse and its determinant. Prove that $\mathsf{F}\,\mathsf{R}_\theta = \mathsf{R}_\theta^{-1}\,\mathsf{F}$.

Problem 4. (*15 points*)

For each of the following matrices:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

1. calculate its determinant,
2. describe its action on the canonical basis of $\mathbb{R}^2$ (for 2×2 matrices) or $\mathbb{R}^3$ (for 3×3 matrices), and
3. give a geometrical interpretation of the corresponding linear transformation.