

Mathematical Techniques III (PHY 317)

Solutions to Problem Set 10

Solution to Problem 1.

Here we work out the Laplace transforms of several functions.

(a) Consider the function

$$f(t) = 3 \cos 2t - 8e^{-2t} .$$

By linearity, the Laplace transform $F(s)$ of $f(t)$ is the sum of the Laplace transforms of each of the terms:

$$F(s) \equiv \mathcal{L} \{f(t)\} = 3\mathcal{L} \{\cos 2t\} - 8\mathcal{L} \{e^{-2t}\} ,$$

which have been worked out in the lecture:

$$\mathcal{L} \{\cos 2t\} (s) = \frac{s}{s^2 + 4} \quad \text{and} \quad \mathcal{L} \{e^{-2t}\} (s) = \frac{1}{s + 2} ;$$

whence

$$F(s) = \frac{3s}{s^2 + 4} - \frac{8}{s + 2} .$$

The transform of the first function is valid for $\text{Re}(s) > 0$ and that of the second for $\text{Re}(s) > -2$. Therefore they are *both* valid for $\text{Re}(s) > 0$.

(b) Consider now the function

$$f(t) = \frac{1}{\sqrt{t}} .$$

By definition its Laplace transform is given by

$$F(s) \equiv \mathcal{L} \{f(t)\} = \int_0^\infty f(t) e^{-st} dt = \int_0^\infty \frac{e^{-st}}{\sqrt{t}} dt .$$

Let us change variables in the integral to $u = \sqrt{st}$. Then, $du = \frac{1}{2} \frac{\sqrt{s}}{\sqrt{t}} dt$, so that

$$F(s) = 2 \frac{1}{\sqrt{s}} \int_0^\infty e^{-u^2} du = \sqrt{\frac{\pi}{s}} .$$

The transform is valid provided that $\text{Re}(s) > 0$, since otherwise the integral would not converge.

(c) Consider now the function

$$f(t) = \begin{cases} 1, & \text{for } t < 1, \text{ and} \\ 0, & \text{for } t \geq 1. \end{cases},$$

whose Laplace transform is given by

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt = \int_0^1 e^{-st} dt = \frac{1}{s} (1 - e^{-s}) .$$

The transform is valid for *all* s : the singularity at $s = 0$ is removable, since $\lim_{s \rightarrow 0} F(s) = 1$.

(d) Now we have

$$f(t) = (\sin t)^2 .$$

Using a trigonometric identity, $(\sin t)^2 = \frac{1}{2}(1 - \cos 2t)$, and linearity of the Laplace transform, we have that

$$F(s) \equiv \mathcal{L} \{ (\sin t)^2 \} (s) = \frac{1}{2} \mathcal{L} \{ 1 \} (s) - \frac{1}{2} \mathcal{L} \{ \cos 2t \} (s) ,$$

which we have already worked out in class:

$$F(s) = \frac{1}{2} \frac{1}{s} - \frac{1}{2} \frac{s}{s^2 + 4} .$$

Again we have condition $\text{Re}(s) > 0$ from the first transform and $\text{Re}(s) > -2$ from the second. Hence both are valid whenever $\text{Re}(s) > 0$.

(e) Finally, we have

$$f(t) = \begin{cases} 0, & \text{for } t < 1, \\ 1, & \text{for } 1 \leq t \leq 2, \text{ and} \\ 0, & \text{for } t > 2. \end{cases},$$

whose Laplace transform is given by

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt = \int_1^2 e^{-st} dt = \frac{e^{-s}}{s} (1 - e^{-s}) .$$

Again the transform is valid for all s . Notice that the singularity at $s = 0$ is removable, since $\lim_{s \rightarrow 0} F(s) = 1$.

Solution to Problem 2.

In this problem we invert some Laplace transforms.

(a) Consider

$$F(s) = \frac{1}{s^2 + 4} .$$

Comparing with the list of transforms we saw in class we notice that

$$F(s) = \frac{1}{2} \frac{2}{s^2 + 4} = \frac{1}{2} \operatorname{Im} \frac{1}{s - 2i} ,$$

whence

$$F(s) = \mathcal{L} \left\{ \frac{1}{2} \operatorname{Im} e^{2it} \right\} = \mathcal{L} \left\{ \frac{1}{2} \sin 2t \right\} .$$

(b) Now consider

$$F(s) = \frac{4}{(s - 1)^2} .$$

Comparing with the results in class,

$$F(s) = 4 \frac{1}{(s - 1)^2} = \mathcal{L} \{ 4t e^t \} .$$

(c) Now we have

$$F(s) = \frac{s}{s^2 + 4s + 4} = \frac{s}{(s + 2)^2} .$$

Into partial fractions,

$$F(s) = \frac{s + 2 - 2}{(s + 2)^2} = \frac{1}{s + 2} - \frac{2}{(s + 2)^2} .$$

We can now read off the inverse transforms:

$$F(s) = \mathcal{L} \{ e^{-2t} - 2t e^{-2t} \} .$$

(d) Consider now

$$F(s) = \frac{1}{s^3 + 3s^2 + 2s} = \frac{1}{s(s + 1)(s + 2)} .$$

Again we expand into partial fractions:

$$F(s) = \frac{1}{2} \frac{1}{s} - \frac{1}{s + 1} + \frac{1}{2} \frac{1}{s + 2} ,$$

whence we can read off the inverse transforms:

$$F(s) = \mathcal{L} \left\{ \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t} \right\} .$$

(e) Finally we have

$$F(s) = \frac{s+3}{s^2+4s+7} .$$

The denominator factorises as $(s+2+i\sqrt{3})(s+2-i\sqrt{3})$, whence

$$\begin{aligned} F(s) &= \frac{s+3}{(s+2+i\sqrt{3})(s+2-i\sqrt{3})} \\ &= \frac{s+2}{(s+2+i\sqrt{3})(s+2-i\sqrt{3})} + \frac{1}{(s+2+i\sqrt{3})(s+2-i\sqrt{3})} \\ &= \frac{s+2}{(s+2)^2+(\sqrt{3})^2} + \frac{1}{\sqrt{3}} \frac{\sqrt{3}}{(s+2)^2+(\sqrt{3})^2} , \end{aligned}$$

from where we can read off the inverse transforms:

$$\begin{aligned} F(s) &= \mathcal{L} \left\{ e^{-2t} \cos \sqrt{3}t + \frac{1}{\sqrt{3}} e^{-2t} \sin \sqrt{3}t \right\} \\ &= \mathcal{L} \left\{ e^{-2t} \left(\cos \sqrt{3}t + \frac{1}{\sqrt{3}} \sin \sqrt{3}t \right) \right\} . \end{aligned}$$

Solution to Problem 3.

In this problem we solve some initial value problems. The method is the same in all cases: we take the Laplace transform of the equation taking into account the initial conditions, solve the corresponding algebraic equation, and invert back. Consider the following differential equation

$$\frac{d^2 f(t)}{dt^2} + a_1 \frac{df(t)}{dt} + a_0 f(t) = u(t) , \quad (1)$$

where a_i are constants and $u(t)$ is some function. Taking the Laplace transform of this differential equation we have

$$(s^2 F(s) - s f(0) - f'(0)) + a_1 (s F(s) - f(0)) + a_0 F(s) = U(s) ,$$

where $F(s) = \mathcal{L} \{f(t)\}$ and $U(s) = \mathcal{L} \{u(t)\}$. We can solve this algebraic equation for $F(s)$ in terms of $U(s)$ and the initial conditions

$$F(s) = \frac{U(s) + (s+a_1)f(0) + f'(0)}{s^2 + a_1 s + a_0} , \quad (2)$$

which we can then try to invert back. Let us apply this to the following differential equations.

(a) Consider first the equation

$$\frac{d^2 f(t)}{dt^2} - 5 \frac{df(t)}{dt} + 6f(t) = 0 ,$$

subject to the initial conditions $f(0) = 1$ and $f'(0) = -1$. This equation is of the form (1) with $a_1 = -5$ and $a_0 = 6$ and $u(t) = 0$. Therefore into (2) we see that

$$F(s) = \frac{(s-5) - 1}{s^2 - 5s + 6} = \frac{s-6}{(s-2)(s-3)} .$$

Decomposition into partial fractions, we have

$$F(s) = \frac{4}{s-2} - \frac{3}{s-3} ,$$

which we recognise as the Laplace transform of the function

$$f(t) = 4 e^{2t} - 3 e^{3t} .$$

Notice that this system is unstable, since $f(t)$ does not vanish in the limit $t \rightarrow \infty$.

(b) Now we have the differential equation

$$\frac{d^2 f(t)}{dt^2} - \frac{df(t)}{dt} - 2f(t) = e^{-t} \sin 2t ,$$

subject to the initial conditions $f(0) = f'(0) = 0$. Comparing with (1), we have $a_1 = -1$, $a_0 = -2$, and $u(t) = e^{-t} \sin 2t$. The Laplace transform of $u(t)$ is given by

$$U(s) = \frac{2}{(s+1)^2 + 4} ,$$

whence into (2), we find

$$F(s) = \frac{2}{((s+1)^2 + 4)(s^2 - s - 2)} = \frac{2}{((s+1)^2 + 4)(s+1)(s-2)} .$$

After a little algebra, we can rewrite this into partial fractions,

$$\begin{aligned} F(s) &= \frac{2}{39} \frac{1}{s-2} - \frac{1}{6} \frac{1}{s+1} + \frac{1}{26} \frac{3s-1}{(s+1)^2 + 4} \\ &= \frac{2}{39} \frac{1}{s-2} - \frac{1}{6} \frac{1}{s+1} + \frac{1}{26} \frac{3(s+1) - 4}{(s+1)^2 + 4} , \end{aligned}$$

whose last term we can rewrite as

$$\frac{1}{26} \frac{3(s+1) - 4}{(s+1)^2 + 4} = \frac{3}{26} \frac{s+1}{(s+1)^2 + 4} - \frac{1}{13} \frac{2}{(s+1)^2 + 4},$$

which allows us to perform the inverse transform. In fact we can easily see that

$$F(s) = \frac{2}{39} \frac{1}{s-2} - \frac{1}{6} \frac{1}{s+1} + \frac{3}{26} \frac{s+1}{(s+1)^2 + 4} - \frac{1}{13} \frac{2}{(s+1)^2 + 4}$$

is the Laplace transform of the function

$$f(t) = \frac{2}{39} e^{2t} - \frac{1}{6} e^{-t} + \frac{3}{26} e^{-t} \cos 2t - \frac{1}{13} e^{-t} \sin 2t.$$

This solution is again unstable, since in the absence of the forcing term $\lim_{t \rightarrow \infty} f(t) \neq 0$.

(c) Finally we have

$$\frac{d^2 f(t)}{dt^2} - 3 \frac{df(t)}{dt} + 2f(t) = \begin{cases} 0, & \text{for } 0 \leq t < 3, \\ 1, & \text{for } 3 \leq t \leq 6, \text{ and} \\ 0, & \text{for } t > 6, \end{cases}$$

subject to the initial conditions $f(0) = f'(0) = 0$. Comparing with (1) we have $a_1 = -3$ and $a_0 = 2$. The Laplace transform $U(s)$ of the function $u(t)$ given above, is given by (cf. Problem 1 (e))

$$U(s) = \frac{1}{s} (e^{-3s} - e^{-6s}).$$

Into (2), we have that

$$F(s) = \frac{e^{-3s} - e^{-6s}}{s(s^2 - 3s + 2)} = \frac{e^{-3s} - e^{-6s}}{s(s-1)(s-2)}.$$

Into partial fractions we have

$$F(s) = (e^{-3s} - e^{-6s}) \left(\frac{1}{2} \frac{1}{s} - \frac{1}{s-1} + \frac{1}{2} \frac{1}{s-2} \right).$$

From the lecture we have that

$$e^{-as} G(s) = \mathcal{L} \{ \theta(t-a) g(t-a) \} (s) \quad \text{where } G(s) = \mathcal{L} \{ g(t) \} (s),$$

where θ is the Heaviside step function. Therefore we can read off the function $f(t)$ whose Laplace transform is $F(s)$:

$$f(t) = \theta(t-3) \left(\frac{1}{2} - e^{t-3} + \frac{1}{2} e^{2(t-3)} \right) - \theta(t-6) \left(\frac{1}{2} - e^{t-6} + \frac{1}{2} e^{2(t-6)} \right) ,$$

or equivalently

$$f(t) = \begin{cases} 0 , & \text{for } t < 3; \\ \frac{1}{2} - e^{t-3} + \frac{1}{2} e^{2(t-3)} , & \text{for } 3 \leq t \leq 6; \text{ and} \\ e^{t-6} + \frac{1}{2} e^{2(t-6)} - e^{t-3} - \frac{1}{2} e^{2(t-3)} , & \text{for } t > 6. \end{cases}$$

Again we see that the system is not stable, since even in the absence of the forcing term $\lim_{t \rightarrow \infty} f(t) \neq 0$.