Mathematical Techniques III
(PHY 317)

Exercise Class 9

Friday, December 11th

You may use the following properties of the Laplace transform, which we (will) have proven in class:

1. $\mathcal{L}\left\{t^n\right\}(s) = \frac{n!}{s^{n+1}}$
2. $\mathcal{L}\left\{\exp(at)\right\}(s) = \frac{1}{s-a}$
3. $\mathcal{L}\left\{\sin at\right\}(s) = \frac{a}{s^2+a^2}$
4. $\mathcal{L}\left\{\cos at\right\}(s) = \frac{s}{s^2+a^2}$
5. $\mathcal{L}\left\{\sinh at\right\}(s) = \frac{s}{s^2-a^2}$
6. $\mathcal{L}\left\{\cosh at\right\}(s) = \frac{s}{s^2-a^2}$
7. $\mathcal{L}\left\{\exp(at)\, f(t)\right\}(s) = F(s-a)$
8. $\mathcal{L}\left\{t^n f(t)\right\}(s) = (-1)^n F^{(n)}(s)$
9. $\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^\infty F(\sigma)\, d\sigma$
10. $\mathcal{L}\left\{f^{(n)}\right\}(s) = s^n\, F(s) - \sum_{j=0}^{n-1} s^{n-1-j} f^{(j)}(0)$
11. $\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\}(s) = \frac{F(s)}{s}$
12. $\mathcal{L}\left\{\int_0^t f(t-\tau)\, g(\tau)\, d\tau\right\}(s) = F(s)\, G(s)$

The notation is such that $F(s) = \mathcal{L}\left\{f(t)\right\}(s)$, and $G(s) = \mathcal{L}\left\{g(t)\right\}(s)$; where

$$\mathcal{L}\left\{f(t)\right\}(s) \equiv \int_0^\infty f(t)\, e^{-st}\, dt\ .$$

Exercise 1.

Suppose that $f(t)$ is a periodic function with period T; that is, $f(t + T) = f(t)$ for all t. Prove that the Laplace transform $F(s)$ of $f(t)$ is given by

$$F(s) = \frac{1}{1-e^{-sT}} \int_0^T f(t)\, e^{-st}\, dt\ ,$$

which converges for $\mathrm{Re}(s) > 0$.
Use this to compute the Laplace transform of the function $f(t)$ which is 1 for t between 0 and 1, 2 and 3, 4 and 5, etcetera and 0 otherwise.

Exercise 2.

Solve the following integral equation:

$$f(t) = 1 - \int_0^t (t-\tau)\, f(\tau)\, d\tau \ .$$

(*Hint*: Take the Laplace transform of the equation, solve for the Laplace transform $F(s)$ of $f(t)$, and finally invert the transform.)

Exercise 3.

Show that the differential equation:

$$f''(t) + \omega^2 f(t) = u(t),$$

subject to the initial conditions $f(0) = f'(0) = 0$, has

$$f(t) = \frac{1}{\omega} \int_0^t u(\tau)\, \sin\omega(t-\tau)\, d\tau$$

as its solution.
(*Hint*: Take the Laplace transform of both sides of the equation, solve for the Laplace transform of f and invert.)

Exercise 4.

We have seen above that the Laplace transform of t^n is $n!/s^{n+1}$. Suppose that we consider the Laplace transform of t^z, where z is a complex number with $\mathrm{Re}(z) > 0$. This is given in terms of the **Euler Γ function**, defined by

$$\Gamma(z) \equiv \int_0^\infty t^{z-1}\, e^{-t}\, dt \ .$$

Prove that

$$\mathcal{L}\left\{t^z\right\}(s) = \frac{\Gamma(z+1)}{s^{z+1}} \ ,$$

provided $\mathrm{Re}(z) > 0$. (This is required for convergence of the integral.)
Prove that

$$\Gamma(z+1) = z\,\Gamma(z) \ ,$$

for $\mathrm{Re}(z) > 0$. Compute $\Gamma(\frac{1}{2})$.