Chapter 2

Complex Analysis

In this part of the course we will study some basic complex analysis. This is an extremely useful and beautiful part of mathematics and forms the basis of many techniques employed in many branches of mathematics and physics. We will extend the notions of derivatives and integrals, familiar from calculus, to the case of complex functions of a complex variable. In so doing we will come across analytic functions, which form the centerpiece of this part of the course. In fact, to a large extent complex analysis is the study of analytic functions. After a brief review of complex numbers as points in the complex plane, we will first discuss analyticity and give plenty of examples of analytic functions. We will then discuss complex integration, culminating with the generalised Cauchy Integral Formula, and some of its applications. We then go on to discuss the power series representations of analytic functions and the residue calculus, which will allow us to compute many real integrals and infinite sums very easily via complex integration.

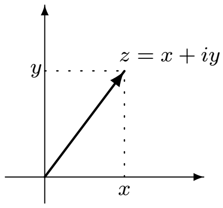
2.1 Analytic functions

In this section we will study complex functions of a complex variable. We will see that differentiability of such a function is a non-trivial property, giving rise to the concept of an analytic function. We will then study many examples of analytic functions. In fact, the construction of analytic functions will form a basic *leitmotif* for this part of the course.

2.1.1 The complex plane

We already discussed complex numbers briefly in Section 1.3.5. The emphasis in that section was on the algebraic properties of complex numbers, and although these properties are of course important here as well and will be used all the time, we are now also interested in more geometric properties of the complex numbers.

The set $\mathbb{C}$ of complex numbers is naturally identified with the plane $\mathbb{R}^2$. This is often called the **Argand plane**.

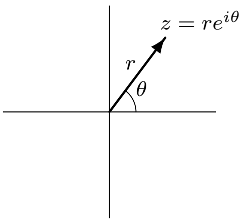


Given a complex number $z = x + i\,y$, its real and imaginary parts define an element (x, y) of $\mathbb{R}^2$, as shown in the figure. In fact this identification is one of real vector spaces, in the sense that adding complex numbers and multiplying them with real scalars mimic the similar operations one can do in $\mathbb{R}^2$. Indeed, if $\alpha \in \mathbb{R}$ is real, then to $\alpha\,z = (\alpha\,x) + i\,(\alpha\,y)$ there corresponds the pair $(\alpha\,x, \alpha\,y) = \alpha\,(x, y)$. Similarly, if $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$ are complex numbers, then $z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)$, whose associated pair is $(x_1 + x_2, y_1 + y_2) = (x_1, y_1) + (x_2, y_2)$. In fact, the identification is even one of euclidean spaces. Given a complex number $z = x + i\,y$, its modulus $|z|$, defined by $|z|^2 = zz^*$, is given by $\sqrt{x^2 + y^2}$ which is precisely the norm $\|(x, y)\|$ of the pair (x, y). Similarly, if $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$, then $\mathrm{Re}(z_1^* z_2) = x_1 x_2 + y_1 y_2$ which is the dot product of the pairs (x_1, y_1) and (x_2, y_2). In particular, it follows from these remarks and the triangle inequality for the norm in $\mathbb{R}^2$, that complex numbers obey a version of the triangle inequality:

$$\boxed{|z_1 + z_2| \leq |z_1| + |z_2|\ .} \tag{2.1}$$

Polar form and the argument function

Points in the plane can also be represented using polar coordinates, and this representation in turn translates into a representation of the complex numbers.



Let (x, y) be a point in the plane. If we define $r = \sqrt{x^2 + y^2}$ and θ by $\theta = \arctan(y/x)$, then we can write $(x, y) = (r\,\cos\theta, r\,\sin\theta) = r\,(\cos\theta, \sin\theta)$. The complex number $z = x + i\,y$ can then be written as $z = r\,(\cos\theta + i\,\sin\theta)$. The real number r, as we have seen, is the modulus $|z|$ of z, and the complex number $\cos\theta + i\,\sin\theta$ has unit modulus. Comparing the Taylor series for the cosine and sine functions and the exponential functions we notice that $\cos\theta + i\sin\theta = e^{i\theta}$. The angle θ is called the **argument** of z and is written $\arg(z)$. Therefore we have the following **polar form** for a complex number z:

$$\boxed{z = |z|\, e^{i \arg(z)}\ .} \tag{2.2}$$

Being an angle, the argument of a complex number is only defined up to the addition of integer multiples of 2π. In other words, it is a **multiple-valued function**. This ambiguity can be resolved by defining the **principal value** Arg of the arg function to take values in the interval $(-\pi, \pi]$; that is, for any complex number z, one has

$$\boxed{-\pi < \operatorname{Arg}(z) \leq \pi\ .} \tag{2.3}$$

Notice, however, that Arg is not a continuous function: it has a discontinuity along the negative real axis. Approaching a point on the negative real axis from the upper half-plane, the principal value of its argument approaches π, whereas if we approach it from the lower half-plane, the principal value of its argument approaches $-\pi$. Notice finally that whereas the modulus is a multiplicative function: $|zw| = |z||w|$, the argument is additive: $\arg(z_1\, z_2) = \arg(z_1) + \arg(z_2)$, provided that we understand the equation to hold up to integer multiples of 2π. Also notice that whereas the modulus is invariant under conjugation $|z^*| = |z|$, the argument changes sign $\arg(z^*) = -\arg(z)$, again up to integer multiples of 2π.

Some important subsets of the complex plane

We end this section with a brief discussion of some very important subsets of the complex plane. Let z_0 be any complex number, and consider all those complex numbers z which are a distance at most ε away from z_0. These points form a disk of radius ε centred at z_0. More precisely, let us define the **open ε-disk around** z_0 to be the subset $D_\varepsilon(z_0)$ of the complex plane defined by

$$D_\varepsilon(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < \varepsilon\}\ . \tag{2.4}$$

Similarly one defines the **closed ε-disk** around z_0 to be the subset

$$\bar{D}_\varepsilon(z_0) = \{z \in \mathbb{C} \mid |z - z_0| \leq \varepsilon\}\ , \tag{2.5}$$

which consists of the open ε-disk and the circle $|z - z_0| = \varepsilon$ which forms its boundary. More generally a subset $U \subset \mathbb{C}$ of the complex plane is said to be **open** if given any $z \in U$, there exists some positive real number $\varepsilon > 0$ (which can depend on z) such that the open ε-disk around z also belongs to U. A set C is said to be **closed** if its complement $C^c = \{z \in \mathbb{C} \mid z \notin C\}$—that is, all those points not in C—is open. One should keep in mind that generic subsets of the complex plane are neither closed nor open. By a **neighbourhood** of a point z_0 in the complex plane, we will mean any open set containing z_0. For example, any open ε-disk around z_0 is a neighbourhood of z_0.

Let us see that the open and closed ε-disks are indeed open and closed, respectively. Let $z \in D_\varepsilon(z_0)$. This means that $|z - z_0| = \delta < \varepsilon$. Consider the disk $D_{\varepsilon-\delta}(z)$. We claim that this disk is contained in $D_\varepsilon(z_0)$. Indeed, if $|w - z| < \varepsilon - \delta$ then,

$$\begin{aligned}
|w - z_0| &= |(w - z) + (z - z_0)| && \text{(adding and subtracting } z\text{)}\\
&\leq |w - z| + |z - z_0| && \text{(by the triangle inequality (2.1))}\\
&< \varepsilon - \delta + \delta\\
&= \varepsilon \;.
\end{aligned}$$

Therefore the disk $D_\varepsilon(z_0)$ is indeed open. Consider now the subset $\bar{D}_\varepsilon(z_0)$. Its complement is the subset of points z in the complex plane such that $|z - z_0| > \varepsilon$. We will show that it is an open set. Let z be such that $|z - z_0| = \eta > \varepsilon$. Then consider the open disk $D_{\eta-\varepsilon}(z)$, and let w be a point in it. Then

$$\begin{aligned}
|z - z_0| &= |(z - w) + (w - z_0)| && \text{(adding and subtracting } w\text{)}\\
&\leq |z - w| + |w - z_0| \;. && \text{(by the triangle inequality (2.1))}
\end{aligned}$$

We can rewrite this as

$$\begin{aligned}
|w - z_0| &\geq |z - z_0| - |z - w|\\
&> \eta - (\eta - \varepsilon) && \text{(since } |z - w| = |w - z| < \eta - \varepsilon\text{)}\\
&= \varepsilon \;.
\end{aligned}$$

Therefore the complement of $\bar{D}_\varepsilon(z_0)$ is open, whence $\bar{D}_\varepsilon(z_0)$ is closed.

We should remark that the closed disk $\bar{D}_\varepsilon(z_0)$ is not open, since any open disk around a point z at the boundary of $\bar{D}_\varepsilon(z_0)$—that is, for which $|z - z_0| = \varepsilon$—contains points which are not included in $D_\varepsilon(z_0)$.

Notice that it follows from this definition that every open set is made out of the union of (a possibly uncountable number of) open disks.

2.1.2 Complex-valued functions

In this section we will discuss complex-valued functions.

We start with a rather trivial case of a complex-valued function. Suppose that f is a complex-valued function of a real variable. That means that if x is a real number, $f(x)$ is a complex number, which can be decomposed into its real and imaginary parts: $f(x) = u(x) + i\,v(x)$, where u and v are real-valued functions of a real variable; that is, the objects you are familiar with from calculus. We say that f is **continuous** at x_0 if u and v are continuous at x_0.

Let us recall the definition of continuity. Let f be a real-valued function of a real variable. We say that f is continuous at x_0, if for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$. A function is said to be continuous if it is continuous at all points where it is defined.

Now consider a complex-valued function f of a complex variable z. We say that f is **continuous** at z_0 if given any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$. Heuristically, another way of saying that f is continuous at z_0 is that $f(z)$ tends to $f(z_0)$ as z approaches z_0. This is equivalent to the continuity of the real and imaginary parts of f thought of as real-valued functions on the complex plane. Explicitly, if we write $f = u + i\,v$ and $z = x + i\,y$, $u(x, y)$ and $v(x, y)$ are real-valued functions on the complex plane. Then the continuity of f at $z_0 = x_0 + i\,y_0$ is equivalent to the continuity of u and v at the point (x_0, y_0).

"Graphing" complex-valued functions

Complex-valued functions of a complex variable are harder to visualise than their real analogues. To visualise a real function $f : \mathbb{R} \to \mathbb{R}$, one simply graphs the function: its graph being the curve $y = f(x)$ in the (x, y)-plane. A complex-valued function of a complex variable $f : \mathbb{C} \to \mathbb{C}$ maps complex numbers to complex numbers, or equivalently points in the (x, y)-plane to points in the (u, v) plane. Hence its graph defines a surface $u = u(x, y)$ and $v = v(x, y)$ in the four-dimensional space with coordinates (x, y, u, v), which is not so easy to visualise. Instead one resorts to investigating what the function does to regions in the complex plane. Traditionally one considers two planes: the z-plane whose points have coordinates (x, y) corresponding to the real and imaginary parts of $z = x + i\,y$, and the w-plane whose points have coordinates (u, v) corresponding to $w = u + i\,v$. Any complex-valued function f of the complex variable z maps points in the z-plane to points in the w-plane via $w = f(z)$. A lot can be learned from a complex function by analysing the image in the w-plane of certain sets in the z-plane. We will have plenty of opportunities to use this throughout the course of these lectures.

With the picture of the z- and w-planes in mind, one can restate the continuity of a function very simply in terms of open sets. In fact, this was the historical reason why the notion of open sets was introduced in mathematics. As we saw, a complex-valued function f of a complex variable z defines a mapping from the complex z-plane to the complex w-plane. The function f is continuous at z_0 if for every neighbourhood U of $w_0 = f(z_0)$ in the w-plane, the set

$$f^{-1}(U) = \{z \mid f(z) \in U\}$$

is open in the z-plane. Checking that both definitions of continuity agree is left as an exercise.

2.1.3 Differentiability and analyticity

Let us now discuss differentiation of complex-valued functions. Again, if $f = u + i\,v$ is a complex-valued function of a *real* variable x, then the derivative of f at the point x_0 is defined by

$$f'(x_0) = u'(x_0) + i\, v'(x_0) \ ,$$

where u' and v' are the derivatives of u and v respectively. In other words, we extend the operation of differentiation complex-linearly. There is nothing novel here.

Differentiability and the Cauchy–Riemann equations

The situation is drastically different when we consider a complex-valued function $f = u + i\, v$ of a complex variable $z = x + i\, y$. As is calculus, let us attempt to define its derivative by

$$\boxed{f'(z_0) \equiv \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \ .} \tag{2.6}$$

The first thing that we notice is that Δz, being a complex number, can approach zero in more than one way. If we write $\Delta z = \Delta x + i\, \Delta y$, then we can approach zero along the real axis $\Delta y = 0$ or along the imaginary axis $\Delta x = 0$, or indeed along any direction. For the derivative to exist, the answer should not depend on how Δz tends to 0. Let us see what this entails. Let us write $f = u + i\, v$ and $z_0 = x_0 + i\, y_0$, so that

$$\begin{aligned} f(z_0) &= u(x_0, y_0) + i\, v(x_0, y_0) \\ f(z_0 + \Delta z) &= u(x_0 + \Delta x, y_0 + \Delta y) + i\, v(x_0 + \Delta x, y_0 + \Delta y) \ . \end{aligned}$$

Then

$$f'(z_0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta u(x_0, y_0) + i\, \Delta v(x_0, y_0)}{\Delta x + i \Delta y} \ ,$$

where

$$\begin{aligned} \Delta u(x_0, y_0) &= u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) \\ \Delta v(x_0, y_0) &= v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0) \ . \end{aligned}$$

Let us first take the limit $\Delta z \to 0$ by first taking $\Delta y \to 0$ and then $\Delta x \to 0$; in other words, we let $\Delta z \to 0$ along the real axis. Then

$$\begin{aligned} f'(z_0) &= \lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \frac{\Delta u(x_0, y_0) + i\, \Delta v(x_0, y_0)}{\Delta x + i \Delta y} \\ &= \lim_{\Delta x \to 0} \frac{\Delta u(x_0, y_0) + i\, \Delta v(x_0, y_0)}{\Delta x} \\ &= \left.\frac{\partial u}{\partial x}\right|_{(x_0, y_0)} + i \left.\frac{\partial v}{\partial x}\right|_{(x_0, y_0)} \ . \end{aligned}$$

Now let us take the limit $\Delta z \to 0$ by first taking $\Delta x \to 0$ and then $\Delta y \to 0$; in other words, we let $\Delta z \to 0$ along the imaginary axis. Then

$$\begin{aligned} f'(z_0) &= \lim_{\Delta y \to 0} \lim_{\Delta x \to 0} \frac{\Delta u(x_0, y_0) + i\,\Delta v(x_0, y_0)}{\Delta x + i \Delta y} \\ &= \lim_{\Delta y \to 0} \frac{\Delta u(x_0, y_0) + i\,\Delta v(x_0, y_0)}{i\,\Delta y} \\ &= -i\, \frac{\partial u}{\partial y}\bigg|_{(x_0,y_0)} + \frac{\partial v}{\partial y}\bigg|_{(x_0,y_0)} \ . \end{aligned}$$

These two expressions for $f'(z_0)$ agree if and only if the following equations are satisfied at (x_0, y_0):

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\ .} \tag{2.7}$$

These equations are called the **Cauchy–Riemann equations**.

We say that the function f is **differentiable** at z_0 if $f'(z_0)$ is well-defined at z_0. For a differentiable function f we have just seen that

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y}\ .$$

We have just shown that a necessary condition for f to be differentiable at z_0 is that its real and imaginary parts obey the Cauchy–Riemann equations at (x_0, y_0). Conversely, it can be shown that this condition is also sufficient provided that the the partial derivatives of u and v are continuous.

We say that the function f is **analytic** in a neighbourhood U of z_0 if it is differentiable everywhere in U. We say that a function is **entire** if it is analytic in the whole complex plane. Often the terms **regular** and **holomorphic** are used as synonyms for analytic.

For example, the function $f(z) = z$ is entire. We can check this either by verifying the Cauchy–Riemann equations or else simply by noticing that

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{z_0 + \Delta z - z_0}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{\Delta z}{\Delta z} \\ &= \lim_{\Delta z \to 0} 1 \\ &= 1\ ; \end{aligned}$$

whence it is well-defined for all z_0.

On the other hand, the function $f(z) = z^*$ is not differentiable anywhere:

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{z_0^* + (\Delta z)^* - z_0^*}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{(\Delta z)^*}{\Delta z} \ ; \end{aligned}$$

whence if we let Δz tend to zero along real values, we would find that $f'(z_0) = 1$, whereas if we would let Δz tend to zero along imaginary values we would find that $f'(z_0) = -1$. We could have reached the same conclusion via the Cauchy–Riemann equations with $u(x,y) = x$ and $v(x,y) = -y$, which violates the first of the Cauchy–Riemann equations.

It is important to realise that analyticity, unlike differentiability, is not a property of a function at a point, but on an open set of points. The reason for this is to able to eliminate from the class of interesting functions, functions which may be differentiable at a point but nowhere else. Whereas this is a rarity in calculus[1], it is a very common occurrence for complex-valued functions of a complex variables. For example, consider the function $f(z) = |z|^2$. This function has $u(x,y) = x^2 + y^2$ and $v(x,y) = 0$. Therefore the Cauchy–Riemann equations are only satisfied at the origin in the complex plane:

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} = 0 \qquad \text{and} \qquad \frac{\partial v}{\partial x} = 0 = -\frac{\partial u}{\partial y} = -2y \ .$$

Relation with harmonic functions

Analytic functions are intimately related to harmonic functions. We say that a real-valued function $h(x,y)$ on the plane is **harmonic** if it obeys **Laplace's equation**:

$$\boxed{\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \ .} \tag{2.8}$$

In fact, as we now show, the real and imaginary parts of an analytic function are harmonic. Let $f = u + i\,v$ be analytic in some open set of the complex

[1]Try to come up with a real-valued function of a real variable which is differentiable only at the origin, for example.

plane. Then,

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} \\
&= \frac{\partial}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial}{\partial y}\frac{\partial v}{\partial x} \qquad \text{(using Cauchy–Riemann)} \\
&= \frac{\partial^2 v}{\partial x\,\partial y} - \frac{\partial^2 v}{\partial y\,\partial x} \\
&= 0\ .
\end{aligned}$$

A similar calculation shows that v is also harmonic. This result is important in applications because it shows that one can obtain solutions of a second order partial differential equation by solving a system of first order partial differential equations. It is particularly important in this case because we will be able to obtain solutions of the Cauchy–Riemann equations without really solving these equations.

Given a harmonic function u we say that another harmonic function v is its **harmonic conjugate** if the complex-valued function $f = u + i\,v$ is analytic. For example, consider the function $u(x,y) = xy - x + y$. It is clearly harmonic since

$$\frac{\partial u}{\partial x} = y - 1 \qquad \text{and} \qquad \frac{\partial u}{\partial y} = x + 1\ ,$$

whence

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} = 0\ .$$

By a harmonic conjugate we mean any function $v(x,y)$ which together with $u(x,y)$ satisfies the Cauchy–Riemann equations:

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -x - 1 \qquad \text{and} \qquad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = y - 1\ .$$

We can integrate the first of the above equations, to obtain

$$v(x,y) = -\tfrac{1}{2}x^2 - x + \psi(y)\ ,$$

for ψ an arbitrary function of y which is to be determined from the second of the Cauchy–Riemann equations. Doing this one finds

$$\psi'(y) = y - 1\ ,$$

which is solved by $\psi(y) = \frac{1}{2}y^2 - y + c$, where c is any constant. Therefore, the function $f = u + i\,v$ becomes

$$f(x,y) = xy - x + y + i\left(-\tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 - x - y + c\right)\ .$$

We can try to write this down in terms of z and z^* by making the substitutions $x = \frac{1}{2}(z + z^*)$ and $y = -i\,\frac{1}{2}(z - z^*)$. After a little bit of algebra, we find

$$f(z) = -iz^2 - (1+i)\,z + i\,c \; .$$

Notice that all the z^* dependence has dropped out. We will see below that this is a sign of analyticity.

2.1.4 Polynomials and rational functions

We now start to build up some examples of analytic functions. We have already seen that the function $f(z) = z$ is entire. In this section we will generalise this to show that so is any polynomial $P(z)$. We will also see that ratios of polynomials are also analytic everywhere but on a finite set of points in the complex plane where the denominator vanishes.

There are many ways to do this, but one illuminating way is to show that complex linear combinations of analytic functions are analytic and that products of analytic functions are analytic functions. Let $f(z)$ be an analytic function on some open subset $U \subset \mathbb{C}$, and let α be a complex number. Then it is easy to see that the function $\alpha\, f(z)$ is also analytic on U. Indeed, from the definition (2.6) of the derivative, we see that

$$(\alpha\, f)'(z_0) = \alpha\, f'(z_0) \; , \tag{2.9}$$

which exists whenever $f'(z_0)$ exists.

Let $f(z)$ and $g(z)$ be analytic functions on the same open subset $U \subset \mathbb{C}$. Then the functions $f(z) + g(z)$ and $f(z)g(z)$ are also analytic. Again from the definition (2.6) of the derivative,

$$(f+g)'(z_0) = f'(z_0) + g'(z_0) \tag{2.10}$$
$$(f\,g)'(z_0) = f'(z_0)\,g(z_0) + f(z_0)\,g'(z_0) \; , \tag{2.11}$$

which exist whenever $f'(z_0)$ and $g'(z_0)$ exist.

The only tricky bit in the above result is that we have to make sure that f and g are analytic in the same open set U. Normally it happens that f and g are analytic in different open sets, say, U_1 and U_2 respectively. Then the sum $f(z) + g(z)$ and product $f(z)\,g(z)$ are analytic in the intersection $U = U_1 \cap U_2$, which is also open. This is easy to see. Let us assume that U is not empty, otherwise the statement is trivially satisfied. Let $z \in U$. This means that $z \in U_1$ and $z \in U_2$. Because each U_i is open there are positive real numbers ε_i such that $D_{\varepsilon_i}(z)$ lies inside U_i. Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ be the smallest of the ε_i. Then $D_\varepsilon(z) \subseteq D_{\varepsilon_i}(z) \subset U_i$ for $i = 1, 2$. Therefore $D_\varepsilon(z) \subset U$, and U is open.

It is important to realise that only finite intersections of open sets will again be open in general. Consider, for example, the open disks $D_{1/n}(0)$ of radius $1/n$ about the origin, for $n = 1, 2, 3, \ldots$. Their intersection consists of the points z with $|z| < 1/n$ for all $n = 1, 2, 3, \ldots$. Clearly, if $z \neq 0$ then there will be some positive integer n for which

$|z| > 1/n$. Therefore the only point in the intersection of all the $D_{1/n}(0)$ is the origin itself. But this set is clearly not open, since it does not contain any open disk with nonzero radius. More generally, sets consisting of a finite number of points are never open; although they are closed.

Therefore we see that (finite) sums and products of analytic functions are analytic with the same domain of analyticity. In particular, sums and products of entire functions are again entire. As a result, from the fact that the function $f(z) = z$ is entire, we see that any polynomial $P(z) = \sum_{n=0}^{N} a_n z^n$ of finite degree N is also an entire function. Indeed, its derivative is given by

$$P'(z_0) = \sum_{n=1}^{N} n\, a_n\, z_0^{n-1} ,$$

as follows from the above formulae for the derivatives of sums and products.

We will see later on in the course that to some extent we will be able to describe all analytic functions (at least locally) in terms of polynomials, provided that we allow the polynomials to have arbitrarily high degree; in other words, in terms of power series.

There are two more constructions which start from analytic functions and yield an analytic function: quotients and composition. Let $f(z)$ and $g(z)$ be analytic functions on some open subset $U \subset \mathbb{C}$. Then the quotient $f(z)/g(z)$ is continuous away from the zeros of $g(z)$, which can be shown (see below) to be an open set. If $g(z_0) \neq 0$, then from the definition of the derivative (2.6), it follows that

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)\, g(z_0) - f(z_0)\, g'(z_0)}{g(z_0)^2} .$$

To see that the subset of points z for which $g(z) \neq 0$ is open, we need only realise that this set is the inverse image $g^{-1}(\{0\}^c)$ under g of the complement of 0. The result then follows because the complement of 0 is open and g is continuous, so that $g^{-1}(\text{open})$ is open.

By a **rational function** we mean the ratio of two polynomials. Let $P(z)$ and $Q(z)$ be two polynomials. Then the rational function

$$R(z) = \frac{P(z)}{Q(z)}$$

is analytic away from the zeros of $Q(z)$.

We have been tacitly assuming that every (non-constant) polynomial $Q(z)$ has zeros. This result is known as the *Fundamental Theorem of Algebra* and although it is of course intuitive and in agreement with our daily experience with polynomials, its proof is surprisingly difficult. There are three standard proofs: one is purely algebraic, but it is long and arduous, one uses algebraic topology and the other uses complex analysis. We will in fact see this third proof later on in Section 2.2.6.

Finally let $g(z)$ be analytic in an open subset $U \subset \mathbb{C}$ and let $f(z)$ be analytic in some open subset containing $g(U)$, the image of U under g. Then the composition $f \circ g$ defined by $(f \circ g)(z) = f(g(z))$ is also analytic in U. In fact, its derivative can be computed using the chain rule,

$$(f \circ g)'(z_0) = f'(g(z_0))\, g'(z_0) \ . \tag{2.12}$$

You may wonder whether $g(U)$ is an open set, for U open and g analytic. This is indeed true: it is called the open mapping property of analytic functions. We may see this later on in the course.

It is clear that if f and g are rational functions so will be its composition $f \circ g$, so one only ever constructs new functions this way when one of the functions being composed is not rational. We will see plenty of examples of this as the lectures progress.

Another look at the Cauchy–Riemann equations

Finally let us mention an a different way to understand the Cauchy–Riemann equations, at least for the case of rational functions. Notice that the above polynomials and rational functions share the property that they do not depend on z^* but only on z. Suppose that one is given a rational function where the dependence on x and y has been made explicit. For example,

$$f(x, y) = \frac{x - 1 - i\,y}{(x-1)^2 + y^2} \ .$$

In order to see whether f is analytic one would have to apply the Cauchy–Riemann equations, which can get rather messy when the rational function is complicated. It turns out that it is not necessary to do this. Instead one can try to re-express the function in terms of z and z^* by the substitutions

$$x = \frac{z + z^*}{2} \qquad \text{and} \qquad y = \frac{z - z^*}{2i} \ .$$

Then, the rational function $f(x, y)$ is analytic if and only if the z^* dependence cancels. In the above example, one can see that this is indeed the case. Indeed, rewriting $f(x, y)$ in terms of z and z^* we see that

$$f(x, y) = \frac{z^* - 1}{zz^* - z - z^* + 1} = \frac{1}{z - 1} \ ,$$

whence the z^* dependence has dropped out. We therefore expect that the Cauchy–Riemann equations will be satisfied. Indeed, one has that

$$u(x, y) = \frac{x - 1}{(x-1)^2 + y^2} \qquad \text{and} \qquad v(x, y) = \frac{-y}{(x-1)^2 + y^2} \ ,$$

and after some algebra,

$$\frac{\partial u}{\partial x} = \frac{-(x-1)^2 + y^2}{\big((x-1)^2 + y^2\big)^2} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = \frac{-2\,(x-1)\;y}{\big((x-1)^2 + y^2\big)^2} = -\frac{\partial v}{\partial x}\ .$$

The reason this works is the following. Let us think formally of z and z^* as independent variables for the plane, like x and y. Then we have that

$$\frac{\partial f}{\partial z^*} = \frac{\partial f}{\partial (x - i\,y)} = \frac{\partial f}{\partial x} + i\,\frac{\partial f}{\partial y}\ .$$

Let us break up f into its real and imaginary parts: $f(x,y) = u(x,y) + i\,v(x,y)$. Then,

$$\begin{aligned}\frac{\partial f}{\partial z^*} &= \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} + i\,\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} \\ &= \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\,\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\ .\end{aligned}$$

Therefore we see that the Cauchy–Riemann equations are equivalent to the condition

$$\boxed{\frac{\partial f}{\partial z^*} = 0\ .} \tag{2.13}$$

2.1.5 The complex exponential and related functions

There are many other analytic functions besides the rational functions. Some of them are related to the exponential function.

Let $z = x + i\,y$ be a complex number and define the **complex exponential** $\exp(z)$ (also written e^z) to be the function

$$\exp(z) = \exp(x + i\,y) \equiv e^x\,(\cos y + i\,\sin y)\ .$$

We will first check that this function is entire. Decomposing it into real and imaginary parts, we see that

$$u(x,y) = e^x\,\cos y \qquad \text{and} \qquad v(x,y) = e^x\,\sin y\ .$$

It is easy to check that the Cauchy–Riemann equations (2.7) are satisfied everywhere on the complex plane:

$$\frac{\partial u}{\partial x} = e^x\,\cos y = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = e^x\,\sin y = -\frac{\partial u}{\partial y}\ .$$

Therefore the function is entire and its derivative is given by

$$\begin{aligned}\exp'(z) &= \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \\ &= e^x \cos y + i\,e^x \sin y \\ &= \exp(z)\ .\end{aligned}$$

The exponential function obeys the following addition property

$$\boxed{\exp(z_1 + z_2) = \exp(z_1)\,\exp(z_2)\ ,} \tag{2.14}$$

which has as a consequence the celebrated **De Moivre's Formula**:

$$(\cos\theta + i\,\sin\theta)^n = \cos(n\theta) + i\,\sin(n\theta)\ ,$$

obtained simply by noticing that $\exp(i\,n\theta) = \exp(i\,\theta)^n$.

The exponential is also a periodic function, with period $2\pi\,i$. In fact from the periodicity of trigonometric functions, we see that $\exp(2\pi\,i) = 1$ and hence, using the addition property (2.14), we find

$$\boxed{\exp(z + 2\pi\,i) = \exp(z)\ .} \tag{2.15}$$

This means that the exponential is not one-to-one, in sharp contrast with the real exponential function. It follows from the definition of the exponential function that

$$\exp(z_1) = \exp(z_2) \quad \text{if and only if} \quad z_1 = z_2 + 2\pi\,i\,k \quad \text{for some integer } k.$$

We can divide up the complex plane into horizontal strips of height 2π in such a way that in each strip the exponential function is one-to-one. To see this define the following subsets of the complex plane

$$\mathcal{S}_k \equiv \{x + i\,y \in \mathbb{C} \mid (2k-1)\pi < y \leq (2k+1)\pi\}\ ,$$

for $k = 0, \pm1, \pm2, \ldots$, as shown in Figure 2.1.

Then it follows that if z_1 and z_2 belong to the same set $\mathcal{S}_k$, then $\exp(z_1) = \exp(z_2)$ implies that $z_1 = z_2$. Each of the sets $\mathcal{S}_k$ is known as a **fundamental region** for the exponential function. The basic property satisfied by a fundamental region of a periodic function is that if one knows the behaviour of the function on the fundamental region, one can use the periodicity to find out the behaviour of the function everywhere, and that it is the smallest region with that property. The periodicity of the complex exponential will have as a consequence that the complex logarithm will not be single-valued.

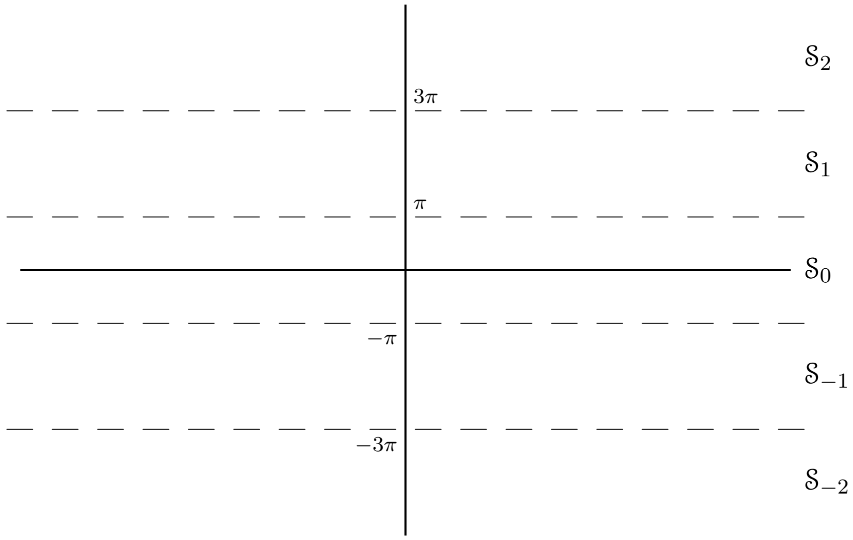


Figure 2.1: Fundamental regions of the complex exponential function.

Complex trigonometric functions

We can also define complex trigonometric functions starting from the complex exponential. Let $z = x + i\,y$ be a complex number. Then we define the complex sine and cosine functions as

$$\sin(z) \equiv \frac{e^{iz} - e^{-iz}}{2i} \qquad \text{and} \qquad \cos(z) \equiv \frac{e^{iz} + e^{-iz}}{2} \ .$$

Being linear combinations of the entire functions $\exp(\pm iz)$, they themselves are entire. Their derivatives are

$$\sin'(z) = \cos(z) \qquad \text{and} \qquad \cos'(z) = -\sin(z) \ .$$

The complex trigonometric functions obey many of the same properties of the real sine and cosine functions, with which they agree when z is real. For example,

$$\cos(z)^2 + \sin(z)^2 = 1 \ ,$$

and they are periodic with period 2π. However, there is one important difference between the real and complex trigonometric functions: whereas the real sine and cosine functions are bounded, their complex counterparts are not. To see this let us break them up into real and imaginary parts:

$$\begin{aligned}\sin(x + i\,y) &= \sin x \, \cosh y + i \, \cos x \, \sinh y \\ \cos(x + i\,y) &= \cos x \, \cosh y - i \, \sin x \, \sinh y \ .\end{aligned}$$

We see that the appearance of the hyperbolic functions means that the complex sine and cosine functions are not bounded.

Finally, let us define the complex hyperbolic functions. If $z = x + i\,y$, then let

$$\sinh(z) \equiv \frac{e^z - e^{-z}}{2} \qquad \text{and} \qquad \cosh(z) \equiv \frac{e^z + e^{-z}}{2} \ .$$

In contrast with the real hyperbolic functions, they are not independent from the trigonometric functions. Indeed, we see that

$$\sinh(iz) = i\sin(z) \qquad \text{and} \qquad \cosh(iz) = \cos(z) \ . \tag{2.16}$$

Notice that one can also define other complex trigonometric functions: $\tan(z)$, $\cot(z)$, $\sec(z)$ and $\csc(z)$ in the usual way, as well as their hyperbolic counterparts. These functions obey many other properties, but we will not review them here. Instead we urge you to play with these functions until you are familiar with them.

2.1.6 The complex logarithm

This section introduces the logarithm of a complex number. We will see that in contrast with the real logarithm function which is only defined for positive real numbers, the complex logarithm is defined for all nonzero complex numbers, but at a price: the function is not single-valued. This has to do with the periodicity (2.15) of the complex exponential or, equivalently, with the multiple-valuedness of the argument $\arg(z)$.

In this course we will use the notation 'log' for the natural logarithm, not for the logarithm base 10. Some people also use the notation 'ln' for the natural logarithm, in order to distinguish it from the logarithm base 10; but we will not be forced to do this since we will only be concerned with the natural logarithm.

By analogy with the real natural logarithm, we define the **complex logarithm** as an inverse to the complex exponential function. In other words, we say that a logarithm of a nonzero complex number z, is any complex number w such that $\exp(w) = z$. In other words, we define the function $\log(z)$ by

$$\boxed{w = \log(z) \quad \text{if} \quad \exp(w) = z \ .} \tag{2.17}$$

From the periodicity (2.15) of the exponential function it follows that if $w = \log(z)$ so is $w + 2\pi\,i\,k$ for any integer k. Therefore we see that $\log(z)$ is a multiple-valued function. We met a multiple-valued function before: the argument function arg(z). Clearly if $\theta = \arg(z)$ then so is $\theta + 2\pi\, k$ for any integer k. This is no accident: the imaginary part of the log(z) function is arg(z). To see this, let us write z in polar form (2.2) $z = |z| \exp(i \arg(z))$ and $w = \log(z) = u + i\, v$. By the above definition and using the addition property (2.14), we have

$$\exp(u + i\, v) = e^u\, e^{i\, v} = |z|\, e^{i \arg(z)} \ ,$$

whence comparing polar forms we see that

$$e^u = |z| \qquad \text{and} \qquad e^{i\, v} = e^{i \arg(z)} \ .$$

Since u is a real number and $|z|$ is a positive real number, we can solve the first equation for u uniquely using the *real* logarithmic function, which in order to distinguish it from the complex function log(z) we will write as Log:

$$u = \operatorname{Log}|z| \ .$$

Similarly, we see that $v = \arg(z)$ solves the second equation. So does $v + 2\pi\, k$ for any integer k, but this is already taken into account by the multiple-valuedness of the arg(z) function. Therefore we can write

$$\boxed{\log(z) = \operatorname{Log}|z| + i \arg(z) \ ,} \tag{2.18}$$

where we see that it is a multiple-valued function as a result of the fact that so is arg(z). In terms of the principal value Arg(z) of the argument function, we can also write the log(z) as follows:

$$\boxed{\log(z) = \operatorname{Log}|z| + i \operatorname{Arg}(z) + 2\pi\, i\, k \qquad \text{for } k = 0, \pm 1, \pm 2, \ldots,} \tag{2.19}$$

which makes the multiple-valuedness manifest.

For example, whereas the real logarithm of 1 is simply 0, the complex logarithm is given by

$$\log(1) = \operatorname{Log}|1| + i \arg(1) = 0 + i\, 2\pi\, k \quad \text{for any integer } k.$$

As promised, we can now take the logarithm of negative real numbers. For example,

$$\log(-1) = \operatorname{Log}|-1| + i \arg(-1) = 0 + i\, \pi + i\, 2\pi\, k \quad \text{for any integer } k.$$

The complex logarithm obeys many of the algebraic identities that we expect from the real logarithm, only that we have to take into account its multiple-valuedness properly. Therefore an identity like

$$\boxed{\log(z_1\, z_2) = \log(z_1) + \log(z_2) \ ,} \tag{2.20}$$

for nonzero complex numbers z_1 and z_2, is still valid in the sense that having chosen a value (out of the infinitely many possible values) for $\log(z_1)$ and for $\log(z_2)$, then there is a value of $\log(z_1\, z_2)$ for which the above equation holds. Or said in a different way, the identity holds up to integer multiples of $2\pi\, i$ or, as it is often said, **modulo** $2\pi\, i$:

$$\log(z_1\, z_2) - \log(z_1) - \log(z_2) = 2\pi\, i\, k \quad \text{for some integer } k.$$

Similarly we have

$$\boxed{\log(z_1/z_2) = \log(z_1) - \log(z_2)\ ,} \tag{2.21}$$

in the same sense as before, for any two nonzero complex numbers z_1 and z_2.

Choosing a branch for the logarithm

We now turn to the discussion of the analyticity properties of the complex logarithm function. In order to discuss the analyticity of a function, we need to investigate its differentiability, and for this we need to be able to take its derivative as in equation (2.6). Suppose we were to try to compute the derivative of the function $\log(z)$ at some point z_0. Writing the derivative as the limit of a quotient,

$$\log'(z_0) = \lim_{\Delta z \to 0} \frac{\log(z_0 + \Delta z) - \log(z_0)}{\Delta z}\ ,$$

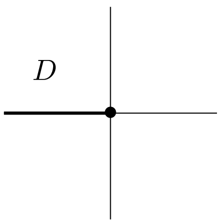
we encounter an immediate obstacle: since the function $\log(z)$ is multiple-valued we have to make sure that the two log functions in the numerator tend to the same value in the limit, otherwise the limit will not exist. In other words, we have to choose one of the infinitely many values for the log function in a consistent way. This way of restricting the values of a multiple-valued function to make it single-valued in some region (in the above example in some neighbourhood of z_0) is called choosing a branch of the function. For example, we define the **principal branch** Log of the logarithmic function to be

$$\mathrm{Log}(z) = \mathrm{Log}\,|z| + i\ \mathrm{Arg}(z)\ ,$$

where $\mathrm{Arg}(z)$ is the principal value of $\arg(z)$. Af first sight it might seem that this notation is inconsistent, since we are using Log both for the real logarithm and the principal branch of the complex logarithm. So let us make sure that this is not the case. If z is a positive real number, then $z = |z|$ and $\mathrm{Arg}(z) = 0$, whence $\mathrm{Log}(z) = \mathrm{Log}\,|z|$. Hence at least the notation is consistent. The function $\mathrm{Log}(z)$ is single-valued, but at a price: it is no longer continuous in the whole complex plane, since $\mathrm{Arg}(z)$ is not continuous in the whole complex plane. As explained in Section 2.1.1, the principal branch $\mathrm{Arg}(z)$ of the argument function is discontinuous along the negative real axis. Indeed, let $z_\pm = -x \pm i\,\varepsilon$ where x and ε are positive numbers. In the limit $\varepsilon \to 0$, z_+ and z_- tend to the same point on the negative real axis from the upper and lower half-planes respectively. Hence whereas $\lim_{\varepsilon\to 0} z_\pm = -x$, the principal value of the logarithm obeys

$$\lim_{\varepsilon \to 0} \mathrm{Log}(z_\pm) = \mathrm{Log}(x) \pm i\,\pi \ ,$$

so that it is not a continuous function anywhere on the negative real axis, or at the origin, where the function itself is not well-defined. The non-positive real axis is known as a **branch cut** for this function and the origin is known as a **branch point**.



Let D denote all the points in the complex plane except for those which are real and non-positive; in other words, D is the complement of the non-positive real axis. It is easy to check that D is an open subset of the complex plane and by construction, $\mathrm{Log}(z)$ is single-valued and continuous for all points in D. We will now check that it is analytic there as well. For this we need to compute its derivative. So let $z_0 \in D$ be any point in D and consider $w_0 = \mathrm{Log}(z_0)$. Letting $\Delta z = z - z_0$, we can write the derivative of $w = \mathrm{Log}(z)$ at z_0 in the following form

$$\begin{aligned}
\mathrm{Log}'(z_0) &= \lim_{z\to z_0} \frac{w - w_0}{z - z_0} \\
&= \lim_{z\to z_0} \frac{1}{\frac{z-z_0}{w-w_0}} \\
&= \lim_{w\to w_0} \frac{1}{\frac{z-z_0}{w-w_0}} \ ,
\end{aligned}$$

where to reach the second line we used the fact that $w = w_0$ implies $z = z_0$ (single-valuedness of the exponential function), and to reach the third line we used the continuity of $\mathrm{Log}(z)$ in D to deduce that $w \to w_0$ as $z \to z_0$. Now using that $z = e^w$ we see that what we have here is the reciprocal of the derivative of the exponential function, whence

$$\mathrm{Log}'(z_0) = \lim_{w\to w_0} \frac{1}{\frac{e^w - e^{w_0}}{w - w_0}} = \frac{1}{\exp'(w_0)} = \frac{1}{\exp(w_0)} = \frac{1}{z_0} \ .$$

Since this is well-defined everywhere but for $z_0 = 0$, which does not belong to D, we see that $\mathrm{Log}(z)$ is analytic in D.

Other branches

The choice of branch for the logarithm is basically that, a choice. It is certainly not the only one. We can make the logarithm function single-valued in other regions of the complex plane by choosing a different branch for the argument function.

For example, another popular choice is to consider the function $\mathrm{Arg}_0(z)$ which is the value of the argument function for which

$$0 \leq \mathrm{Arg}_0(z) < 2\pi \ .$$

This function, like $\mathrm{Arg}(z)$, is single-valued but discontinuous; however the discontinuity is now along the positive real axis, since approaching a positive real number from the upper half-plane we would conclude that its argument tends to 0 whereas approaching it from the lower half-plane the argument would tend to 2π. We can therefore define a branch $\mathrm{Log}_0(z)$ of the logarithm by

$$\mathrm{Log}_0(z) = \mathrm{Log}\,|z| + i\ \mathrm{Arg}_0(z) \ .$$

This branch then has a branch cut along the non-negative real axis, but it is continuous in its complement D_0 as shown in Figure 2.2. The same argument as before shows that $\mathrm{Log}_0(z)$ is analytic in D_0 with derivative given by

$$\mathrm{Log}_0'(z_0) = \frac{1}{z_0} \qquad \text{for all } z_0 \text{ in } D_0.$$

Figure 2.2: Two further branches of the logarithm.

There are of course many other branches. For example, let τ be any real number and define the branch $\mathrm{Arg}_\tau(z)$ of the argument function to take the values

$$\tau \leq \mathrm{Arg}_\tau(z) < \tau + 2\pi \ .$$

This gives rise to a branch $\mathrm{Log}_\tau(z)$ of the logarithm function defined by

$$\mathrm{Log}_\tau(z) = \mathrm{Log}\,|z| + i\ \mathrm{Arg}_\tau(z) \ ,$$

which has a branch cut emanating from the origin and consisting of all those points z with $\arg(z) = \tau$ modulo 2π. Again the same arguments show that $\mathrm{Log}_\tau(z)$ is analytic everywhere on the complement D_τ of the branch cut, as shown in Figure 2.2, and its derivative is given by

$$\mathrm{Log}'_\tau(z_0) = \frac{1}{z_0} \qquad \text{for all } z_0 \text{ in } D_\tau.$$

The choice of branch is immaterial for many properties of the logarithm, although it is important that a choice be made. Different applications may require choosing one branch over another. Provided one is consistent this should not cause any problems.

As an example suppose that we are faced with computing the derivative of the function $f(z) = \log(z^2 + 2iz + 2)$ at the point $z = i$. We need to choose a branch of the logarithm which is analytic in a region containing a neighbourhood of the point $i^2 + 2i\,i + 2 = -1$. The principal branch is not analytic there, so we have to choose another branch. Suppose that we choose $\mathrm{Log}_0(z)$. Then, by the chain rule

$$f'(i) = \left.\frac{2z + 2i}{z^2 + 2iz + 2}\right|_{z=i} = \frac{2\,i + 2\,i}{i^2 + 2i^2 + 2} = -4\,i\ .$$

Any other valid branch would of course give the same result.

2.1.7 Complex powers

With the logarithm function at our disposal, we are able to define complex powers of complex numbers. Let α be a complex number. The for all $z \neq 0$, we define the α-**th power** z^α of z by

$$\boxed{z^\alpha \equiv e^{\alpha\,\log(z)} = e^{\alpha\,\mathrm{Log}\,|z| + i\,\alpha\,\arg(z)}\ .} \tag{2.22}$$

The multiple-valuedness of the argument means that generically there will be an infinite number of values for z^α. We can rewrite the above expression a little to make this manifest:

$$z^\alpha = e^{\alpha\,\mathrm{Log}\,|z| + i\,\alpha\,\mathrm{Arg}(z) + i\,\alpha\,2\pi\,k} = e^{\alpha\,\mathrm{Log}(z)} e^{i\,\alpha\,2\pi\,k}\ ,$$

for $k = 0, \pm 1, \pm 2, \ldots$.

Depending on α we will have either one, finitely many or infinitely many values of $\exp(i\,2\pi\,\alpha\,k)$. Suppose that α is real. If $\alpha = n$ is an integer then so is $\alpha\, k = nk$ and $\exp(i\, 2\pi\, \alpha\, k) = \exp(i\, 2\pi\, nk) = 1$. There is therefore only one value for z^n. This is as we expect, since in this case we have

$$z^n = \begin{cases} 1 & \text{for } n = 0, \\ \underbrace{z\, z\, \cdots\, z}_{n \text{ times}} & \text{for } n > 0, \\ \frac{1}{z^{-n}} & \text{for } n < 0. \end{cases}$$

If $\alpha = p/q$ is a rational number, where we have chosen the integers p and q to have no common factors (i.e., to be **coprime**), then $z^{p/q}$ will have a finite number of values. Indeed consider $\exp(i\, 2\pi\, kp/q)$ as k ranges over the integers. It is clear that this exponential takes the same values for k and for $k + q$:

$$e^{i\, 2\pi\, (k+q)p/q} = e^{i\, 2\pi\, (k(p/q)+p)} = e^{i\, 2\pi\, k(p/q)+i\, 2\pi\, p} = e^{i\, 2\pi\, kp/q} \,,$$

where we have used the addition and periodicity properties (2.14) and (2.15) of the exponential function. Therefore $z^{p/q}$ will have at most q distinct values, corresponding to the above formula with, say, $k = 0, 1, 2, \ldots, q - 1$. In fact, it will have precisely q distinct values, as we will see below. Finally, if α is irrational, then z^α will possess an infinite number of values. To see this notice that if there are integers k and k' for which $e^{i\, \alpha\, 2\pi\, k} = e^{i\, \alpha\, 2\pi\, k'}$, then we must have that $e^{i\, \alpha\, 2\pi\, (k-k')} = 1$, which means that $\alpha\, (k - k')$ must be an integer. Since α is irrational, this can only be true if $k = k'$.

For example, let us compute $1^{1/q}$. According to the formula,

$$1^{1/q} = e^{\mathrm{Log}(1)/q}\, e^{i\, 2\pi\, (k/q)} = e^{i\, 2\pi\, (k/q)} \,,$$

as k ranges over the integers. As discussed above only the q values $k = 0, 1, 2, \ldots, q - 1$ will be different. The values of $1^{1/q}$ are known as **q-th roots of unity**. They each have the property that their q-th power is equal to 1: $(1^{1/q})^q = 1$, as can be easily seen from the above expression. Let $\omega = \exp(i\, 2\pi/q)$ correspond to the $k = 1$ value of $1^{1/q}$. Then the q-th roots of unity are given by $1, \omega, \omega^2, \ldots, \omega^{q-1}$, and there are q of them. The q-th roots of unity lie in the unit circle $|z| = 1$ in the complex plane and define the vertices of a regular q-gon. For example, in Figure 2.3 we depict the q-th roots of unity for $q = 3, 5, 7, 11$.

Let z be a nonzero complex number and suppose that we are after its q-th roots. Writing z in polar form $z = |z|\, \exp(i\, \theta)$, we have

$$z^{1/q} = |z|^{1/q}\, e^{i\, \theta/q} \omega^k \qquad \text{for } k = 0, 1, 2, \ldots, q - 1.$$

In other words the q different values of $z^{1/q}$ are obtained from any one value by multiplying it by the q powers of the q-th roots of unity. If p is any integer,

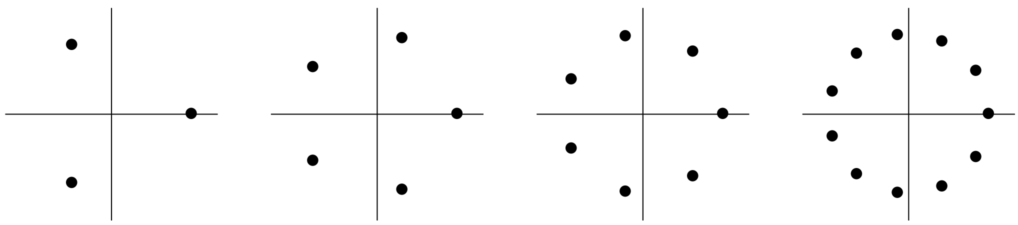


Figure 2.3: Some roots of unity.

we can then take the p-th power of the above formula:

$$z^{p/q} = |z|^{p/q} \, e^{i\,p\,\theta/q} \omega^{pk} \qquad \text{for } k = 0, 1, 2, \ldots, q-1.$$

If p and q are coprime, the ω^{pk} for $k = 0, 1, 2, \ldots, q-1$ are different. Indeed, suppose that $\omega^{pk} = \omega^{pk'}$, for k and k' between 0 and $q-1$. Then $\omega^{p(k-k')} = 1$, which means that $p(k-k')$ has to be a multiple of q. Because p and q are coprime, this can only happen when $k = k'$. Therefore we see that indeed a rational power p/q (with p and q coprime) of a complex number has precisely q values.

Let us now consider complex powers. If $\alpha = a + i\,b$ is not real (so that $b \neq 0$), then z^α will always have an infinite number of values. Indeed, notice that the last term in the following expression takes a different value for each integer k:

$$e^{i\,\alpha\,2\pi\,k} = e^{i\,(a+i\,b)\,2\pi\,k} = e^{i\,2\pi\,k\,a} e^{-2\pi\,k\,b} \ .$$

For examples, let us compute i^i. By definition,

$$i^i = e^{i\,\log(i)} = e^{i\,(\mathrm{Log}(i)+i\,2\pi\,k)} = e^{i\,(i\pi/2+i\,2\pi\,k)} = e^{-\pi/2}\,e^{-2\pi\,k} \ ,$$

for $k = 0. \pm 1, \pm 2, \ldots$, which interestingly enough is real.

Choosing a branch for the complex power

Every branch of the logarithm gives rise to a branch of z^α. In particular we define the **principal branch** of z^α to be $\exp(\alpha\ \mathrm{Log}(z))$. Since the exponential function is entire, the principal branch of z^α is analytic in the domain D where $\mathrm{Log}(z)$ is analytic. We can compute its derivative for any point z_0 in D using the chain rule (2.12):

$$\frac{d}{dz}\left(e^{\alpha\ \mathrm{Log}(z)}\right)\Big|_{z=z_0} = e^{\alpha\ \mathrm{Log}(z_0)}\,\frac{\alpha}{z_0} \ .$$

Given any nonzero z_0 in the complex plane, we can choose a branch of the logarithm so that the function z^α is analytic in a neighbourhood of z_0. We can compute its derivative there and we see that the following equation holds

$$\frac{d}{dz}\left(z^{\alpha}\right)\Big|_{z=z_0} = \alpha\, z_0^{\alpha}\, \frac{1}{z_0}\ ,$$

provided that we use the same branch of z^{α} on both sides of the equation.

One might be tempted to write the right-hand side of the above equation as $\alpha\, z_0^{\alpha-1}$, and indeed this is correct, since the complex powers satisfy many of the identities that we are familiar with from real powers. For example, one can easily show that for any complex numbers α and β

$$z^{\alpha}\, z^{\beta} = z^{\alpha+\beta}\ ,$$

provided that the same branch of the logarithm, and hence of the complex power, is chosen on both sides of the equation. Nevertheless, there is one identity that does *not* hold. Suppose that α is a complex number and let z_1 and z_2 be nonzero complex numbers. Then it is *not* true that $z_1^{\alpha}\, z_2^{\alpha}$ and $(z_1\, z_2)^{\alpha}$ agree, even if, as we always should, we choose the same branch of the complex power on both sides of the equation.

We end this section with the observation that the function z^z is analytic wherever the chosen branch of the logarithm function is defined. Indeed, $z^z = \exp(z\ \log(z))$ and its principal branch can is defined to be the function $\exp(z\ \mathrm{Log}(z))$, which as we now show is analytic in D. Taking the derivative we see that

$$\frac{d}{dz}\left(e^{z\ \mathrm{Log}(z)}\right)\Big|_{z=z_0} = e^{z_0\ \mathrm{Log}(z_0)}\ \left(\mathrm{Log}(z_0) + 1\right)\ ,$$

which exists everywhere on D. Again a similar result holds for any other branch provided we are consistent and take the same branches of the logarithm in both sides of the following equation:

$$\frac{d}{dz}\left(z^{z}\right)\Big|_{z=z_0} = z_0^{z_0}\left(\log(z_0) + 1\right)\ .$$

2.2 Complex integration

Having discussed differentiation of complex-valued functions, it is time to now discuss integration. In real analysis differentiation and integration are roughly speaking inverse operations. We will see that something similar also happens in the complex domain; but in addition, and this is unique to complex analytic functions, differentiation and integration are also roughly equivalent operations, in the sense that we will be able to take derivatives by performing integrals.

2.2.1 Complex integrals

There is a sense in which the integral of a complex-valued function is a trivial extension of the standard integral one learns about in calculus. Suppose that f is a complex-valued function of a *real* variable t. We can decompose $f(t)$ into its real and imaginary parts $f(t) = u(t) + i\,v(t)$, where u and v are now real-valued functions of a real variable. We can therefore define the integral $\int_a^b f(t)\,dt$ of $f(t)$ on the interval $[a,b]$ as

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt\ ,$$

provided that the functions u and v are integrable. We will not develop a formal theory of integrability in this course. You should nevertheless be aware of the fact that whereas not every function is integrable, a continuous function always is. Hence, for example, if f is a continuous function in the interval $[a,b]$ then the integral $\int_a^b f(t)\,dt$ will always exist, since u and v are continuous and hence integrable.

This integral satisfies many of the properties that real integrals obey. For instance, it is (complex) linear, so that if α and β are complex numbers and f and g are complex-valued functions of t, then

$$\int_a^b \left(\alpha\,f(t) + \beta\,g(t)\right)\,dt = \alpha\int_a^b f(t)\,dt + \beta\int_a^b g(t)\,dt\ .$$

It also satisfies a complex version of the **Fundamental Theorem of Calculus**. This theorem states that if $f(t)$ is continuous in $[a,b]$ and there exists a function $F(t)$ also defined on $[a,b]$ such that $\dot{F}(t) = f(t)$ for all $a \leq t \leq b$, where $\dot{F}(t) \equiv \frac{dF}{dt}$, then

$$\int_a^b f(t)\,dt = \int_a^b \frac{dF(t)}{dt}\,dt = F(b) - F(a)\ . \qquad (2.23)$$

This follows from the similar theorem for real integrals, as we now show. Indeed, let us decompose both f and F into real and imaginary parts: $f(t) = u(t) + i\,v(t)$ and $F(t) = U(t) + i\,V(t)$. Then since F is an antiderivative $\dot{F}(t) = \dot{U}(t) + i\,\dot{V}(t) = f(t) = u(t) + i\,v(t)$, whence $\dot{U}(t) = u(t)$ and $\dot{V}(t) = v(t)$. Therefore, by definition

$$\begin{aligned}\int_a^b f(t)\,dt &= \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt\\ &= U(b) - U(a) + i\,\left(V(b) - V(a)\right)\\ &= U(b) + i\,V(b) - \left(U(a) + i\,V(a)\right)\\ &= F(b) - F(a)\ ,\end{aligned}$$

where to reach the second line we used the real version of the fundamental theorem of calculus for the real and imaginary parts of the integral.

A final useful property of the complex integral is that

$$\left| \int_a^b f(t)\, dt \right| \leq \int_a^b |f(t)| \; dt \; . \tag{2.24}$$

This result makes sense intuitively because in integrating $f(t)$ one might encounter cancellations which do not occur while integrating the non-negative quantity $|f(t)|$.

This last property follows from the similar property of real integrals. Let us see this. Write the complex integral $\int_a^b f(t)\, dt$ in polar form:

$$\int_a^b f(t)\, dt = R\, e^{i\,\theta} \; ,$$

where

$$R = \int_a^b f(t)\, dt \quad .$$

On the other hand,

$$R = \int_a^b e^{-i\,\theta} f(t)\, dt \; .$$

Write $e^{-i\,\theta} f(t) = U(t) + i\, V(t)$ where $U(t)$ and $V(t)$ are real-valued functions. Then because R is real,

$$R = \int_a^b U(t)\, dt \; .$$

But now,

$$U(t) = \mathrm{Re} \;\; e^{-i\,\theta} f(t) \;\; \leq \;\; e^{-i\,\theta} f(t) \;\; = |f(t)| \; .$$

Therefore, from the properties of real integrals,

$$\int_a^b U(t)\, dt \leq \int_a^b |f(t)|\, dt \; ,$$

which proves the desired result.

2.2.2 Contour integrals

Much more interesting is the integration of complex-valued functions of a *complex* variable. We would like to be able to make sense out of something like

$$\int_{z_0}^{z_1} f(z)\, dz \; ,$$

where z_0 and z_1 are complex numbers. We are immediately faced with a difficulty. Unlike the case of an interval $[a, b]$ when it is fairly obvious how to go from a to b, here z_0 and z_1 are points in the complex plane and there are many ways to go from one point to the other. Therefore as it stands, the above integral is ambiguous. The way out of this ambiguity is to specify a path joining z_0 to z_1 and then integrate the function along the path. In order to do this we will have to introduce some notation.

The integral along a parametrised curve

Let z_0 and z_1 be two points in the complex plane. One has an intuitive notion of what one means by a curve joining z_0 and z_1. Physically, we can think of a point-particle moving in the complex plane, starting at some time t_0 at the point z_0 and ending at some later time t_1 at the point z_1. At any given instant in time $t_0 \leq t \leq t_1$, the particle is at the point $z(t)$ in the complex plane. Therefore we see that a curve joining z_0 and z_1 can be defined by a function $z(t)$ taking points t in the interval $[t_0, t_1]$ to points $z(t)$ in the complex plane in such a way that $z(t_0) = z_0$ and $z(t_1) = z_1$. Let us make this a little more precise. By a **(parametrised) curve** joining z_0 and z_1 we shall mean a continuous function $z : [t_0, t_1] \to \mathbb{C}$ such that $z(t_0) = z_0$ and $z(t_1) = z_1$. We can decompose z into its real and imaginary parts, and this is equivalent to two continuous real-valued functions $x(t)$ and $y(t)$ defined on the interval $[t_0, t_1]$ such that $x(t_0) = x_0$ and $x(t_1) = x_1$ and similarly for $y(t)$: $y(t_0) = y_0$ and $y(t_1) = y_1$, where $z_0 = x_0 + i\,y_0$ and $z_1 = x_1 + i\,y_1$. We say that the curve is **smooth** if its velocity $\dot{z}(t)$ is a continuous function $[t_0, t_1] \to \mathbb{C}$ which is never zero.

Let Γ be a smooth curve joining z_0 to z_1, and let $f(z)$ be a complex-valued function which is continuous on Γ. Then we define the **integral of f along** Γ by

$$\boxed{\int_\Gamma f(z)\,dz \equiv \int_{t_0}^{t_1} f(z(t))\,\dot{z}(t)\,dt\ .} \tag{2.25}$$

By hypothesis, the integrand, being a product of continuous functions, is itself continuous and hence the integral exists.

Let us compute some examples. Consider the function $f(z) = x^2 + i\,y^2$ integrated along the smooth curve parametrised by $z(t) = t + i\,t$ for $0 \leq t \leq 1$. As shown in Figure 2.4 this is the straight line segment joining the origin and the point $1+i$. Decomposing $z(t) = x(t) + i\,y(t)$ into real and imaginary parts, we see that $x(t) = y(t) = t$. Therefore $f(z(t)) = t^2 + i\,t^2$ and $\dot{z}(t) = 1 + i$. Putting it all together, using complex linearity of the integral and performing the elementary real integral, we find the following result

$$\int_\Gamma f(z)\,dz = \int_0^1 (t^2 + i\,t^2)(1+i)\,dt = \int_0^1 (1+i)^2\,t^2\,dt = 2i\ \frac{t^3}{3}\bigg|_0^1 = \frac{2i}{3}\ .$$

Consider now the function $f(z) = 1/z$ integrated along the smooth curve Γ parametrised by $z(t) = R\exp(i\,2\pi\,t)$ for $0 \leq t \leq 1$, where $R \neq 0$. As shown in Figure 2.4, the resulting curve is the circle of radius R centred about the origin. Here $f(z(t)) = (1/R)\exp(-i\,2\pi\,t)$ and $\dot{z}(t) = 2\pi\,i\,R\exp(i\,2\pi\,t)$.

Figure 2.4: Two parametrised curves.

Putting it all together we obtain

$$\int_\Gamma f(z)\,dz = \int_0^1 \frac{2\pi\, i\, Re^{i\,2\pi\, t}}{Re^{i\,2\pi\, t}}\,dt = 2\pi\, i \int_0^1 dt = 2\pi\, i\ . \tag{2.26}$$

Notice that the result is independent of the radius. This is in sharp contrast with real integrals, which we are used to interpret physically in terms of area. In fact, the above integral behaves more like a charge than like an area.

Finally let us consider the function $f(z) \equiv 1$ along *any* smooth curve Γ parametrised by $z(t)$ for $0 \leq t \leq 1$. It may seem that we do not have enough information to compute the integral, but let us see how far we can get with the information given. The integral becomes

$$\int_\Gamma f(z)\,dz = \int_0^1 \dot{z}(t)\,dt\ .$$

Using the complex version of the fundamental theorem of calculus, we have

$$\int_0^1 \dot{z}(t)\,dt = z(1) - z(0)\ ,$$

independent of the actual curve used to join the two points! Notice that this integral is therefore *not* the length of the curve as one might think from the notation.

The length of a curve and a useful estimate

The length of the curve can be computed, but the integral is not related to the complex dz but the real $|dz|$. Indeed, if Γ is a curve parametrised by $z(t) = x(t) + i\,y(t)$ for $t \in [t_0, t_1]$, consider the real integral

$$\begin{aligned}\int_\Gamma |dz| &\equiv \int_{t_0}^{t_1} |\dot{z}(t)|\,dt \\ &= \int_{t_0}^{t_1} \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}\,dt\ ,\end{aligned}$$

which is the integral of the infinitesimal line element $\sqrt{dx^2 + dy^2}$ along the curve. Therefore, the integral is the (arc)length $\ell(\Gamma)$ of the curve:

$$\boxed{\int_\Gamma |dz| = \ell(\Gamma) \ .} \tag{2.27}$$

This immediately yields a useful estimate on integrals along curves, analogous to equation (2.24). Indeed, suppose that Γ is a curve parametrised by $z(t)$ for $t \in [t_0, t_1]$. Then,

$$\begin{aligned}
\left| \int_\Gamma f(z)\, dz \right| &= \left| \int_{t_0}^{t_1} f(z(t))\, \dot{z}(t)\, dt \right| \\
&\leq \int_{t_0}^{t_1} |f(z(t))|\, |\dot{z}(t)|\, dt \qquad \text{(using (2.24))} \\
&\leq \max_{z\in\Gamma} |f(z)| \int_{t_0}^{t_1} |\dot{z}(t)|\, dt \ .
\end{aligned}$$

But this last integral is simply the length $\ell(\Gamma)$ of the curve, whence we have

$$\boxed{\left| \int_\Gamma f(z)\, dz \right| \leq \int_\Gamma |f(z)|\, |dz| \leq \max_{z\in\Gamma} |f(z)|\, \ell(\Gamma) \ .} \tag{2.28}$$

Results of this type are the bread and butter of analysis and in this part of the course we will have ample opportunity to use this particular one.

Some further properties of the integrals along a curve

We have just seen that one of the above integrals does not depend on the actual path but just on the endpoints of the contour. We will devote the next two sections to studying conditions for complex integrals to be independent of the path; but before doing so, we discuss some general properties of the integrals $\int_\Gamma f(z)\, dz$.

The first important property is that the integral is complex linear. That is, if α and β are complex numbers and f and g are functions which are continuous on Γ, then

$$\int_\Gamma (\alpha\, f(z) + \beta\, g(z))\ dz = \alpha \int_\Gamma f(z)\, dz + \beta \int_\Gamma g(z)\, dz \ .$$

The proof is routine and we leave it as an exercise.

The first nontrivial property is that the integral $\int_\Gamma f(z)\, dz$ does not depend on the actual parametrisation of the curve Γ. In other words, it is a "physical" property of the curve itself, meaning the set of points $\Gamma \subset \mathbb{C}$ together with the direction along the curve, and not of the way in which we go about traversing them.

The only difficult thing in showing this is coming up with a mathematical statement to prove. Let $z(t)$ for $t_0 \leq t \leq t_1$ and $z'(t)$ for $t'_0 \leq t \leq t'_1$ be two smooth parametrisations of the same curve Γ. This means that $z(t_0) = z'(t'_0)$ and $z(t_1) = z'(t'_1)$. We will say that the parametrisations $z(t)$ and $z'(t)$ are **equivalent** if there exists a one-to-one differentiable function $\lambda : [t'_0, t'_1] \to [t_0, t_1]$ such that $z'(t) = z(\lambda(t))$. In particular, this means that $\lambda(t'_0) = t_0$ and $\lambda(t'_1) = t_1$. (It is possible to show that this is indeed an equivalence relation.)

The condition of **reparametrisation invariance** of $\int_\Gamma f(z)\, dz$ can then be stated as follows. Let z and z' be two equivalent parametrisations of a curve Γ. Then for any function $f(z)$ continuous on Γ, we have

$$\int_{t'_0}^{t'_1} f(z'(t))\, \dot{z}'(t)\, dt = \int_{t_0}^{t_1} f(z(t))\, \dot{z}(t)\, dt\ .$$

Let us prove this.

$$\begin{aligned}\int_{t'_0}^{t'_1} f(z'(t))\, \dot{z}'(t)\, dt &= \int_{t'_0}^{t'_1} f(z(\lambda(t)))\, \dot{z}(\lambda(t))\, dt \\ &= \int_{\lambda(t'_0)}^{\lambda(t'_1)} f(z(\lambda))\, \frac{dz(\lambda)}{d\lambda}\, d\lambda \\ &= \int_{t_0}^{t_1} f(z(\lambda))\, \frac{dz(\lambda)}{d\lambda}\, d\lambda\ ,\end{aligned}$$

which after changing the name of the variable of integration from λ to t (Shakespeare's Theorem!), is seen to agree with

$$\int_{t_0}^{t_1} f(z(t))\, \dot{z}(t)\, dt\ .$$

Because of reparametrisation invariance, we can always parametrise a curve in such a way that the initial time is $t = 0$ and the final time is $t = 1$. Indeed, let $z(t)$ for $t_0 \leq t \leq t_1$ be any smooth parametrisation of a curve Γ. Then define the parametrisation $z'(t) = z(t_0 + t(t_1 - t_0))$. Clearly, $z'(0) = z(t_0)$ and $z'(1) = z(t_1)$, and moreover $\dot{z}'(t) = (t_1 - t_0)\dot{z}(t_0 + t(t_1 - t_0))$ hence z' is also smooth.

Now let us notice that parametrised curves Γ have a natural notion of direction: this is the direction in which we traverse the curve. Choosing a parametrisation $z(t)$ for $0 \leq t \leq 1$, as we go from $z(0)$ to $z(1)$, we trace the points in the curve in a given order, which we depict by an arrowhead on the curve indicating the direction along which t increases, as in the curves in Figure 2.4. A curve with such a choice of direction is said to be **directed**.

Given any directed curve Γ, we let $-\Gamma$ denote the directed curve with the opposite direction; that is, with the arrow pointing in the opposite direction. The final interesting property of the integral $\int_\Gamma f(z)\,dz$ is that

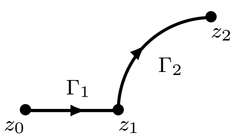
$$\boxed{\int_{-\Gamma} f(z)\,dz = -\int_\Gamma f(z)\,dz\ .} \tag{2.29}$$

To prove this it is enough to find two parametrisations for Γ and $-\Gamma$ and compute the integrals. By reparametrisation independence it does not matter which parametrisations we choose. If $z(t)$ for $0 \leq t \leq 1$ is a parametrisation for Γ, then $z'(t) = z(1-t)$ for $0 \leq t \leq 1$ is a parametrisation for $-\Gamma$. Indeed, $z'(0) = z(1)$ and $z'(1) = z(0)$ and they trace the same set of points. Let us compute:

$$\begin{aligned}
\int_{-\Gamma} f(z)\,dz &= \int_0^1 f(z'(t))\,\dot{z}'(t)\,dt \\
&= -\int_0^1 f(z(1-t))\,\dot{z}(1-t)\,dt \\
&= \int_1^0 f(z(t'))\,\dot{z}(t')\,dt' \\
&= -\int_0^1 f(z(t'))\,\dot{z}(t')\,dt' \\
&= -\int_\Gamma f(z)\,dz\ .
\end{aligned}$$

Piecewise smooth curves and contour integrals

Finally we have to generalise the integral $\int_\Gamma f(z)\,dz$ to curves which are not necessarily smooth, but which are made out of smooth curves. Curves can be composed: if Γ_1 is a curve joining z_0 to z_1 and Γ_2 is a curve joining z_1 to z_2, then we can make a curve Γ joining z_0 to z_2 by first going to the intermediate point z_1 via Γ_1 and then from there via Γ_2 to our destination z_2. The resulting curve Γ is still continuous, but it will generally fail to be smooth, since the velocity need not be continuous at the intermediate point z_1, as shown in the figure.



However such curve is **piecewise smooth**: which means that it is made out of smooth components by the composition procedure just outlined. In terms of parametrisations, if $z_1(t)$ and $z_2(t)$, for $0 \leq t \leq 1$, are smooth parametrisations for Γ_1 and Γ_2 respectively, then

$$z(t) = \begin{cases} z_1(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ z_2(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

is a parametrisation for Γ. Notice that it is well-defined and continuous at $t = \frac{1}{2}$ precisely because $z_1(1) = z_2(0)$; however it need not be smooth there since $\dot{z}_1(1) \neq \dot{z}_2(0)$ necessarily. We can repeat this procedure and construct curves which are not smooth but which are made out of a finite number of smooth curves: one curve ending where the next starts. Such a piecewise smooth curve will be called a **contour** from now on. If a contour Γ is made out of composing a finite number of smooth curves $\{\Gamma_j\}$ we will say that each Γ_j is a **smooth component** of Γ.

Let Γ be a contour with n smooth components $\{\Gamma_j\}$ for $j = 1, 2, \ldots, n$. If $f(z)$ is a function continuous on Γ, then the **contour integral of f along** Γ is defined as

$$\int_\Gamma f(z)\,dz = \sum_{j=1}^{n} \int_{\Gamma_j} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz + \cdots + \int_{\Gamma_n} f(z)\,dz \ ,$$

with each of the $\int_{\Gamma_i} f(z)\,dz$ is defined by (2.25) relative to any smooth parametrisation.

2.2.3 Independence of path

In this section we will investigate conditions under which a contour integral only depends on the endpoints of the contour, and not not the contour itself. This is necessary preparatory material for Cauchy's integral theorem which will be discussed in the next section.

We will say that an open subset U of the complex plane is **connected**, if every pair of points in U can be joined by a contour. A connected open subset of the complex plane will be called a **domain**.

What we have called connected here is usually called **path-connected**. We can allow ourselves this abuse of notation because path-connectedness is easier to define and it can be shown that the two notions agree for subsets of the complex plane.

Fundamental Theorem of Calculus: contour integral version

First we start with a contour integral version of the fundamental theorem of calculus. Let D be a domain and let $f : D \to \mathbb{C}$ be a continuous complex-valued function defined on D. We say that f has an **antiderivative** in D if there exists some function $F : D \to \mathbb{C}$ such that

$$F'(z) = \frac{dF(z)}{dz} = f(z) \ .$$

Notice that F is therefore analytic in D. Now let Γ be any contour in D with endpoints z_0 and z_1. If f has an antiderivative F on D, the contour integral is given by

$$\int_\Gamma f(z)\,dz = F(z_1) - F(z_0) \ . \tag{2.30}$$

Let us first prove this for Γ a smooth curve, parametrised by $z(t)$ for $0 \le t \le 1$. Then

$$\int_\Gamma f(z)\,dz = \int_0^1 F'(z(t))\dot{z}(t)dt = \int_0^1 \frac{dF(z(t))}{dt}\,dt\ .$$

Using the complex version of the fundamental theorem of calculus (2.23), we see that

$$\int_\Gamma f(z)\,dz = F(z(1)) - F(z(0)) = F(z_1) - F(z_0)\ .$$

Now we consider the general case: Γ a contour with smooth components $\{\Gamma_j\}$ for $j = 1, 2, \ldots, n$. The curve Γ_1 starts in z_0 and ends in some intermediate point τ_1, Γ_2 starts in τ_1 and ends in a second intermediate point τ_2, and so so until Γ_n which starts in the intermediate point τ_{n-1} and ends in z_1. Then

$$\begin{aligned}
\int_\Gamma f(z)dz &= \sum_{j=1}^n \int_{\Gamma_j} f(z)\,dz \\
&= \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz + \cdots + \int_{\Gamma_n} f(z)\,dz \\
&= F(\tau_1) - F(z_0) + F(\tau_2) - F(\tau_1) + \cdots + F(z_1) - F(\tau_{n-1}) \\
&= F(z_1) - F(z_0)\ ,
\end{aligned}$$

where we have used the definition of the contour integral and the result proven above for each of the smooth components.

This result says that if a function f has an antiderivative, then its contour integrals do not depend on the precise path, but only on the endpoints. Path independence can also be rephrased in terms of closed contour integrals. We say that a contour is **closed** if its endpoints coincide. The contour integral along a closed contour Γ is sometimes denoted $\oint_\Gamma$ when we wish to emphasise that the contour is closed.

The path-independence lemma

As a corollary of the above result, we see that if Γ is a closed contour in some domain D and $f : D \to \mathbb{C}$ has an antiderivative in D, then

$$\oint_\Gamma f(z)\,dz = 0\ .$$

This is clear because if the endpoints coincide, so that $z_0 = z_1$, then $F(z_1) - F(z_0) = 0$.

In fact, let $f : D \to \mathbb{C}$ be a continuous function on some domain D. Then the following three statements are equivalent:

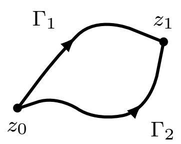
(a) f has an antiderivative F in D;

(b) The closed contour integral $\oint_\Gamma f(z)\,dz$ vanishes for all closed contours Γ in D; and

(c) The contour integrals $\int_\Gamma f(z)\,dz$ are independent of the path.

We shall call this result the **Path-independence Lemma**.

We have already proven that (a) implies (b) and (c). We will now show that in fact (b) and (c) are equivalent.



Let Γ_1 and Γ_2 be any two contours in D sharing the same initial and final endpoints: z_0 and z_1, say. Then consider the contour Γ obtained by composing Γ_1 with $-\Gamma_2$. This is a closed contour with initial and final endpoint z_0. Therefore, using (2.29) for the integral along $-\Gamma_2$,

$$\begin{aligned}\oint_\Gamma f(z)\,dz &= \int_{\Gamma_1} f(z)\,dz + \int_{-\Gamma_2} f(z)\,dz \\ &= \int_{\Gamma_1} f(z)\,dz - \int_{\Gamma_2} f(z)\,dz\ ,\end{aligned}$$

whence $\oint_\Gamma f(z)\,dz = 0$ if and only if $\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$. This shows that (b) implies (c). Now we prove that, conversely, (c) implies (b). Let Γ be any closed contour with endpoints $z_1 = z_0$. By path-independence, we can evaluate the integral by taking the trivial contour which remains at z_0 for all $0 \leq t \leq 1$. This parametrisation is strictly speaking not smooth since $\dot{z}(t) = 0$ for all t, but the integrand $f(z(t))\dot{z}(t) = 0$ is certainly continuous, so that the integral exists and is clearly zero. Hence $\oint_\Gamma f(z)\,dz = 0$ for all closed contours Γ. Alternatively, we can pick any point τ in the contour not equal to $z_0 = z_1$. We can think of the contour as made out of two contours: Γ_1 from z_0 to τ and Γ_2 from τ to $z_1 = z_0$. We can therefore go from $z_0 = z_1$ to τ in two ways: one is along Γ_1 and the other one is along $-\Gamma_2$. Path-independence says that the result is the same:

$$\int_{\Gamma_1} f(z)\,dz = \int_{-\Gamma_2} f(z)\,dz = -\int_{\Gamma_2} f(z)\,dz\ ,$$

where we have used equation (2.29). Therefore,

$$0 = \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz = \int_\Gamma f(z)\,dz\ .$$

Finally we finish the proof of the path-independence lemma by showing that (c) implies (a); that is, if all contour integrals are path-independence, then the function f has an antiderivative. The property of path-independence suggests a way to define the antiderivative. Let us fix once and for all a point z_0 in the domain D. Let z be an arbitrary point in D. Because D is connected there will be a contour Γ joining z_0 and z. Define a function $F(z)$ by

$$F(z) \equiv \int_\Gamma f(\zeta)\, d\zeta \ ,$$

where we have changed notation in the integral (Shakespeare's Theorem again) not to confuse the variable of integration with the endpoint z of the contour. By path-independence this integral is independent of the contour and is therefore well-defined as a function of the endpoint z. We must now check that it is an antiderivative for f.

The derivative of $F(z)$ is computed by

$$F'(z) = \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left[\int_{\Gamma'} f(\zeta)\, d\zeta - \int_\Gamma f(\zeta)\, d\zeta \right] \ ,$$

where Γ' is any contour from z_0 to $z+\Delta z$. Since we are interested in the limit of $\Delta z \to 0$, we can assume that Δz is so small that $z+\Delta z$ is contained in some open ε-disk about z which also belongs to D.[2] This means that the straight-line segment Γ'' from z to $z + \Delta z$ belongs to D. By path-independence we are free to choose the contour Γ', and we exercise this choice by taking Γ' to be the composition of Γ with this straight-line segment Γ''. Therefore,

$$\begin{aligned} \int_{\Gamma'} f(\zeta)\, d\zeta - \int_\Gamma f(\zeta)\, d\zeta &= \int_\Gamma f(\zeta)\, d\zeta + \int_{\Gamma''} f(\zeta)\, d\zeta - \int_\Gamma f(\zeta)\, d\zeta \\ &= \int_{\Gamma''} f(\zeta)\, d\zeta \ , \end{aligned}$$

whence

$$F'(z) = \lim_{\Delta z \to 0} \frac{1}{\Delta z} \int_{\Gamma''} f(\zeta)\, d\zeta \ .$$

We parametrise the contour Γ'' by $\zeta(t) = z + t\Delta z$ for $0 \le t \le 1$. Then we

[2] In more detail, since D is open we know that there exists some $\varepsilon > 0$ small enough so that $D_\varepsilon(z)$ belongs to D. We then simply take $|\Delta z| < \varepsilon$, which we can do since we are interested in the limit $\Delta z \to 0$.

have

$$\begin{aligned} F'(z) &= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \int_0^1 f(z + t\Delta z)\, \dot{\zeta}(t)\, dt \\ &= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \int_0^1 f(z + t\Delta z)\, \Delta z\, dt \\ &= \lim_{\Delta z \to 0} \int_0^1 f(z + t\Delta z)\, dt \ . \end{aligned}$$

One might be tempted now to simply sneak the limit inside the integral, use continuity of f and obtain

$$F'(z) \stackrel{?}{=} \int_0^1 \lim_{\Delta z \to 0} f(z + t\Delta z)\, dt = \int_0^1 f(z)\, dt = f(z) \ ,$$

which would finish the proof. However sneaking the limit inside the integral is not always allowed since integration itself is a limiting process and limits cannot always be interchanged.

A simple example showing that the order in which one takes limits matters is the following. Consider the following limit

$$\lim_{\substack{n \to \infty \\ m \to \infty}} \frac{m+n}{m} \ .$$

We can take this limit in two ways. On the one hand,

$$\lim_{n \to \infty} \lim_{m \to \infty} \frac{m}{m+n} = \lim_{n \to \infty} 1 = 1 \ ;$$

yet on the other

$$\lim_{m \to \infty} \lim_{n \to \infty} \frac{m}{m+n} = \lim_{m \to \infty} 0 = 0 \ .$$

Nevertheless, as we sketch below, in this case interchanging the limits turns out to be a correct procedure due to the continuity of the integrand.

We want to prove here that indeed

$$\lim_{\Delta z \to 0} \int_0^1 f(z + t\Delta z)\, dt = f(z) \ .$$

We do this by showing that in this limit, the quantity

$$\int_0^1 f(z + t\Delta z)\, dt \quad - f(z) = \int_0^1 \left[f(z + t\Delta z) - f(z) \right]\, dt$$

goes to zero. We will prove that its modulus goes to zero, which is clearly equivalent. By equation (2.24), we have

$$\left| \int_0^1 \left[f(z + t\Delta z) - f(z) \right]\, dt \right| \le \int_0^1 \left| f(z + t\Delta z) - f(z) \right|\, dt \ .$$

By continuity of f we know that given any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(z + t\Delta z) - f(z)| < \varepsilon \qquad \text{whenever} \qquad |\Delta z| < \delta \ .$$

Since we are taking the limit $\Delta z \to 0$, we can take $|\Delta z| < \delta$, whence

$$\lim_{\Delta z \to 0} \int_0^1 [f(z + t\Delta z) - f(z)] \ dt \ \leq \lim_{\Delta z \to 0} \int_0^1 |f(z + t\Delta z) - f(z)| \ dt < \int_0^1 \varepsilon \, dt = \varepsilon \ ,$$

for any $\varepsilon > 0$, where we have used equation (2.24) to arrive at the last inequality. Hence,

$$\lim_{\Delta z \to 0} \int_0^1 [f(z + t\Delta z) - f(z)] \ dt \ = 0 \ ,$$

so that

$$\lim_{\Delta z \to 0} \int_0^1 [f(z + t\Delta z) - f(z)] \ dt = 0 \ .$$

2.2.4 Cauchy's Integral Theorem

We have now laid the groundwork to be able to discuss one of the key results in complex analysis. The path-independence lemma tells us that a continuous function $f : D \to \mathbb{C}$ in some domain D has an antiderivative if and only if all its closed contour integrals vanish. Unfortunately it is impractical to check this hypothesis explicitly, so one would like to be able to conclude the vanishing of the closed contour integrals some other way. Cauchy's integral theorem will tell us that, under some conditions, this is true if f is analytic. These conditions refer to the topology of the domain, so we have to first introduce a little bit of notation.

Let us say that a contour is **simple** if it has no self-intersections. We define a **loop** to be a closed simple contour. We start by mentioning the celebrated **Jordan curve lemma**, a version of which states that any loop in the complex plane separates the plane into two domains with the loop as common boundary: one of which is bounded and is called the **interior** and one of which is unbounded and is called the **exterior**.

This is a totally obvious statement and as most such statements extremely hard to prove, requiring techniques of algebraic topology.

We say that a domain D is **simply-connected** if the interior domain of every loop in D lies wholly in D. Hence for example, a disk is simply connected, while a punctured disk is not: any circle around the puncture contains the puncture in its interior, but this has been excised from the disk. Intuitively speaking, a domain is simply-connected if any loop in the domain can be continuously shrunk to a point without any point of the loop ever leaving the domain.

We are ready to state the **Cauchy Integral Theorem**: Let $D \subset \mathbb{C}$ be a *simply-connected* domain and let $f : D \to \mathbb{C}$ be an analytic function, then for any loop Γ, the contour integral vanishes:

$$\oint_\Gamma f(z)dz = 0 \ .$$

As an immediate corollary of this theorem and of the path-independence lemma, we see that an analytic function in a simply-connected domain has an antiderivative, which is itself analytic in D.

We will actually prove a slightly weaker version of the theorem which requires the stronger hypothesis that $f'(z)$ be continuous in D. Recall that analyticity only requires $f'(z)$ to exist. The proof uses a version of Green's theorem which is valid in the complex plane. This theorem states that if $\boldsymbol{V}(x,y) = P(x,y)\,dx + Q(x,y)\,dy$ is a continuously differentiable vector field in a simply-connected domain D in the complex plane, and if Γ is any positively oriented loop in D, then the line integral of $\boldsymbol{V}$ along Γ can be written as the area integral of the function $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ on the interior $\mathrm{Int}(\Gamma)$ of Γ:

$$\oint_\Gamma (P(x,y)\,dx + Q(x,y)\,dy) = \iint\limits_{\mathrm{Int}(\Gamma)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \ . \qquad (2.31)$$

We will sketch a proof of this theorem below; but now let us use it to prove the Cauchy Integral Theorem. Let Γ be a loop in a simply-connected domain D in the complex plane, and let $f(z)$ be a function which is analytic in D. Computing the contour integral, we find

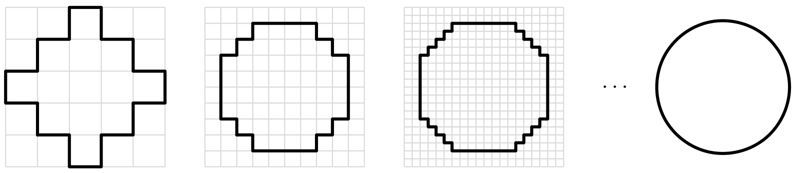
$$\begin{aligned}\oint_\Gamma f(z)\,dz &= \int_\Gamma (u(x,y) + i\,v(x,y))\,(dx + i\,dy) \\ &= \int_\Gamma (u(x,y)\,dx - v(x,y)\,dy) + i\int_\Gamma (v(x,y)\,dx + u(x,y)\,dy) \ .\end{aligned}$$

By hypothesis, $f'(z)$ is continuous, which means that the vector fields $u\,dx - v\,dy$ and $v\,dx + u\,dy$ are continuously differentiable, whence we can use Green's Theorem (2.31) to deduce that

$$\oint_\Gamma f(z)\,dz = \iint\limits_{\mathrm{Int}(\Gamma)} \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + \iint\limits_{\mathrm{Int}(\Gamma)} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \ ,$$

which vanishes by the Cauchy–Riemann equations (2.7).

Here we will sketch a proof of Green's Theorem (2.31). The strategy will be the following. We will approximate the interior of the loop by tiny squares (*plaquettes*) in such a way that the loop itself is approximated by the straight line segments which make up the edges of the squares. As the size of the plaquettes decreases, the approximation becomes better and better. In the picture we have illustrated this by showing three approximations to the unit disk. For each we show the value of the length ℓ of the contour and of the area A of its interior.



$\ell = 9.6$, $A = 2.9952$ — $\ell = 7.68$, $A = 2.9952$ — $\ell = 7.68$, $A = 3.1104$ — … — $\ell = 2\pi$, $A = \pi$

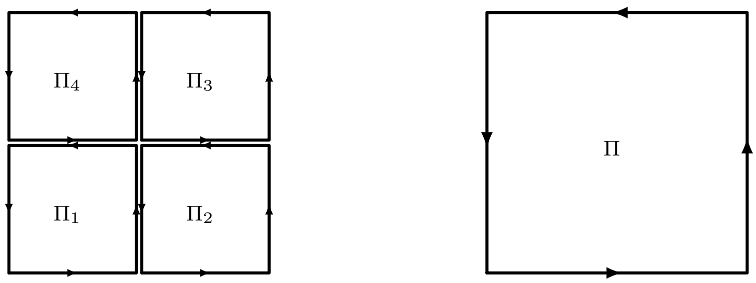
In fact, it is a simple matter of careful bookkeeping to prove that in the limit,

$$\iint_{\mathrm{Int}(\Gamma)} = \lim_{\text{size}\to 0} \sum_{\text{plaquettes }\Pi} \iint_{\mathrm{Int}(\Pi)} .$$

Similarly for the contour integral,

$$\oint_{\Gamma} = \lim_{\text{size}\to 0} \sum_{\text{plaquettes }\Pi} \oint_{\Pi} .$$

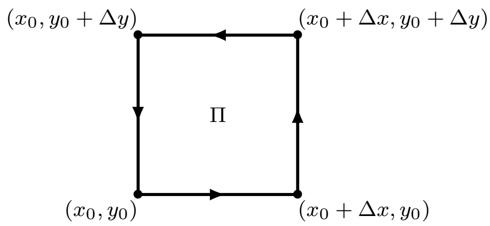
To see this notice that the contour integrals along internal edges common to two adjacent plaquettes cancel because of equation (2.29) and the fact that we integrated twice along them: once for each plaquette but in the opposite orientation, as shown in the picture below. Therefore we only receive contributions from the external edges. Since the region is simply-connected this means that boundary of the region covered by the plaquettes.



In the notation of the picture, then, one has

$$\oint_{\Pi_1} + \oint_{\Pi_2} + \oint_{\Pi_3} + \oint_{\Pi_4} = \oint_{\Pi} .$$

Therefore it is sufficient to prove formula (2.31) for the special case of one plaquette. To this effect we will choose our plaquette Π to have size $\Delta x \times \Delta y$ and whose lower left-hand corner is at the point (x_0, y_0):



Performing the contour integral we have for $\boldsymbol{V}(x, y) = P(x, y)\, dx + Q(x, y)\, dy$,

$$\oint_{\Pi} \boldsymbol{V}(x, y) = \int_{(x_0, y_0)}^{(x_0+\Delta x, y_0)} \boldsymbol{V}(x, y) + \int_{(x_0+\Delta x, y_0)}^{(x_0+\Delta x, y_0+\Delta y)} \boldsymbol{V}(x, y) + \int_{(x_0+\Delta x, y_0+\Delta y)}^{(x_0, y_0+\Delta y)} \boldsymbol{V}(x, y) + \int_{(x_0, y_0+\Delta y)}^{(x_0, y_0)} \boldsymbol{V}(x, y) \; .$$

Along the first and third contour integrals the value of y is constant, whereas along the second and fourth integrals it is the value of x which is constant. Taking this into account, we can rewrite the integrals as follows

$$\oint_{\Pi} \boldsymbol{V}(x, y) = \int_{x_0}^{x_0+\Delta x} P(x, y_0)\, dx + \int_{y_0}^{y_0+\Delta y} Q(x_0 + \Delta x, y)\, dy + \int_{x_0+\Delta x}^{x_0} P(x, y_0 + \Delta y)\, dx + \int_{y_0+\Delta y}^{y_0} Q(x_0, y)\, dy \; .$$

Exchanging the limits of integration in the third and fourth integrals, and picking up a sign in each, we can rewrite the integrals as follows:

$$\oint_{\Pi} \boldsymbol{V}(x, y) = \int_{y_0}^{y_0+\Delta y} \left[Q(x_0 + \Delta x, y) - Q(x_0, y) \right] dy - \int_{x_0}^{x_0+\Delta x} \left[P(x, y_0 + \Delta y) - P(x, y_0) \right] dx \; .$$

But now we make use of the facts that

$$Q(x_0 + \Delta x, y) - Q(x_0, y) = \int_{x_0}^{x_0+\Delta x} \frac{\partial Q(x, y)}{\partial x}\, dx$$

$$P(x, y_0 + \Delta y) - P(x, y_0) = \int_{y_0}^{y_0+\Delta y} \frac{\partial P(x, y)}{\partial y}\, dy \; ;$$

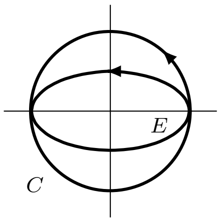
whence the integrals become

$$\begin{aligned} \oint_{\Pi} \boldsymbol{V}(x, y) &= \int_{y_0}^{y_0+\Delta y} \int_{x_0}^{x_0+\Delta x} \frac{\partial Q(x, y)}{\partial x}\, dx\, dy - \int_{x_0}^{x_0+\Delta x} \int_{y_0}^{y_0+\Delta y} \frac{\partial P(x, y)}{\partial y}\, dy\, dx \\ &= \int_{x_0}^{x_0+\Delta x} \int_{y_0}^{y_0+\Delta y} \left(\frac{\partial Q(x, y)}{\partial x} - \frac{\partial P(x, y)}{\partial y} \right) dx\, dy \\ &= \iint\limits_{\mathrm{Int}(\Pi)} \left(\frac{\partial Q(x, y)}{\partial x} - \frac{\partial P(x, y)}{\partial y} \right) dx\, dy \; , \end{aligned}$$

which proves the formula for the plaquette Π.

Deforming the contour without changing the integral

The Cauchy Integral Theorem has a very important consequence for the computation of contour integrals. It basically says that contours can be moved about (or deformed) without changing the result of the integral, provided that in doing so we never cross a point where the integrand ceases to be analytic. Let us illustrate this with a few examples.



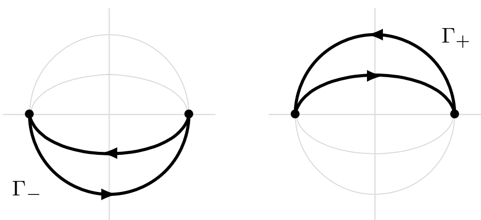
Let us compute the contour integral

$$\oint_E \frac{1}{z}\, dz ,$$

where E is the positively-oriented ellipse $x^2 + 4y^2 = 1$ depicted in the figure. Earlier we computed the same integral around a circular contour C of radius 1, centred at the origin, and we obtained

$$\oint_C \frac{1}{z}\, dz = 2\pi\, i .$$

We will argue, using the Cauchy Integral Theorem, that we get the same answer whether we integrate along E or along C. Consider the two domains in the interior of the circle C but in the exterior of the ellipse E. The integrand is analytic everywhere in the complex plane except for the origin, which lies outside these two regions. The Cauchy Integral Theorem says that the contour integral vanishes along either of the two contours which make up the boundary of these domains. Let us be more explicit and let us call these contours $\Gamma_\pm$ as in the figure below.



Then it is clear that

$$\oint_C \frac{1}{z}\, dz = \oint_{\Gamma_+} \frac{1}{z}\, dz + \oint_{\Gamma_-} \frac{1}{z}\, dz + \oint_E \frac{1}{z}\, dz .$$

Since the interior $\Gamma_\pm$ is simply-connected and the integrand $\frac{1}{z}$ is analytic in and on $\Gamma_\pm$, the Cauchy Integral Theorem says that

$$\oint_{\Gamma_\pm} \frac{1}{z}\, dz = 0 ,$$

whence

$$\oint_E \frac{1}{z}\,dz = \oint_C \frac{1}{z}\,dz = 2\pi\, i\ .$$

In other words, we could deform the contour from E to C without altering the result of the integral because in doing so we do not pass over any point where the integrand ceases to be analytic.

Let us illustrate this with another example, which generalises this one. Let Γ be any positively-oriented loop in the complex plane, let z_0 be any complex number which does *not* lie on Γ, and consider the following contour integral

$$\oint_\Gamma \frac{1}{z - z_0}\,dz\ .$$

We must distinguish two possibilities: z_0 is in the interior of Γ or in the exterior. In the latter case, the integral is zero because the integrand is analytic everywhere but at z_0, hence if z_0 lies outside Γ, Cauchy's Integral Theorem applies. On the other hand, if z_0 is in the interior of Γ we expect that we should obtain a nonzero answer—after all, if Γ were the circle $|z - z_0| = R > 0$, then the same calculation as in (2.26) yields a value of $2\pi\, i$ for the integral. In fact, as we will now show this is the answer we get for *any* positively-oriented loop containing z_0 in its interior.

In Figure 2.5 we have depicted the contour Γ and a circular contour C of radius r about the point z_0. We have also depicted two pairs of points (P_1, P_2) and (P_3, P_4): each pair having one point in each contour, as well as straight line segments joining the points in each pair.

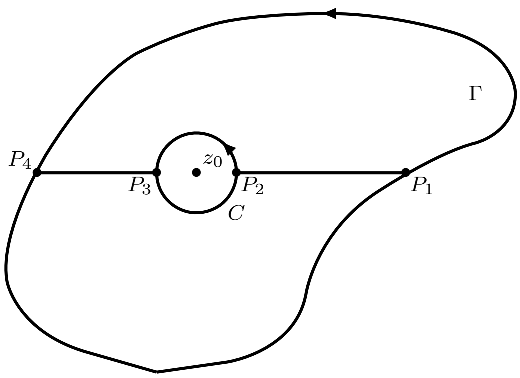


Figure 2.5: The contours Γ and C and some special points.

Now consider the following loop Γ_1 starting and ending at P_1, as illustrated in Figure 2.6. We start at P_1 and go to P_4 via the top half of Γ, call this, Γ_+; then we go to P_3 along the straight line segment joining them, call it $-\gamma_{34}$; then to P_2 via the upper half of C in the negative sense, call it $-C_+$; and then back to P_1 via the straight line segment joining P_2 and P_1, call it $-\gamma_{12}$. The interior of this contour is simply-connected and does not contain the point z_0. Therefore Cauchy's Integral Theorem says that

$$\begin{aligned}\oint_{\Gamma_1} \frac{1}{z-z_0}\, dz &= \left(\int_{\Gamma_+} + \int_{-\gamma_{34}} + \int_{-C_+} + \int_{-\gamma_{12}}\right) \frac{1}{z-z_0}\, dz \\ &= \left(\int_{\Gamma_+} - \int_{\gamma_{34}} - \int_{C_+} - \int_{\gamma_{12}}\right) \frac{1}{z-z_0}\, dz \\ &= 0 \ ,\end{aligned}$$

from where we deduce that

$$\int_{\Gamma_+} \frac{1}{z-z_0}\, dz = \left(\int_{\gamma_{34}} + \int_{C_+} + \int_{\gamma_{12}}\right) \frac{1}{z-z_0}\, dz \ .$$

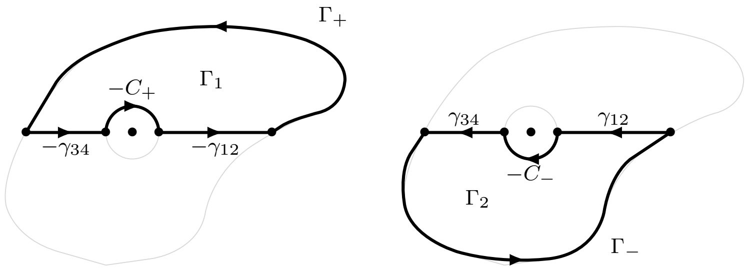


Figure 2.6: The contours Γ_1 and Γ_2.

Similarly consider the loop Γ_2 starting and ending at P_4. We start at P_4 and go to P_1 along the lower half of Γ, call it Γ_-; then we go to P_2 along γ_{12}; then to P_3 via the lower half of the circular contour in the negative sense $-C_-$; and then finally back to P_4 along γ_{34}. By the same argument as above, the interior of Γ_2 is simply-connected and z_0 lies in its exterior domain. Therefore by the Cauchy Integral Theorem,

$$\begin{aligned}\oint_{\Gamma_2} \frac{1}{z-z_0}\, dz &= \left(\int_{\Gamma_-} + \int_{\gamma_{34}} + \int_{-C_-} + \int_{-\gamma_{12}}\right) \frac{1}{z-z_0}\, dz \\ &= \left(\int_{\Gamma_-} + \int_{\gamma_{34}} - \int_{C_-} + \int_{-\gamma_{12}}\right) \frac{1}{z-z_0}\, dz \\ &= 0 \ ,\end{aligned}$$

from where we deduce that

$$\int_{\Gamma_-} \frac{1}{z-z_0}\, dz = \left(-\int_{\gamma_{34}} + \int_{C_+} - \int_{\gamma_{12}}\right) \frac{1}{z-z_0}\, dz \ .$$

Putting the two results together, we find that

$$
\begin{aligned}
\int_\Gamma \frac{1}{z-z_0}\,dz &= \int_{\Gamma_+} \frac{1}{z-z_0}\,dz + \int_{\Gamma_-} \frac{1}{z-z_0}\,dz \\
&= \int_{C_+} \frac{1}{z-z_0}\,dz + \int_{C_-} \frac{1}{z-z_0}\,dz \\
&= \int_{C} \frac{1}{z-z_0}\,dz \\
&= 2\pi\, i \ .
\end{aligned}
$$

In summary, we find that if Γ is any positively-oriented loop in the complex plane and z_0 a point *not* in Γ, then

$$
\int_\Gamma \frac{1}{z-z_0}\,dz = \begin{cases} 2\pi\, i & \text{for } z_0 \text{ in the interior of } \Gamma\text{; and} \\ 0 & \text{otherwise.} \end{cases} \tag{2.32}
$$

In the following section we will generalise this formula in a variety of ways.

2.2.5 Cauchy's Integral Formula

In this section we present several generalisations of the formula (2.32). Let $f(z)$ be analytic in a simply-connected domain D, and let Γ be a positively-oriented loop in D. Let z_0 be any point in the interior of Γ. Then the **Cauchy Integral Formula** reads

$$
\boxed{f(z_0) = \frac{1}{2\pi\, i} \oint_\Gamma \frac{f(z)}{z-z_0}\,dz \ .} \tag{2.33}
$$

This is a remarkable formula. It says that an analytic function in a simply-connected domain is determined by its behaviour on the boundary. In other words, if two analytic functions $f(z)$ and $g(z)$ agree on the boundary of a simply-connected domain they agree everywhere in the domain.

Cauchy's Integral Formula is a mathematical analogue of a notion that is very much in vogue in today's theoretical physics, namely 'holography'. You all know what the idea of an optical hologram is: it is a two-dimensional film which contains enough information to reconstruct (optically) a three-dimensional object. In theoretical physics, holography is exemplified in the celebrated formula of Beckenstein–Hawking for the entropy of a black hole. On the one hand, we know from Boltzmann's formula that the entropy of a statistical mechanical system is a measure of the density of states of the system. The black-hole entropy formula says that the entropy is a black hole is proportional to the area of the horizon. In simple terms, the horizon of the black hole is the surface within which light can no longer escape the gravitational attraction of the black hole. The entropy formula is holographic because it tells us that the degrees of freedom of a three-dimensional object like a black hole is determined from the properties of a two-dimensional system: the horizon, just like with the optical hologram. The "Holographic Principle" roughly states that any theory of quantum gravity, i.e., a theory which can explain the microscopic origin of the entropy of the black hole, must be able to explain the entropy formula and hence be holographic. The Cauchy Integral Formula is holographic in the sense that an analytic function in the plane (which is two-dimensional) is determined by its behaviour on contours (which are one-dimensional).

Notice that by equation (2.32), we have that

$$f(z_0) = \frac{1}{2\pi\, i} \oint_\Gamma \frac{f(z_0)}{z - z_0}\, dz \ ,$$

whence we will have proven the Cauchy Integral Formula if we can show that

$$\oint_\Gamma \frac{f(z) - f(z_0)}{z - z_0}\, dz = 0 \ .$$

As a first step in proving this result, let us use the Cauchy Integral Theorem to conclude that the above integral can be computed along a small circle C_r of radius r about z_0 without changing its value:

$$\oint_\Gamma \frac{f(z) - f(z_0)}{z - z_0}\, dz = \oint_{C_r} \frac{f(z) - f(z_0)}{z - z_0}\, dz \ .$$

Moreover since the radius of the circle does not matter, we are free to take the limit in which the radius goes to zero, so that:

$$\oint_\Gamma \frac{f(z) - f(z_0)}{z - z_0}\, dz = \lim_{r \to 0} \oint_{C_r} \frac{f(z) - f(z_0)}{z - z_0}\, dz \ .$$

Let us parametrise C_r by $z(t) = z_0 + r \exp(2\pi\, i\, t)$ for $t \in [0, 1]$. Then

$$\begin{aligned}\oint_{C_r} \frac{f(z) - f(z_0)}{z - z_0}\, dz &= \int_0^1 \frac{f(z) - f(z_0)}{re^{2\pi\, i\, t}}\, 2\pi\, i\, re^{2\pi\, i\, t}\, dt \\ &= 2\pi\, i \int_0^1 (f(z) - f(z_0))\, dt \ .\end{aligned}$$

Let us estimate the integral. Using (2.24) we find

$$\left| \int_0^1 (f(z) - f(z_0))\, dt \right| \leq \int_0^1 |f(z) - f(z_0)|\, dt \leq \max_{|z - z_0| = r} |f(z) - f(z_0)| \ .$$

Because f is continuous at z_0—that is, $f(z) \to f(z_0)$ as $z \to z_0$—the limit as $r \to 0$ of $|f(z) - f(z_0)|$ is zero, whence

$$\lim_{r \to 0} \oint_{C_r} \frac{f(z) - f(z_0)}{z - z_0}\, dz = 0 \ .$$

Formally, continuity of f at z_0 says that given any $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$. Since we are interested in the limit $r \to 0$, we can always take δ small enough so that $|f(z) - f(z_0)|$ is smaller than any ε. Therefore, $\lim_{r\to 0} |f(z) - f(z_0)| = 0$.

Now let us do something "deep." We will change notation in the Cauchy Integral Formula (2.33) and rewrite it as

$$f(z) = \frac{1}{2\pi\, i} \oint_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta \ .$$

All we have done is change the name of the variable of integration (Shakespeare's Theorem again!); but as a result we have obtained an integral representation of an analytic function which suggests a way to take its derivative simply by sneaking the derivative inside the integral:

$$\begin{aligned}
f'(z) &\stackrel{?}{=} \frac{1}{2\pi\, i} \oint_\Gamma \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \\
f''(z) &\stackrel{?}{=} \frac{2}{2\pi\, i} \oint_\Gamma \frac{f(\zeta)}{(\zeta - z)^3}\, d\zeta \\
&\vdots \\
f^{(n)}(z) &\stackrel{?}{=} \frac{n!}{2\pi\, i} \oint_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \ .
\end{aligned}$$

Of course such manipulations have to be justified, and we will see that indeed this is correct. Given that we are going to spend the effort in justifying this procedure, let us at least get something more out of it.

Integral representation for analytic functions

We already have at our disposal quite a number of analytic functions: rational functions, exponential and related functions, logarithm and complex powers. To some extent these are complex versions of functions with which we are familiar from real calculus. In this section we will learn of yet another way of constructing analytic functions. Functions constructed in this way usually do not have names, since anonymity is the fate which befalls most functions. But by the same token, this means that the method below is a powerful way to construct new analytic functions, or to determine that a function is analytic.

Let g be a function which is continuous in some contour Γ which need *not* be closed. Let z be any complex number not contained in Γ, and define the following function:

$$G(z) = \int_\Gamma \frac{g(\zeta)}{\zeta - z}\, d\zeta \ . \tag{2.34}$$

We claim that $G(z)$ is analytic except possible on Γ, and

$$G'(z) = \int_\Gamma \frac{g(\zeta)}{(\zeta - z)^2}\, d\zeta \ . \tag{2.35}$$

This generalises the above discussion in two important ways: g need *not* be analytic (just continuous) and the contour need not be closed.

To see if $G(z)$ is analytic we need to investigate whether the derivative $G'(z)$ exists and is well-defined. By definition,

$$\begin{aligned} G'(z) &= \lim_{\Delta z \to 0} \frac{G(z + \Delta z) - G(z)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \int_\Gamma \left(\frac{g(\zeta)}{\zeta - z - \Delta z} - \frac{g(\zeta)}{\zeta - z} \right) d\zeta \\ &= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \int_\Gamma \frac{g(\zeta)\Delta z}{(\zeta - z - \Delta z)(\zeta - z)}\, d\zeta \\ &= \lim_{\Delta z \to 0} \int_\Gamma \frac{g(\zeta)}{(\zeta - z - \Delta z)(\zeta - z)}\, d\zeta \ . \end{aligned}$$

Again, we would be done if we could simply take the limit inside the integral:

$$G'(z) \stackrel{?}{=} \int_\Gamma \lim_{\Delta z \to 0} \frac{g(\zeta)}{(\zeta - z - \Delta z)(\zeta - z)}\, d\zeta = \int_\Gamma \frac{g(\zeta)}{(\zeta - z)^2}\, d\zeta \ .$$

This can be justified (see below), so we are allowed to do so and recover what we were after. The formula (2.34) defines an **integral representation** for the analytic function $G(z)$.

Let us show that one can take the limit inside the integral, so that

$$\lim_{\Delta z \to 0} \int_\Gamma \frac{g(\zeta)}{(\zeta - z - \Delta z)(\zeta - z)}\, d\zeta = \int_\Gamma \frac{g(\zeta)}{(\zeta - z)^2}\, d\zeta \ .$$

Equivalently we would like to show that in the limit $\Delta z \to 0$, the difference

$$\int_\Gamma \quad \frac{g(\zeta)}{(\zeta - z - \Delta z)(\zeta - z)} - \frac{g(\zeta)}{(\zeta - z)^2} \quad d\zeta$$

vanishes. We can rewrite this difference as

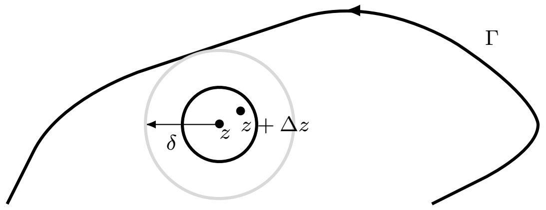
$$\Delta z \int_\Gamma \frac{g(\zeta)}{(\zeta - z - \Delta z)(\zeta - z)^2}\, d\zeta \ ,$$

which we would like to vanish as $\Delta z \to 0$. By equation (2.24), we have that

$$\begin{aligned} \int_\Gamma \frac{g(\zeta)}{(\zeta - z - \Delta z)(\zeta - z)^2}\, d\zeta \ &\leq \int_\Gamma \quad \frac{g(\zeta)}{(\zeta - z - \Delta z)(\zeta - z)^2} \quad |d\zeta| \\ &= \max_{\zeta \in \Gamma} \quad \frac{g(\zeta)}{(\zeta - z - \Delta z)(\zeta - z)^2} \quad \ell(\Gamma) \ , \end{aligned}$$

where we have used equation (2.27) for the length $\ell(\Gamma)$ of the contour.

Since $g(\zeta)$ is continuous on Γ, $|g(\zeta)|$ is bounded there: $|g(\zeta)| \leq M$, for some positive real M.



Because z is not on Γ, any point ζ on Γ is at least a certain distance δ from z: $|\zeta - z| \geq \delta > 0$, as shown in the above figure. Now by the triangle inequality (2.1),

$$|\zeta - z| = |\zeta - z - \Delta z + \Delta z| \leq |\zeta - z - \Delta z| + |\Delta z| \ ,$$

whence

$$|\zeta - z - \Delta z| \geq |\zeta - z| - |\Delta z| \ .$$

Since we are taking the limit $\Delta z \to 0$, we can choose $|\Delta z| \leq \frac{1}{2}\delta$ so that

$$|\zeta - z - \Delta z| \geq \delta - \tfrac{1}{2}\delta = \tfrac{1}{2}\delta \ .$$

Therefore putting it all together we find that

$$\int_\Gamma \frac{g(\zeta)}{(\zeta - z - \Delta z)(\zeta - z)^2}\, d\zeta \ \leq \frac{2M\ell(\Gamma)}{\delta^3} \ .$$

Therefore

$$\lim_{\Delta z \to 0} \ \Delta z \int_\Gamma \frac{g(\zeta)}{(\zeta - z - \Delta z)(\zeta - z)^2}\, d\zeta \ \leq \lim_{\Delta z \to 0} |\Delta z| \frac{2M\ell(\Gamma)}{\delta^3} = 0 \ .$$

This is as good a place as any to mention another way of writing the triangle inequality (2.1), which is sometimes more useful and which was used above:

$$|z + w| \geq |z| - |w| \ . \tag{2.36}$$

To obtain the second version of the triangle inequality from the first we simply make the following substitution: $z_1 + z_2 = z$, and $z_2 = -w$, so that $z_1 = z + w$. Then we find from the (2.1), that $|z| \leq |z + w| + |-w| = |z + w| + |w|$, which can be rewritten as (2.36).

The same argument shows that if we define

$$H(z) = \int_\Gamma \frac{g(\zeta)}{(\zeta - z)^n}\, d\zeta \ , \tag{2.37}$$

where n is a positive integer, then H is analytic and its derivative is given by

$$H'(z) = n \int_\Gamma \frac{g(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \ . \tag{2.38}$$

The generalised Cauchy Integral Formula

This has as an important consequence: if f is analytic in a neighbourhood of z_0, then *so are all its derivatives* $f^{(n)}$. To prove this simply notice that if f is analytic in a neighbourhood of z_0, there is some $\varepsilon > 0$ such that f is analytic in and on the circle C of radius ε centred at z_0; that is, the closed disk $|\zeta - z_0| \leq \varepsilon$. Therefore for any z in the interior of the circle—that is, such that $|z - z_0| < \varepsilon$—we have the Cauchy Integral representation

$$f(z) = \frac{1}{2\pi\, i} \oint_C \frac{f(\zeta)}{\zeta - z}\, d\zeta \ .$$

But this integral representation is of the form (2.34), whence its derivative is given by the analogue of equation (2.35):

$$f'(z) = \frac{1}{2\pi\, i} \oint_C \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \ .$$

But this is of the general form (2.37) (with $n = 2$), whence by the above results, $f'(z)$ is an analytic function and its derivative is given by the analogue of (2.38):

$$f''(z) = \frac{2}{2\pi\, i} \oint_C \frac{f(\zeta)}{(\zeta - z)^3}\, d\zeta \ ,$$

which again follows the pattern (2.37). Continuing in this fashion we deduce that f', f'', ... are analytic in the open ε-disk about z_0.

In summary, *an analytic function is infinitely differentiable*, its derivatives being given by the **generalised Cauchy Integral Formula**:

$$\boxed{f^{(n)}(z) = \frac{n!}{2\pi\, i} \oint_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \ .} \tag{2.39}$$

Notice that if we put $n = 0$ in this formula, define $0! = 1$ and understand the zeroth derivative $f^{(0)}$ as the function f itself, then this is precisely the Cauchy Integral Formula.

Infinite differentiability of harmonic functions.

The generalised Cauchy Integral Formula can also be turned around in order to compute contour integrals. Hence if f is analytic in and on a positively oriented loop Γ, and if z_0 is a point in the interior of Γ, then

$$\boxed{\oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz = \frac{2\pi\, i}{n!} f^{(n)}(z_0) \ .} \tag{2.40}$$

For example, let us compute the following contour integral

$$\oint_\Gamma \frac{e^{5z}}{z^3}\, dz \ ,$$

where Γ is the positively oriented unit circle $|z| = 1$. This integral is of the form (2.40) with $n = 2$, $f(z) = e^{5z}$, which is entire and hence, certainly analytic in and on the contour, and with $z_0 = 0$, which lies in the interior of the contour. Therefore by (2.40) we have

$$\oint_\Gamma \frac{e^{5z}}{z^3}\, dz = 2\pi\, i\, \frac{1}{2!}\, \frac{d^2}{dz^2}\left(e^{5z}\right)\bigg|_{z=0} = 2\pi\, i\, \frac{1}{2!}\, 25 = 25\pi\, i \ .$$

Let us consider a more complicated example. Let us compute the contour integral

$$\int_\Gamma \frac{2z+1}{z(z-1)^2}\, dz \ ,$$

where Γ is the contour depicted in Figure 2.7. Two things prevent us from applying the generalised Cauchy Integral Formula: the contour is not a loop—indeed it is not simple—and the integrand is not of the form $g(z)/(z-z_0)^n$ where $g(z)$ is analytic inside the contour. This last problem could be solved by rewriting the integrand using partial fractions:

$$\frac{2z+1}{z(z-1)^2} = \frac{3}{(z-1)^2} - \frac{1}{z-1} + \frac{1}{z} \ . \tag{2.41}$$

However we are still faced with a contour which is not simple.

Figure 2.7: The contour Γ and an equivalent pair of contours $\{\Gamma_0, \Gamma_1\}$.

This problem can be circumvented by noticing that the smooth contour Γ can be written as a piecewise smooth contour with two smooth components: both starting and ending at the point of self-intersection of Γ. The first such contour is the left lobe of Γ, which is a negatively oriented loop about $z = 0$, and the second is the right lobe of Γ, which is a positively oriented loop about $z = 1$. Because the integrand is analytic everywhere but at $z = 0$ and $z = 1$, the Cauchy Integral Theorem tells us that we get the same result by integrating around the circular contours Γ_0 and Γ_1 in Figure 2.7. In other words,

$$\int_\Gamma \frac{2z+1}{z(z-1)^2}\,dz = \oint_{\Gamma_0} \frac{2z+1}{z(z-1)^2}\,dz + \oint_{\Gamma_1} \frac{2z+1}{z(z-1)^2}\,dz \ .$$

We can now evaluate this in either of two ways. Using the partial fraction decomposition (2.41) of the integrand, one finds

$$\oint_{\Gamma_0} \frac{2z+1}{z(z-1)^2}\,dz = \oint_{\Gamma_0} \frac{1}{z}\,dz = -\oint_{-\Gamma_0} \frac{1}{z}\,dz = -2\pi\, i \ ,$$
$$\oint_{\Gamma_1} \frac{2z+1}{z(z-1)^2}\,dz = \oint_{\Gamma_1} \frac{3}{(z-1)^2}\,dz - \oint_{\Gamma_1} \frac{1}{z-1}\,dz = 0 - 2\pi\, i = -2\pi\, i \ ;$$

whence

$$\int_\Gamma \frac{2z+1}{z(z-1)^2}\,dz = -4\pi\, i \ .$$

Alternatively we notice that

$$\oint_{\Gamma_0} \frac{2z+1}{z(z-1)^2}\,dz = \oint_{\Gamma_0} \frac{\frac{2z+1}{(z-1)^2}}{z}\,dz = -2\pi\, i \ ,$$

where we have used the fact that $\frac{2z+1}{(z-1)^2}$ is analytic in and on Γ_0 and the Cauchy Integral Formula after taking into account that Γ_0 is negatively oriented. Similarly, one has

$$\oint_{\Gamma_1} \frac{2z+1}{z(z-1)^2}\,dz = \oint_{\Gamma_1} \frac{\frac{2z+1}{z}}{(z-1)^2}\,dz = 2\pi\, i\ \frac{d}{dz}\left(\frac{2z+1}{z}\right)\bigg|_{z=1} = -2\pi\, i \ ,$$

where we have used that $\frac{2z+1}{z}$ is analytic in and on Γ_1, and the generalised Cauchy Integral formula (with $n = 1$). Therefore again

$$\int_\Gamma \frac{2z+1}{z(z-1)^2}\,dz = -4\pi\, i \ .$$

Morera's Theorem

Finally we discuss a converse of the Cauchy Integral Theorem, known as Morera's Theorem. Suppose that f is continuous in a domain D and has an antiderivative F in D. This means that F is analytic, and by what we have just shown, so is $f(z) = F'(z)$. Therefore we have just shown that if $f(z)$ is continuous with an antiderivative, then f is analytic. Now from the path independence lemma, f has an antiderivative if and only if all its loop integrals in D vanish:

$$\oint_\Gamma f(z)dz = 0 \ .$$

Therefore we arrive at **Morera's Theorem** which states that: if $f(z)$ is continuous in D and all the loop integrals of $f(z)$ in D vanish, then f is analytic. This theorem will be of use in Section 2.3.

2.2.6 Liouville's Theorem and its applications

The generalised Cauchy Integral Formula is one of the cornerstones of complex analysis, as it has a number of very useful corollaries. An immediate application of the generalised Cauchy Integral Formula is the so-called *Cauchy estimates* for the derivatives of an analytic function. These estimates will play an important role in the remainder of this section.

Suppose that $f(z)$ is analytic in some domain D containing a circle C of radius R centred about z_0. Suppose moreover that $|f(z)| \leq M$ for all z on the circle C. We can then use the generalised Cauchy Integral Formula (2.39) to obtain a bound for the derivatives of f at z_0:

$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi\, i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz \right| \leq \frac{n!}{2\pi} \oint_C \frac{|f(z)|}{|z-z_0|^{n+1}}\, |dz| \ ,$$

where we have used (2.28) to arrive at the inequality. On the circle, $|z-z_0| = R$ and $|f(z)| \leq M$, whence

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \frac{M}{R^{n+1}} \oint_C |dz| \ ,$$

which, using that the length of the contour is $2\pi\, R$, can be rewritten neatly as

$$\boxed{|f^{(n)}(z_0)| \leq \frac{n!\, M}{R^n}} \ . \tag{2.42}$$

This inequality is known as the **Cauchy estimate**.

As an immediate corollary of this estimate suppose that f is analytic in whole complex plane (i.e., that f is an entire function) and that it is bounded, so that $|f(z)| \leq M$ for all z. Then from the Cauchy estimate, at *any* point z_0 in the complex plane, its derivative is bounded by $|f'(z_0)| \leq M/R$. But because the function is entire, we can take R as large as we wish. Now given any number $\varepsilon > 0$, however small, there is always an R large enough for which $M/R < \varepsilon$, so that $|f'(z_0)| < \varepsilon$. Therefore $|f'(z_0)| = 0$, whence

$f'(z_0) = 0$. Since this is true for all z_0 in the complex plane, we have proven **Liouville's theorem**:

a bounded entire function is constant.

This does not violate our experience since the only entire functions we have met are polynomials and the exponential and functions we can make out of them by multiplication, linear combinations and compositions, and these functions are all clearly not bounded.

Indeed, suppose that $P(z)$ is a polynomial of order N; that is,

$$P(z) = z^N + a_{N-1}z^{N-1} + \cdots + a_1 z + a_0 \ .$$

Then intuitively, for large z we expect that $P(z)$ should go as z^N, since the largest power dominates the other ones. The precise statement, to be proven below, is that there exists $R > 0$ large enough such that for $|z| \geq R$, $|P(z)| \geq c|z|^N$, where $0 < c < 1$ depends on R in such a way that as R tends to ∞, c tends to 1.

Let $P(z)$ be the above polynomial and let $A \geq 1$ denote the largest of the moduli of coefficients of the polynomial: $A = \max\{|a_0|, |a_1|, \ldots, |a_{N-1}|, 1\}$. Then let us rewrite the polynomial as $P(z) = z^N \;\; 1 + a_{N-1}/z + \cdots a_0/z^N$. Now by the triangle inequality (2.36),

$$1 + \frac{a_{N-1}}{z} + \cdots + \frac{a_1}{z^{N-1}} + \frac{a_0}{z^N} \;\geq 1 - \;\; \frac{a_{N-1}}{z} + \cdots + \frac{a_1}{z^{N-1}} + \frac{a_0}{z^N} \quad .$$

Using the triangle inequality again,

$$\begin{aligned} \frac{a_{N-1}}{z} + \cdots + \frac{a_1}{z^{N-1}} + \frac{a_0}{z^N} \;&\leq\; \frac{a_{N-1}}{z} \;+ \cdots +\; \frac{a_1}{z^{N-1}} \;+\; \frac{a_0}{z^N} \\ &\leq \frac{A}{|z|} + \cdots + \frac{A}{|z|^{N-1}} + \frac{A}{|z|^N} \ . \end{aligned}$$

Now take $|z| \geq 1$ so that $|z|^N \geq |z|^{N-1} \geq \cdots \geq |z|$. Then,

$$\frac{a_{N-1}}{z} + \cdots + \frac{a_1}{z^{N-1}} + \frac{a_0}{z^N} \;\leq \frac{NA}{|z|} \ .$$

Therefore,

$$1 + \frac{a_{N-1}}{z} + \cdots + \frac{a_1}{z^{N-1}} + \frac{a_0}{z^N} \;\geq 1 - \frac{NA}{|z|} \ .$$

Hence if we take z such that $|z| \geq R \geq NA \geq 1$, then

$$1 + \frac{a_{N-1}}{z} + \cdots + \frac{a_1}{z^{N-1}} + \frac{a_0}{z^N} \;\geq 1 - \frac{NA}{R} = \frac{R - NA}{R} \ .$$

Finally then,

$$|P(z)| = |z|^N \;\; 1 + \frac{a_{N-1}}{z} + \cdots + \frac{a_1}{z^{N-1}} + \frac{a_0}{z^N} \;\geq \frac{R - NA}{R}|z|^N \ .$$

Hence $c = (R - NA)/R < 1$ and as $R \to \infty$, $c \to 1$.

We are now able to prove the **Fundamental Theorem of Algebra** which states that

every nonconstant polynomial has at least one zero.

Indeed, let $P(z)$ be a polynomial and suppose that it does not have any zeros. Then $1/P(z)$ is an entire function. If we manage to prove that this function is bounded, then we can use Liouville's theorem and conclude that $1/P(z)$, and hence $P(z)$, would have to be constant. So let us try to prove that it is bounded. Without loss of generality we can assume that the polynomial has the form $P(z) = z^N + a_{N-1}z^{N-1} + \cdots + a_1 z + a_0$ for some N. Let R be such that $|z| \geq R$, $|P(z)| \geq c|z|^N$, where $0 < c < 1$. Then, for $|z| \geq R$,

$$\left|\frac{1}{P(z)}\right| = \frac{1}{|P(z)|} \leq \frac{1}{c|z|^N} \leq \frac{1}{cR^N} \; .$$

While for $|z| \leq R$, then the function $1/P(z)$, being continuous, is bounded in this disk by some $M = \max_{|z|\leq R} 1/|P(z)|$. Therefore $1/|P(z)|$ is bounded above for all z by the largest of M and $1/(cR^N)$. Hence $1/P(z)$ is bounded.

I have always found this proof of the Fundamental Theorem of Algebra quite remarkable. It is compelling evidence in favour of the vision of mathematics as a coherent whole, that a purely algebraic statement like the Fundamental Theorem of Algebra can be proven in a relatively elementary fashion using complex analysis. I hope that as physicists we can be forgiven the vanity of thinking that this unity of mathematics stems from it being the language of nature.

2.3 Series expansions for analytic functions

This section ushers in the second half of this part of the course. The purpose of this section is to learn about the series representations for analytic functions. We will see that every function analytic in a disk can be approximated by polynomials: the partial sums of its Taylor series. Similarly every function analytic in a punctured disk can be described by a Laurent series, a generalisation of the notion of a power series, where we also allow for negative powers. This will allow us to discuss the different types of singularities that an analytic function can have. This section is organised as follows: we start with a study of sequences and series of complex numbers and of complex functions and of different notions of convergence and methods of establishing convergence. We will then show that a function analytic in the neighbourhood of a point can be approximated there by a power series: its Taylor series. We will then discuss power series and prove that every power series converges to an analytic function in its domain of convergence, and in fact is the Taylor series of that function. Therefore the power series representation of an analytic function is unique. We then introduce Laurent series: which allows us to represent analytic functions around an isolated singularity. We also prove that they are unique in a sense. We end the section with a discussion of the different isolated singularities which an analytic function can have.

2.3.1 Sequences and Series

In this section we discuss sequences and series and the rudiments of the theory of convergence. This is necessary groundwork to be able to discuss the Taylor and Laurent series representations for analytic functions.

Sequences

By a **sequence** we mean an infinite set $\{z_0, z_1, z_2, z_3, \dots\}$ of complex numbers. It is often denoted $\{z_n\}$ where the index is understood to run over the non-negative integers. Intuitively, a sequence $\{z_n\}$ converges to a complex number z if as n increases, z_n remains ever closer to z. A precise definition is the following. A sequence $\{z_n\}$ is said to **converge** to z (written $z_n \to z$ or $\lim_{n\to\infty} z_n = z$) if given any $\varepsilon > 0$, there exists an integer N, which may depend on ε, such that for all $n \geq N$, $|z_n - z| < \varepsilon$. In other words, the "tail" of the sequence remains arbitrary close to z provided we go sufficiently far into it. A sequence which converges to some point is said to be **convergent**. Convergence is clearly a property only of the tail of the sequence, in the sense that two sequences which differ only in the first N terms (any finite N) but are identical afterwards will have the same convergence properties.

For example, the sequence $\{z_n = 1/n\}$ clearly converges to 0: $|z_n| = 1/n$ and we can make this as small as we like by taking n as large as needed.

A sequence $\{z_n\}$ is said to satisfy the **Cauchy criterion** (or be a **Cauchy sequence**) if it satisfies the following property: given any $\varepsilon > 0$ there exists N (again, depending on ε) such that $|z_n - z_m| < \varepsilon$ for all $n, m \geq N$. This criterion simply requires that the elements in the sequence remain ever closer to each other, not that they should converge to any point. Clearly, if a sequences converges it is Cauchy: simply notice that adding and subtracting z,

$$|z_n - z_m| = |(z_n - z) - (z_m - z)| \leq |z_n - z| + |z_m - z|$$

by the triangle inequality (2.1). Hence if we want z_n and z_m to remain within ε of each other for n, m larger than some N, we need just choose N such that $|z_n - z| < \varepsilon/2$ for all $n \geq N$.

What is a relatively deep result, is that every Cauchy sequence is convergent. This is essentially the fact that the complex numbers are **complete**. To prove this requires a more careful axiomatisation of the real number system than we have time for.

Series

By a **series** we mean a formal sum

$$c_0 + c_1 + c_2 + \cdots + c_j + \cdots$$

of complex numbers, c_j, called the **coefficients**. We say *formal* since just because we can write something down does not mean it makes any sense: it does not make much sense to add an infinite number of terms. What does make sense is the following: define the n**-th partial sum**

$$S_n \equiv \sum_{j=0}^{n} c_j = c_0 + c_1 + \cdots + c_{n-1} + c_n \ .$$

This defines a sequence $\{S_n\}$. Then we can analyse the limit as $n \to \infty$ of this sequence. If one exists, say $S_n \to S$, then we say that the series **converges** to or **sums** to S, and we write

$$S = \sum_{j=0}^{\infty} c_j \ .$$

Otherwise we say that the series is **divergent**. Applying the Cauchy criterion to the sequence of partial sums, we see that a necessary condition for the convergence of a series is that the sequence of coefficients converge to 0. Indeed, if $\{S_n\}$ is convergent, it is Cauchy, whence given any $\varepsilon > 0$, there exists N such that for all $n, m \geq N$, $|S_n - S_m| < \varepsilon$. Taking $m = n - 1$, we see that

$$\left| \sum_{j=0}^{n} c_j - \sum_{j=0}^{n-1} c_j \right| = |c_n| < \varepsilon \ ,$$

for every $n \geq N$. Therefore the sequence $\{c_j\}$ converges to 0. We can summarise this as follows

$$\boxed{\text{If } \sum_{j=0}^{\infty} c_j \text{ converges, then } \lim_{j \to \infty} c_j = 0 \ .}$$

This is a necessary criterion for the convergence of a series, so it can be used to conclude that a series is divergent, but not to conclude that it is convergent. For example, consider the series

$$\sum_{j=0}^{\infty} \frac{j}{2j+1} \ . \tag{2.43}$$

It is clearly divergent because $j/(2j+1) \to \frac{1}{2}$. On the other hand consider the series (we start at $j = 1$ for obvious reasons)

$$\sum_{j=1}^{\infty} \frac{1}{j} \ . \tag{2.44}$$

Now the coefficients do converge to zero, but this series is actually divergent. One way to see this is to notice that for every $n \geq 1$,

$$\sum_{j=1}^{n} \frac{1}{j} = \sum_{j=1}^{n} \int_{j}^{j+1} \frac{dx}{j} > \sum_{j=1}^{n} \int_{j}^{j+1} \frac{dx}{x} = \int_{1}^{n+1} \frac{dx}{x} = \log(n+1) \ ,$$

and $\lim_{n\to\infty} \log(n+1) = \infty$. On the other hand, the series

$$\sum_{j=1}^{\infty} \frac{1}{j^2}$$

does converge. One can argue in a similar style. Notice that for $j \geq 2$,

$$\frac{1}{j^2} = \int_{j-1}^{j} \frac{dx}{j^2} < \int_{j-1}^{j} \frac{dx}{x^2} = \frac{1}{j(j-1)} \ .$$

Hence, for all $n \geq 2$,

$$\sum_{j=1}^{n} \frac{1}{j^2} = 1 + \sum_{j=2}^{n} \frac{1}{j^2} < 1 + \sum_{j=2}^{n} \int_{j-1}^{j} \frac{dx}{x^2} = 1 + \int_{1}^{n} \frac{dx}{x^2} = 2 - \frac{1}{n} \ ,$$

so that in the limit,

$$\sum_{j=1}^{\infty} \frac{1}{j^2} < 2 \ .$$

Indeed, we will be able to compute this sum very easily using contour integration and it will turn out that $\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6} \simeq 1.6449341$. Similarly, one can show in the same way that the series

$$\sum_{j=1}^{\infty} \frac{1}{j^p}$$

converges for any $p > 1$. In fact, p can be any real number.

Establishing convergence

There are two useful tests for establishing the convergence of a series. The first one is known as the **Comparison Test**: Suppose that $\sum_{j=0}^{\infty} M_j$ is a convergent series whose coefficients are non-negative real numbers: $M_j \geq 0$. Let $\sum_{j=0}^{\infty} c_j$ be such that $|c_j| \leq M_j$ for all sufficiently large j. Then $\sum_{j=0}^{\infty} c_j$ also converges.

Prove the Comparison Test.

Of course, in order to apply this test we need to have some examples of convergent series to compare with. We have already seen the series $\sum_{j=1}^{\infty} 1/j^p$, for $p > 1$, but perhaps the most useful series we will come across is the **geometric series** $\sum_{j=0}^{\infty} c^j$, where c is some complex number. To investigate the convergence of this series, simply notice that $|c^j| = |c|^j$ and hence the coefficient sequence $\{c^j\}$ converges to 0 if and only if $|c| < 1$. Thus we let $|c| < 1$ from now on. We proceed as follows:

$$(1-c)S_n = (1-c)(1+c+\cdots+c^n) = 1 - c^{n+1} ,$$

whence

$$S_n = \frac{1-c^{n+1}}{1-c} \qquad \text{or} \qquad S_n - \frac{1}{1-c} = -\frac{c^{n+1}}{1-c} .$$

Therefore taking the modulus, we see that

$$\left| S_n - \frac{1}{1-c} \right| = \frac{|c|^{n+1}}{|1-c|} ,$$

which converges to 0 as $n \to \infty$ since $|c| < 1$. Therefore

$$\boxed{\sum_{j=0}^{\infty} c^j = \frac{1}{1-c} \qquad \text{if } |c| < 1.} \tag{2.45}$$

As an example, let us consider the following series

$$\sum_{j=0}^{\infty} \frac{3+2i}{(j+1)^j} . \tag{2.46}$$

Its coefficient sequence converges to zero. Notice also that

$$\left| \frac{3+2i}{(j+1)^j} \right| = \frac{\sqrt{13}}{(j+1)^j} < \frac{4}{(j+1)^j} .$$

Hence for $j \geq 3$,

$$\left|\frac{3+2i}{(j+1)^j}\right| < \frac{1}{2^j} \ .$$

But since $\frac{1}{2} < 1$, the geometric series

$$\sum_{j=0}^{\infty} \frac{1}{2^j} = 2$$

converges. Hence by the comparison test, the original series (2.46) converges as well.

A further convergence criterion is the **Ratio Test**: Let $\sum_{j=0}^{\infty} c_j$ be such that the limit

$$L \equiv \lim_{j\to\infty} \left|\frac{c_{j+1}}{c_j}\right|$$

exists. Then if $L < 1$ the series converges, and if $L > 1$ the series diverges. (Alas, if $L = 1$ we cannot conclude anything.)

Prove the Ratio Test.

The Ratio Test does not contradict our experience so far: for the geometric series $L = |c|$, and we certainly needed $|c| < 1$ for convergence. Moreover in this case $L \geq 1$ implies divergence. Similarly, the series (2.44) has $L = 1$, so that the test tells us nothing. The same goes for the series (2.43). Notice that there are series for which the Ratio Test cannot even be applied, since the limit L may not exist.

Sequences and series of functions: uniform convergence

Our primary interest in series and sequences being the construction of analytic functions, let us now turn our attention to the important case of sequences and series of *functions*. Consider a sequence $\{f_n\}$ whose elements are functions $f_n(z)$ defined on some domain in the complex plane. For a fixed point z we can study the sequence of complex numbers $\{f_n(z)\}$ and analyse its convergence. If it does converge, let us call the limit $f(z)$; that is, $f_n(z) \to f(z)$. This procedure defines a function f for those z such that the sequence $\{f_n(z)\}$ converges. If this is the case we say that the sequence $\{f_n\}$ converges **pointwise** to f. Now suppose that each f_n is continuous (or analytic) will f be continuous (or analytic)? It turns out that pointwise convergence is too weak in order to guarantee that the limit function shares some of these properties of the f_n.

For instance, it is easy to cook up a pointwise limit of analytic functions which is not even continuous. Consider the functions $f_n(z) = \exp(-nz^2)$. Clearly these functions are analytic for each n. Let us now consider the functions restricted to the real axis: $z = x$, and consider the limit function $f(x)$. For all n, $f_n(0) = 1$, whence in the limit $f(0) = 1$. On the other hand, let $x \neq 0$. Then given any $\varepsilon > 0$, however small, there will be N such that $\exp(-nx^2) < \varepsilon$ for $n \geq N$. Hence

$$f(x) = \begin{cases} 1 & \text{for } x = 0; \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the limit function has a discontinuity at the origin. Continuity would require $f(0) = 0$. To understand what is going on here, notice that to make $f_n(x) < \varepsilon$ we require that

$$e^{-nx^2} < \varepsilon \qquad \Longrightarrow \qquad n > \frac{\log(1/\varepsilon)}{x^2} ,$$

as can be easily seen by taking the logarithm of both sides of the first inequality. Hence as x becomes smaller, the value of n has to be larger and larger to the extent that in the limit as $x \to 0$, there is no n for which this is the case.

The above "post mortem" analysis prompts the following definition. A sequence of functions $\{f_n\}$ is said to **converge to a function f uniformly in a subset U** if given any $\varepsilon > 0$ there exists an N such that for all $n \geq N$,

$$|f_n(z) - f(z)| < \varepsilon \qquad \text{for } \textit{all } z \in U.$$

In other words, N can depend on ε but *not* on z.

Similarly one says that a series of functions

$$\sum_{j=0}^{\infty} f_j(z) ,$$

converges pointwise or uniformly if the sequence of partial sums does.

To show that this definition takes care of the kind of pathologies encountered above, let us first of all prove that the uniform limit of continuous functions is again continuous. Indeed, let $\{f_n(z)\}$ be a sequence of functions which are continuous at z_0, and let it converge to a function $f(z)$ uniformly in a neighbourhood of z_0. We claim that $f(z)$ is continuous at z_0. This means that given any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$

whenever $|z - z_0| < \delta$. To prove this we will employ a device known as the $\varepsilon/3$ *trick*. Let us rewrite $|f(z) - f(z_0)|$ as follows

$$\begin{aligned} |f(z) - f(z_0)| &= |f(z) - f_n(z) + f_n(z) - f_n(z_0) + f_n(z_0) - f(z_0)| \\ &\leq |f(z) - f_n(z)| + |f_n(z) - f_n(z_0)| + |f_n(z_0) - f(z_0)| , \end{aligned}$$

by the triangle inequality. Now, because $f_n(z) \to f(z)$ uniformly, we can choose n above so large that $|f(z) - f_n(z)| < \varepsilon/3$ for all z, so in particular for $z = z_0$. Similarly, because $f_n(z)$ is continuous at z_0, there exists δ such that $|f_n(z) - f_n(z_0)| < \varepsilon/3$ whenever $|z - z_0| < \delta$. Therefore,

$$|f(z) - f(z_0)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon .$$

In other words, we have shown that

the uniform limit of continuous functions is continuous.

Similarly we will see that the uniform limit of analytic functions is analytic. Uniform convergence is sufficiently strong to allow us to manipulate sequences of functions naively and yet sufficiently weak to allow for many examples. For instance we will see that if a series converges uniformly to a function, then the series can be differentiated and integrated termwise and it will converge to the derivative or integral of the limit function.

In practice, the way one checks that a sequence $\{f_n\}$ of functions converges uniformly in U to a function f is to write

$$f_n(z) = f(z) + R_n(z)$$

and then to see whether the remainder $R_n(z)$ can be made arbitrarily small for some large enough n independently of z in U. Let us see this for the geometric series:

$$\sum_{j=0}^{\infty} z^j . \tag{2.47}$$

The partial sums are the functions

$$f_n(z) = \sum_{j=0}^{n} z^j = 1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} .$$

We claim that this geometric series converges uniformly to the function $1/(1-z)$ on every closed disk $|z| \leq R$ with $R < 1$. Indeed, we have the following estimate for the remainder:

$$\left| f_n(z) - \frac{1}{1-z} \right| = \frac{|z|^{n+1}}{|1-z|} \leq \frac{R^{n+1}}{|1-z|} .$$

Now, using the triangle inequality (2.36),

$$|z-1| = |1-z| \geq 1-|z| \qquad \text{whence} \qquad \frac{1}{|1-z|} \leq \frac{1}{1-|z|} \leq \frac{1}{1-R} \; .$$

In other words,

$$\left| f_n(z) - \frac{1}{1-z} \right| = \frac{|z|^{n+1}}{|1-z|} \leq \frac{R^{n+1}}{1-R} \; .$$

This bound is independent of z and can be made as small as desired since $R < 1$, whence the convergence is uniform.

Another way to check for uniform convergence is the **Weierstrass M-test**, which generalises the Comparison Test. Suppose that $\sum_{j=0}^{\infty} M_j$ is a convergent series with real non-negative terms $M_j \geq 0$. Suppose further that for all z in some subset U of the complex plane and for all sufficiently large j, $|f_j(z)| \leq M_j$. Then the series $\sum_{j=0}^{\infty} f_j(z)$ converges uniformly in U. (Notice that the Comparison Test is obtained as a special case, when $f_j(z)$ are constant functions.)

Proof of the Weierstrass M-test.

Using the Weierstrass M-test we can prove the uniform convergence of the geometric series on any closed disk $|z| \leq R < 1$. Indeed, notice that $|z^j| = |z|^j \leq R^j$ and that since $R < 1$, the geometric series $\sum_{j=0}^{\infty} R^j$ converges.

2.3.2 Taylor series

In this section we will prove the remarkable result that a function analytic in the neighbourhood of a point can be approximated by a sequence of polynomials, namely by its Taylor series. Moreover we will see that convergence is uniform inside the largest open disk over which the function is analytic.

The Taylor series of a function is the result of successive approximations of the function by polynomials. Suppose that $f(z)$ is analytic in a neighbourhood of z_0. Then as we saw in Section 2.2.5 f is infinitely differentiable around z_0. Let us then write down a polynomial function f_n such that it agrees with f at z_0 up to an including its n-th derivative. In other words, $f_n^{(j)}(z_0) = f^{(j)}(z_0)$ for $j = 0, 1, \ldots, n$. The polynomial function of least order which satisfies this condition is

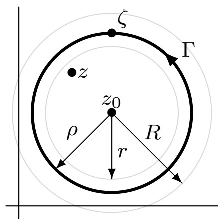
$$f_n(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2}(z-z_0)^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \; .$$

The sequence $\{f_n\}$, if it converges, does so to the **Taylor series around** z_0 of the function f:

$$\sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!} (z - z_0)^j \ . \tag{2.48}$$

(If $z_0 = 0$ this series is also called the **Maclaurin series** of f.)

We will now prove the following important result: Let $f(z)$ be analytic in the disk $|z - z_0| < R$ centred at z_0. Then the Taylor series for f around z_0 converges to $f(z)$ for all z in the disk and moreover the convergence is uniform on any closed subdisk $|z - z_0| \leq r < R$.



The proof uses the generalised Cauchy Integral Formula with an appropriate choice of contour, as shown in the diagram. Let Γ denote the positively oriented circle centred at z_0 with radius ρ where $r < \rho < R$. By hypothesis, f is analytic in and on the contour Γ, whence for any z satisfying $|z - z_0| \leq r$, we have the Cauchy Integral Formula:

$$f(z) = \frac{1}{2\pi\, i} \oint_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta \ .$$

Now we rewrite the integrand:

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \ ,$$

and use the geometric series to write

$$\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \sum_{j=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^j \ ,$$

which is valid because $|z - z_0| = r < \rho = |\zeta - z_0|$. Putting it all together, we have

$$\frac{1}{\zeta - z} = \sum_{j=0}^{\infty} \frac{(z - z_0)^j}{(\zeta - z_0)^{j+1}} \ .$$

Inserting it into the Cauchy Integral Formula,

$$f(z) = \frac{1}{2\pi\, i} \oint_\Gamma \sum_{j=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} (z - z_0)^j\, d\zeta \ .$$

Now we would be tempted to interchange the order of the integral and the summation and arrive at

$$
\begin{aligned}
f(z) &\stackrel{?}{=} \sum_{j=0}^{\infty} \left[\frac{1}{2\pi\, i} \oint_\Gamma \frac{f(\zeta)}{(\zeta - z_0)^{j+1}}\, d\zeta\right] (z-z_0)^j \\
&= \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}\, (z-z_0)^j \ ,
\end{aligned}
$$

where we have used the generalised Cauchy Integral Formula. This manipulation turns out to be allowed, but doing it this way we do not see the uniform convergence. This is done with more care below.

Let us prove the Taylor series theorem carefully. It is not hard, but it takes a bit more bookkeeping. Rather than using the geometric series in its entirety, let us use its n-th partial sum:

$$
\frac{1}{1-\frac{z-z_0}{\zeta-z_0}} = \sum_{j=0}^{n} {\frac{z-z_0}{\zeta-z_0}}^{j} + \frac{{\frac{z-z_0}{\zeta-z_0}}^{n+1}}{1-\frac{z-z_0}{\zeta-z_0}} \ ,
$$

whence

$$
\frac{1}{\zeta - z} = \sum_{j=0}^{n} \frac{(z-z_0)^j}{(\zeta-z_0)^{j+1}} + \frac{{\frac{z-z_0}{\zeta-z_0}}^{n+1}}{\zeta - z} \ .
$$

Into the Cauchy Integral Formula, we have

$$
f(z) = \frac{1}{2\pi\, i} \oint_\Gamma f(\zeta) \left[\sum_{j=0}^{n} \frac{(z-z_0)^j}{(\zeta-z_0)^{j+1}} + \frac{{\frac{z-z_0}{\zeta-z_0}}^{n+1}}{\zeta - z}\right] d\zeta \ .
$$

Now this is only a finite sum, so by linearity we can integrate it term by term. Using the generalised Cauchy Integral Formula we have

$$
f(z) = \sum_{j=0}^{n} \frac{f^{(j)}(z_0)}{j!}\, (z-z_0)^j + R_n(z) \ ,
$$

where

$$
R_n(z) \equiv \frac{1}{2\pi\, i} \oint_\Gamma \frac{f(\zeta)}{\zeta - z}\ {\frac{z-z_0}{\zeta-z_0}}^{n+1} d\zeta \ .
$$

In other words,

$$
f(z) - \sum_{j=0}^{n} \frac{f^{(j)}(z_0)}{j!}\, (z-z_0)^j = R_n(z) \ ,
$$

whence in order to prove uniform convergence of the Taylor series, we only have to show that we can make $|R_n(z)|$ as small as desired for all z by simply taking n sufficiently large. Let us estimate $|R_n(z)|$. Using (2.28)

$$
|R_n(z)| \leq \frac{1}{2\pi} \oint_\Gamma \frac{|f(\zeta)|}{|\zeta - z|}\ {\frac{z-z_0}{\zeta-z_0}}^{n+1} |d\zeta| \ .
$$

We now use that $|z-z_0| \leq r$, $|\zeta - z_0| = \rho$, $|f(\zeta)| \leq M$ for some M, $\ell(\Gamma) = 2\pi\,\rho$, and the triangle inequality (2.36),

$$
|\zeta - z| = |(\zeta - z_0) - (z - z_0)| \geq |\zeta - z_0| - |z - z_0| \geq \rho - r \ ,
$$

whence

$$\frac{1}{|\zeta - z|} \leq \frac{1}{\rho - r} \; .$$

Therefore,

$$|R_n(z)| \leq \frac{\rho M}{\rho - r} \left(\frac{r}{\rho}\right)^{n+1} \; .$$

This is what we wanted, because the right-hand side does not depend on z and can be made as small as desired by taking n large, since $r/\rho < 1$. This proves uniform convergence of the Taylor series.

Notice that this result implies that *the Taylor series will converge to $f(z)$ everywhere inside the largest open disk, centred at z_0, over which f is analytic.*

As an example, let us compute the Taylor series for the functions $\operatorname{Log} z$ around $z_0 = 1$ and also $1/(1-z)$ around $z_0 = 0$. The derivatives of the principal branch of the logarithm are:

$$\frac{d^j \operatorname{Log} z}{dz^j} = (-1)^{j+1}(j-1)! \frac{1}{z^j} \; .$$

Evaluating at $z = 1$ and constructing the Taylor series, we have

$$\operatorname{Log} z = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} (z-1)^j \; .$$

This series is valid for $|z-1| < 1$ which is the largest open disk centred at $z = 1$ over which $\operatorname{Log} z$ is analytic, as seen in Figure 2.8. Similarly,

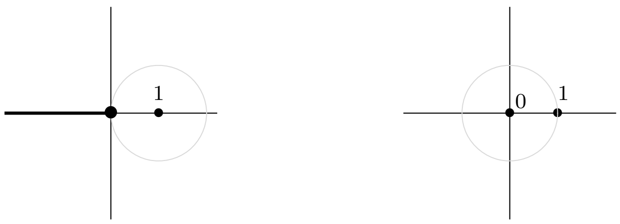


Figure 2.8: Analyticity disks for the Taylor series of $\operatorname{Log} z$ and $1/(1-z)$.

$$\frac{d^j}{dz^j} \frac{1}{1-z} = \frac{j!}{(1-z)^{j+1}} \; ,$$

whence evaluating at $z = 0$ and building the Taylor series we find the geometric series

$$\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j \; ,$$

which is valid for $|z| < 1$ since that is the largest open disk around the origin over which $1/(1-z)$ is analytic, as seen in Figure 2.8. *Now notice something remarkable.* We have two a priori different series representations for the function $1/(1-z)$ around the origin: one is the Taylor series and another is the geometric series. Yet we have shown that these series are the same. This is not a coincidence and we will see in Section 2.3.3 that series representations for analytic functions are unique: they are all essentially Taylor series.

Basic properties of Taylor series

Taking the derivative of the Taylor series for $\operatorname{Log} z$ about $z_0 = 1$ term by term, we find the series

$$\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{}(z-1)^{j-1} = \sum_{j=0}^{\infty}(-1)^j\,(z-1)^j = \sum_{j=0}^{\infty}(1-z)^j \ .$$

This is a geometric series which for $|z-1| < 1$ converges to

$$\frac{1}{1-(1-z)} = \frac{1}{z} \ ,$$

which is precisely the derivative of $\operatorname{Log} z$. This might not seem at all remarkable, but it is. There is no reason *a priori* why the termwise differentiation of an infinite series which converges to a function $f(z)$, should converge to the derivative $f'(z)$ of the function. This is because there are two limits involved: the limit in the definition of the derivative and the one which we take to approach the function $f(z)$, and we know from previous experience that the order in which one takes limits matters in general. On the other hand, what we have just seen is that for the case of the $\operatorname{Log} z$ function, these two limits commute; that is, they can be taken in any order. It turns out that this is not just a property of $\operatorname{Log} z$ but indeed of any analytic function.

To see this recall that we saw in Section 2.2.5 that if a function $f(z)$ is analytic in a disk $|z-z_0| < R$, then so are all its derivatives. In particular $f(z)$ and $f'(z)$ have Taylor series in the disk which converge uniformly on any closed subdisk. The Taylor series for $f'(z)$ is given by equation (2.48) applied to f' instead of f:

$$\sum_{j=0}^{\infty} \frac{(f')^{(j)}(z_0)}{j!}\,(z-z_0)^j \ .$$

But notice that the j-th derivative of f' is just the $(j+1)$-st derivative of f:

$(f')^{(j)} = f^{(j+1)}$. Therefore we can rewrite the above Taylor series as

$$\sum_{j=0}^{\infty} \frac{f^{(j+1)}(z_0)}{j!} (z - z_0)^j \ . \tag{2.49}$$

On the other hand, differentiating the Taylor series (2.48) for f termwise, we get

$$\begin{aligned} \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!} j (z - z_0)^{j-1} &= \sum_{j=1}^{\infty} \frac{f^{(j)}(z_0)}{(j-1)!} (z - z_0)^{j-1} \\ &= \sum_{k=0}^{\infty} \frac{f^{(k+1)}(z_0)}{k!} (z - z_0)^k \ , \end{aligned}$$

where we have reindexed the last sum by introducing $k = j - 1$. Finally, Shakespeare's Theorem tells us that this last series is the same as the one in equation (2.49). In other words, we have proven that if $f(z)$ is analytic around z_0, the Taylor series for $f'(z)$ around z_0 is obtained by termwise differentiation of the Taylor series for $f(z)$ around z_0.

Similarly one can show that Taylor series have additional properties. Let $f(z)$ and $g(z)$ be analytic around z_0. That means that there is some disk $|z - z_0| < R$ in which the two functions are analytic. Then as shown in Section 2.1.4, $\alpha f(z)$, for α any complex number, and $f(z) + g(z)$ are also analytic in the disk. Then one can show

- The Taylor series for $\alpha f(z)$ is the series obtained by multiplying each term in the Taylor series for $f(z)$ by α:

$$\sum_{j=0}^{\infty} \frac{\alpha\, f^{(j)}(z_0)}{j!} (z - z_0)^j \ .$$

- The Taylor series of $f(z) + g(z)$ is the series obtained by adding the terms for the Taylor series of $f(z)$ and $g(z)$:

$$\sum_{j=0}^{\infty} \frac{f^{(j)}(z_0) + g^{(j)}(z_0)}{j!} (z - z_0)^j \ .$$

These results follow from equations (2.9) and (2.10).

Finally, let $f(z)$ and $g(z)$ be analytic in a disk $|z - z_0| < R$ around z_0. We also saw in Section 2.1.4 that their product $f(z)g(z)$ is analytic there. Therefore it has a Taylor series which converges uniformly in any closed subdisk. What is the relation between this series and the Taylor series for $f(z)$ and $g(z)$? Let us compute the first couple of terms. We have that the first few derivatives of fg are

$$(fg)(z_0) = f(z_0)g(z_0) \qquad (fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$$
$$(fg)''(z_0) = f''(z_0)g(z_0) + 2f'(z_0)g'(z_0) + f(z_0)g''(z_0) \ ,$$

so that the first few terms of the Taylor series for fg are

$$f(z_0)g(z_0) + (f'(z_0)g(z_0) + f(z_0)g'(z_0))\ (z - z_0) + \frac{f''(z_0)g(z_0) + 2f'(z_0)g'(z_0) + f(z_0)g''(z_0)}{2}\ (z - z_0)^2 + \cdots$$

Notice that this can be rewritten as follows:

$$\left(f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2}(z - z_0)^2 + \cdots\right) \times \left(g(z_0) + g'(z_0)(z - z_0) + \frac{g''(z_0)}{2}(z - z_0)^2 + \cdots\right) \ ,$$

which looks like the product of the first few terms in the Taylor series of f and g. Appearances do not lie in this case and one can show that the Taylor series for the product fg of any two analytic functions is the product of their Taylor series, provided one defines the product of the Taylor series appropriately.

Let us see this. To save some writing let me write the Taylor series for $f(z)$ as $\sum_{j=0}^{\infty} a_j(z - z_0)^j$ and for $g(z)$ as $\sum_{j=0}^{\infty} b_j(z - z_0)^j$. In other words, I have introduced abbreviations $a_j = f^{(j)}(z_0)/j!$ and $b_j = g^{(j)}(z_0)/j!$. The **Cauchy product** of these two series is defined by multiplying the series formally and collecting terms with the same power of $z - z_0$. In other words,

$$\left(\sum_{j=0}^{\infty} a_j(z - z_0)^j\right) \times \left(\sum_{j=0}^{\infty} b_j(z - z_0)^j\right) = \sum_{j=0} c_j(z - z_0)^j \ ,$$

where

$$c_j = \sum_{\substack{k,\ell=0 \\ k+\ell=j}}^{\infty} a_k b_\ell = \sum_{k=0}^{j} a_k b_{j-k} = \sum_{k=0}^{j} \frac{f^{(k)}(z_0)}{k!} \frac{g^{(j-k)}(z_0)}{(j-k)!} \ .$$

On the other hand, the Taylor series for fg can be written an

$$\sum_{j=0}^{\infty} \frac{(fg)^{(j)}(z_0)}{j!}\ (z - z_0)^j \ ,$$

where one can use the **generalised Leibniz rule** to obtain

$$(fg)^{(j)}(z_0) = \sum_{k=0}^{j} \binom{j}{k} f^{(k)}(z_0)g^{(j-k)}(z_0) \ ,$$

where $\binom{j}{k}$ is the binomial coefficient

$$\binom{j}{k} = \frac{j!}{k!(j-k)!} .$$

Therefore the Taylor series for fg can be written as

$$\sum_{j=0}^{\infty} \frac{1}{j!} \sum_{k=0}^{j} \binom{j}{k} f^{(k)}(z_0) g^{(j-k)}(z_0) (z - z_0)^j$$
$$= \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{1}{k!(j-k)!} f^{(k)}(z_0) g^{(j-k)}(z_0) (z - z_0)^j = \sum_{j=0}^{\infty} c_j (z - z_0)^j ,$$

with the c_j being the same as above.

2.3.3 Power series

Taylor series are examples of a more general type of series, called power series, whose study is the purpose of this section. We will see that power series are basically always the Taylor series of some analytic function. This shows that series representations of analytic functions are in some sense unique, so that if we manage to cook up, by whatever means, a power series converging to a function in some disk, we know that this series will be its Taylor series of the function around the centre of the disk.

By a **power series** around z_0 we mean a series of the form

$$\sum_{j=0}^{\infty} a_j (z - z_0)^j ,$$

and where $\{a_j\}$ are known as the **coefficients** of the power series. A power series is clearly determined by its coefficients and by the point z_0. Given a power series one can ask many questions: For which z does it converge? Is the convergence uniform? Will it converge to an analytic function? Will the power series be a Taylor series?

We start the section with the following result, which we will state without proof. It says that to any power series $\sum_{j=0}^{\infty} a_j (z - z_0)^j$ one can associate a number $0 \leq R \leq \infty$, called the **radius of convergence**, depending only on the coefficients $\{a_j\}$, such that the series converges in the disk $|z - z_0| < R$, uniformly on any closed subdisk, and the series diverges in $|z - z_0| > R$.

Introduce lim sup, root test and the proof of this theorem.

One can actually give a formula for the number R in terms of the coefficients $\{a_j\}$ but we will not do so here in general. Instead we will give a formula which is valid only in those cases when the Ratio Test can be used. Recall that the Ratio Test says that if the limit

$$L \equiv \lim_{j\to\infty} \left| \frac{c_{j+1}}{c_j} \right| \tag{2.50}$$

exists, then the series $\sum_{j=0}^{\infty} c_j$ converges for $L < 1$ and diverges for $L > 1$. In the case of a power series, we have

$$L = \lim_{j\to\infty} \left| \frac{a_{j+1}(z-z_0)^{j+1}}{a_j(z-z_0)^j} \right| = \lim_{j\to\infty} \left| \frac{a_{j+1}}{a_j} \right| |z-z_0| \ .$$

Therefore convergence is guaranteed if $L < 1$, which is equivalent to

$$|z-z_0| < \lim_{j\to\infty} \left| \frac{a_j}{a_{j+1}} \right|$$

and divergence is guaranteed for $L > 1$, which is equivalent to

$$|z-z_0| > \lim_{j\to\infty} \left| \frac{a_j}{a_{j+1}} \right| \ .$$

Therefore if the limit (2.50) exists, we have that the radius of convergence is given by

$$\boxed{R = \lim_{j\to\infty} \left| \frac{a_j}{a_{j+1}} \right| \ .} \tag{2.51}$$

Notice that this agrees with our experience with the geometric series (2.47), which is clearly a power series around the origin. Since all the coefficients are equal, the limit exists and $R = 1$, which is precisely the radius of convergence we had established previously.

Power series are Taylor series

We are now going to prove the main result of this section: that a power series is the Taylor series of the functions it approximates. This is a very useful result, because it says that in order to compute the Taylor series of a function it is enough to produce any power series which converges to that function. The proof will follow two steps. The first is to show that a power series converges to an analytic function and the second step will use the Cauchy Integral formula to relate the coefficients of the power series with those of the Taylor series. The first step will itself require two preliminary results, which we state in some more generality.

Suppose that $\{f_n\}$ is a sequence of continuous functions which converges uniformly to a function $f(z)$ in the closed disk $|z - z_0| \leq R$. Let Γ be any contour (not necessarily closed) inside the disk, and let ℓ be the length of the contour. Then we claim that the sequence $\int_\Gamma f_n(z)\, dz$ converges to the integral $\int_\Gamma f(z)\, dz$. To see this, let $\varepsilon > 0$. Then because of uniform convergence, there exists N depending only on ε such that for all $n \geq N$, one has $|f(z) - f_n(z)| < \varepsilon/\ell$ for all z in the disk. Then

$$\begin{aligned}\left|\int_\Gamma f(z)\, dz - \int_\Gamma f_n(z)\, dz\right| &= \left|\int_\Gamma (f(z) - f_n(z))\ dz\right| \\ &\leq \max_{z\in\Gamma} |f(z) - f_n(z)|\, \ell \qquad \text{(using (2.28))} \\ &< (\varepsilon/\ell)\ell = \varepsilon\ .\end{aligned}$$

Now suppose that the sequence $\{f_n\}$ is the sequence of partial sums of some infinite series of functions. Then the above result says that one can integrate the series termwise, since for any partial sum, the integral of the sum is the sum of the integrals. In other words, when integrating an infinite series which converges uniformly in some region U along any contour in U, we can interchange the order of the summation and the integration.

Now suppose that the functions $\{f_n\}$ are not just continuous but actually analytic, and let Γ be any loop; that is, a closed simple contour. Then by the Cauchy Integral Theorem, $\oint_\Gamma f_n(z)\, dz = 0$, whence by what we have just shown

$$\oint_\Gamma f(z)\, dz = \lim_{n\to\infty} \oint_\Gamma f_n(z)\, dz = 0\ .$$

Therefore by Morera's theorem, $f(z)$ is also analytic. Therefore we have shown that

the uniform limit of analytic functions is analytic.

In particular, let $\sum_{j=0}^\infty a_j(z - z_0)^j$ be a power series with circle of convergence $|z - z_0| = R > 0$. Since each of the partial sums, being a polynomial function, is analytic in the disk (in fact, in the whole plane), the limit is also analytic in the disk. In other words, a power series converges to an analytic function inside its disk of convergence.

Now that we know that $\sum_{j=0}^\infty a_j(z - z_0)^j$ defines an analytic function, call it $f(z)$, in its disk of convergence, we can compute its Taylor series and compare it with the original series. The Taylor series of $f(z)$ around z_0 has coefficients given by the generalised Cauchy Integral Formula:

$$\frac{f^{(j)}(z_0)}{j!} = \frac{1}{2\pi\, i} \oint_\Gamma \frac{f(z)}{(z - z_0)^{j+1}}\, dz\ ,$$

where Γ is any positively oriented loop inside the disk of convergence of the power series which contains the point z_0 in its interior. Because the power series converges uniformly, we can now substitute the power series for $f(z)$ inside the integral and compute the integral termwise:

$$\begin{aligned} \frac{f^{(j)}(z_0)}{j!} &= \frac{1}{2\pi\, i} \oint_\Gamma \frac{f(z)}{(z-z_0)^{j+1}}\, dz \\ &= \sum_{k=0}^{\infty} \frac{1}{2\pi\, i} \oint_\Gamma a_k \frac{(z-z_0)^k}{(z-z_0)^{j+1}}\, dz \\ &= \sum_{k=0}^{\infty} a_k \frac{1}{2\pi\, i} \oint_\Gamma (z-z_0)^{k-j-1}\, dz\ . \end{aligned}$$

But now, from the generalised Cauchy Integral Formula,

$$\oint_\Gamma (z-z_0)^{k-j-1}\, dz = \begin{cases} 2\pi\, i & \text{if } j=k, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{2.52}$$

Therefore, only one term contributes to the $\sum_k$, namely the term with $k=j$, and hence we see that

$$\frac{f^{(j)}(z_0)}{j!} = a_j\ .$$

In other words, the power series *is* the Taylor series. Said differently, any power series is the Taylor series of a function analytic in the disk of convergence $|z-z_0| < R$.

For example, let us compute the Taylor series of the function

$$\frac{1}{(z-1)(z-2)}$$

in the disk $|z| < 1$. This is the largest disk centred at the origin where we could hope to find a convergent power series for this function, since it has singularities at $z=1$ and $z=2$. The naive solution to this problem would be to take derivatives and evaluate them at the origin and build the Taylor series this way. However from our discussion above, it is enough to exhibit any power series which converges to this function in the specified region. We use partial fractions to rewrite the function as a sum of simple fractions:

$$\frac{1}{(z-1)(z-2)} = \frac{1}{1-z} - \frac{1}{2-z}\ .$$

Now we use geometric series for each of them. For the first fraction we have

$$\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j \qquad \text{valid for } |z| < 1;$$

whereas for the second fraction we have

$$\frac{-1}{2-z} = \frac{-1/2}{1-(z/2)} = -\frac{1}{2}\sum_{j=0}^{\infty}\frac{z^j}{2^j} = -\sum_{j=0}^{\infty}\frac{z^j}{2^{j+1}}\ ,$$

which is valid for $|z| < 2$, which contains the region of interest. Therefore, putting the two series together,

$$\frac{1}{(z-1)(z-2)} = \sum_{j=0}^{\infty}\left(1 - \frac{1}{2^{j+1}}\right) z^j\ , \qquad \text{for } |z| < 1.$$

2.3.4 Laurent series

In the previous section we saw that any function which is analytic in some neighbourhood of a point z_0 can be approximated by a power series (its Taylor series) about that point. How about a function which has a "mild" singularity at z_0? For example, how about a function of the form $g(z)/(z-z_0)$? Might we not expect to be able to approximate it by some sort of power series? It certainly could not be a power series of the type we have been discussing because these series are analytic at z_0. There is, however, a simple yet useful generalisation of the notion of power series which can handle these cases. These series are known as Laurent series and consist of a sum of two power series.

A **Laurent series** about the point z_0 is a sum of two power series one consisting of positive powers of $z - z_0$ and the other of negative powers:

$$\sum_{j=0}^{\infty} a_j (z-z_0)^j + \sum_{j=1}^{\infty} a_{-j}(z-z_0)^{-j}\ .$$

Laurent series are often abbreviated as

$$\sum_{j=-\infty}^{\infty} a_j (z-z_0)^j,$$

but we should keep in mind that this is only an abbreviation: conceptually a Laurent series is the sum of two independent power series.

A Laurent series is said to converge if each of the power series converges. The first series, being a power series in $z - z_0$ converges inside some circle of convergence $|z - z_0| = R$, for some $0 \leq R \leq \infty$. The second series, however, is a power series in $w = 1/(z - z_0)$. Hence it will converge inside a circle of convergence $|w| = R'$; that is, for $|w| < R'$. If we let $R' = 1/r$, then this condition translates into $|z - z_0| > r$. In other words, such a Laurent series will converge in an annulus: $r < |z - z_0| < R$. (Of course for this to make sense, we need $r < R$. If this is not the case, then the Laurent series does not converge anywhere.)

It turns out that the results which are valid for Taylor series have generalisations for Laurent series. The first main result that we will prove is that any function analytic in an open annulus $r < |z - z_0| < R$ centred at z_0 has a Laurent series around z_0 which converges to it everywhere inside the annulus and uniformly on closed sub-annuli $r < R_1 \leq |z - z_0| \leq R_2 < R$. Moreover the coefficients of the Laurent series are given by

$$a_j = \frac{1}{2\pi\, i} \oint_\Gamma \frac{f(z)}{(z - z_0)^{j+1}}\, dz \ , \qquad \text{for } j = 0, \pm 1, \pm 2, \ldots,$$

where Γ is any positively oriented loop lying in the annulus and containing z_0 in its interior.

Notice that this result generalises the result proven in Section 2.3.2 for functions analytic in the disk. Indeed, if $f(z)$ were analytic in $|z - z_0| < R$, then by the Cauchy Integral Theorem and the above formula for a_j, it would follow that that $a_{-j} = 0$ for $j = 1, 2, \ldots$, and hence that the Laurent series is the Taylor series. Notice also that the Laurent series is a nontrivial generalisation of the Taylor series in that the coefficients a_{-j} for $j = 1, 2, \ldots$ are not just simply derivatives of the function, but rather require contour integration.

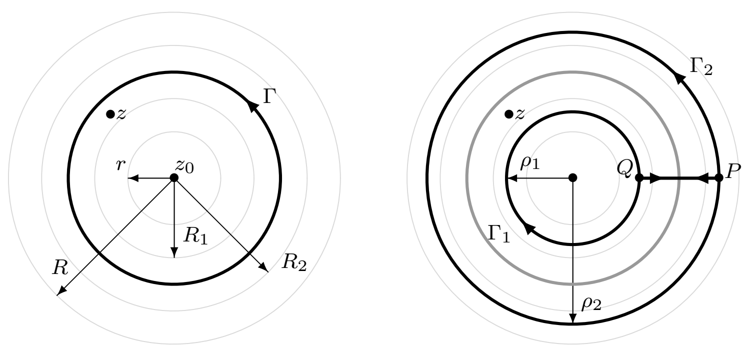


Figure 2.9: Contours Γ, Γ_1 and Γ_2.

In order to follow the logic of the proof, it will be convenient to keep Figure 2.9 in mind. The left-hand picture shows the annuli $r < R_1 \leq |z - z_0| \leq R_2 < R$ and the contour Γ. The right-hand picture shows the equivalent contours Γ_1 and Γ_2, circles with radii ρ_1 and ρ_2 satisfying the inequalities $r < \rho_1 < R_1$ and $R_2 < \rho_2 < R$.

Consider the closed contour C, starting and ending at the point P in the Figure, and defined as follows: follow Γ_2 all the way around until P again, then go to Q via the 'bridge' between the two circles, then all the way along Γ_1 until Q, then back to P along the 'bridge.' This contour encircles the point z once in the positive sense, hence by the Cauchy Integral Formula we have that

$$f(z) = \frac{1}{2\pi\, i} \oint_C \frac{f(\zeta)}{\zeta - z}\, d\zeta \ .$$

On the other hand, because the 'bridge' is traversed twice in opposite directions, their contribution to the integral cancels and we are left with

$$f(z) = \frac{1}{2\pi\, i} \oint_{\Gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi\, i} \oint_{\Gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta \ .$$

We now treat each integral at a time.

The integral along Γ_2 can be treated *mutatis mutandis* as we did the similar integral in the proof of the Taylor series theorem in Section 2.3.2. We simply quote the result:

$$\frac{1}{2\pi\, i} \oint_{\Gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_{j=0}^{\infty} a_j (z - z_0)^j \ ,$$

where

$$a_j = \frac{1}{2\pi\, i} \oint_{\Gamma_2} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}}\, d\zeta = \frac{f^{(j)}(z_0)}{j!} \ . \tag{2.53}$$

Moreover the series converges uniformly in the closed disk $|z - z_0| \leq R_2$, as was shown in that section.

The integral along Γ_1 can be treated along similar lines, except that because $|z - z_0| > |\zeta - z_0|$, we must expand the integrand differently. We will be brief, since the idea is very much the same as what was done for the Taylor series. We start by rewriting $1/(\zeta - z)$ appropriately:

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = -\frac{1}{z - z_0} \frac{1}{1 - \frac{\zeta - z_0}{z - z_0}} \ .$$

Let us write this now as a geometric series:

$$\frac{1}{\zeta - z} = -\frac{1}{z - z_0} \left[\sum_{j=0}^{n} \left(\frac{\zeta - z_0}{z - z_0} \right)^j + \frac{\left(\frac{\zeta - z_0}{z - z_0} \right)^{n+1}}{1 - \frac{\zeta - z_0}{z - z_0}} \right] \ ;$$

whence

$$\frac{1}{2\pi\, i} \oint_{\Gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_{j=1}^{m+1} a_{-j} (z - z_0)^{-j} + S_n(z) \ ,$$

where

$$a_{-j} = -\frac{1}{2\pi\, i} \oint_{\Gamma_1} \frac{f(\zeta)}{(\zeta - z_0)^{-j+1}}\, d\zeta \ , \tag{2.54}$$

and where

$$S_n(z) = \frac{1}{2\pi\, i} \oint_{\Gamma_1} \frac{f(\zeta)}{\zeta - z} \frac{(\zeta - z_0)^{n+1}}{(z - z_0)^{n+1}}\, d\zeta \ .$$

Now, for ζ in Γ_1 we have that $|\zeta - z_0| = \rho_1$ and from the triangle inequality (2.36), that $|\zeta - z| \geq R_1 - \rho_1$. We also note that $|z - z_0| \geq R_1$. Furthermore, $f(\zeta)$, being continuous, is bounded so that $|f(\zeta)| \leq M$ for some M and all ζ on Γ_1. Therefore using (2.28) and the above inequalities,

$$|S_n(z)| \leq \frac{M\, \rho_1}{R_1 - \rho_1} \left(\frac{\rho_1}{R_1} \right)^{n+1} \ ,$$

which is independent of z and, because $\rho_1 < R_1$, can be made arbitrarily small by choosing n large. Hence $S_n(z) \to 0$ as $n \to \infty$ uniformly in $|z - z_0| \geq R_1$, and

$$\frac{1}{2\pi\, i} \oint_{\Gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_{j=1}^{\infty} a_{-j} (z - z_0)^{-j} \ ,$$

where the a_{-j} are still given by (2.54). In other words,

$$\frac{1}{2\pi\, i} \oint_{\Gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_{j=1}^{\infty} a_{-j} (z - z_0)^{-j} \ ,$$

and the series converges uniformly to the integral for $|z - z_0| \leq R_1$. In summary, we have that proven that $f(z)$ is approximated by the Laurent series

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j \ ,$$

everywhere on $r < |z - z_0| < R$ and uniformly on any closed sub-annulus, where the coefficients a_j are given by (2.53) for $j \geq 0$ and by (2.54) for $j < 0$.

We are almost done, except that in the statement of the theorem the coefficients a_j are given by contour integrals along Γ and what we have shown is that they are given by contour integrals along Γ_1 or Γ_2. But notice that the integrand in (2.53) is analytic in the domain bounded by the contours Γ and Γ_2; and similarly for the integrand in (2.54) in the region bounded by the contours Γ and Γ_1. Therefore we can deform the contours Γ_1 and Γ_2 to $-\Gamma$ and Γ respectively, in the integrals

$$a_{-j} = -\frac{1}{2\pi\, i}\oint_{\Gamma_1} \frac{f(\zeta)}{(\zeta - z_0)^{-j+1}}\, d\zeta = \frac{1}{2\pi\, i}\oint_{\Gamma} \frac{f(\zeta)}{(\zeta - z_0)^{-j+1}}\, d\zeta$$
$$a_j = \frac{1}{2\pi\, i}\oint_{\Gamma_2} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}}\, d\zeta = \frac{1}{2\pi\, i}\oint_{\Gamma} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}}\, d\zeta\ ,$$

which proves the theorem.

Laurent series are unique

We saw in Section 2.3.3 that any power series is the Taylor series of the analytic function it converges to. In other words, the power series representation of an analytic function is unique (in the domain of convergence of the series, of course). Since Laurent series are generalisations of the Taylor series and agree with them when the function is analytic not just in the annulus but in fact in the whole disk, we might expect that the same is true and that the Laurent series representation of a function analytic in an annulus should also be unique. This turns out to be true and the proof follows basically from that of the uniqueness of the power series.

More precisely, one has the following result. Let

$$\sum_{j=0}^{\infty} c_j (z - z_0)^j \qquad \text{and} \qquad \sum_{j=1}^{\infty} c_{-j} (z - z_0)^{-j}$$

be any two power series converging in $|z - z_0| < R$ and $|z - z_0| > r$, respectively, with $R > r$. Then there is a function $f(z)$ analytic in the annulus $r < |z - z_0| < R$, such that

$$\sum_{j=0}^{\infty} c_j (z - z_0)^j + \sum_{j=1}^{\infty} c_{-j} (z - z_0)^{-j}$$

is its Laurent series. We shall omit the proof, except to notice that this follows from the uniqueness of the power series applied to each of the series in turn.

Do this in detail.

This is a very useful result because it says that no matter how we obtain the power series, their sum is guaranteed to be the Laurent series of the analytic function in question. Let us illustrate this in order to compute the Laurent series of some functions.

For example, let us compute the Laurent series of the rational function $(z^2-2z+3)(z-2)$ in the region $|z-1|>1$. Let us first rewrite the numerator as a power series in $(z-1)$:

$$z^2 - 2z + 3 = (z-1)^2 + 2 \; .$$

Now we do the same with the denominator:

$$\frac{1}{z-2} = \frac{1}{(z-1)-1} = \frac{1}{z-1}\frac{1}{1-\frac{1}{z-1}} \; ,$$

where we have already left it in a form which suggests that we try a geometric series in $1/(z-1)$, which converges in the specified region $|z-1|>1$. Indeed, we have that in this region,

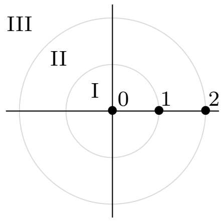
$$\frac{1}{z-1}\frac{1}{1-\frac{1}{z-1}} = \frac{1}{z-1}\sum_{j=0}^{\infty}\frac{1}{(z-1)^j} = \sum_{j=0}^{\infty}\frac{1}{(z-1)^{j+1}} \; .$$

Putting the two series together,

$$\begin{aligned}\frac{z^2-2z+3}{z-2} &= \left((z-1)^2+2\right)\sum_{j=0}^{\infty}\frac{1}{(z-1)^{j+1}} \\ &= (z-1) + 1 + \sum_{j=0}^{\infty}\frac{3}{(z-1)^{j+1}} \; .\end{aligned}$$

By the uniqueness of the Laurent series, this is *the* Laurent series for the function in the specified region.

As a final example, consider the function $1/(z-1)(z-2)$. Let us find its Laurent expansions in the regions: $|z|<1$, $1<|z|<2$ and $|z|>2$, which we have labelled I, II and III in the figure. We start by decomposing the function into partial fractions:



$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} \; .$$

In region I, we have the following geometric series:

$$-\frac{1}{z-1} = \frac{1}{1-z} = \sum_{j=0}^{\infty} z^j \quad \text{valid for } |z|<1\text{; and}$$

$$\frac{1}{z-2} = \frac{-\frac{1}{2}}{1-(z/2)} = -\tfrac{1}{2}\sum_{j=0}^{\infty}\left(\frac{z}{2}\right)^j = \sum_{j=0}^{\infty}\frac{-1}{2^{j+1}}\,z^j \quad \text{valid for } |z|<2.$$

Therefore in their common region of convergence, namely region I, we have that

$$\frac{1}{(z-1)(z-2)} = \sum_{j=0}^{\infty} \left(1 - \frac{1}{2^{j+1}}\right) z^j \ .$$

In region II, the first of the geometric series above is not valid, but the second one is. Because in region II, $|z| > 1$, this means that $|1/z| < 1$, whence we should try and use a geometric series in $1/z$. This is easy:

$$-\frac{1}{z-1} = -\frac{1}{z}\,\frac{1}{1-(1/z)} = -\frac{1}{z}\sum_{j=0}^{\infty}\left(\frac{1}{z}\right)^j = \sum_{j=0}^{\infty}\frac{-1}{z^{j+1}} \quad \text{valid for } |z| > 1.$$

Therefore in region II we have that

$$\frac{1}{(z-1)(z-2)} = \sum_{j=0}^{\infty}\frac{-1}{z^{j+1}} + \sum_{j=0}^{\infty}\frac{-1}{2^{j+1}}\, z^j \ .$$

Finally in region III, we have that $|z| > 2$, so that we will have to find another series converging to $1/(z-2)$ in this region. Again, since now $|2/z| < 1$ we should try to use a geometric series in $2/z$. This is once again easy:

$$\frac{1}{z-2} = \frac{1}{z}\,\frac{1}{1-(2/z)} = \frac{1}{z}\sum_{j=0}^{\infty}\left(\frac{2}{z}\right)^j = \sum_{j=0}^{\infty}\frac{2^j}{z^{j+1}} \quad \text{valid for } |z| > 2.$$

Therefore in region III we have that

$$\frac{1}{(z-1)(z-2)} = \sum_{j=0}^{\infty}\left(-1+2^j\right)\frac{1}{z^{j+1}} \ .$$

Again by the uniqueness of the Laurent series, we know that these are *the* Laurent series for the function in the specified regions.

2.3.5 Zeros and Singularities

As a consequence of the existence of power and Laurent series representations for analytic functions we are able to characterise the possible singularities that an analytic function can have, and this is the purpose of this section.

A point z_0 is said to be a **singularity** for a function $f(z)$, if f ceases to be analytic at z_0. Singularities can come in two types. One says that a a point z_0 is an **isolated singularity** for a function $f(z)$, if f is analytic in some punctured disk around the singularity; that is, in $0 < |z - z_0| < R$ for some

$R > 0$. We have of course already encountered isolated singularities; e.g., the function $1/(z - z_0)$ has an isolated singularity at z_0. In fact, we will see below that the singularities of a rational function are always isolated. Singularities need not be isolated, of course. For example, any point $-x$ in the non-positive real axis is a singularity for the principal branch $\operatorname{Log} z$ of the logarithm function which is *not* isolated, since any disk around $-x$, however small, will contain other singularities. In this section we will concentrate on isolated singularities. We will see that there are three types of isolated singularities, distinguished by the behaviour of the function as it approaches the singularity. Before doing so we will discuss the singularities of rational functions. As these occur at the zeros of the denominators, we will start by discussing zeros.

Zeros of analytic functions

Let $f(z)$ be analytic in a neighbourhood of a point z_0. This means that there is an open disk $|z - z_0| < R$ in which f is analytic. We say that z_0 is a **zero** of f if $f(z_0) = 0$. More precisely we say that z_0 is a **zero of order** m, for $m = 1, 2, \ldots$, if

$$f(z_0) = f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0 \qquad \textit{but } f^{(m)}(z_0) \neq 0.$$

(A zero of order $m = 1$ is often called a **simple zero**.) Because $f(z)$ is analytic in the disk $|z - z_0| < R$, it has a power series representation there: namely the Taylor series:

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!} (z - z_0)^j \ .$$

But because z_0 is a zero of order m, the first m terms in the Taylor series vanish, whence

$$f(z) = \sum_{j=m}^{\infty} \frac{f^{(j)}(z_0)}{j!} (z - z_0)^j = (z - z_0)^m g(z) \ ,$$

where $g(z)$ has a power series representation

$$g(z) = \sum_{j=0}^{\infty} \frac{f^{(j+m)}(z_0)}{(j+m)!} (z - z_0)^j$$

in the disk, whence it is analytic there and moreover, by hypothesis, $g(z_0) = f^{(m)}(z_0)/m! \neq 0$. It follows from this that the zeros of an analytic function are isolated. Because $g(z)$ is analytic, and hence continuous, in the disk $|z - z_0| < R$ and $g(z_0) \neq 0$, it means that there is a disk $|z - z_0| < \varepsilon < R$ in which $g(z) \neq 0$, and hence neither is $f(z) = (z - z_0)^m g(z)$ zero there.

Now let $P(z)/Q(z)$ be a rational function. Its singularities will be the zeroes of $Q(z)$ and we have just seen that these are isolated, whence the singularities of a rational function are isolated.

Isolated singularities

Now let z_0 be an isolated singularity for a function $f(z)$. This means that f is analytic in some punctured disk $0 < |z - z_0| < R$, for some $R > 0$. The punctured disk is a degenerate case of an open annulus $r < |z - z_0| < R$, corresponding to $r = 0$. By the results of the previous section, we know that $f(z)$ has a Laurent series representation there. We can distinguish three types of singularities depending on the Laurent expansion:

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j \ .$$

Let us pay close attention to the negative powers in the Laurent expansion: we can either have no negative powers—that is, $a_j = 0$ for all $j < 0$; a finite number of negative powers—that is, $a_j = 0$ for all but a finite number of $j < 0$; or an infinite number of negative powers—that is, $a_j \neq 0$ for an infinite number of $j < 0$. This trichotomy underlies the following definitions:

- We say that z_0 is a **removable singularity** of f, if the Laurent expansion of f around z_0 has no negative powers; that is,

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j \ .$$

- We say that z_0 is a **pole of order** m for f, if the Laurent expansion of f around z_0 has a_j for all $j < -m$ and $a_{-m} \neq 0$; that is powers; that is,

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \cdots + a_0 + a_1 (z - z_0) + \cdots \qquad \text{with } a_{-m} \neq 0.$$

  A pole of order $m = 1$ is often called a **simple pole**.

- Finally we say that z_0 is an **essential singularity** of f if the Laurent expansion of f around z_0 has an infinite number of nonzero terms with negative powers of $(z - z_0)$.

The different types of isolated singularities can be characterised by the way the function behaves in the neighbourhood of the singularity. For a removable singularity the function is clearly bounded as $z \to z_0$, since the power series representation

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j = a_0 + a_1(z - z_0) + \cdots$$

certainly has a well-defined limit as $z \to z_0$: namely, a_0. This is *not* the same thing as saying that $f(z_0) = a_0$. If this were the case, then the function would not have a singularity at z_0, but it would be analytic there as well. Therefore, removable singularities are due to f being incorrectly or "peculiarly" defined at z_0. For example, consider the following bizarre-looking function:

$$f(z) = \begin{cases} e^z & \text{for } z \neq 0; \\ 26 & \text{at } z = 0. \end{cases}$$

This function is clearly analytic in the punctured plane $|z| > 0$, since it agrees with the exponential function there, which is an entire function. This means that in the punctured plane, $f(z)$ has a power series representation which agrees with the Taylor series of the exponential function:

$$f(z) = \sum_{j=0}^{\infty} \frac{1}{j!} z^j \ .$$

However this series has the limit 1 as $z \to 0$, which is the value of the exponential for $z = 0$, and this does not agree with the value of f there. Hence the function has a singularity, but one which is easy to cure: we simply redefine f at the origin so that $f(z) = \exp(z)$ throughout the complex plane. Other examples of removable singularities are

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \ ; \tag{2.55}$$

and

$$\frac{z^2 - 1}{z - 1} = \frac{1}{z - 1}\left((z-1)^2 + 2(z-1)\right) = (z - 1) + 2 \ .$$

Of course in this last example we could have simply noticed that $z^2 - 1 = (z - 1)(z + 1)$ and simplified the rational function to $z + 1 = (z - 1) + 2$. In summary, at a removable singularity the function is bounded and can be redefined at the singularity so that the new function is analytic there, in effect removing the singularity.

In contrast, a pole is a true singularity for the function f. Indeed, around a pole z_0 of order m, the Laurent series for f looks like

$$f(z) = \frac{1}{(z-z_0)^m}\, h(z) \ ,$$

where $h(z)$ has a series expansion around z_0 given by

$$h(z) = \sum_{j=0}^{\infty} a_{j-m}(z-z_0)^j = a_{-m} + a_{-m+1}(z-z_0) + \cdots$$

This means that $h(z)$ has at most a removable singularity at z_0. We have already seen many examples of functions with poles throughout these lectures, so we will not give more examples. Let us however pause to discuss the singularities of a rational function.

Let $f(z) = P(z)/Q(z)$ be a rational function. Then we claim that $f(z)$ has either a pole or a removable singularities at the zeros of $Q(z)$. Let us be a little bit more precise. Suppose that z_0 is a zero of $Q(z_0)$, and assume that it is a zero of order m. This means that

$$Q(z) = (z-z_0)^m\, q(z) \ ,$$

where $q(z)$ is an analytic function around z_0 and such that $q(z_0) \neq 0$. If z_0 is *not* a zero of $P(z)$, then z_0 is a pole of f of order m. If z_0 is a zero of order k of $P(z)$, then we have that

$$P(z) = (z-z_0)^k\, p(z) \ ,$$

where $p(z)$ is analytic and $p(z_0) \neq 0$. Therefore we have that

$$f(z) = \frac{(z-z_0)^k\, p(z)}{(z-z_0)^m\, q(z)} = \frac{1}{(z-z_0)^{m-k}}\, \frac{p(z)}{q(z)} \ ;$$

whence $f(z)$ has a pole of order $m-k$ if $m > k$ and has a removable singularity otherwise.

How about essential singularities? A result known as **Picard's Theorem** says that a function takes all possible values (with the possible exception of one) in any neighbourhood of an essential singularity. This is a deep result in complex analysis and one we will not even attempt to prove. Let us however verify this for the function $f(z) = \exp(1/z)$. This function is analytic in the punctured plane $|z| > 0$ since the exponential function is entire. For any finite w we have seen that the exponential function has a power series expansion:

$$e^w = \sum_{j=0}^{\infty} \frac{1}{j!}\, w^j \ .$$

Therefore for $|z| > 0$, we have that

$$e^{1/z} = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{1}{z^j} \ ,$$

whence $z_0 = 0$ is an essential singularity. According to Picard's theorem, the function $\exp(1/z)$ takes every possible value (except possibly one) in *any* neighbourhood of the origin. Clearly, the value 0 is never attainable, but we can easily check that any other value is obtained. Let $c \neq 0$ be any nonzero complex number, and let us solve for those z such that $\exp(1/z) = c$. The multiple-valuedness of the logarithm says that there are infinitely many such z, satisfying:

$$\frac{1}{z} = \log(c) = \mathrm{Log}\,|c| + i\,\mathrm{Arg}(c) + 2\pi\, i\, k \ ,$$

for $k = 0, \pm 1\ \pm 2, \ldots$, whose moduli are given by

$$|z| = \frac{\mathrm{Log}\,|c| - i\,\mathrm{Arg}(c) - 2\pi\, i\, k}{(\mathrm{Log}\,|c|)^2 + (\mathrm{Arg}(c) + 2\pi\, k)^2} \ ,$$

which can be as small as desired by taking k as large as necessary. Therefore in any neighbourhood of the origin, there are an infinite number of points for which the function $\exp(1/z)$ takes as value a given nonzero complex number.

2.4 The residue calculus and its applications

We now start the final section of this part of the course. It is the culmination of a lot of hard work and formalism but one which is worth the effort and the time spent developing the necessary vocabulary. In this section we will study the theory of residues. The theory itself is very simple and is basically a matter of applying what we have learned already in the appropriate way. Most of the sections are applications of the theory to the computation of real integrals and infinite sums. These are problems which are simple to state in the context of real calculus but whose solutions (at least the elementary ones) take us to the complex plane. In a sense they provide the simplest instance of a celebrated phrase by the French mathematician Hadamard, who said that the shortest path between two real truths often passes by a complex domain.

2.4.1 The Cauchy Residue Theorem

Let us consider the behaviour of an analytic function around an isolated singularity. To be precise let z_0 be an isolated singularity for an analytic function $f(z)$. The function is analytic in some punctured disk $0 < |z-z_0| < R$, for some $R > 0$, and has a Laurent series there of the form

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z-z_0)^j \ .$$

Consider the contour integral of the function $f(z)$ along a positively oriented loop Γ contained in the punctured disk and having the singularity z_0 in its interior. Because the Laurent series converges uniformly, we can integrate the series term by term:

$$\oint_\Gamma f(z)\, dz = \sum_{j=-\infty}^{\infty} a_j \oint_\Gamma (z-z_0)^j\, dz \ .$$

From the (generalised) Cauchy Integral Formula or simply by deforming the contour to a circle of radius $\rho < R$, we have that (c.f., equation (2.52))

$$\oint_\Gamma (z-z_0)^j\, dz = \begin{cases} 2\pi\, i & \text{for } j=-1, \text{ and} \\ 0 & \text{otherwise;} \end{cases}$$

whence only the $j = -1$ term contributes to the sum, so that

$$\oint_\Gamma f(z)\, dz = 2\pi\, i\, a_{-1} \ .$$

This singles out the coefficient a_{-1} in the Laurent series, and hence we give it a special name. We say that a_{-1} is the **residue** of f at z_0, and we write this as $\mathrm{Res}(f; z_0)$ or simply as $\mathrm{Res}(z_0)$ when f is understood.

For example, consider the function $z \exp(1/z)$. This function has an essential singularity at the origin and is analytic everywhere else. The residue can be computed from the Laurent series:

$$z e^{1/z} = z \sum_{j=0}^{\infty} \frac{1}{j!} \frac{1}{z^j} = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{1}{z^{j-1}} = z + 1 + \frac{1}{2z} + \cdots \ ,$$

whence the residue is given by $\mathrm{Res}(0) = \frac{1}{2}$.

It is often not necessary to calculate the Laurent expansion in order to extract the residue of a function at a singularity. For example, the residue of a function at a removable singularity vanishes, since there are no negative powers in the Laurent expansion. On the other hand, if the singularity happens to be a pole, we will see that the residue can be computed by differentiation.

Suppose, for simplicity, that $f(z)$ has a simple pole at z_0. Then the Laurent series of $f(z)$ around z_0 has the form

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots ,$$

whence the residue can be computed by

$$\begin{aligned} \operatorname{Res}(f; z_0) &= \lim_{z \to z_0} (z - z_0)\, f(z) \\ &= \lim_{z \to z_0} \left(a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + \cdots\right) \\ &= a_{-1} + 0 . \end{aligned}$$

For example, the function $f(z) = e^z/z(z+1)$ has simple poles at $z = 0$ and $z = -1$; therefore,

$$\begin{aligned} \operatorname{Res}(f; 0) &= \lim_{z \to 0} z\, f(z) = \lim_{z \to 0} \frac{e^z}{z+1} = 1 \\ \operatorname{Res}(f; -1) &= \lim_{z \to -1} (z+1)\, f(z) = \lim_{z \to -1} \frac{e^z}{z} = -\frac{1}{e} . \end{aligned}$$

Suppose that $f(z) = P(z)/Q(z)$ where P and Q are analytic at z_0 and Q has a simple zero at z_0 whereas $P(z_0) \neq 0$. Clearly f has a simple pole at z_0, whence the residue is given by

$$\operatorname{Res}(f; z_0) = \lim_{z \to z_0} (z - z_0) \frac{P(z)}{Q(z)} = \lim_{z \to z_0} \frac{P(z)}{\frac{Q(z) - Q(z_0)}{z - z_0}} = \frac{P(z_0)}{Q'(z_0)} ,$$

where we have used that $Q(z_0) = 0$ and the definition of the derivative, which exists since Q is analytic at z_0.

We can use this to compute the residues at each singularity of the function $f(z) = \cot z$. Since $\cot z = \cos z / \sin z$, the singularities occur at the zeros of the sine function: $z = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. These zeros are simple because $\sin'(n\pi) = \cos(n\pi) = (-1)^n \neq 0$. Therefore we can apply the above formula to deduce that

$$\operatorname{Res}(f; n\pi) = \left.\frac{\cos z}{(\sin z)'}\right|_{z = n\pi} = \frac{\cos(n\pi)}{\cos(n\pi)} = 1 .$$

This result will be crucial for the applications concerning infinite series later on in this section.

Now suppose that f has a pole of order m at z_0. The Laurent expansion is then

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$$

Let us multiply this by $(z - z_0)^m$ to obtain

$$(z - z_0)^m f(z) = a_{-m} + \cdots + a_{-1}(z - z_0)^{m-1} + a_0(z - z_0)^m + \cdots \ ,$$

whence taking $m - 1$ derivatives, we have

$$\frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z)\right] = (m-1)!\, a_{-1} + m!\, a_0(z - z_0) + \cdots \ .$$

Finally if we evaluate this at $z = z_0$, we obtain $(m-1)!\, a_{-1}$, which then gives a formula for the residue of f at a pole of order m:

$$\boxed{\operatorname{Res}(f; z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z)\right] \ .} \tag{2.56}$$

For example, let us compute the residues of the function

$$f(z) = \frac{\cos z}{z^2 (z - \pi)^3} \ .$$

This function has a pole of order 2 at the origin and a pole of order 3 at $z = \pi$. Therefore, applying the above formula, we find

$$\begin{aligned}
\operatorname{Res}(f; 0) &= \lim_{z \to 0} \frac{1}{1!} \frac{d}{dz} \left[z^2 f(z)\right] \\
&= \lim_{z \to 0} \frac{d}{dz} \left[\frac{\cos z}{(z - \pi)^3}\right] \\
&= \lim_{z \to 0} \left[\frac{-\sin z}{(z - \pi)^3} - \frac{3 \cos z}{(z - \pi)^4}\right] \\
&= -\frac{3}{\pi^4} \ , \\
\operatorname{Res}(f; \pi) &= \lim_{z \to \pi} \frac{1}{2!} \frac{d^2}{dz^2} \left[(z - \pi)^3 f(z)\right] \\
&= \lim_{z \to \pi} \frac{1}{2} \frac{d^2}{dz^2} \left[\frac{\cos z}{z^2}\right] \\
&= \lim_{z \to \pi} \frac{1}{2} \left[\frac{6 \cos z}{z^4} + \frac{4 \sin z}{z^3} - \frac{\cos z}{z^2}\right] \\
&= -\frac{6 - \pi^2}{2\pi^4} \ .
\end{aligned}$$

We are now ready to state the main result of this section, which concerns the formula for the integral of a function $f(z)$ which is analytic on a positively-oriented loop Γ and has only a finite number of isolated singularities $\{z_k\}$ in the interior of the loop. Because of the analyticity of the function, and using a contour deformation argument, we can express the integral of $f(z)$ along Γ as the sum of the integrals of $f(z)$ along positively-oriented loops Γ_k, each one encircling one of the isolated singularities. But we have just seen that the integral along each of these loops is given by $2\pi\, i$ times the residue of the function at the singularity. In other words, we have

$$\oint_\Gamma f(z)\, dz = \sum_k \oint_{\Gamma_k} f(z)\, dz = \sum_k 2\pi\, i\ \mathrm{Res}(f; z_k)\ .$$

In other words, we arrive at the **Cauchy Residue Theorem**, which states that the integral of $f(z)$ along Γ is equal to $2\pi\, i$ times the sum of the residues of the singularities in the interior of the contour:

$$\boxed{\oint_\Gamma f(z)\, dz = 2\pi\, i \sum_{\substack{\text{singularities}\\ z_k \in \mathrm{Int}\,\Gamma}} \mathrm{Res}(f; z_k)\ .}$$

For example, let us compute the integral

$$\oint_\Gamma \frac{1-2z}{z(z-1)(z-3)}\, dz\ ,$$

along the positively oriented circle of radius 2: $|z| = 2$. The integrand $f(z)$ has simple poles at $z = 0$, $z = 1$ and $z = 2$, but only the first two lie in the interior of the contour. Thus by the residue theorem,

$$\oint_\Gamma \frac{1-2z}{z(z-1)(z-3)}\, dz = 2\pi\, i \left[\mathrm{Res}(f;0) + \mathrm{Res}(f;1)\right]\ ,$$

and

$$\begin{aligned}
\mathrm{Res}(f;0) &= \lim_{z\to 0} z\, f(z) \\
&= \lim_{z\to 0} \frac{(1-2z)}{(z-1)(z-3)} \\
&= \frac{1}{3}\ , \\
\mathrm{Res}(f;1) &= \lim_{z\to 1} (z-1)\, f(z) \\
&= \lim_{z\to 1} \frac{(1-2z)}{z(z-3)} \\
&= \frac{1}{2}\ ;
\end{aligned}$$

so that

$$\oint_\Gamma \frac{1-2z}{z(z-1)(z-3)}\, dz = 2\pi\, i \left(\frac{1}{3}+\frac{1}{2}\right) = \frac{5\pi\, i}{3} \ .$$

Notice something curious. Computing the residue at $z=3$, we find,

$$\begin{aligned} \operatorname{Res}(f;3) &= \lim_{z\to 3}(z-3)\, f(z) \\ &= \lim_{z\to 3}\frac{(1-2z)}{z(z-1)} \\ &= \frac{-5}{6} \ ; \end{aligned}$$

whence the sum of all three residues is 0. This can be explained by introducing the Riemann sphere model for the extended complex plane, and thus noticing that a contour which would encompass all three singularities can be deformed to surround the point at infinity but in the opposite sense. Since the integrand is analytic at infinity, the Cauchy Integral Theorem says that the integral is zero, but (up to factors) this is equal to the sum of the residues.

2.4.2 Application: trigonometric integrals

The first of the applications of the residue theorem is to the computation of trigonometric integrals of the form

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta \ ,$$

where R is a rational function of its arguments and such that it is finite in the range $0 \le \theta \le 2\pi$. We want to turn this into a complex contour integral so that we can apply the residue theorem. One way to do this is the following. Consider the contour Γ parametrised by $z = \exp(i\theta)$ for $\theta \in [0, 2\pi]$: this is the unit circle traversed once in the positive sense. On this contour, we have $z = \cos\theta + i\,\sin\theta$ and $1/z = \cos\theta - i\,\sin\theta$. Therefore we can solve for $\cos\theta$ and $\sin\theta$ in terms of z and $1/z$ as follows:

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \text{and} \qquad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right) \ .$$

Similarly, $dz = d\exp(i\theta) = iz\, d\theta$, whence $d\theta = \frac{dz}{iz}$. Putting it all together we have that

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = \oint_\Gamma \frac{1}{iz} R\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right)\, dz \ ,$$

which is the contour integral of a rational function of z, and hence can be computed using the residue theorem:

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = 2\pi \sum_{\substack{\text{singularities} \\ |z_k|<1}} \operatorname{Res}(f; z_k) \ , \tag{2.57}$$

where $f(z)$ is the rational function

$$f(z) \equiv \frac{1}{z} R\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right) \ . \tag{2.58}$$

As an example, let us compute the integral

$$I = \int_0^{2\pi} \frac{(\sin\theta)^2}{5+4\cos\theta}\, d\theta \ .$$

First of all notice that the denominator never vanishes, so that we can go ahead. The rational function $f(z)$ given in (2.58) is

$$f(z) = \frac{1}{z} \frac{\left(\frac{1}{2i}\left(z-\frac{1}{z}\right)\right)^2}{5+4\frac{1}{2}\left(z+\frac{1}{z}\right)} = -\frac{1}{4} \frac{(z^2-1)^2}{z^2(2z^2+5z+2)} = -\frac{1}{8} \frac{(z^2-1)^2}{z^2(z+\frac{1}{2})(z+2)} \ ,$$

whence it has a double pole at $z = 0$ and single poles at $z = -\frac{1}{2}$ and $z = -2$. Of these, only the poles at $z = 0$ and $z = -\frac{1}{2}$ lie inside the unit disk, whence

$$I = 2\pi \left[\operatorname{Res}(f; 0) + \operatorname{Res}(f; -\tfrac{1}{2})\right] \ .$$

Let us compute the residues. The singularity at $z = 0$ is a pole of order 2, whence by equation (2.56), we have

$$\operatorname{Res}(f; 0) = \lim_{z\to 0} \frac{d}{dz} \left[-\frac{1}{8} \frac{(z^2-1)^2}{(z+\frac{1}{2})(z+2)}\right] = \frac{5}{16} \ .$$

The pole at $z = -\frac{1}{2}$ is simple, so that its residue is even simpler to compute:

$$\operatorname{Res}(f; -1/2) = \lim_{z\to -\frac{1}{2}} \left[-\frac{1}{8} \frac{(z^2-1)^2}{z^2(z+2)}\right] = -\frac{3}{16} \ .$$

Therefore, the integral becomes

$$I = 2\pi \left(\frac{5}{16} - \frac{3}{16}\right) = \frac{\pi}{4} \ .$$

As a mild check on our result, we notice that it is real whence it is not obviously wrong.

Let us do another example:

$$I = \int_0^\pi \frac{d\theta}{2-\cos\theta} \ .$$

This time the integral is only over $[0,\pi]$, so that we cannot immediately use the residue theorem. However in this case we notice that because $\cos(2\pi - \theta) = \cos\theta$, we have that

$$\int_\pi^{2\pi} \frac{d\theta}{2-\cos\theta} = \int_\pi^0 \frac{d(2\pi-\theta)}{2-\cos(2\pi-\theta)} = -\int_\pi^0 \frac{d\theta}{2-\cos\theta} = \int_0^\pi \frac{d\theta}{2-\cos\theta} \ .$$

Therefore,

$$I = \tfrac{1}{2}\int_0^{2\pi} \frac{d\theta}{2-\cos\theta} \ ,$$

which using equation (2.57) and paying close attention to the factor of $\frac{1}{2}$, becomes π times the sum of the residues of the function

$$f(z) = \frac{1}{z}\frac{1}{2-\frac{1}{2}(z+\frac{1}{z})}$$

lying inside the unit disk. After a little bit of algebra, we find that

$$f(z) = -\frac{2}{z^2-4z+1} = -\frac{2}{(z-2+\sqrt{3})(z-2-\sqrt{3})} \ .$$

Of the two simple poles of this function only the one at $z = 2-\sqrt{3}$ lies inside the unit disk, hence

$$\operatorname{Res}(f;2-\sqrt{3}) = \lim_{z\to 2-\sqrt{3}} \frac{-2}{z-2-\sqrt{3}} = \frac{1}{\sqrt{3}} \ ,$$

and thus the integral becomes

$$I = \frac{\pi}{\sqrt{3}} \ .$$

2.4.3 Application: improper integrals

In this section we consider improper integrals of rational functions and of products of rational and trigonometric functions.

Let $f(x)$ be a function of a real variable, which is continuous in $0 \leq x < \infty$. Then by the improper integral $\int_0^\infty f(x)\,dx$, we mean the limit

$$\int_0^\infty f(x)\,dx \equiv \lim_{R\to\infty} \int_0^R f(x)\,dx \ ,$$

if such a limit exists. Similarly, if $f(x)$ is continuous in $-\infty < x \leq 0$, then the improper integral $\int_{-\infty}^0 f(x)\,dx$ is defined by the limit

$$\int_{-\infty}^0 f(x)\,dx \equiv \lim_{r\to-\infty} \int_r^0 f(x)\,dx \ ,$$

again provided that it exists. If $f(x)$ is continuous on the whole real line and both of the above limits exists, we define

$$\int_{-\infty}^\infty f(x)\,dx \equiv \lim_{\substack{R\to\infty \\ r\to-\infty}} \int_r^R f(x)\,dx \ . \tag{2.59}$$

If such limits exist, then we get the same result by symmetric integration:

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{\rho\to\infty} \int_{-\rho}^\rho f(x)\,dx \ . \tag{2.60}$$

Notice however that the symmetric integral may exist even if the improper integral (2.59) does not. For example consider the function $f(x) = x$. Clearly the integrals $\int_0^\infty x\,dx$ and $\int_{-\infty}^0 x\,dx$ do not exist, yet because x is an odd function, $\int_{-\rho}^\rho x dx = 0$ for all ρ, whence the limit is 0. In cases like this we say that equation (2.60) defines the **Cauchy principal value** of the integral, and we denote this by

$$\text{p. v.}\int_{-\infty}^\infty f(x)\,dx \equiv \lim_{\rho\to\infty} \int_{-\rho}^\rho f(x)\,dx \ .$$

We stress to point out that whenever the improper integral (2.59) exists it agrees with its principal value (2.60).

Improper integrals of rational functions over $(-\infty, \infty)$

Let us consider as an example the improper integral

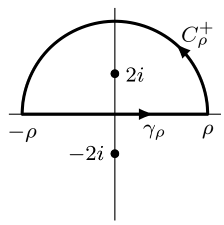
$$I = \text{p. v.}\int_{-\infty}^\infty \frac{dx}{x^2+4} = \lim_{\rho\to\infty} \int_{-\rho}^\rho \frac{dx}{x^2+4} \ .$$

The integral for finite ρ can be interpreted as the complex integral of the function $f(z) = 1/(z^2+4)$,

$$\int_{\gamma_\rho} \frac{dz}{z^2+4} \ ,$$

where γ_ρ is the straight line segment on the real axis: $y = 0$ and $-\rho \leq x \leq \rho$. In order to use the residue theorem we need to *close the contour*; that is, we must produce a closed contour along which we can apply the residue theorem. Ot f course, in so doing we are introducing a further integral, and the success of the method depends on whether the extra integral is computable. We will see that in this case, the extra integral, if chosen judiciously, vanishes.

Let us therefore complete the contour γ_ρ to a closed contour. One suggestion is to consider the semicircular contour C_ρ^+ in the upper half plane, parametrised by $z(t) = \rho \exp(it)$, for $t \in [0, \pi]$. Let Γ_ρ be the composition of both contours: it is a closed contour as shown in the figure. Then, according to the residue theorem,



$$\int_{\Gamma_\rho} \frac{dz}{z^2+4} = \int_{\gamma_\rho} \frac{dz}{z^2+4} + \int_{C_\rho^+} \frac{dz}{z^2+4} = 2\pi\, i \sum_{\substack{\text{singularities} \\ z_k \in \operatorname{Int}\Gamma_\rho}} \operatorname{Res}(f; z_k) \ ;$$

whence

$$\int_{\gamma_\rho} \frac{dz}{z^2+4} = 2\pi\, i \sum_{\substack{\text{singularities} \\ z_k \in \operatorname{Int}\Gamma_\rho}} \operatorname{Res}(f; z_k) - \int_{C_\rho^+} \frac{dz}{z^2+4} \ .$$

We will now argue that the integral along C_ρ^+ vanishes in the limit $\rho \to \infty$. Of course, this is done using (2.28):

$$\left| \int_{C_\rho^+} \frac{dz}{z^2+4} \right| \leq \int_{C_\rho^+} \frac{|dz|}{|z^2+4|} \ . \tag{2.61}$$

Using the triangle inequality (2.36), we have that on C_ρ^+,

$$|z^2+4| \geq |z^2| - 4 = |z|^2 - 4 = \rho^2 - 4 \ ,$$

whence

$$\frac{1}{|z^2+4|} \leq \frac{1}{\rho^2-4} \ .$$

Plugging this into (2.61), and taking into account that the length of the semicircle C_ρ^+ is $\pi\rho$,

$$\left| \int_{C_\rho^+} \frac{dz}{z^2+4} \right| \leq \frac{\pi\rho}{\rho^2-4} \to 0 \qquad \text{as } \rho \to \infty.$$

Therefore in the limit,

$$\int_{\gamma_\rho} \frac{dz}{z^2+4} = 2\pi\, i \sum_{\substack{\text{singularities} \\ z_k \in \operatorname{Int}\Gamma_\rho}} \operatorname{Res}(f; z_k) \ .$$

The function $f(z)$ has poles at $z = \pm 2i$, of which only the one at $z = 2i$ lies inside the closed contour Γ_ρ, for large ρ. Computing the residue there, we find from (2.56) that

$$\operatorname{Res}(f; 2i) = \lim_{z \to 2i} \left[\frac{1}{z + 2i} \right] = \frac{1}{4i} \ ,$$

and hence the integral is given by

$$I = 2\pi\, i \, \frac{1}{4i} = \frac{\pi}{2} \ .$$

There is no reason why we chose to close the contour using the top semicircle C_ρ^+ instead of using the bottom semicircle C_ρ^- parametrised by $z(t) = \rho \exp(it)$ for $t \in [\pi, 2\pi]$. The same argument shows that in the limit $\rho \to \infty$ the integral along C_ρ^- vanishes. It is now the pole at $-2i$ that we have to take into account, and one has that $\operatorname{Res}(f; -2i) = -1/4i$. Notice however that the closed contour is negatively-oriented, which produces an extra $-$ sign from the residue formula, in such a way that the final answer is again

$$I = -2\pi\, i \, \frac{-1}{4i} = \frac{\pi}{2} \ .$$

The technique employed in the calculation of the above integral can be applied in more general situations. All that we require is for the integral along the large semicircle C_ρ^+ to vanish and this translates into a condition on the behaviour of the integrand for large $|z|$.

We will now show the following general result. Let $R(x) = P(x)/Q(x)$ be a rational function of a real variable satisfying the following two criteria:

- $Q(x) \neq 0$; and
- $\deg Q - \deg P \geq 2$.

Then the improper integral of $R(x)$ along the real line is given by considering the residues of the complex rational function $R(z)$ at its singularities in the upper half-plane. Being a rational function the only singularities are either removable or poles, and only these latter ones contribute to the residue. In summary,

$$\boxed{\text{p. v.} \int_{-\infty}^{\infty} R(x)\, dx = 2\pi\, i \sum_{\substack{\text{poles } z_k \\ \operatorname{Im}(z_k) > 0}} \operatorname{Res}(R; z_k) \ .} \tag{2.62}$$

The proof of this relation follows the same steps as in the computation of the integral I above. The trick is to close the contour using the upper semicircle C_ρ^+ and then argue that the integral along the semicircle vanishes. This is guaranteed by the the behaviour of $R(z)$ for large $|z|$.

Let us do this in detail. The integral to be computed is

$$I = \text{p. v.}\int_{-\infty}^{\infty} R(x)\, dx = \lim_{\rho\to\infty}\int_{-\rho}^{\rho} R(x)\, dx = \lim_{\rho\to\infty}\int_{\gamma_\rho} R(z)\, dz \ .$$

Closing the contour with C_ρ^+ to Γ_ρ, we have

$$\int_{\gamma_\rho} R(z)\, dz = \oint_{\Gamma_\rho} R(z)\, dz - \int_{C_\rho^+} R(z)\, dz \ .$$

The first integral in the right-hand side can be easily dispatched using the residue theorem. In the limit $\rho \to \infty$, one finds

$$\lim_{\rho\to\infty}\oint_{\Gamma_\rho} R(z)\, dz = 2\pi\, i \sum_{\substack{\text{poles } z_k \\ \text{Im}(z_k)>0}} \text{Res}(R; z_k) \ .$$

All that remains then is to show that the second integral vanishes in the limit $\rho \to \infty$. We can estimate it using (2.28) as usual:

$$\int_{C_\rho^+} R(z)\, dz \ \le \int_{C_\rho^+} |R(z)|\, |dz| = \int_{C_\rho^+} \frac{|P(z)|}{|Q(z)|}\, |dz| \ .$$

Let the degree of the polynomial $P(z)$ be p and that of $Q(z)$ be q, where by hypothesis we have that $q - p \ge 2$. Recall from our discussion in Section 2.2.6 that for large $|z|$ a polynomial $P(z)$ of degree N behaves like $|P(z)| \sim c|z|^N$ for some c. Similar considerations in this case show that the rational function $R(z) = P(z)/Q(z)$ with $q = \deg Q > \deg P = p$ obeys

$$|R(z)| \le \frac{c}{|z|^{q-p}} \ ,$$

for some constant c independent of $|z|$. Using this into the estimate of the integral along C_ρ^+, and using that the semicircle has length $\pi\rho$,

$$\int_{C_\rho^+} R(z)\, dz \ \le \frac{c\pi\rho}{\rho^{q-p}} \ .$$

Since $q - p \ge 2$, we have that this goes to zero in the limit $\rho \to \infty$, as desired.

As an example, let us compute the following integral

$$I = \text{p. v.}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\, dx \ .$$

The integrand is rational and obeys the two criteria above: it is always finite and the degree of the denominator is 4 whereas that of the numerator is 2, whence $4 - 2 \geq 2$. In order to compute the integral it is enough to compute the residues of the rational function

$$f(z) = \frac{z^2}{(z^2+1)^2} \ ,$$

at the poles in the upper half-plane. This function has poles of order 2 at the points $z = \pm i$, of which only $z = +i$ is in the upper half-plane, hence from (2.62) we have

$$I = 2\pi\, i \lim_{z\to i} \frac{d}{dz} \left[\frac{z^2}{(z+i)^2}\right] = 2\pi\, i \lim_{z\to i} \left[\frac{2i\, z}{(z+i)^3}\right] = 2\pi\, i\, \frac{-i}{4} = \frac{\pi}{2} \ .$$

Improper integrals of rational and trigonometric functions

The next type of integrals which can be be handled by the method of residues are of the kind

$$\text{p. v.}\int_{-\infty}^{\infty} R(x)\cos(ax)\, dx \qquad \text{and} \qquad \text{p. v.}\int_{-\infty}^{\infty} R(x)\sin(ax)\, dx \ ,$$

where $R(x)$ is a rational function which is continuous everywhere in the real line (except maybe at the zeros of $\cos(ax)$ and $\sin(ax)$, depending on the integral), and where a is a nonzero real number.

As an example, consider the integral

$$I = \text{p. v.}\int_{-\infty}^{\infty} \frac{\cos(3x)}{x^2+4}\, dx = \lim_{\rho\to\infty} \int_{-\rho}^{\rho} \frac{\cos(3x)}{x^2+4}\, dx \ .$$

From the discussion in the previous section, we are tempted to try to express the integral over $[-\rho, \rho]$ as a complex contour integral, close the contour and use the residue theorem. Notice however that we cannot use the function $\cos(3z)/(z^2+4)$ because $|\cos(3z)|$ is not bounded for large values of $|\operatorname{Im}(z)|$. Instead we notice that we can write the integral as the real part of a complex integral $I = \operatorname{Re}(I_0)$, where

$$I_0 = \lim_{\rho\to\infty} \int_{-\rho}^{\rho} \frac{e^{i3x}}{x^2+4}\, dx \ .$$

Therefore let us consider the integral

$$\int_{-\rho}^{\rho} \frac{e^{i3x}}{x^2+4}\, dx = \int_{\gamma_\rho} \frac{e^{i3z}}{z^2+4}\, dz \ ,$$

where γ_ρ is the line segment on the real axis from $-\rho$ to ρ. We would like to close this contour to be able to use the residue theorem, and in such a way that the integral vanishes on the extra segment that we must add to close it. Let us consider the upper semicircle C_ρ^+. There we have that

$$\left|\frac{e^{i3z}}{z^2+4}\right| = \frac{e^{-3\,\mathrm{Im}(z)}}{|z^2+4|} \leq \frac{e^{-3\,\mathrm{Im}(z)}}{\rho^2-4} \ ,$$

where to reach the inequality we used (2.36) as was done above. The function $e^{-3\,\mathrm{Im}(z)}$ is bounded above by 1 in the *upper* half-plane, and in particular along C_ρ^+, hence we have that on the semicircle,

$$\left|\frac{e^{i3z}}{z^2+4}\right| \leq \frac{1}{\rho^2-4} \ .$$

Therefore the integral along the semicircle is bounded above by

$$\left|\int_{C_\rho^+} \frac{e^{i3z}}{z^2+4}\,dz\right| \leq \frac{\pi\rho}{\rho^2-4} \to 0 \qquad \text{as } \rho\to\infty.$$

Therefore we can use the residue theorem to express I_0 in terms of the residues of the function $f(z) = \exp(i3z)/(z^2+4)$ at the poles in the upper half-plane. This function has simple poles at $z=\pm 2i$, but only $z = 2i$ lies in the upper half-plane, whence

$$I_0 = 2\pi\, i\ \mathrm{Res}(f; 2i) = 2\pi\, i \lim_{z\to 2i}\left[\frac{e^{i3z}}{z+2i}\right] = 2\pi\, i\,\frac{e^{-6}}{4i} = \frac{\pi}{2e^6} \ ,$$

which is already real. (One could have seen this because the imaginary part is the integral of $\sin(3x)/(x^2+4)$ which is an odd function and hence integrates to zero under symmetric integration.) Therefore,

$$I = \mathrm{Re}(I_0) = \frac{\pi}{2e^6} \ .$$

Suppose instead that we had wanted to compute the integral

$$\text{p. v.}\int_{-\infty}^{\infty} \frac{e^{-i3x}}{x^2+4}\,dx \ .$$

Of course, now we could do it because this is the complex conjugate of the integral we have just computed, but let us assume that we had not yet done the other integral. We would follow the same steps as before, but notice that now,

$$\left|\frac{e^{-i3z}}{z^2+4}\right| = \frac{e^{3\,\mathrm{Im}(z)}}{|z^2+4|} \ ,$$

which is no longer bounded in the upper half-plane. In this case we would be forced the close the contour using the lower semicircle C_ρ^-, keeping in mind that the closed contour is now negatively oriented. The lesson to learn from this is that there is some choice in how to close to contour and that one has to exercise this choice judiciously for the calculation to work out.

This method of course generalises to compute integrals of the form

$$\text{p. v.} \int_{-\infty}^{\infty} R(x) e^{iax}\, dx \tag{2.63}$$

where a is real. Surprisingly the conditions on the rational function $R(x)$ are now slightly weaker. Indeed, we have the following general result.

Let $R(x) = P(x)/Q(x)$ be a rational function satisfying the following conditions:

- $Q(x) \neq 0$,[3] and
- $\deg Q - \deg P \geq 1$.

Then the improper integral (2.63) is given by considering the residues of the function $f(z) = R(z)e^{iaz}$ at its singularities in the upper (if $a > 0$) or lower (if $a < 0$) half-planes. These singularities are either removable or poles, and again only the poles contribute to the residues. In summary,

$$\boxed{\text{p. v.} \int_{-\infty}^{\infty} R(x) e^{iax}\, dx = \begin{cases} 2\pi i \sum_{\substack{\text{poles } z_k \\ \operatorname{Im}(z_k)>0}} \operatorname{Res}(f; z_k) & \text{if } a > 0; \\ -2\pi i \sum_{\substack{\text{poles } z_k \\ \operatorname{Im}(z_k)<0}} \operatorname{Res}(f; z_k) & \text{if } a < 0. \end{cases}} \tag{2.64}$$

This result is similar to (2.62) with two important differences. The first is that we have to choose the contour appropriately depending on the integrand; that is, depending on the sign of a. The second one is that the condition on the rational function is less restrictive than before: now we simply demand that the degree of Q be greater than the degree of P. This will therefore require a more refined estimate of the integral along the semicircle, which goes by the name of the **Jordan lemma**, which states that

$$\lim_{\rho\to\infty} \int_{C_\rho^+} e^{iaz} \frac{P(z)}{Q(z)}\, dz = 0 \ ,$$

whenever $a > 0$ and $\deg Q > \deg P$. Of course an analogous result holds for $a < 0$ and along C_ρ^-.

[3] This could in principle be relaxed provided the zeros of Q at most gave rise to removable singularities in the integrand.

Let us prove this lemma. Parametrise the semicircle C_ρ^+ by $z(t) = \rho \exp(it)$ for $t \in [0, \pi]$. Then by (2.25)

$$\int_{C_\rho^+} e^{iaz} \frac{P(z)}{Q(z)}\, dz = \int_0^\pi e^{ia\rho e^{it}} \frac{P(\rho e^{it})}{Q(\rho e^{it})}\, \rho\, i\, dt \ .$$

Let us now estimate the integrand term by term. First we have that

$$e^{ia\rho e^{it}} \quad = \quad e^{ia\rho(\cos t + i \sin t)} \quad = e^{-a\rho \sin t} \ .$$

Similarly, since $\deg Q - \deg P \geq 1$, we have that

$$\frac{P(\rho e^{it})}{Q(\rho e^{it})} \ \leq \frac{c}{\rho} \ ,$$

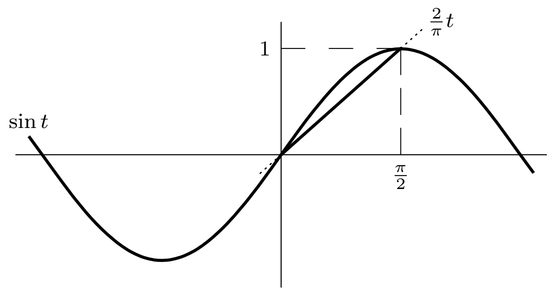
for ρ large, for some $c > 0$. Now using (2.24) on the t-integral together with the above (in)equalities,

$$\int_{C_\rho^+} e^{iaz} \frac{P(z)}{Q(z)}\, dz \ = \int_0^\pi e^{ia\rho e^{it}} \frac{P(\rho e^{it})}{Q(\rho e^{it})}\, \rho\, i\, dt \ \leq c \int_0^\pi e^{-a\rho \sin t}\, dt \ .$$

We need to show that this latter integral goes to zero in the limit $\rho \to \infty$. First of all notice that $\sin t = \sin(\pi - t)$ for $t \in [0, \pi]$, whence

$$\int_0^\pi e^{-a\rho \sin t}\, dt = 2 \int_0^{\pi/2} e^{-a\rho \sin t}\, dt \ .$$

Next notice that for $t \in [0, \pi/2]$, $\sin t \geq 2t/\pi$. This can be seen pictorially as in the following picture, which displays the function $\sin t$ in the range $t \in [-\pi, \pi]$ and the function $2t/\pi$ in the range $t \in [0, \pi/2]$ and makes the inequality manifest.



Therefore,

$$\int_0^{\pi/2} e^{-a\rho \sin t}\, dt \leq \int_0^\pi e^{-2a\rho\, t/\pi}\, dt = \frac{\pi}{2a\rho} \quad 1 - e^{-2a\rho\pi} \quad .$$

Putting this all together, we see that

$$\int_{C_\rho^+} e^{iaz} \frac{P(z)}{Q(z)}\, dz \ \leq \frac{c\pi}{a\rho} \quad 1 - e^{-2a\rho\pi} \quad ,$$

which clearly goes to 0 in the limit $\rho \to \infty$, proving the lemma.

As an example, let us compute the integral

$$I = \text{p. v.} \int_{-\infty}^{\infty} \frac{x \sin x}{1 + x^2}\, dx \ .$$

This is the imaginary part of the integral

$$I_0 = \mathrm{p.\,v.} \int_{-\infty}^{\infty} \frac{x e^{ix}}{1+x^2}\, dx \ ,$$

which satisfies the conditions which permit the use of (2.64) with $a = 1$ and $R(z) = z/(1+z^2)$. This rational function has simple poles for $z = \pm i$, but only $z = i$ lies in the upper half-plane. According to (2.64) then, and letting $f(z) = R(z)e^{iz}$, we have

$$I_0 = 2\pi\, i \ \mathrm{Res}(f; i) = 2\pi\, i \lim_{z \to i} \left[\frac{z\, e^{iz}}{z+i} \right] = 2\pi\, i \frac{ie^{-1}}{2i} = \frac{i\pi}{e} \quad \Longrightarrow \quad I = \frac{\pi}{e} \ .$$

Improper integrals of rational functions on $(0, \infty)$

The next type of integrals which can be tackled using the residue theorem are integrals of rational functions but over the half line; that is, integrals of the form:

$$\int_0^{\infty} R(x)\, dx \ ,$$

where $R(x)$ is continuous for $x \geq 0$. Of course, if $R(x)$ were an even function, i.e., $R(-x) = R(x)$, then we would have $\int_0^\infty R(x)\, dx = \frac{1}{2} \int_{-\infty}^{\infty} R(x)\, dx$, and we could use the method discussed previously. However for more general integrands, this does not work and we have to do something different.

The following general result is true. Let $R(x) = P(x)/Q(x)$ be a rational function of a real variable satisfying the following two conditions

- $Q(x) \neq 0$; and
- $\deg Q - \deg P \geq 2$.

Further let $f(z) = \log(z)\, R(z)$ with the branch of the logarithm chosen to be analytic at the poles $\{z_k\}$ of R; for example, we can choose the branch $\mathrm{Log}_0(z)$ which has the cut along the positive real axis, since $Q(x)$ has no zeros there. Then,

$$\boxed{\int_0^{\infty} R(x)\, dx = - \sum_{\text{poles } z_k} \mathrm{Res}(f; z_k) \ , \quad \text{for } f(z) = \log(z)\, R(z).} \tag{2.65}$$

The details.

This same method can be applied to integrals of the form

$$\int_a^\infty R(x)\, dx \ ,$$

where the rational function $R(x) = P(x)/Q(x)$ satisfies the same conditions as above except that now $Q(x) \neq 0$ only for $x \geq a$. In this case we must consider the function $f(z) = \log(z - a)\, R(z)$.

Details?

Similarly, since $\int_0^a = \int_0^\infty - \int_a^\infty$, we can use this method to compute *indefinite* integrals of rational functions.

2.4.4 Application: improper integrals with poles

Suppose that we want to compute the principal value integral

$$I = \text{p. v.} \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx \ .$$

This integral should converge: the singularity at $x = 0$ is removable, as we saw in equation (2.55), so that the integrand is continuous for all x, and the rational function $1/x$ satisfies the conditions of the Jordan Lemma. Following the ideas in the previous section, we would be write

$$I = \operatorname{Im}(I_0) \qquad \text{where} \qquad I_0 = \text{p. v.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx \ , \tag{2.66}$$

and compute I_0. However notice that now the integrand of I_0 has a pole at $x = 0$. Until now we have always assumed that integrands have no poles along the contour, so the methods developed until now are not immediately applicable to perform the above integral. We therefore need to make sense out of integrals whose integrands are not continuous everywhere in the region of integration.

Let $f(x)$ be a function of a real variable, which is continuous in the interval $[a, b]$ except for a discontinuity at some point c, $a < c < b$. Then the **improper integrals** of f over the intervals $[a, c]$, $[c, b]$ and $[a, b]$ are defined by

$$\int_a^c f(x)\, dx \equiv \lim_{r \searrow 0} \int_a^{c-r} f(x)\, dx \ ,$$
$$\int_c^b f(x)\, dx \equiv \lim_{s \searrow 0} \int_{c+s}^{b} f(x)\, dx \ ,$$

and

$$\int_a^b f(x)\,dx \equiv \lim_{r\searrow 0}\int_a^{c-r} f(x)\,dx + \lim_{s\searrow 0}\int_{c+s}^b f(x)\,dx\ , \tag{2.67}$$

provided the appropriate limit(s) exist. We have used the notation $r \searrow 0$ to mean that r approaches 0 from above; that is, $r > 0$ as we take the limit. As an example, consider the function $1/\sqrt{x}$ integrated on $[0,1]$:

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{s\searrow 0}\int_s^1 \frac{dx}{\sqrt{x}} = \lim_{s\searrow 0} 2\sqrt{x}\Big|_s^1 = \lim_{s\searrow 0}\left[2 - 2\sqrt{s}\right] = 2\ .$$

If the limits in (2.67) exist, then we can calculate the integral using symmetric integration, which defines the **principal value** of the integral,

$$\text{p. v.}\int_a^b f(x)\,dx \equiv \lim_{r\searrow 0}\left[\int_a^{c-r} f(x)\,dx + \int_{c+r}^b f(x)\,dx\right]\ .$$

However the principal value integral may exist even when the improper integral does not. Take, for instance,

$$\begin{aligned}\text{p. v.}\int_1^4 \frac{dx}{x-2} &= \lim_{r\searrow 0}\left[\int_1^{2-r} + \int_{2+r}^4\right]\frac{dx}{x-2}\\ &= \lim_{r\searrow 0}\left[\operatorname{Log}|x-2|\Big|_1^{2-r} + \operatorname{Log}|x-2|\Big|_{2+r}^4\right]\\ &= \lim_{r\searrow 0}\left[\operatorname{Log} r + \operatorname{Log} 2 - \operatorname{Log} r\right] = \operatorname{Log} 2\ ,\end{aligned}$$

whereas it is clear that the improper integral $\int_1^4 \frac{dx}{x-2}$ does not exist.

When the function $f(x)$ is continuous everywhere in the real line except at the point c we define the principal value integral by

$$\text{p. v.}\int_{-\infty}^{\infty} f(x)\,dx \equiv \lim_{\substack{\rho\to\infty\\ r\searrow 0}}\left[\int_{-\rho}^{c-r} f(x)\,dx + \int_{c+r}^{\rho} f(x)\,dx\right]\ , \tag{2.68}$$

provided the limits $\rho \to \infty$ and $r \searrow 0$ exist independently. In the case of several discontinuities $\{c_i\}$ we extend the definition of the improper integral in the obvious way: excising a small symmetric interval $(c_i - r_i, c_i + r_i)$ about each discontinuity and then taking the limits $r_i \searrow 0$ and, if applicable, $\rho \to \infty$.

It turns out that principal value integrals of this type can often be evaluated using the residue theorem. The residue theorem applies to closed contours, so in computing a principal value integral we need to close the contour, not just ρ to $-\rho$ as in the previous session, but also $c - r$ to $c + r$.

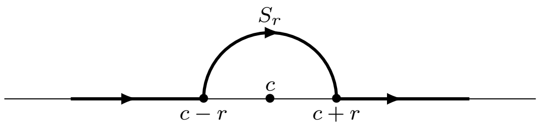


Figure 2.10: Closing the contour around a singularity.

One way to do this is to consider a small semicircle S_r of radius r around the singular point c, as in Figure 2.10.

Because we are interested in the limit $r \searrow 0$, we will have to consider the integral

$$\lim_{r \searrow 0} \int_{S_r} f(z)\, dz \ .$$

When the singularity of $f(z)$ at $z = c$ is a simple pole, this integral can be evaluated using the following result, which we state in some generality.

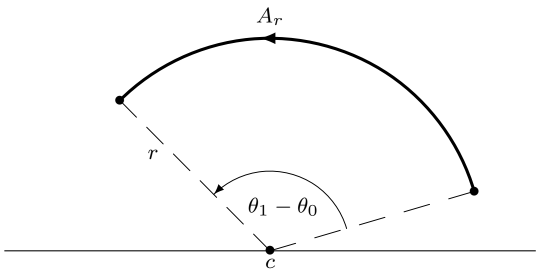


Figure 2.11: A small circular arc.

Let $f(z)$ have a simple pole at $z = c$ and let A_r be the circular arc in Figure 2.11, parametrised by $z(\theta) = c + r\exp(i\theta)$ with $\theta_0 \leq \theta \leq \theta_1$. Then

$$\lim_{r \searrow 0} \int_{A_r} f(z)\, dz = i\,(\theta_1 - \theta_0)\ \mathrm{Res}(f;c) \ .$$

Therefore for the semicircle S_r in Figure 2.10, we have

$$\lim_{r \searrow 0} \int_{S_r} f(z)\, dz = -i\,\pi\ \mathrm{Res}(f;c) \ . \tag{2.69}$$

Let us prove this result. Since $f(z)$ has a simple pole at c, its Laurent expansion in a punctured disk $0 < |z-c| < R$ has the form

$$f(z) = \frac{a_{-1}}{z-c} + \sum_{k=0}^{\infty} a_k (z-c)^k \ ,$$

where

$$g(z) \equiv \sum_{k=0}^{\infty} a_k (z-c)^k$$

defines an analytic function in the disk $|z-c| < R$. Now let $0 < r < R$ and consider the integral

$$\int_{A_r} f(z)\, dz = a_{-1} \int_{A_r} \frac{dz}{z-c} + \int_{A_r} g(z)\, dz \ .$$

Because $g(z)$ is analytic it is in particular bounded on some neighbourhood of c, so that $|g(z)| \leq M$ for some M and all $|z-c| < R$. Then we can estimate its integral by using (2.28):

$$\left| \int_{A_r} g(z)\, dz \right| \leq \int_{A_r} |g(z)|\, |dz| \leq M\ell(A_r) = Mr(\theta_1 - \theta_0) \ ,$$

whence

$$\lim_{r \searrow 0} \int_{A_r} g(z)\, dz = 0 \ .$$

On the other hand,

$$\int_{A_r} \frac{dz}{z-c} = \int_{\theta_0}^{\theta_1} \frac{rie^{i\theta}}{re^{i\theta}}\, d\theta = i \int_{\theta_0}^{\theta_1} d\theta = i\ (\theta_1 - \theta_0) \ .$$

Therefore

$$\lim_{r \searrow 0} \int_{A_r} f(z)\, dz = i\,(\theta_1 - \theta_0) a_{-1} + 0 = i\,(\theta_1 - \theta_0)\ \mathrm{Res}(f;c) \ .$$

Having discussed the basic theory, let us go back to the original problem: the computation of the integral I_0 given in (2.66):

$$I_0 = \lim_{\substack{\rho \to \infty \\ r \searrow 0}} \left[\int_{-\rho}^{-r} \frac{e^{ix}}{x}\, dx + \int_{r}^{\rho} \frac{e^{ix}}{x}\, dx \right] \ ,$$

which for finite ρ and nonzero r can be understood as a contour integral in the complex plane along the subset of the real axis consisting of the intervals $[-\rho, -r]$ and $[r, \rho]$. In order to use the residue theorem we must close this contour. The Jordan lemma forces us to join ρ and $-\rho$ via a large semicircle C_ρ^+ of radius ρ in the *upper* half-plane. In order to join $-r$ and r we choose a small semicircle S_r also in the upper half-plane. The resulting closed contour is depicted in Figure 2.12.

Because the function is analytic on and inside the contour, the Cauchy Integral Theorem says that the contour integral vanishes. Splitting this contour integral into its different pieces, we have that

$$\left[\int_{-\rho}^{-r} + \int_{S_r} + \int_{r}^{\rho} + \int_{C_\rho^+} \right] \frac{e^{iz}}{z}\, dz = 0 \ ,$$

which remains true in the limits $\rho \to \infty$ and $r \searrow 0$. By the Jordan lemma, the integral along C_ρ^+ vanishes in the limit $\rho \to \infty$, whence, using (2.69),

$$I_0 = -\lim_{r \searrow 0} \int_{S_r} \frac{e^{iz}}{z}\, dz = \lim_{r \searrow 0} \int_{-S_r} \frac{e^{iz}}{z}\, dz = i\pi\ \mathrm{Res}(0) = i\pi \ ,$$

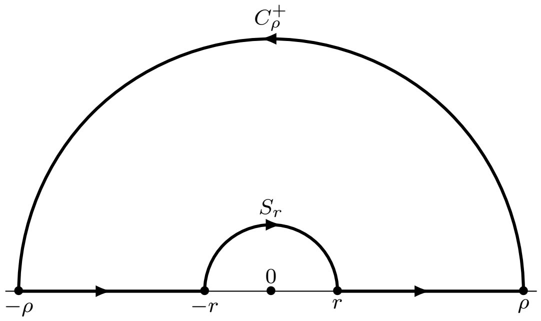


Figure 2.12: The contour in the calculation of I_0 in (2.66).

since the residue of e^{iz}/z at $z = 0$ is equal to 1. Therefore, we have that

$$\text{p. v.}\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \operatorname{Im}(i\pi) = \pi \ .$$

There are plenty of other integrals which can be calculated using the residue theorem; e.g., integrals involving multi-valued functions. We will not have time to discuss them all, but the lesson to take home from this cursory introduction to residue techniques is that when faced with a real integral, one should automatically think of this as a parametrisation of a contour integral in the complex plane, where we have at our disposal the powerful tools of complex analysis.

2.4.5 Application: infinite series

The final section of this part of the course is a beautiful application of the theory of residues to the computation of infinite sums.

How can one use contour integration in order to calculate sums like the following one:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \ ? \tag{2.70}$$

The idea is to exhibit this sum as part of the right-hand side of the Cauchy Residue Theorem. For this we need a function $F(z)$ which has only simple poles at the integers and whose residue is 1 there. We already met a function which has an infinite number of poles which are integrally spaced: the function $\cot z$ has simple poles for $z = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$ with residues equal to 1. Therefore the function $F(z) = \pi\cot(\pi z)$ has simple poles at $z = n$, n an integer, and the residue is still 1:

$$\operatorname{Res}(F; n) = \lim_{z\to n} \frac{\pi\cos(\pi z)}{(\sin(\pi z))'} = \lim_{z\to n} \frac{\pi\cos(\pi z)}{\pi\cos(\pi z)} = 1 \ .$$

Now let $R(z) = P(z)/Q(z)$ be any rational function such that $\deg Q - \deg P \geq 2$. Consider the function $f(z) = \pi \cot(\pi z) R(z)$ and let us integrate this along the contour Γ_N, for N a positive integer, defined as the positively oriented square with vertices $(N + \frac{1}{2})(1+i)$, $(N + \frac{1}{2})(-1+i)$, $(N + \frac{1}{2})(-1-i)$ and $(N + \frac{1}{2})(1-i)$, as shown in Figure 2.13. Notice that the contour misses the poles of $\pi \cot(\pi z)$. Assuming that N is taken to be large enough, and since $R(z)$ has a finite number of poles, one can also guarantee that the contour will miss the poles of $R(z)$.

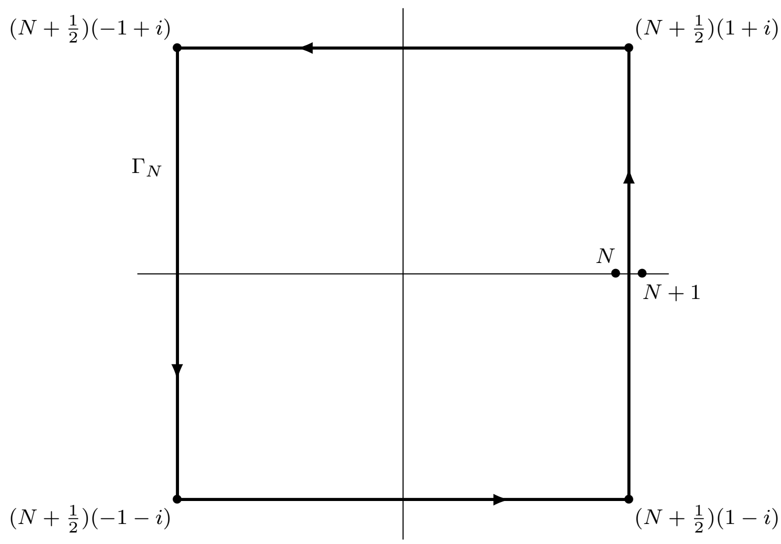


Figure 2.13: The contour Γ_N.

Let us compute the integral of the function $f(z)$ along this contour,

$$\int_{\Gamma_N} \pi \cot(\pi z) R(z)\, dz \ ,$$

in two ways. On the one hand we can use the residue theorem to say that the integral will be $(2\pi i)$ times the sum of the residues of the poles of $f(z)$. These poles are of two types: the poles of $R(z)$ and the poles of $\pi \cot(\pi z)$, which occur at the integers. Let us assume for simplicity that $R(z)$ has no poles at integer values of z, so that the poles of $R(z)$ and $\pi \cot(\pi z)$ do not coincide. Therefore we see that

$$\int_{\Gamma_N} \pi \cot(\pi z) R(z)\, dz = 2\pi\, i \left(\sum_{n=-N}^{N} \operatorname{Res}(f; n) + \sum_{\substack{\text{poles } z_k \text{ of } R \\ \text{inside } \Gamma_N}} \operatorname{Res}(f; z_k) \right) .$$

The residue of $f(z)$ at $z - n$ is easy to compute. Since by assumption $R(z)$ is analytic there and $\pi \cot(\pi z)$ has a simple pole with residue 1, we see that around $z = n$, we have

$$f(z) = R(z)\pi \cot(\pi z) = R(z) \left(\frac{1}{z-n} + \cdots \right) = \frac{R(z)}{z-n} + h(z) \ ,$$

where $h(z)$ is analytic at $z = n$. Therefore,

$$\mathrm{Res}(f; n) = \lim_{z \to n} \left[(z-n) f(z) \right] = R(n) + 0 \ ,$$

and as a result,

$$\int_{\Gamma_N} \pi \cot(\pi z) R(z)\, dz = 2\pi\, i \left(\sum_{n=-N}^{N} R(n) + \sum_{\substack{\text{poles } z_k \text{ of } R \\ \text{inside } \Gamma_N}} \mathrm{Res}(f; z_k) \right) \ . \tag{2.71}$$

On the other hand we can estimate the integral for large enough N as follows. First of all because of the condition on $R(z)$, we have that for large $|z|$,

$$|R(z)| \leq \frac{c}{|z|^2} \ .$$

Similarly, it can be shown that the function $\pi \cot(\pi z)$ is bounded along the contour, so that $|\pi \cot(\pi z)| \leq K$ for some K independent of N.

Indeed, notice that

$$|\cot(\pi z)| = \left| \frac{\cos(\pi z)}{\sin(\pi z)} \right| = \left| \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right| = \left| \frac{1 + e^{-2i\pi z}}{1 - e^{-2i\pi z}} \right| \ .$$

Therefore along the segment of the contour parametrised by $z(t) = (N + \frac{1}{2}) + it$ for $t \in [-N - \frac{1}{2}, N + \frac{1}{2}]$, we have that

$$\begin{aligned} |\cot(\pi z(t))| &= \left| \frac{1 + e^{i2\pi((N+\frac{1}{2})+it)}}{1 - e^{i2\pi((N+\frac{1}{2})+it)}} \right| \\ &= \left| \frac{1 - e^{\pi(2N+1)\,t}}{1 + e^{\pi(2N+1)\,t}} \right| < 1 \ ; \end{aligned}$$

whereas along the segments of the contour parametrised by $z(t) = t - i(N + \frac{1}{2})$ for $t \in [-N - \frac{1}{2}, N + \frac{1}{2}]$, we have that

$$\begin{aligned} |\cot(\pi z(t))| &= \left| \frac{1 + e^{-i\pi(2N+1)(t-i)}}{1 - e^{-i\pi(2N+1)(t-i)}} \right| \\ &= \left| \frac{1 + e^{2\pi t} e^{-\pi(2N+1)}}{1 - e^{2\pi t} e^{-\pi(2N+1)}} \right| \\ &\leq \frac{1 + e^{-\pi(2N+1)}}{1 - e^{-\pi(2N+1)}} \end{aligned}$$

where we have used the triangle inequalities (2.1) on the numerator and (2.36) on the denominator. But

$$\frac{1+e^{-\pi(2N+1)}}{1-e^{-\pi(2N+1)}}$$

is maximised for $N = 0$, whence it is bounded.

Since the length of the contour Γ_N is given by $4(2N+1)$, equation (2.28) gives the following estimate for the integral

$$\left|\int_{\Gamma_N} \pi\cot(\pi z)R(z)\,dz\right| \leq \frac{Kc}{(N+\frac{1}{2})^2}\,4(2N+1)\ ,$$

which vanishes in the limit $N \to \infty$. Therefore, taking the limit $N \to \infty$ of equation (2.71), and using that the left-hand side vanishes, one finds

$$\sum_{n=-\infty}^{\infty} R(n) = -\sum_{\text{poles } z_k \text{ of } R} \mathrm{Res}(f; z_k)\ .$$

More generally, if $R(z)$ does have some poles for integer values of z, then we have to take care not to over-count these poles in the sum of the residues. We will count them as poles of $R(z)$ and not as poles of $\pi\cot(\pi z)$, and the same argument as above yields the general formula:

$$\boxed{\sum_{\substack{n=-\infty \\ n\neq z_k}}^{\infty} R(n) = -\sum_{\substack{\text{poles} \\ z_k \text{ of } R}} \mathrm{Res}(f; z_k)\ , \quad \text{for } f(z) = \pi\cot(\pi z)\,R(z).} \tag{2.72}$$

Let us compute then the sum (2.70). Notice that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n^2}\ .$$

The function $R(z) = 1/z^2$ has a double pole at $z = 0$, hence by (2.72)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{1}{2}\lim_{z\to 0}\frac{d}{dz}\left[\pi\cot(\pi z)\right]\ .$$

Now, the Laurent expansion of $\pi\cot(\pi z)$ around $z = 0$ is given by

$$\frac{1}{z} - \frac{\pi^2 z}{3} - \frac{\pi^4 z^3}{45} + O(z^5)\ , \tag{2.73}$$

whence

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{1}{2}\left[-\frac{\pi^2}{3}\right] = \frac{\pi^2}{6}\ .$$

This sum has an interesting history. Its computation was an open problem in the 18th century for quite some time. It was known that the series was convergent (proven in fact by one of the Bernoullis) but it was up to Euler to calculate it. His "proof" is elementary and quite clever. Start with the Taylor series for the sine function:

$$\sin x = x \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right) ,$$

and treat the expression in parenthesis as an algebraic equation in x^2. Its solutions are known: $n^2\pi^2$ for $n = 1, 2, 3, \ldots$. Suppose we could factorise the expression in parenthesis:

$$\begin{aligned} &\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{(2\pi)^2}\right)\left(1 - \frac{x^2}{(3\pi)^2}\right)\cdots \\ &\qquad = 1 - x^2\left(\frac{1}{\pi^2} + \frac{1}{(2\pi)^2} + \frac{1}{(3\pi)^2} + \cdots\right) + O(x^4) . \end{aligned}$$

Therefore, comparing the coefficient of x^2, we see that

$$\frac{1}{3!} = \frac{1}{\pi^2} + \frac{1}{(2\pi)^2} + \frac{1}{(3\pi)^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{(n\pi)^2} ,$$

which upon multiplication by π^2 yields the sum.

Similarly, we can compute the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = -\tfrac{1}{2}\operatorname{Res}(f;0) ,$$

where $f(z) = \pi\cot(\pi z)/z^4$, whose Laurent series about $z = 0$ is can be read off from (2.73) above:

$$\frac{1}{z^5} - \frac{\pi^2}{3z^3} - \frac{\pi^4}{45\,z} + O(z) ,$$

whence

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} .$$

Infinite alternating sums

The techniques above can be extended to the computation of infinite alternating sums of the form

$$\sum_{n=-\infty}^{\infty} (-1)^n R(n) ,$$

where $R(z) = P(z)/Q(z)$ is a rational function with $\deg Q - \deg P \geq 2$. Now what is needed is a function $G(z)$ which has a simple pole at $z = n$, for n

an integer, and whose residue there is $(-1)^n$. We claim that this function is $\pi\csc(\pi z)$. Indeed, the Laurent expansion about $z=0$ is given by

$$\pi\csc(\pi z) = \frac{1}{z} + \frac{\pi^2 z}{6} + \frac{7\,\pi^4\, z^3}{360} + O(z^5)\ ; \tag{2.74}$$

whence its residue at 0 is 1. Because of the periodicity $\csc(\pi(z+2k)) = \csc(\pi z + 2k\pi) = \csc(\pi z)$ for any integer k, this is also the residue about every even integer. Now from the periodicity $\csc(\pi(z+1)) = \csc(\pi z+\pi) = -\csc(\pi z)$, we notice that the residue at every odd integer is -1. Therefore we conclude that for $G(z) = \pi\csc(\pi z)$, $\operatorname{Res}(G;n) = (-1)^n$.

The trigonometric identity

$$(\csc(\pi z))^2 = 1 + (\cot(\pi z))^2\ ,$$

implies that $\csc(\pi z)$ is also bounded along the contour Γ_N, with a bound which is independent of N just like for $\cot(\pi z)$. Just as was done above for the cotangent function, we can now prove that the integral of the function $f(z) = \pi\csc(\pi z)R(z)$ along Γ_N vanishes in the limit $N\to 0$. This proof is virtually identical to the one given above. Therefore we can conclude that

$$\boxed{\sum_{\substack{n=-\infty\\ n\neq z_k}}^{\infty} (-1)^n R(n) = -\sum_{\substack{\text{poles}\\ z_k \text{ of } R}} \operatorname{Res}(f;z_k)\ ,\quad \text{for } f(z) = \pi\csc(\pi z)\,R(z).} \tag{2.75}$$

As an example, let us compute the alternating sums

$$S_1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \qquad \text{and} \qquad S_2 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}\ .$$

For the first sum we have that

$$S_1 = -\tfrac{1}{2}\operatorname{Res}(f;0)\ ,$$

where $f(z) = \pi\csc(\pi z)/z^2$, whose Laurent expansion about $z=0$ can be read off from (2.74):

$$f(z) = \frac{1}{z^3} + \frac{\pi^2}{6\,z} + \frac{7\,\pi^4\, z}{360} + O(z^3)\ ,$$

whence the residue is $\pi^2/6$ and the sum

$$S_1 = -\frac{\pi^2}{12}\ .$$

For the second sum we also have that

$$S_2 = -\tfrac{1}{2}\,\mathrm{Res}(f;0)\ ,$$

where the function $f(z) = \pi \csc(\pi z)/z^4$ has now a Laurent series

$$f(z) = \frac{1}{z^5} + \frac{\pi^2}{6\,z^3} + \frac{7\,\pi^4}{360\,z} + O(z^1)\ ,$$

whence the residue is $7\pi^4/360$ and the sum

$$S_2 = -\frac{7\,\pi^4}{720}\ .$$

Sums involving binomial coefficients

There are other types of sums which can also be performed or at least estimated using residue techniques, particularly sums whose coefficients are related to the binomial coefficients, as in $\sum_{n=1}^{\infty}\binom{2n}{n}R(n)$. By definition, the binomial coefficient $\binom{n}{k}$ is the coefficient of z^k in the binomial expansion of $(1+z)^n$. In other words, using the residue theorem,

$$\binom{n}{k} = \frac{1}{2\pi i}\oint_\Gamma \frac{(1+z)^n}{z^{k+1}}\,dz\ ,$$

where Γ is any positively oriented loop surrounding the origin.

Suppose that we wish to compute the sum

$$S = \sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{5^n}\ .$$

We can substitute the integral representation for the binomial coefficient,

$$S = \sum_{m=0}^{\infty}\left[\frac{1}{2\pi i}\int_\Gamma \frac{(1+z)^{2n}}{z^{n+1}}\,dz\right]\frac{1}{5^n} = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_\Gamma \frac{(1+z)^{2n}}{(5z)^n}\,\frac{dz}{z}\ .$$

Now provided that we choose Γ inside the domain of convergence of the series $\sum_{n=0}^{\infty}\frac{(1+z)^{2n}}{(5z)^n}$ then we would obtain that by uniform convergence, the integral of the sum is the sum of the termwise integrals. Being a geometric series, its convergence is uniform in the region

$$\left|\frac{(1+z)^2}{5z}\right| < 1\ ,$$

so choose the contour Γ inside this region. For definiteness we can choose the unit circle, since on the unit circle:

$$\left| \frac{(1+z)^2}{5z} \right| \leq \frac{4}{5} \ .$$

In this case, we can interchange the order of the summation and the integration:

$$S = \frac{1}{2\pi i} \int_{|z|=1} \sum_{n=0}^{\infty} \frac{(1+z)^{2n}}{(5z)^n} \frac{dz}{z} = \frac{5}{2\pi i} \int_{|z|=1} \frac{1}{3z - 1 - z^2} \, dz \ .$$

Now the integral can be performed using the residue theorem. The integrand has simple poles at $(3 \pm \sqrt{5})/2$ of which only the $(3 - \sqrt{5})/2$ lies inside the contour. Therefore,

$$S = 5 \operatorname{Res}\left(f; \frac{3 - \sqrt{5}}{2} \right) \qquad \text{where} \qquad f(z) = \frac{1}{3z - 1 - z^2} \ .$$

Computing the residue, we find $\operatorname{Res}((3 - \sqrt{5})/2) = 1/\sqrt{5}$, whence $S = \sqrt{5}$.