Mathematical Techniques III
(PHY 317)

Solutions to Problem Set 1

Solution to Problem 1.

This problem and the next are asking you to prove that certain sets with certian operations are vector spaces. Recall from lecture that a vector space consists of a set of vectors ($\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$,...), a set of scalars (λ, μ, ...) which in this case are real numbers, and two operations (vector addition and scalar multiplication) obeying a set of eight axioms, which we recall here for convenience:

1. $(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w} = \boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$;
2. $\boldsymbol{u}+\boldsymbol{v} = \boldsymbol{v}+\boldsymbol{u}$;
3. There exists a zero vector $\mathbf{0}$ which obeys $\mathbf{0}+\boldsymbol{v} = \boldsymbol{v}$;
4. For any given $\boldsymbol{v}$, there exists a vector $-\boldsymbol{v}$ such that $\boldsymbol{v}+(-\boldsymbol{v}) = \mathbf{0}$;
5. $\lambda\,(\mu\,\boldsymbol{v}) = (\lambda\,\mu)\,\boldsymbol{v}$;
6. $1\,\boldsymbol{v} = \boldsymbol{v}$ for all $\boldsymbol{v}$;
7. $(\lambda+\mu)\,\boldsymbol{v} = \lambda\,\boldsymbol{v}+\mu\,\boldsymbol{v}$; and
8. $\lambda\,(\boldsymbol{u}+\boldsymbol{v}) = \lambda\,\boldsymbol{u}+\lambda\,\boldsymbol{v}$.

Notice that the real numbers satisfy these eight axioms; that is, if we take the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, ... to be real numbers themselves, then the eight axioms become familiar properties of the real numbers.

Now notice that the way that vector addition and scalar multiplication are defined in $\mathbb{R}^N$, we are performing these operations in each slot of the N-tuple independently. Since each slot is a real number, the eight axioms for $\mathbb{R}^N$ follow from the ones for the real numbers. Therefore $\mathbb{R}^N$ is a vector space.

For each of the three subsets of $\mathbb{R}^N$ in the Problem we have to determine whether they are closed under vector addition and scalar multiplication. Clearly the first subset is closed under both because if

$$v_1+v_2+\cdots+v_N = 0 \quad \text{and} \quad w_1+w_2+\cdots+w_N = 0\ ,$$

then

$$(v_1+w_1)+(v_2+w_2)+\cdots+(v_N+w_N) = 0\ ;$$

and, similarly, if $v_1 + v_2 + \cdots + v_N = 0$ then

$$(\lambda\, v_1) + (\lambda\, v_2) + \cdots + (\lambda\, v_N) = 0 \; .$$

Therefore the first subset is a vector subspace.

The second subset is not closed under either operation, e.g., if

$$v_1 + v_2 + \cdots + v_N = 1 \quad \text{and} \quad w_1 + w_2 + \cdots + w_N = 1 \; ,$$

then

$$(v_1 + w_1) + (v_2 + w_2) + \cdots + (v_N + w_N) = 2 \; ;$$

hence it is not a vector subspace.

Finally, the third subset is closed under vector addition because the sum of two rational numbers is again rational (Prove it!). However if λ is irrational and v_i rational, then $\lambda\, v_i$ is irrational. Therefore it is not closed under scalar multiplication, so that it is not a vector subspace.

Solution to Problem 2.

In this case, the set $\mathcal{F}$ of real-valued functions in the interval inherits the operations of vector addition and scalar multiplication from those in $\mathbb{R}$ by defining them pointwise. Therefore the axioms follow again from those of $\mathbb{R}$.

As for the three subsets: the first one is closed under both addition and scalar multiplication: if $f(0) = 0$ and $g(0) = 0$, $(f + g)(0) = f(0) + g(0) = 0 + 0 = 0$. Similarly if $f(0) = 0$, then $(\lambda\, f)(0) = \lambda\, f(0) = 0$. So it is a vector subspace.

The second subset is not closed under either vector addition or scalar multiplication since if $f(0) = 1$ and $g(0) = 1$, then

$$(f + g)(0) = f(0) + g(0) = 1 + 1 = 2 \; .$$

The third subset is closed under both operations. Indeed, if $f(-1) = f(1)$ and $g(-1) = g(1)$ then,

$$\begin{gathered}(f + g)(-1) = f(-1) + g(-1) = f(1) + g(1) = (f + g)(1) \\ (\lambda\, f)(-1) = \lambda\, f(-1) = \lambda\, f(1) = (\lambda\, f)(1) \; .\end{gathered}$$

Finally the fourth subset is also closed under both vector addition and scalar multiplication. Let $f'' + 2f' - 5f = 0$ and $g'' + 2g' - 5g = 0$.

Then using the fact that $(f+g)' = f' + g'$ and similarly for the second derivative $(f+g)'' = f'' + g''$, we see that

$$(f+g)'' + 2(f+g)' - 5(f+g) = f'' + g'' + 2f' + 2g' - 5f - 5g = 0 \ .$$

Finally scalar multiplication works the same way after noticing that $(\lambda f)' = \lambda f'$, etc.

Solution to Problem 3.

We can think of the plane as $\mathbb{R}^2$ as follows. A displacement in the plane with vector $\boldsymbol{r} = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}}$ has coordinates $(x,y) \in \mathbb{R}^2$. Under this identification, therefore the basis $\{\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}\}$ for the plane correspond to the canonical basis for $\mathbb{R}^2$, which we write as $\{\mathsf{e}_1, \mathsf{e}_2\}$.

A rotation by an angle θ on the plane has the following effect on the coordinates:

$$x \mapsto x' = \cos\theta\, x - \sin\theta\, y \quad \text{and} \quad y \mapsto y' = \sin\theta\, x + \cos\theta\, y \ ,$$

which is clearly linear. Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ denote the corresponding linear map. A linear transformation is uniquely characterised by how it acts on a basis, and on the canonical basis R_θ acts as:

$$\begin{aligned} \mathsf{e}_1 \mapsto R_\theta(\mathsf{e}_1) &= \cos\theta\, \mathsf{e}_1 + \sin\theta\, \mathsf{e}_2 \\ \mathsf{e}_2 \mapsto R_\theta(\mathsf{e}_2) &= -\sin\theta\, \mathsf{e}_1 + \cos\theta\, \mathsf{e}_2 \ . \end{aligned}$$

Now the matrix R_θ representing R_θ in this basis has the form

$$\mathsf{R}_\theta = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \ ,$$

where the entries R_{ij} are defined by

$$R_\theta(\mathsf{e}_j) = \sum_{i=1}^{2} R_{ij}\, \mathsf{e}_i = R_{1j}\, \mathsf{e}_1 + R_{2j}\, \mathsf{e}_2 \ .$$

From the action of R_θ on the canonical basis, we can then read off the matrix:

$$\mathsf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \ ,$$

whose determinant is given by

$$\det \mathsf{R}_\theta = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = (\cos\theta)^2 + (\sin\theta)^2 = 1 \ .$$

In Lecture we saw that matrix multiplication is simply composition of linear maps. Therefore the matrix product $\mathsf{R}_\theta \mathsf{R}_\phi$ is the matrix of the composition of two rotations: one by angle ϕ followed by one by angle θ. Geometrically it is clear that the composition corresponds to a single rotation by angle $\theta+\phi$. Therefore we know that $\mathsf{R}_\theta \mathsf{R}_\phi = \mathsf{R}_{\theta+\phi}$. We can check this explicitly, of course:

$$\begin{aligned}
\mathsf{R}_\theta \mathsf{R}_\phi &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix} \\
&= \mathsf{R}_{\theta+\phi} \ .
\end{aligned}$$

Notice that we could have turned this calculation around and used it to *derive* the trigonometric identities for addition of angles.

Similarly, the inverse of a matrix is the matrix of the inverse linear transformation. Now the inverse of a rotation by θ is a rotation by $-\theta$, therefore $\mathsf{R}_\theta^{-1} = \mathsf{R}_\theta$. Let us nevertheless check this explicitly. The inverse of an arbitrary (invertible) 2×2 matrix is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \ .$$

Applying this to R_θ, we find

$$\mathsf{R}_\theta^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta & \cos(-\theta) \end{pmatrix} \ ,$$

where we have used that the cosine and sine functions are even and odd, respectively:

$$\cos(-\theta) = \cos\theta \quad \text{and} \quad \sin(-\theta) = -\sin\theta \ .$$

A reflection in the x-axis acts on the coordinates on the plane as follows:

$$x \mapsto x' = x \qquad \text{and} \qquad y \mapsto y' = -y \ ,$$

which is again clearly linear. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ denote the corresponding linear transformation. Its action on the canonical basis is

$$\mathsf{e}_1 \mapsto F(\mathsf{e}_1) = \mathsf{e}_1 \qquad \text{and} \qquad \mathsf{e}_2 \mapsto F(\mathsf{e}_2) = -\mathsf{e}_2 \ ,$$

from where we can read off the matrix F as was done above for the rotation:

$$\mathsf{F} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Its determinant is clearly -1. Because $\mathsf{F}^2 = \mathbb{1}$, the identity matrix, we see that F is its own inverse: $\mathsf{F}^{-1} = \mathsf{F}$. One can understand this geometrically by drawing a picture and contemplating what a reflection does. A linear transformation which is its own inverse is called an **involution**.

Finally we are asked to prove that $\mathsf{F}\mathsf{R}_\theta = \mathsf{R}_\theta^{-1}\mathsf{F}$. This is simply matrix multiplication, but one can understand this geometrically without having to do any calculation. One simply notices that this equation can be written as

$$\mathsf{R}_\theta^{-1} = \mathsf{F}\mathsf{R}_\theta\mathsf{F} .$$

The right-hand side is the composition of three linear transformations: a reflection on the x-axis, a rotation by an angle θ and a second reflection on the x-axis. If you draw a picture, the equation is obvious.

Solution to Problem 4.

Let us take each of the eight matrices in turn:

- $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

  It has determinant 1 and acts as follows on the canonical basis:

  $$\mathsf{e}_1 \mapsto -\mathsf{e}_1 \quad \text{and} \quad \mathsf{e}_2 \mapsto \mathsf{e}_2 .$$

  It can be interpreted geometrically as a reflection on the e_1-axis.

- $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

  It has determinant 1, acts as follows on the canonical basis:

  $$\mathsf{e}_1 \mapsto \mathsf{e}_2 \quad \text{and} \quad \mathsf{e}_2 \mapsto -\mathsf{e}_1 ;$$

  and can be interpreted geometrically as a $\pi/2$ rotation in the plane.

- $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

  It has determinant 6, acts as follows on the canonical basis:

  $$\mathsf{e}_1 \mapsto 2\mathsf{e}_1 \quad \text{and} \quad \mathsf{e}_2 \mapsto 3\mathsf{e}_2 ;$$

and can be interpreted geometrically as stretching the plane by a factor of 2 in the direction e_1 and by a factor of 3 in the direction e_2.

- $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

  It has determinant 0 and acts as follows on the canonical basis:

  $$\mathsf{e}_1 \mapsto \mathsf{e}_1 \quad \text{and} \quad \mathsf{e}_2 \mapsto \mathsf{0} \ ,$$

  where $\mathsf{0}$ is the zero column vector. It can be interpreted geometrically as an orthogonal projection onto the e_1-axis along the e_2 direction.

- $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$

  It has determinant 1 and acts as follows on the canonical basis:

  $$\mathsf{e}_1 \mapsto \mathsf{e}_1 \quad \text{and} \quad \mathsf{e}_2 \mapsto a\,\mathsf{e}_1 + \mathsf{e}_2 \ .$$

  It can be interpreted geometrically as a shear of the plane.

- $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

  It has determinant 1 and it acts as follows on the canonical basis:

  $$\mathsf{e}_1 \mapsto \cos\theta\,\mathsf{e}_1 + \sin\theta\,\mathsf{e}_2 \ , \mathsf{e}_2 \mapsto -\sin\theta\,\mathsf{e}_1 + \cos\theta\,\mathsf{e}_2 \ , \text{and } \mathsf{e}_3 \mapsto \mathsf{e}_3 \ .$$

  It can be interpreted geometrically as a rotation by an angle θ about the e_3-axis.

- $\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$

  It has determinant 1 and it acts as follows on the canonical basis:

  $$\mathsf{e}_1 \mapsto \cos\theta\,\mathsf{e}_1 + \sin\theta\,\mathsf{e}_3 \ , \mathsf{e}_2 \mapsto \mathsf{e}_2 \ , \text{and } \mathsf{e}_3 \mapsto -\sin\theta\,\mathsf{e}_1 + \cos\theta\,\mathsf{e}_3 \ .$$

  It can be interpreted geometrically as a rotation by an angle $-\theta$ about e_2-axis.

- $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

It has determinant -1 and acts as follows on the canonical basis:

$$\mathsf{e}_1 \mapsto \mathsf{e}_1 \ , \mathsf{e}_2 \mapsto \mathsf{e}_2 \ , \text{and } \mathsf{e}_3 \mapsto -\mathsf{e}_3 \ .$$

It can can be interpreted geometrically as a reflection on the $(\mathsf{e}_1, \mathsf{e}_2)$-plane.