

Mathematical Techniques III (PHY 317)

Solutions to Problem Set 9

Solution to Problem 1.

In this problem we are asked to compute the Fourier series of certain periodic functions. In many cases, the task is therefore first to determine the periodicity T of each of the functions, and then expand the function in a Fourier series with the correct period:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi}{T}nt} ,$$

where the Fourier coefficients $\{c_n\}$ are given by

$$c_n = \frac{1}{T} \int_{\text{one period}} f(t) e^{-i\frac{2\pi}{T}nt} dt .$$

- (a) The function is $f(t) = (\sin t)^3$ which has period 2π . Therefore we seek an expansion of the form

$$(\sin t)^3 = \sum_{n=-\infty}^{\infty} c_n e^{int} .$$

In this case, it is not necessary to perform an integration in order to obtain the $\{c_n\}$. It is sufficient to expand the function $(\sin t)^3$ in terms of complex exponentials. This can be done by using the trigonometric identities suggested in the problem set or else simply by expanding $(\sin t)^3$ directly:

$$(\sin t)^3 = \frac{1}{(2i)^3} (e^{it} - e^{-it})^3 = \frac{i}{8} (e^{3it} - 3e^{it} + 3e^{-it} - e^{-3it}) ,$$

from where we can read off the Fourier coefficients directly:

$$c_n = \begin{cases} \pm\frac{i}{8} & \text{for } n = \pm 3; \\ \mp\frac{3i}{8} & \text{for } n = \pm 1; \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

- (b) The function $|\cos(t/3)|^3$ has period 3π , whence we expect a Fourier series of the form

$$|\cos(t/3)|^3 = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2n}{3}t} .$$

We notice that $|\cos(t/3)|^3 = \cos(t/3)^3$ for the period $-\frac{3}{2}\pi \leq t \leq \frac{3}{2}\pi$, so that the coefficients are given by

$$c_n = \frac{1}{3\pi} \int_{-3\pi/2}^{3\pi/2} \cos(t/3)^3 e^{-i\frac{2n}{3}t} dt .$$

Expanding $\cos(t/3)^3$ or using the trigonometric identity suggested, one finds

$$\cos(t/3)^3 = \frac{1}{8} (e^{it} + 3e^{it/3} + 3e^{-it/3} + e^{-it}) .$$

Therefore c_n receives four contributions:

$$c_n = \frac{1}{8}c_n^{(1)} + \frac{3}{8}c_n^{(1/3)} + \frac{3}{8}c_n^{(-1/3)} + \frac{1}{8}c_n^{(-1)} ,$$

where

$$c_n^{(\alpha)} = \frac{1}{3\pi} \int_{-3\pi/2}^{3\pi/2} e^{i\alpha t} e^{-i\frac{2n}{3}t} dt = \frac{2}{\pi} \frac{(-1)^n}{3\alpha - 2n} \sin\left(\frac{3\pi}{2}\alpha\right) .$$

Adding up all the contributions, one finds

$$c_n = \frac{12}{\pi} \frac{(-1)^n}{(4n^2 - 9)(4n^2 - 1)} .$$

Notice that this is actually a Fourier cosine series, since the coefficients satisfy $c_n = c_{-n}$, so that

$$|\cos(t/3)|^3 = \frac{4}{3\pi} + \sum_{n=1}^{\infty} \frac{24(-1)^n}{\pi(4n^2 - 1)(4n^2 - 9)} \cos\left(\frac{2nt}{3}\right) .$$

- (c) The next function is $f(t) = t^2$ for $-\pi < t < \pi$. One can extend this function to the whole real line by making it periodic with period 2π . We therefore seek a series of the form

$$t^2 = \sum_{n=-\infty}^{\infty} c_n e^{int} ,$$

agreeing with t^2 for $t \in [-\pi, \pi]$ and elsewhere with its periodic extension. The Fourier coefficients are obtained by integrating

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} t^2 e^{-int} dt .$$

This integral is easy to perform. If $n = 0$, then $c_0 = \pi^2/3$. Otherwise, we can integrate by parts to obtain,

$$t^2 e^{-int} = -\frac{2}{n^2} e^{-int} + \frac{2}{n^2} \frac{d}{dt} (te^{-int}) + \frac{i}{n} \frac{d}{dt} (t^2 e^{-int}) .$$

This can now be easily integrated and one finds

$$c_n = \begin{cases} \frac{\pi^2}{3} & \text{if } n = 0; \text{ and} \\ \frac{2(-1)^n}{n^2} & \text{otherwise.} \end{cases}$$

Again, notice that this is a Fourier cosine series, since the function is even:

$$t^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nt , \quad \text{valid for } t \in [-\pi, \pi].$$

Notice the following interesting fact: if we evaluate this at $t = \pi$, then we have

$$\pi^2 - \frac{\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{1}{n^2} ,$$

from where we can work out the value of the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} .$$

Similarly evaluating the series at $\pi/2$, we find that only even $n = 2\ell$, say, contribute to the sum, whence

$$\frac{\pi^2}{4} - \frac{\pi^2}{3} = \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{\ell^2} \quad \text{or equivalently} \quad \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{\ell^2} = -\frac{\pi^2}{12} .$$

- (d) Finally we consider the function $f(t) = t|t|$ in $t \in (-\pi, \pi)$, extended periodically with period 2π . Notice that unlike the other functions we have dealt with in this problem, the periodic extension of this function is not continuous. In fact, $\lim_{t \rightarrow \pi} t|t| = \pi^2$,

whereas $\lim_{t \rightarrow -\pi} t|t| = -\pi^2$. We shall comment on this below. We look for a series expansion of the form

$$t|t| = \sum_{n=-\infty}^{\infty} c_n e^{int} ,$$

valid for $t \in (-\pi, \pi)$, where the Fourier coefficients are given by

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} t|t| e^{-int} dt \\ &= \frac{1}{2\pi} \int_{-\pi}^0 -t^2 e^{-int} dt + \frac{1}{2\pi} \int_0^{\pi} t^2 e^{-int} dt \\ &= \frac{1}{2\pi} \int_0^{\pi} t^2 (e^{-int} - e^{int}) dt \\ &= \frac{1}{i\pi} \int_0^{\pi} t^2 \sin nt dt . \end{aligned}$$

For $n = 0$ the integrand is zero, so that $c_0 = 0$. This is as expected because the function being odd will be expanded in a Fourier sine series. For $n \neq 0$, we can integrate this by parts using

$$t^2 \sin nt = -\frac{2}{n^2} \sin nt + \frac{2}{n^2} \frac{d}{dt} (t \sin nt) - \frac{1}{n} \frac{d}{dt} (t^2 \cos nt) .$$

The result is

$$c_n = \begin{cases} 0 & \text{for } n = 0; \text{ and} \\ \frac{i\pi}{n^2} (-1)^n + \frac{2i}{n^3\pi} (1 - (-1)^n) & \text{otherwise.} \end{cases}$$

As expected this assembles itself into a sine series:

$$t|t| = \sum_{n=1}^{\infty} \left[-\frac{2\pi(-1)^n}{n} - \frac{4(1 - (-1)^n)}{n^3\pi} \right] \sin nt .$$

Now notice the following. We saw above that the periodic extension of the function $t|t|$ is not continuous at $\pm\pi, \pm3\pi, \pm5\pi, \dots$. Evaluating the series at these points yields 0, which does not agree with either of the two possible limits of the function at those points: $\pm\pi^2$. In fact, it is the average of the two possible values. This is a general fact of Fourier series: *at a discontinuity the series converges to the average of the left and right limiting values.*

Solution to Problem 2.

The heat flow in a uniform rod of length L is governed by the partial differential equation

$$\frac{\partial^2 \Theta(x, t)}{\partial x^2} = \frac{1}{\kappa} \frac{\partial \Theta(x, t)}{\partial t}, \quad (1)$$

whose solution $\Theta(x, t)$ correspond to the temperature at a time t at the position x along the rod. In order to solve the equation, we must supplement it with suitable boundary and initial conditions. The boundary conditions are such that the ends of the rod are kept at zero temperature:

$$\Theta(0, t) = \Theta(L, t) = 0 \quad \text{for all } t.$$

The initial condition is that the initial temperature distribution is given by a function $f(x)$:

$$\Theta(x, 0) = f(x).$$

Of course, $f(0) = f(L) = 0$ for consistency with the boundary conditions.

We will employ separation of variables in order to solve the equation. Hence, let

$$\Theta(x, t) = u(x)v(t).$$

Into the equation,

$$u''(x)v(t) = \frac{1}{\kappa} u(x)\dot{v}(t) \quad \text{equivalently} \quad \frac{u''(x)}{u(x)} = \frac{1}{\kappa} \frac{\dot{v}(t)}{v(t)}.$$

Since the right-hand side is independent of t and the left-hand side is independent of x , and they are equal, they must be constant, $-\lambda$, say. Therefore the equations decouple into two equations:

$$u''(x) = -\lambda u(x) \quad \text{and} \quad \dot{v}(t) = -\kappa \lambda v(t).$$

The t -equation can be solved by

$$v(t) = v(0)e^{-\kappa \lambda t}.$$

The solution of the x -equation depends on whether λ is zero, positive or negative. As we explained in lecture for the similar equation appearing in the solution of the wave equation, the boundary condition $u(0) = u(L) = 0$ force $\lambda = \left(\frac{n\pi}{L}\right)^2 > 0$, for $n = 1, 2, \dots$. Therefore the most general solution subject to the boundary conditions is

$$\Theta(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) \exp\left(-\frac{\kappa n^2 \pi^2}{L^2}t\right), \quad (2)$$

for some coefficients c_n , which can be fixed by the initial conditions:

$$f(x) = \Theta(x, 0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) .$$

As we saw in class for the wave equation, we can solve for the coefficients c_n using the orthonormality of the functions $\sin(n\pi x/L)$:

$$\frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 1 & \text{if } n = m; \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Indeed,

$$\begin{aligned} c_n &= \frac{2}{L} \int_0^L \sum_{m=1}^{\infty} c_m \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx \\ &= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx . \end{aligned}$$

In the limit $t \rightarrow \infty$, the exponential in equation (2) goes to zero since $\kappa > 0$. Therefore,

$$\lim_{t \rightarrow \infty} \Theta(x, t) = 0 .$$

This is simply thermal equilibrium. For example, one way one can think of maintaining the ends of the rod always at zero temperature is to have them in contact with a much larger system (a thermal bath) at zero temperature. As time evolves, the rod transmits its heat to the thermal bath, which being very large does not undergo an appreciable increase in temperature.

Solution to Problem 3.

In this problem we are asked to compute the Fourier transforms of several functions. If $f(t)$ is square-integrable, so that

$$\int_{-\infty}^{\infty} |f(t)|^2 dt < \infty ,$$

then its Fourier transform is defined by

$$\hat{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt . \quad (3)$$

Let us compute this integral for several functions.

(a) Consider the function $f(t) = e^{-|t|}$ which obeys

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^0 e^{2t} dt + \int_0^{\infty} e^{-2t} dt = 2 \int_0^{\infty} e^{-2t} dt = 1 ,$$

so that it is square-integrable. Its Fourier transform is given by

$$\begin{aligned} \hat{f}(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|t|} e^{-i\omega t} dt \\ &= \frac{1}{2\pi} \left(\int_{-\infty}^0 e^{t(1-i\omega)} dt + \int_0^{\infty} e^{-t(1+i\omega)} dt \right) \\ &= \frac{1}{2\pi} \left(\int_0^{\infty} e^{-t(1-i\omega)} dt + \int_0^{\infty} e^{-t(1+i\omega)} dt \right) \\ &= \frac{1}{2\pi} \left(\frac{1}{1-i\omega} + \frac{1}{1+i\omega} \right) \\ &= \frac{1}{\pi} \frac{1}{1+\omega^2} . \end{aligned}$$

(b) Now consider $f(t) = e^{-t^2}$, which obeys

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} e^{-2t^2} dt = \sqrt{\frac{\pi}{2}} .$$

Its Fourier transform was worked out in the class exercises. Indeed, in Exercise 8.1 we worked out the Fourier transform of $e^{-\alpha t^2/2}$ to be

$$\frac{1}{\sqrt{2\pi\alpha}} e^{-\omega^2/2\alpha} .$$

In this problem we have the case $\alpha = 2$, whence the Fourier transform of $f(t) = e^{-t^2}$ is given by

$$\hat{f}(\omega) = \frac{1}{2\sqrt{\pi}} e^{-\omega^2/4} .$$

(c) This case can be worked out from the previous one. Notice that $f(t) = te^{-t^2}$ is again square-integrable:

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} t^2 e^{-2t^2} dt = \frac{1}{4} \sqrt{\frac{\pi}{2}} . \quad (4)$$

It is instructive to do this integral in detail, since it shows a useful trick. We start with

$$\int_{-\infty}^{\infty} e^{-\alpha t^2} dt = \sqrt{\frac{\pi}{\alpha}} .$$

Now take a derivative with respect to α on both sides of the equation:

$$\int_{-\infty}^{\infty} (-t^2) e^{-\alpha t^2} dt = -\frac{1}{2} \sqrt{\frac{\pi}{\alpha^3}} .$$

Therefore, setting $\alpha = 2$ we obtain (4).

In order to compute the Fourier transform, we do a similar trick:

$$\hat{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} t e^{-t^2} e^{-i\omega t} dt = i \frac{d}{d\omega} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-t^2} e^{-i\omega t} dt \right) .$$

But now the integral in the right-hand side is simply the Fourier transform of e^{-t^2} , which was computed in part (b) above; whence

$$\hat{f}(\omega) = i \frac{d}{d\omega} \left(\frac{1}{2\sqrt{\pi}} e^{-\omega^2/4} \right) = \frac{-i\omega}{4\sqrt{\pi}} e^{-\omega^2/4} .$$

- (d) Consider now the function $f(t) = (\sin t)/t$, which is square-integrable:

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} \frac{(\sin t)^2}{t^2} dt = \pi ,$$

a calculation which can be made using residue techniques, but which we will omit. The Fourier transform is given by

$$\begin{aligned} \hat{f}(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin t}{t} e^{-i\omega t} dt \\ &= \frac{1}{4\pi i} \int_{-\infty}^{\infty} \left(\frac{e^{-i(\omega-1)t}}{t} - \frac{e^{-i(\omega+1)t}}{t} \right) dt . \end{aligned}$$

It is therefore necessary to compute the following integral:

$$I(\alpha) = \int_{-\infty}^{\infty} \frac{e^{i\alpha t}}{t} dt , \quad \text{for } \alpha \text{ real,}$$

of which we did the case $\alpha = 1$ in the lecture:

$$I(1) = \int_{-\infty}^{\infty} \frac{e^{it}}{t} dt = i\pi .$$

Let us suppose that $\alpha \neq 0$, and let $x = \alpha t$, so that

$$\begin{aligned} I(\alpha) &= \begin{cases} \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx & \text{for } \alpha > 0; \text{ and} \\ -\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx & \text{for } \alpha < 0. \end{cases} \\ &= \begin{cases} +i\pi & \text{for } \alpha > 0; \text{ and} \\ -i\pi & \text{for } \alpha < 0. \end{cases} \end{aligned}$$

Therefore we have that

$$\hat{f}(\omega) = \frac{1}{4\pi i} (I(-\omega + 1) - I(-\omega - 1)) = \begin{cases} \frac{1}{2} & \text{for } |\omega| < 1; \text{ and} \\ 0 & \text{for } |\omega| > 1 . \end{cases}$$

This is, of course, not the end of the story, since we have not been able to compute $\hat{f}(\pm 1)$. This can be done from the Fourier inversion theorem, which says that at a discontinuity we obtain the average of the limiting values, hence $\hat{f}(\pm 1) = \frac{1}{4}$.

- (e) Finally consider the function $f(t) = (\sin \pi t)/(1 - t^2)$, which is square-integrable:

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} \frac{(\sin \pi t)^2}{(1 - t^2)^2} dt = \frac{\pi^2}{2} .$$

Its Fourier transform is given by

$$\hat{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin \pi t}{1 - t^2} e^{-i\omega t} dt = \frac{1}{4\pi i} \int_{-\infty}^{\infty} \frac{e^{i(\pi-\omega)t} - e^{-i(\pi+\omega)t}}{1 - t^2} dt .$$

This means that we have to compute integrals of the form

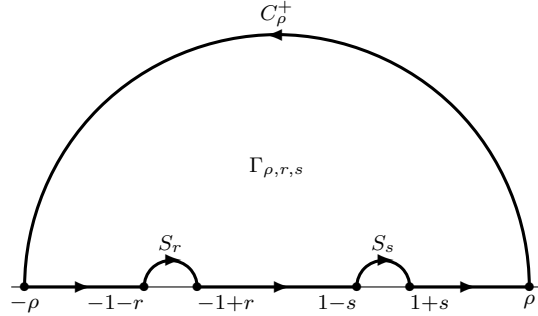
$$I(\alpha) = \int_{-\infty}^{\infty} \frac{e^{-i\alpha t}}{1 - t^2} dt .$$

First of all, one notices that the integrand has discontinuities at $t = \pm 1$, whence (the principal value of) the integral is defined by

$$I(\alpha) = \lim_{\substack{\rho \rightarrow \infty \\ r, s \searrow 0}} \left[\int_{-\rho}^{-1-r} + \int_{-1+r}^{1-s} + \int_{1+s}^{\rho} \right] \frac{e^{-i\alpha t}}{1 - t^2} dt ,$$

which as we saw in the lectures can be computed using the residue theorem. In order to apply the residue theorem we must close the contour. This involves using small semicircles to connect $-1 - r$ to $-1 + r$ and $1 - s$ to $1 + s$, as well as a large semicircle connecting ρ and $-\rho$. If $\alpha > 0$ (resp. $\alpha < 0$) the large semicircle has to be chosen to lie in the lower (resp. upper) half-plane, so that the exponential is bounded. If $\alpha = 0$ either choice will do. We will treat the case $\alpha \leq 0$ first. Consider the closed contour $\Gamma_{\rho, r, s}$

depicted below:



Because the integrand is analytic in and on the contour, the integral theorem says that

$$\left[\int_{-\rho}^{-1-r} + \int_{S_r} + \int_{-1+r}^{1-s} + \int_{S_s} + \int_{1+s}^{\rho} + \int_{C_{\rho}^+} \right] \frac{e^{-i\alpha z}}{1-z^2} dz = 0 ,$$

and this remains zero in the limits $\rho \rightarrow \infty$ and $r, s \searrow 0$. Now, the Jordan lemma says that in this limit the integral over C_{ρ}^+ vanishes. Therefore,

$$\begin{aligned} I(\alpha) &= \lim_{r,s \searrow 0} \left[- \int_{S_r} - \int_{S_s} \right] \frac{e^{-i\alpha z}}{1-z^2} dz \\ &= \lim_{r,s \searrow 0} \left[\int_{-S_r} + \int_{-S_s} \right] \frac{e^{-i\alpha z}}{1-z^2} dz . \end{aligned}$$

From the result derived in class about the integrals around small circular arcs about simple poles, we have that

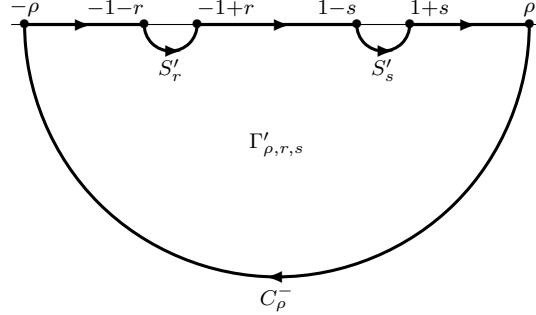
$$\begin{aligned} \lim_{r \searrow 0} \int_{-S_r} \frac{e^{-i\alpha z}}{1-z^2} dz &= i\pi \operatorname{Res}(-1) = \frac{i\pi}{2} e^{i\alpha} \\ \lim_{s \searrow 0} \int_{-S_s} \frac{e^{-i\alpha z}}{1-z^2} dz &= i\pi \operatorname{Res}(1) = \frac{-i\pi}{2} e^{-i\alpha} ; \end{aligned}$$

whence, for $\alpha < 0$,

$$I(\alpha) = \frac{i\pi}{2} (e^{i\alpha} - e^{-i\alpha}) = -\pi \sin \alpha .$$

Similarly, if $\alpha \geq 0$, we must close the contour in the lower half-plane, and we choose to integrate over the closed contour $\Gamma'_{\rho,r,s}$

depicted below:



Again, the integral the closed contour vanishes since the integrand is analytic in and on the contour. Using the Jordan lemma, the integral along C_ρ^- vanishes in the limit $\rho \rightarrow \infty$, so that

$$I(\alpha) = \lim_{r,s \searrow 0} - \left[\int_{S'_r} + \int_{S'_s} \right] \frac{e^{-i\alpha z}}{1-z^2} dz .$$

Again, we can appeal to the result derived in class about the integrals along small circular arcs about simple poles, to obtain

$$\begin{aligned} \lim_{r \searrow 0} \int_{S'_r} \frac{e^{-i\alpha z}}{1-z^2} dz &= i\pi \operatorname{Res}(-1) = \frac{i\pi}{2} e^{i\alpha} \\ \lim_{s \searrow 0} \int_{S'_s} \frac{e^{-i\alpha z}}{1-z^2} dz &= i\pi \operatorname{Res}(1) = \frac{-i\pi}{2} e^{-i\alpha} ; \end{aligned}$$

whence, for $\alpha > 0$,

$$I(\alpha) = -\frac{i\pi}{2} (e^{i\alpha} - e^{-i\alpha}) = \pi \sin \alpha .$$

In summary we have that for all (real) α ,

$$I(\alpha) = \pi \sin |\alpha| .$$

Finally, we can compute the Fourier transform:

$$\begin{aligned} \hat{f}(\omega) &= \frac{1}{4\pi i} (I(\omega - \pi) - I(\omega + \pi)) \\ &= \frac{1}{4i} (\sin |\omega - \pi| - \sin |\omega + \pi|) , \end{aligned}$$

or equivalently,

$$\hat{f}(\omega) = \begin{cases} \frac{1}{2i} \sin \omega & \text{for } |\omega| \leq \pi; \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Solution to Problem 4.

In this problem we will compute some Fourier transforms and verify the inversion formula. We do this for three functions.

(a) Consider the function

$$f(t) = \frac{1}{1+t^4},$$

whose Fourier transform is defined by

$$\hat{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{1+t^4} dt.$$

We can perform this integral using the residue theorem after we close the contour with a large semicircle and use the Jordan lemma to argue that the integral along the semicircle can be neglected in the infinite radius limit. For $\omega < 0$ (resp. $\omega > 0$) we must close the contour in the upper (resp. lower) half-plane, whereas for $\omega = 0$ either one will do. The integrand has simple poles at the points $\lambda = \exp(i\pi/4)$, λ^3 , $\lambda^5 = -\lambda^3$ and $\lambda^7 = -\lambda$, of which λ and λ^3 are in the upper half-plane and λ^5 and λ^7 are in the lower half-plane. Therefore using the residue theorem we have that

$$\hat{f}(\omega) = \begin{cases} +i [\text{Res}(F; \lambda) + \text{Res}(F; \lambda^3)] & \text{for } \omega \leq 0, \text{ and} \\ -i [\text{Res}(F; -\lambda) + \text{Res}(F; -\lambda^3)] & \text{for } \omega \geq 0, \end{cases} \quad (5)$$

where the function $F(z)$ is defined by

$$F(z) = \frac{e^{i\omega z}}{1+z^4}.$$

Computing the residues we find that

$$\text{Res}(\pm\lambda) = \pm \frac{e^{\mp i\omega\lambda}}{4\lambda^3} \quad \text{and} \quad \text{Res}(\pm\lambda^3) = \pm \frac{e^{\mp i\omega\lambda^3}}{4\lambda}.$$

Into equation (5)

$$\begin{aligned} \hat{f}(\omega) &= \begin{cases} \frac{ie^{-i\omega\lambda}}{4\lambda^3} + \frac{ie^{-i\omega\lambda^3}}{4\lambda} & \text{for } \omega \leq 0; \text{ and} \\ \frac{ie^{i\omega\lambda}}{4\lambda^3} + \frac{ie^{i\omega\lambda^3}}{4\lambda} & \text{for } \omega \geq 0. \end{cases} \\ &= \frac{i}{4} \left[\frac{e^{i|\omega|\lambda}}{\lambda^3} + \frac{e^{i|\omega|\lambda^3}}{\lambda} \right]. \end{aligned}$$

Using that $\lambda = (1 + i)/\sqrt{2}$ and that $\lambda^3 = (-1 + i)/\sqrt{2}$, and after a little bit of algebra, we find that

$$\begin{aligned}\hat{f}(\omega) &= \frac{1}{2\sqrt{2}}e^{-|\omega|/\sqrt{2}} \left[\sin\left(\frac{|\omega|}{\sqrt{2}}\right) + \cos\left(\frac{|\omega|}{\sqrt{2}}\right) \right] \\ &= \frac{1}{2}e^{-|\omega|/\sqrt{2}} \sin\left(\frac{|\omega|}{\sqrt{2}} + \frac{\pi}{4}\right).\end{aligned}$$

Let us now verify the inversion formula. Because the original function $f(t) = 1/(1 + t^4)$ is continuous, we expect that

$$\int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega = f(t),$$

or in this case

$$I = \frac{1}{2} \int_{-\infty}^{\infty} e^{-|\omega|/\sqrt{2}} \sin\left(\frac{|\omega|}{\sqrt{2}} + \frac{\pi}{4}\right) e^{i\omega t} d\omega \stackrel{?}{=} \frac{1}{1 + t^4}.$$

Let us verify that this is the case. Splitting the integral I into two parts: $\int_{-\infty}^0$ and \int_0^{∞} and changing variables to $-\omega$ in the first integral, and adding the two integrals (now both \int_0^{∞} , we arrive at

$$I = \int_0^{\infty} e^{-\omega/\sqrt{2}} \sin\left(\frac{\omega}{\sqrt{2}} + \frac{\pi}{4}\right) \cos \omega t d\omega.$$

If we expand the trigonometric functions into exponentials, this is simply a sum of integrals of exponential functions. Let us introduce the notation $E(\alpha)$ for the exponential integral

$$E(\alpha) \equiv \int_0^{\infty} e^{-\alpha\omega} d\omega = \frac{1}{\alpha}, \quad \text{for } \operatorname{Re}(\alpha) > 0. \quad (6)$$

Then it is clear that

$$\begin{aligned}I &= \frac{1}{4i}e^{i\pi/4} \left[E\left(\frac{1-i}{\sqrt{2}} - it\right) + E\left(\frac{1-i}{\sqrt{2}} + it\right) \right] \\ &\quad - \frac{1}{4i}e^{-i\pi/4} \left[E\left(\frac{1+i}{\sqrt{2}} - it\right) + E\left(\frac{1+i}{\sqrt{2}} + it\right) \right],\end{aligned}$$

which is simply the real part of

$$\begin{aligned}\frac{1}{2i}e^{i\pi/4} \left[E\left(\frac{1-i}{\sqrt{2}} - it\right) + E\left(\frac{1-i}{\sqrt{2}} + it\right) \right] \\ = \frac{1}{2i}e^{i\pi/4} \left[\frac{1}{\frac{1-i}{\sqrt{2}} - it} + \frac{1}{\frac{1-i}{\sqrt{2}} + it} \right].\end{aligned}$$

We can simplify this as follows:

$$\frac{1}{\frac{1-i}{\sqrt{2}} - it} + \frac{1}{\frac{1-i}{\sqrt{2}} + it} = \frac{2(1-i)/\sqrt{2}}{-i + t^2} = \frac{2e^{-i\pi/4}}{-i + t^2} ,$$

whence

$$I = \operatorname{Re} \left(\frac{1}{i} \frac{1}{-i + t^2} \right) = \operatorname{Re} \left(\frac{1}{1 + it^2} \right) = \frac{1}{1 + t^4} .$$

(b) Now consider the continuous function

$$f(t) = \frac{t}{1 + t^4} ,$$

whose Fourier transform is given by

$$\hat{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{t}{1 + t^4} e^{-i\omega t} dt .$$

In this case we don't have to perform the integral anew, since we can use the result of part (a) after noticing that

$$\hat{f}(\omega) = i \frac{d}{d\omega} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1 + t^4} e^{-i\omega t} dt \right) ,$$

which we computed in part (a). Indeed,

$$\hat{f}(\omega) = i \frac{d}{d\omega} \left[\frac{1}{2} e^{-|\omega|/\sqrt{2}} \sin \left(\frac{|\omega|}{\sqrt{2}} + \frac{\pi}{4} \right) \right] .$$

Taking care with the absolute values, that is, breaking the calculation of the derivative into two cases: $\omega \leq 0$ and $\omega \geq 0$, we find after using standard trigonometric addition identities,

$$\hat{f}(\omega) = \frac{-i}{2} e^{-|\omega|/\sqrt{2}} \sin \frac{\omega}{\sqrt{2}} .$$

The verification of the inversion formula can be done as before. We will be brief. It is easy to see that

$$I = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega = \int_0^{\infty} e^{-\omega/\sqrt{2}} \sin \frac{\omega}{\sqrt{2}} \sin \omega t d\omega ,$$

which upon expansion of the sines into exponentials can be written in terms of the exponential integrals (6):

$$I = -\frac{1}{4} \left[E \left(\frac{1-i}{\sqrt{2}} - it \right) - E \left(\frac{1-i}{\sqrt{2}} + it \right) + E \left(\frac{1+i}{\sqrt{2}} + it \right) - E \left(\frac{1+i}{\sqrt{2}} - it \right) \right] ,$$

or equivalently

$$I = -\frac{1}{4} \left[\frac{1}{\frac{1-i}{\sqrt{2}} - it} - \frac{1}{\frac{1-i}{\sqrt{2}} + it} + \frac{1}{\frac{1+i}{\sqrt{2}} + it} - \frac{1}{\frac{1+i}{\sqrt{2}} - it} \right] .$$

Simplifying this expression we see that

$$I = -\frac{1}{4} \left[\frac{2it}{-i+t^2} + \frac{-2it}{i+t^2} \right] = -\frac{1}{4} 2it \frac{2i}{1+t^4} = \frac{t}{1+t^4} .$$

(c) Finally, we consider the function

$$f(t) = e^{-t^2} ,$$

whose Fourier transform was computed in Problem 3 (b):

$$\hat{f}(\omega) = \frac{1}{2\sqrt{\pi}} e^{-\omega^2/4} .$$

To verify the inversion formula it is necessary to compute the integral

$$I = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi}} e^{-\omega^2/4} e^{i\omega t} dt .$$

Change variables to $u = \omega/2$, so that

$$I = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} e^{-iu(-2t)} du , \quad (7)$$

which is easy to compute from the calculation in Problem 3 (b). Indeed, we know that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-t^2} e^{-i\omega t} dt = \frac{1}{2\sqrt{\pi}} e^{-\omega^2/4} ,$$

whence

$$\int_{-\infty}^{\infty} e^{-t^2} e^{-i\omega t} dt = \sqrt{\pi} e^{-\omega^2/4} .$$

Comparing with the expression for I in (7), we see that

$$I = \frac{1}{\sqrt{\pi}} \sqrt{\pi} e^{-(-2t)^2/4} = e^{-t^2} .$$

Solution to Problem 5.

In this problem we are going to use the Fourier transform to solve differential equations. The idea is to take the Fourier transform of the differential equation, then solve the resulting algebraic equation and invert the solution. This leaves the solution in terms of an integral which can be computed either by elementary means or with the use of the residue theorem. We will however stop short of computing the integrals. Let $\hat{f}(\omega)$ be the Fourier transform of a function $f(t)$. Then the Fourier transform of $\dot{f}(t)$ is $i\omega\hat{f}(\omega)$ and that of $\ddot{f}(t)$ is $-\omega^2\hat{f}(\omega)$. Consider a general second order inhomogeneous ordinary differential equation with constant coefficients $\{a_i\}$:

$$a_2 \frac{d^2 f(t)}{dt^2} + a_1 \frac{df(t)}{dt} + a_0 f(t) = g(t) .$$

Taking the Fourier transform of both sides of the equation we find the algebraic equation:

$$(-a_2\omega^2 + ia_1\omega + a_0) \hat{f}(\omega) = \hat{g}(\omega) .$$

This equation can be solved for $\hat{f}(\omega)$:

$$\hat{f}(\omega) = \frac{\hat{g}(\omega)}{-a_2\omega^2 + ia_1\omega + a_0} ,$$

which can be inverted to solve for $f(t)$:

$$f(t) = \int_{-\infty}^{\infty} \frac{\hat{g}(\omega)}{-a_2\omega^2 + ia_1\omega + a_0} e^{i\omega t} d\omega .$$

Now let us apply this to three differential equations.

(a) Consider the differential equation

$$\frac{d^2 f(t)}{dt^2} + \frac{df(t)}{dt} + f(t) = e^{-t^2} ;$$

that is, $a_0 = a_2 = 1$, $a_1 = 1$ and $g(t) = \exp(-t^2)$. Therefore as we saw in Problem 3 (b),

$$\hat{g}(\omega) = \frac{1}{2\sqrt{\pi}} e^{-\omega^2/4} ,$$

whence the solution of the differential equation is

$$f(t) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi}} \frac{e^{-\omega^2/4}}{1+i\omega-\omega^2} e^{i\omega t} d\omega .$$

(b) Now consider the equation

$$\frac{d^2 f(t)}{dt^2} + 4 \frac{df(t)}{dt} + f(t) = \begin{cases} 0, & \text{for } t < 0; \\ e^{-t}, & \text{for } t \geq 0, \end{cases}$$

so that $a_0 = a_2 = 1$, $a_1 = 4$ and

$$g(t) = \begin{cases} 0, & \text{for } t < 0; \\ e^{-t}, & \text{for } t \geq 0, \end{cases}$$

whose Fourier transform is

$$\hat{g}(\omega) = \frac{1}{2\pi} \int_0^{\infty} e^{-(1+i\omega)t} dt = \frac{1}{2\pi} \frac{1}{1+i\omega} .$$

Therefore the solution of the equation is

$$f(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{e^{i\omega t}}{(1+i\omega)(-\omega^2+4i\omega+1)} d\omega .$$

(c) Finally, consider the equation

$$\frac{d^2 f(t)}{dt^2} + 2 \frac{df(t)}{dt} + 3f(t) = \begin{cases} 1, & \text{for } |t| < 1; \\ 0, & \text{otherwise,} \end{cases}$$

where $a_2 = 1$, $a_1 = 2$, $a_0 = 3$ and

$$g(t) = \begin{cases} 1, & \text{for } |t| < 1; \\ 0, & \text{otherwise,} \end{cases}$$

whose Fourier transform is

$$\hat{g}(\omega) = \frac{1}{2\pi} \int_{-1}^1 e^{-i\omega t} dt = \frac{\sin \omega}{\pi \omega} .$$

Therefore the solution to the equation is

$$f(t) = \int_{-\infty}^{\infty} \frac{\sin \omega e^{i\omega t}}{\pi \omega (-\omega^2 + 2i\omega + 3)} d\omega .$$

Solution to Problem 6.

Consider again the heat equation (1), but this time with an infinitely long rod. Suppose that at $t = 0$ the temperature distribution is given by a continuous function $f(x)$ with Fourier transform $\hat{f}(k)$. Then let

$$\Theta(x, t) = \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} e^{-\kappa k^2 t} dk .$$

We claim that $\Theta(x, t)$ solves the differential equation with the initial condition $\Theta(x, 0) = f(x)$. Notice that

$$\Theta(x, 0) = \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk ,$$

which by the Fourier inversion theorem is equal to $f(x)$. (We are using that $f(x)$ is continuous.) Now let us see that $\Theta(x, t)$ solves the heat equation:

$$\frac{\partial^2 \Theta(x, t)}{\partial x^2} = \int_{-\infty}^{\infty} (-k^2) \hat{f}(k) e^{ikx} e^{-\kappa k^2 t} dk ,$$

whereas

$$\frac{\partial \Theta(x, t)}{\partial t} = \int_{-\infty}^{\infty} (-\kappa k^2) \hat{f}(k) e^{ikx} e^{-\kappa k^2 t} dk .$$

Therefore the heat equation is satisfied. Let us now insert the definition of the Fourier transform,

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) e^{-iky} dy ,$$

into the integral expression for $\Theta(x, t)$:

$$\Theta(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(y) e^{-iky} dy \right] e^{ikx} e^{-\kappa k^2 t} dk .$$

Inverting the order of integration, we have

$$\Theta(x, t) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-y)} e^{-\kappa k^2 t} dk \right] f(y) dy .$$

Let $K(x - y, t)$ denote the integral within brackets. Notice that it is formally the same integral as the Fourier transform of the gaussian function which was computed in Exercise 8.1. We can therefore read off the answer from the solution to that exercise and obtain that

$$K(x - y, t) = \frac{1}{2\sqrt{\pi\kappa t}} e^{-(x-y)^2/2\kappa t} , \quad \text{for } t > 0. \quad (8)$$

It is easy to show that $K(x, t)$ obeys the heat equation (1). Indeed,

$$\frac{\partial^2 K(x, t)}{\partial x^2} = \frac{e^{-x^2/4\kappa t}}{2\sqrt{\pi\kappa t}} \left[\frac{x^2}{4\kappa^2 t^2} - \frac{1}{2\kappa t} \right],$$

whereas

$$\frac{\partial K(x, t)}{\partial t} = \frac{e^{-x^2/4\kappa t}}{2\sqrt{\pi\kappa t}} \left[\frac{x^2}{4\kappa t^2} - \frac{1}{2t} \right],$$

which shows that $K(x, t)$ obeys the heat equation. Finally, we determine the initial condition at $t = 0$. The expression (8) is only valid for $t > 0$. This makes sense because for $t = 0$ the heat kernel $K(x, 0)$ is not a function. Indeed, we are asked to show that $K(x, 0) = \delta(x)$, or equivalently that

$$\lim_{t \searrow 0} \int_{-\infty}^{\infty} K(x, t) f(x) dx = f(0). \quad (9)$$

We can argue indirectly by the fact that $\delta(x)$ is the Fourier transform of the constant function 1 and hence to see that the Fourier transform of $K(x, t)$ is indeed 1 for $t = 0$. Alternatively, we can try to prove (9) directly. The following derivation is not meant to be rigorous, but let us do it nonetheless. Suppose that we expand $f(x)$ in a Taylor series around 0:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) x^n$$

and that we integrate it term by term:

$$\begin{aligned} \int_{-\infty}^{\infty} K(x, t) f(x) dx &= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) \int_{-\infty}^{\infty} K(x, t) x^n dx \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi\kappa t}} e^{-x^2/2\kappa t} x^n dx. \end{aligned}$$

For any $t > 0$ introduce $y = x/2\sqrt{\kappa t}$. Then we can rewrite the above as

$$\begin{aligned} \int_{-\infty}^{\infty} K(x, t) f(x) dx &= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) \int_{-\infty}^{\infty} K(x, t) x^n dx \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(2\sqrt{\kappa t})^n}{\sqrt{\pi}} f^{(n)}(0) \int_{-\infty}^{\infty} e^{-y^2} y^n dy. \end{aligned}$$

The integral vanishes for n odd and for n even it is a number I_n which is independent of t . Therefore, we find that

$$\int_{-\infty}^{\infty} K(x, t) f(x) dx = \sum_{\ell=0}^{\infty} \frac{1}{(2\ell)!} \frac{(4\kappa t)^\ell}{\sqrt{\pi}} f^{(2\ell)}(0) I_{2\ell} .$$

Taking the limit $t \searrow 0$, we see that only the $\ell = 0$ term remains:

$$\lim_{t \searrow 0} \int_{-\infty}^{\infty} K(x, t) f(x) dx = \frac{1}{\sqrt{\pi}} f(0) I_0 = f(0) ,$$

since $I_0 = \sqrt{\pi}$.