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Fields Medallists' Lectures

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PREFACE

Although the Fields Medal does not have the same public recognition as the Nobel Prizes, they share a similar intellectual standing. It is restricted to one field — that of mathematics — and an age limit of 40 has become an accepted tradition. Mathematics has in the main been interpreted as pure mathematics, and this is not so unreasonable since major contributions in some applied areas can be (and have been) recognized with Nobel Prizes. The restriction to 40 years is of marginal significance, since most mathematicians have made their mark long before this age.

A list of Fields Medallists and their contributions provides a bird's eye view of mathematics over the past 60 years. It highlights the areas in which, at various times, greatest progress has been made. This volume does not pretend to be comprehensive, nor is it a historical document. On the other hand, it presents 22 Fields Medallists and so provides a highly interesting and varied picture.

The contributions themselves represent the choice of the individual Medallists. They are either reproductions of already published works, or are new articles produced for this volume. In some cases they relate directly to the work for which the Fields Medals were awarded. In other cases they relate to more current interests of the Medallists. This indicates that while Fields Medallists must be under 40 at the time of the award, their mathematical development goes well past this age. In fact the age limit of 40 was chosen so that young mathematicians would be encouraged in their future work.

The contribution of each medallist is in most cases preceded by the introductory speech given by another leading mathematician during the prize ceremony, a photograph and an up-to-date biographical notice. The introductory speech outlines the basic works of the medallist at the time of the medal and the reasons why it was awarded.

The Editors

RECIPIENTS OF FIELDS MEDALS

1936
Lars V. Ahlfors
Jesse Douglas

1950
Laurent Schwartz
Atle Selberg

1954
Kunihiko Kodaira
Jean-Pierre Serre

1958
Klaus F. Roth
René Thom

1962
Lars Hörmander
John W. Milnor

1966
Michael F. Atiyah
Paul J. Cohen
Alexander Grothendieck
Stephen Smale

1970
Alan Baker
Heisuke Hironaka
Sergei P. Novikov
John G. Thompson

1974
Enrico Bombieri
David B. Mumford

1978
Pierre R. Deligne
Charles L. Fefferman
Grigorii A. Margulis
Daniel G. Quillen

1982
Alain Connes
William P. Thurston
Shing-Tung Yau

1986
Simon K. Donaldson
Gerd Faltings
Michael H. Freedman

1990
Vladimir G. Drinfeld
Vaughan F. R. Jones
Shigefumi Mori
Edward Witten

1994
Jean Bourgain
Pierre-Louis Lions
Jean-Christophe Yoccoz
Efim I. Zelmanov

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Lars V. Ahlfors

Reprinted from L. V. Ahlfors: Collected Papers, Volume 1

AUTOBIOGRAPHY OF LARS V. AHLFORS

I was born on the eighteenth of April 1907 in Helsingfors, Finland. My father was a professor of mechanical engineering at the Polytechnical Institute. My mother died in childbirth when I was born.

At the time of my early childhood Finland was under Russian sovereignty, but with a certain degree of autonomy, sometimes observed and sometimes disregarded by the czar who was, by today's standards, a relatively benevolent despot. Civil servants, including professors, were able to enjoy a fairly high standard of living, a condition that was to change radically during World War I and the Russian revolution that followed.

As a child I was fascinated by mathematics without understanding what it was about, but I was by no means a child prodigy. As a matter of fact I had no access to any mathematical literature except in the highest grades. Having seen many prodigies spoiled by ambitious parents, I can only be thankful to my father for his restraints. The high school curriculum did not include any calculus, but I finally managed to learn some on my own, thanks to clandestine visits to my father's engineering library.

I entered Helsingfors University in 1924 and soon realized how very fortunate I was to have two truly outstanding mathematicians as teachers, Ernst Lindelöf and Rolf Nevanlinna. At that time the university was still run on the system of one professor for each subject. Fortunately, the need for more professors had become acute and Nevanlinna soon occupied the second chair in mathematics. The elementary teaching was in the hands of two "adjunct professors" and several part-time teachers with the title of "docent". There were essentially no courses on the graduate level; advanced reading was done under the supervision of Lindelöf.

Ernst Lindelöf is rightly considered the father of mathematics in Finland. In the 1920s *all* Finnish mathematicians were his students. He was essentially self-taught and found much of his inspiration in the works of Cauchy. His worldwide reputation as a leading complex analyst was well founded, but when I knew him he had given up research in favor of teaching, at which he was a master. I still remember many Saturday mornings when I had to visit him in his home at 8 a.m. to be praised or scolded — as the case may have been.

In the spring of 1928 I earned my degree of "fil. kand", a title that was to be changed to the equivalent "fil. mag." at a public "promotion". In the fall term of 1928 Hermann Weyl was on leave from ETH in Zürich and Rolf Nevanlinna had been

invited to take his place. At the urging of Lindelöf, my father agreed to let me go along to Zürich, no doubt at a nontrivial sacrifice. He may have been moved by the fact that he himself spent some time at ETH as a young man. This was my first trip abroad, and I found myself suddenly transported from the periphery to the center of Europe.

It was also my first exposure to live mathematics. Nevanlinna was a young man of 33 who had already won widespread acclaim, and I was a very immature 21. The course covered contemporary function theory, including the main parts of Nevanlinna's theory of meromorphic functions, and was essentially a forerunner of his famous "Eindeutige analytische Funktionen". Among other things, Nevanlinna introduced the class to Denjoy's conjecture on the number of asymptotic values of an entire function, including Carleman's partial proof. I had the incredible luck of hitting upon a new approach, based on conformal mapping, which with very considerable help from Nevanlinna and Pólya led to a proof of the full conjecture. With unparalleled generosity they forbade me to mention the part they had played, and Pólya, who rightly did not trust my French, wrote the Comptes Rendus note. For my part I have tried to repay my debt by never accepting to appear as coauthor with a student.

With a small grant from a student organization I was able to follow Nevanlinna to Paris for three more months. There I discovered a geometric interpretation of the Nevanlinna characteristic, which, as it turned out, had been found independently by Shimizu in Japan. Nevertheless, this was the upbeat to an intense involvement with meromorphic functions.

On my return to Finland I entered my first teaching assignment as lecturer (lektor) at Åbo Akademi, the Swedish-language university in Åbo (Turku). At the same time I began work on my thesis, which I defended in the spring of 1930. For formal reasons my degree of Ph.D. was delayed until 1932.

During 1930–1932 I made several trips to continental Europe, including a longer stay in Paris, with a grant from the Rockefeller Institute. My name was becoming known, and I met many of the leading mathematicians. In 1933 I was able to return to Helsingfors as adjunct professor. That same year I married Erna Lehnert, a girl from Vienna, who with her parents had settled first in Sweden and then in Finland. This was the happiest and most important event in my life.

Our life in Helsingfors was pleasant, but uneventful. Quite unexpectedly I received a letter from Harvard University, which to me was hardly more than a name. It turned out that the Mathematics Department was looking for a young mathematician, and Carathéodory, whom I had met in Munich, had recommended me warmly. I was hesitant at first, but after persuasion by my sponsor W. C. Graustein I agreed to a trial period of three years beginning in the fall of 1935. We found life in Cambridge very rewarding, although the social life seemed somewhat Victorian. However, the friendliness of my colleagues was absolutely disarming. I also found that my knowledge of mathematics was still rather spotty, and I learned a lot during these three years.

I was in for the surprise of my life when in 1936, at the International Congress in Oslo, I was told only hours before the ceremony that I was to receive one of the first two Fields medals ever awarded. The prestige was perhaps not yet the same as it is now, but in any case I felt singled out and greatly honored. The citation by Carathéodory mentions explicitly my paper "Zur Theorie der Überlagerungsflächen", which threw some new light on Nevanlinna's theory of meromorphic functions. The award contributed in great measure to the confidence I felt in my work.

In the spring of 1938 I had to decide whether to stay at Harvard or return to Finland, where I was offered a professorship at the University of Helsinki. Lindelöf, who was already retired, urged me to return home as a "partriotic duty". In the end I think it was plain homesickness that decided my return, and back in Finland we enjoyed a very happy year together with our firstborn child.

Alas, the happiness did not last. The war broke out, and it became clear that Finland would not be spared. My wife and children — the second a newborn infant — were evacuated and found refuge with relatives in Sweden. The university was closed for lack of male students, but otherwise life went on, partly in air-raid shelters. Because of an earlier physical condition, I had never been called to military duty, and my only part in the war effort was as an insignificant link in a communications setup.

Soon after the end of the Winter War, my family was able to return home and resume a seemingly normal life. Politics in Finland took an unfortunate turn, however, and when Hitler attacked the Soviet Union in 1941 Finland was his ally. When the Russians were finally able to repulse the attack, they could also intensify the war in Finland with foreseeable results. The Finnish–Russian war ended with a separate armistice in September 1944, whereupon Finland was able to expulse the German troops stationed there. The harsh terms of the armistice left Finland in a very difficult position.

Although the uncertainty of the future and the suffering of the bereaved were on everybody's mind, the wartime was not a complete loss. It unified the nation and paved the way for a return to relatively stable conditions. Paradoxically, I was able to do a lot of work during the war, although without the benefit of accessible libraries.

During the summer of 1944, I received an offer from the University of Zürich. Because I saw this as my only opportunity to be reunited with my family and in view of the bleak future in Finland, I accepted in principle although for the moment I saw no physical possibility of following through the invitation. My health had declined, and, because I had no military duties, I was allowed to go to Sweden to recuperate. The problem now was to get from Sweden to Switzerland. By that time Finland was at war with Germany and, at least on paper, also with England. An appeal to the German legation in Stockholm proved fruitless, but the British were willing to let us pass through Great Britain if an opportunity arose. The Swedes had organized some semiregular "stratospheric flights" on moonless nights from Stockholm to Prestwick, Scotland. With the help of diplomatic channels we managed to be placed on the list

of potential passengers. Obtaining permission from the British was still necessary and depended on the military situation. Unfortunately, when our low priority had made us eligible, the Battle of the Bulge put a temporary stop to our hopes. Finally, one day in March 1945 we were told to be ready to leave, weather permitting. It is difficult to forget that flight. The plane was reconditioned Flying Fortress, with perhaps a dozen passengers. It was not pressurized, and breathing was accomplished by individual oxygen masks. Swim vests were worn by all. Our children, ages 5 and 6, were quite capable of understanding the implications of danger.

We left Sweden with feelings of deep gratitude. Virtually penniless, we had been taken care of not only by close relatives, but also by mathematical colleagues, who made it possible to stay in Uppsala for months. I am forever indebted to Arne Beurling, who showed what true friendship can be.

An arduous train ride took us from Glasgow to London, where we had to wait several days for the Channel to be cleared for a ferry to Dieppe. Much of the time was spent in the London zoo, but the frequent explosions of V2 rockets made us wonder if this was wise.

To cut a long story short, we were finally shipped to Paris, where the Swiss legation was unforgettably helpful in securing lodgings in a luxury hotel — and in providing Swiss cigars to the station master. On arriving at the Swiss border, somewhat disheveled and again penniless, we were met by the Red Cross, who lent us Swiss money. Although there was some rationing even in Switzerland, we were overwhelmed by the chocolates, cakes, and other foods, the likes of which we had not seen for years.

The University of Zürich was ready to begin the summer term. I was met by the Director of the Mathematics Institute, Professor R. Fueter, in the single room that served as office for the institute and all the professors. My first disappointment came when I learned that I would be responsible for Descriptive Geometry, a subject that for some reason had survived in the Swiss high school and undergraduate curriculum. My second shock was that it was to be taught from 7 to 9 o'clock in the morning.

Nevertheless, I slowly adjusted to my work, which even included some serious, although not very advanced, mathematics. Professor Fueter and his colleague Professor Finsler were getting on in years, and it became clear that the reason for inviting me was that no competent native successor was in sight. I took over a class of students in their formative years, and I am happy to say that many have remained my friends and are now important mathematicians in their own right.

I cannot honestly say that I was happy in Zürich. The postwar era was not a good time for a stranger to take root in Switzerland. The whole nation, although spared from war, was in a state of suspension, with, understandably, quite a bit of xenophobia in the lower classes. My wife and I did not feel welcome outside the circle of our immediate colleagues.

I was therefore very pleased when 1946 I was asked if I would like to return to Harvard. My Swiss colleagues and the "Erziehungsdirektion des Kantons Zürch"

did their best to persuade me to stay, but I was convinced that for the sake of my mathematical career I should go to America, and I was consoled when I learned that my teacher Rolf Nevanlinna had agreed to become my successor. This arrangement turned out to be a great success both for Nevanlinna and for the university.

In the fall of 1946 I took up my duties at Harvard, where I was to stay until my retirement in 1977 and as emeritus to this day. My relationship with Harvard, with both the administration and my friends in the department, has been singularly happy. I have enjoyed the many excellent students Harvard has had to offer, many of whom I have watched become leaders in my own or some other field. Harvard has also offered me an optimal milieu for my research, and it has been a source of great satisfaction that the Mathematics Department has been able to maintain the high standards that have always been its hallmark.

Reprinted from L. V. Ahlfors: Collected Papers, Vol. 1

COMMENTARY ON: ZUR THEORIE DER ÜBERLAGERUNGSFLÄCHEN (1935)

by

LARS V. AHLFORS

In this work, I am no longer primarily interested in the problem of type although the connection with my earlier work is quite evident. My aim was to interpret the Nevanlinna theorems as geometric properties of covering surfaces. To set the stage, suppose first that $\mathcal{W}_o$ is a compact surface, with or without boundary, and that $\mathcal{W}$ is a ramified covering surface of $\mathcal{W}_o$, which covers $\mathcal{W}_o$ evenly N times, branch-points counted according to the local degree of the projection map. If the Euler characteristics of $\mathcal{W}_o$ and $\mathcal{W}$ are ρ_o and ρ, the Hurwitz relation reads $\rho = N\rho_o + \varpi$, where ϖ is the sum of the ramification orders. It was known that Nevanlinna's second main theorem was in some way connected with the Hurwitz relation, and I wanted to make this connection explicit.

In itself the Hurwitz relation is purely topological, and it presupposes that no boundary point of $\mathcal{W}$ projects on an interior point of $\mathcal{W}_o$. Consider now the more general situation where part of the boundary projects on the interior of $\mathcal{W}_o$. We suppose that in some metric, given on $\mathcal{W}_o$ and lifted to $\mathcal{W}$, the length of this relative boundary is L. On the other hand let S denote the average time a point of $\mathcal{W}_o$ is covered by $\mathcal{W}$; in other words, S is the area of $\mathcal{W}$ divided by that of $\mathcal{W}_o$. In these circumstances it is shown that the Hurwitz relation survives as an inequality $\overset{+}{\rho} \geqslant S\rho_o - kL$ with a constant k that depends only on $\mathcal{W}_o$ and $\overset{+}{\rho} = \max\,(\rho, 0)$. The result is meaningful only if $\rho_o > 0$ and L/S is small. The proof is elementary but not obvious.

The crucial fact is that k does not depend on the covering. This means that we can pass to the case of an open covering $\mathcal{W}$, which we exhaust by compact subsurfaces $\mathcal{W}_n$. If S_n and L_n refer to $\mathcal{W}_n$, the exhaustion is said to be regular if $L_n/S_n \to 0$, and in that case the inequalities $\overset{+}{\rho}(\mathcal{W}_n) \geqslant S_n\rho_o - kL_n$ give pertinent information about the covering properties of $\mathcal{W}$.

How does this apply to meromorphic functions? Let f be a meromorphic function that we regard as mapping the complex plane on a covering of the Riemann sphere. Because the Euler characteristic of the sphere is -2, we cannot choose the sphere itself as ground surface $\mathcal{W}_o$. Instead we let $\mathcal{W}_o$ be the sphere from which, in

the simplest case, $q \geqslant 3$ disjoint disks Δ_i have been removed. Now $\rho_o = q - 2 > 0$, and the Hurwitz relation becomes $\overset{+}{\rho}(\mathcal{W}_n) \geqslant (q-2)S_n - kL_n$. The conformality of the mapping implies that the exhaustion by the images $\mathcal{W}_r$ of the concentric disks of radius r is regular in the sense that $\liminf L(r)/S(r) = 0$. Here $S(r)$ can be interpreted as the Shimizu–Ahlfors characteristic before integration, and $\rho(\mathcal{W}_r)$ is essentially the number of "islands" over the Δ_i inside the circle $|z| = r$. In easily recognizable notation this is expressed through

$$\sum_{i=1}^{q} n(r, \Delta_i) \geqslant (q-2)S(r) - kL(r)\,.$$

If the Δ_i are replaced by points $a_i \varepsilon \Delta_i$, it is still true that

$$\sum_{i=1}^{q} n(r, a_i) \geqslant (q-2)S(r) - kL(r)\,,$$

and one recognizes that this is an unintegrated form of Nevanlinna's second main theorem.

It is almost obvious that $\liminf L(r)/S(r) = 0$ remains true if the conformal mapping is replaced by a quasiconformal mapping. I included this more general situation in my paper, but with pangs of conscience because I considered it rather cheap padding. Quasiconformal mappings had been introduced already in 1928 by Grötzsch, who called them "nichtkonforme Abbildungen" and almost simultaneously with my paper by M. A. Lavrentiev, who used the term "fonctions presque analytiques". I have been credited with being the first to have used the name "quasiconformal", which has become standard. The truth is that I cannot recollect having invented the name, but I have also not been able to locate it elsewhere. Little did I know at the time what an important role quasiconformal mappings would come to play in my own work.

My paper met with immediate recognition and earned me the Fields medal. It has been incorporated in the leading textbooks: R. Nevanlinna's *Eindeutige Analytische Funktionen*, W. Hayman's *Meromorphic Functions*, K. Noshiro's *The Modern Theory of Functions* (in Japanese). Attempts have been made to generalize the method to several dimensions, notably by Mme. Hélène Schwartz, but he difficulties were too overwhelming for an easily accessible theory.

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QUASICONFORMAL MAPPINGS, TEICHMÜLLER SPACES, AND KLEINIAN GROUPS

by

LARS V. AHLFORS*

I am extremely grateful to the Committee to select hour speakers for the great honor they have bestowed on me, and above all for this opportunity to address the mathematicians of the whole world from the city of my birth. The city has changed a great deal since my childhood, but I still get a thrill each time I return to this place that holds so many memories for me. I assure you that today is even a more special event for me.

I have interpreted the invitation as a mandate to report on the state of knowledge in the fields most directly dominated by the theory and methods of quasiconformal mappings. I was privileged to speak on the same topic once before, at the Congress in Stockholm 1962, and it has been suggested that I could perhaps limit myself to the developments after that date. But I feel that this talk should be directed to a much wider audience. I shall therefore speak strictly to the non-specialists and let the experts converse among themselves on other occasions.

The whole field has grown so rapidly in the last years that I could not possibly do justice to all recent achievements. A mere list of the results would be very dull and would not convey any sense of perspective. What I shall try to do, in the limited time at my disposal, is to draw your attention to the rather dramatic changes that have taken place in the theory of functions as a direct of the inception and development of quasiconformal mappings. I should also like to make it clear that I am not reporting on my own work; I have done my share in the early stages, and I shall refer to it only when needed for background.

1. Historial remarks. In classical analysis the theory of analytic functions of complex variables, and more particularly functions of one variable, have played a dominant role ever since the middle of the nineteenth century. There was an obvious peak around the turn of the century, centering around names like Poincaré, Klein, Picard, Borel, and Hadamard. Another blossoming took place in the 1920s with the arrival of Nevanlinna's theory. The next decade seemed at the time as a

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slackening of the pace, but this deceptive; many of the ideas that were later to be fruitful were conceived at that time.

The war and the first post-war years were of course periods of stagnation. The first areas of mathematics to pick up momentum after the war were topology and functions of several complex variables. Big strides were taken in these fields, and under the leadership of Henri Cartan, Behnke, and many others, the more-dimensional theory of analytic functions and manifolds acquired an almost entirely new structure affiliated with algebra and topology. As a result of this development the gap between the conservative analysts who were still doing conformal mapping and the more radical ones involved with sheaf-theory became even wider, and for some time it looked as if the one-dimensional theory had lost out and was in danger of becoming a rehash of old ideas. The gap is still there, but I shall try to convince you that in the long run the old-fashioned theory has recovered and is doing quite well.

The theory of quasiconformal mappings is almost exactly fifty years old. They were introduced in 1928 by Herbert Grötzsch in order to formulate and prove a generalization of Picard's theorem. More important is his paper of 1932 in which he discusses the most elementary but at the same time most typical cases of extremal quasiconformal mappings, for instance the most nearly conformal mapping of one doubly connected region on another. Grötzsch's contribution is twofold: (1) to have been the first to introduce non-conformal mappings in a discipline that was so exclusively dominated by analytic functions, (2) to have recognized the importance of measuring the degree of quasiconformality by the maximum of the dilatation rather than by some integral mean (this was recently pointed out by Lipman Bers).

Grötzch's papers remained practically unnoticed for a long time. In 1935 essentially the same class of mappings was introduced by M. A. Lavrentiev in the Soviet Union whose work was connected more closely with partial differential equations than with function theory proper. In any case, the theory of quasiconformal mappings, which at that time had also acquired its name, slowly gained recognition, originally as a useful and flexible tool, but inevitably also as an interesting piece of mathematics in its own right.

Nevertheless, quasiconformal mappings might have remained a rather obscure and peripheral object of study if it had not been for Oswald Teichmüller, an exceptionally gifted and intense young mathematician and political fanatic, who suddenly made a fascinating and unexpected discovery. At that time, many special extremal problems in quasiconformal mapping had already been solved, but these were isolated results without a connecting general idea. In 1939 he presented to the Prussian Academy a now famous paper which marks the rebirth of quasiconformal mappings as a new discipline which completely overshadows the rather modest beginnings of the theory. With remarkable intuition he made a synthesis of what was known and proceeded to announce a bold outline of a new program which he presents, rather dramatically, as the result of a sudden revelation that occured to him at night. His main discovery was that the extremal problem of quasiconformal mapping, when applied to Riemann surfaces,

leads automatically to an intimate connection with the holomorphic quadratic differentials on the surface. With this connection the whole theory takes on a completely different complexion: A problem concerned with non-conformal mappings turns out to have a solution which is expressed in terms of holomorphic differentials, so that in reality the problem belongs to classical function theory. Even if some of the proofs were only heuristic, it was clear from the start that this paper would have a tremendous impact, although actually its influence was delayed due to the poor communications during the war. In the same paper Teichmüller lays the foundations for what later has become known as the theory of Teichmüller spaces.

2. Beltrami coefficients. It is time to become more specific, and I shall start by recalling the definition and main properties of quasiconformal (q.c.) mappings. To begin with I shall talk only about the two-dimensional case. There is a corresponding theory in several dimensions, necessarily less developed, but full of interesting problems. One of the reasons for considering q.c. mappings, although not the most compelling one, is precisely that the theory does not fall apart when passing to more than two dimensions. I shall return to this at the end of the talk.

Today it can be assumed that even a non-specialist knows roughly what is meant by a q.c. mapping. Intuitively, a homeomorphism is q.c. if small circles are carried into small ellipses with a bounded ratio of the axes; more precisely, it is K-q.c. if the ratio is $\leqslant K$. For a diffeomorphism f this means that the complex derivatives $f_z = \frac{1}{2}(f_x - if_y)$ and $f_{\bar{z}} = \frac{1}{2}(f_x + if_y)$ satisfy $|f_{\bar{z}}| \leqslant k|f_z|$ with $k = (K-1)(K+1)$.

Already at an early stage it became clear that it would not do to consider only diffeomorphisms, for the class of diffeomorphisms lacks compactness. In the beginning rather arbitrary restrictions were introduced, but in time they narrowed down to two conditions, one geometric and one analytic, which eventually were found to be equivalent. The easiest to formulate is the analytic condition which says that f is K-q.c. if it is a weak L^2-solution of a *Beltrami equation*

$$f_{\bar{z}} = \mu f_z \tag{1}$$

where $\mu = \mu_f$, known as a Beltrami coefficient, is a complex-valued measurable function with $\|\mu\|_\infty \leqslant k$.

The equation is classical for smooth μ, but there is in fact a remarkably strong existence and uniqueness theorem without additional conditions. If μ is defined in the whole complex plane, with $|\mu| \leqslant k < 1$ a.e., then (1) has a homeomorphic solution which maps the plane on itself, and the solution is unique up to conformal mappings. Simple uniform estimates, depending only on k, show that the class of K-q.c. mappings is compact.

It must be clear that I am condensing years of research into minutes. The fact is that the post-Teichmüller era of quasiconformal mappings did not start seriously until 1954. In 1957 I. N. Vekua in the Soviet Union proved the existence and uniqueness theorem for the Beltrami equation, and in the same year L. Bers discovered

that the theorem had been proved already in 1938 by C. Morrey. The great difference in language and emphasis had obscured the relevance of Morrey's paper for the theory of q.c. mappings. The simplest version of the proof is due to B. V. Boyarski who made it a fairly straightforward application of the Calderón–Zygmund theory of singular integral transforms.

As a consequence of the chain rule the Beltrami coefficients obey a simple composition law:

$$\mu_{g\circ f^{-1}} = \left[\frac{\mu_g - \mu_f}{1 - \bar{\mu}_f \mu_g}\,(f_z/|f_z|)^2\right] \circ f^{-1}\,.$$

The interesting thing about this formula is that for any fixed z and f the dependence on $\mu_g(z)$ is complex analytic, and a conformal mapping of the unit disk on itself. This simple fact turns out to be crucial for the study of Teichmüller space.

3. Extremal length. The geometric definition is conceptually even more important than the analytic definition. It makes important use of the theory of extremal length, first developed by A. Beurling for conformal mappings. Let me recall this concept very briefly. If L is a set of locally rectifiable arcs in R^2, then a Borel measurable function $\varrho : R^2 \to R^+$ is said to be *admissible* for L if $\int_\gamma \varrho\, ds \geqslant 1$ for all $\gamma \in L$. The *module* $M(L)$ is defined as $\inf \int \varrho^2 dx$ for all admissible ϱ; its reciprocal is the *extremal length* of L. It is connected with q.c. mappings in the following way: If f is a K-q.c. mapping (according to the analytic definition), then $M(fL) \leqslant KM(L)$. Conversely, this property may be used as a geometric definition of K-q.c. mappings, and it is sufficient that the inequality hold for a rather restrictive class of families L that can be chosen in various ways. This definition has the advantage of having an obvious generalization to several dimensions.

Inasmuch as extremal length was first introduced for conformal mappings, its connection with q.c. mappings, even in more than two dimensions, is another indication of the close relationship between q.c. mappings and classical function theory.

4. Teichmüller's theorem. The problem of extremal q.c. mappings has dominated the subject from the start. Given a family of homeomorphisms, usually defined by some specific geometric or topological conditions, it is required to find a mapping f in the family such that the maximal dilatation, and hence the norm $\|\mu_f\|_\infty$ is a minimum. Because of compactness the existence is usually no problem, but the solution may or may not be unique, and if it is there remains the problem of describing and analyzing the solution.

It is quite obvious that the notion of q.c. mappings generalizes at once to mappings from one Riemann surface to another, each with its own conformal structure, and that the problem of extremal mapping continues to make sense. The Beltrami coefficient becomes a Beltrami differential $\mu(z)d\bar{z}/dz$ of type $(-1,1)$. Note that $\mu(z)$ does not depend on the local parameter on the target surface.

Teichmüller considers topological maps $f : S_0 \to S$ from one compact Riemann surface to another. In addition he requires f to belong to a prescribed homotopy class, and he wishes to solve the extremal problem separately for each such class. Teichmüller asserted that there is always an extremal mapping, and that it is unique. Moreover, either there is a unique conformal mapping in the given homotopy class, or there is a constant k, $0 < k < 1$, and a *holomorphic quadratic differential* $\varphi(z)\, dz^2$ on S_0 such that the Beltrami coefficient of the extremal mapping is $\mu_f = k\bar{\varphi}/|\varphi|$. It is thus a mapping with constant dilatation $K = (1+k)/(1-k)$. The inverse f^{-1} is simultaneously extremal for the mappings $S \to S_0$, and it determines an associated quadratic differential $\psi(w)\, dw^2$ on S. In local coordinates the mapping can be expressed through

$$\sqrt{\psi(w)}\, dw = \sqrt{\varphi(z)}\, dz + k\sqrt{\bar{\varphi}}\,(z)\, d\bar{z}\,.$$

Naturally, there are singularities at the zeros of φ, which are mapped on zeros of ψ of the same order, but these singularities are of a simple explicit nature. The integral curves along which $\sqrt{\varphi}\, dz$ is respectively real or purely imaginary are called horizontal and vertical trajectories, and the extremal mapping maps the horizontal and vertical trajectories on S_0 to corresponding trajectories on S. At each point the stretching is maximal in the direction of the horizontal trajectory and minimal along the vertical trajectory.

This is a beautiful and absolutely fundamental result which, as I have already tried to emphasize, throws a completely new light on the theory of q.c. mappings. In his 1939 paper Teichmüller gives a complete proof of the uniqueness part of his theorem, and it is still essentially the only known proof. His existence proof, which appeared later, is not so transparent, but it was put in good shape by Bers; the result itself was never in doubt. Today, the existence can be proved more quickly than the uniqueness, thanks to a fruitful idea of Hamilton. Unfortunately, times does not permit me to indicate how and why these proofs work, except for saying that the proofs are variational and make strong use of the chain rule for Beltrami coefficients.

5. Teichmüller spaces. Teichmuüller goes on to consider the slightly more general case of compact surfaces with a finite number of punctures. Specifically, we say that S is of finite type (p, m) if it is an oriented topological surface of genus p with m points removed. It becomes a Riemann surface by giving it a conformal structure. Following Bers we shall define a conformal structure as a sense-preserving topological mapping σ on a Riemann surface. Two conformal structures σ_1 and σ_2 are equivalent if there is a conformal mapping g of $\sigma_1(S)$ on $\sigma_2(S)$ such that $\sigma_s^{-1} \circ g \circ \sigma_1$ is homotopic to the identity. The equivalence classes $[\sigma]$ are the points of the Teichmüller space $T(p, m)$, and the distance between $[\sigma_1]$ and $[\sigma_2]$ is defined to be

$$d([\sigma_1],\ [\sigma_2]) = \log \inf K(f)$$

where $K(f)$ is the maximal dilatation of f, and f ranges over all mappings homotopic to $\sigma_2 \circ \sigma_1^{-1}$. It is readily seen that the infimum is actually a minimum, and that the extremal mapping from $\sigma_1(S)$ to $\sigma_2(S)$ is as previously described, except that the quadratic differentials are now allowed to have simple poles at the punctures.

With this metric $T(p,m)$ is a complete metric space, and already Teichmüller showed that it is homeomorphic to $R^{6p-6+2m}$ (provided that $2p-2+m>0$).

Let f be a self-mapping of S. It defines an isometry $\tilde{f}$ of $T(p,m)$ which takes $[\sigma]$ to $[\sigma \circ f]$. This isometry depends only on the homotopy class of f and is regarded as an element of the *modular group* $\mathrm{Mod}(p,m)$. It follows from the definition that two Riemann surfaces $\sigma_1(S)$ and $\sigma_2(S)$ are conformally equivalent if and only if $[\sigma_2]$ is the image of $[\sigma_1]$ under an element of the modular group. The quotient space $T(p,m)/\mathrm{Mod}\ (p,m)$ is the Riemann space of algebraic curves or moduli. The Riemann surfaces that allow conformal self-mappings are branch-points of the covering.

6. Fuchsian and quasifuchsian groups. The universal covering of any Riemann surface S, with a few obvious exceptions, is conformally equivalent to the unit disk U. The self-mappings of the covering surface correspond to a group G of fractional linear transformations, also referred to as Möbius transformations, which map U conformally on itself. More generally one can allow coverings with a signature, that is to say regular covering surfaces which are branched to a prescribed order over certain isolated points. In this case G includes elliptic transformations of finite order. It is always discrete.

Any discrete group of Möbius transformations that preserves a disk or a half-plane, for instance U, is called a Fuchsian group. It is a recent theorem, due to Jørgensen, that a nonelementary group which maps U on itself is discrete, and hence Fuchsian, if and only if every elliptic transformation in the group is of finite order. As soon as this condition is fulfilled the quotient U/G is a Riemann surface S, and U appears as a covering of S with a signature determined by the orders of the elliptic transformations. The group acts simultaneously on the exterior U^* of U, and $S^* = U^*/G$ is a mirror image of S. G is determined by S up to conjugation.

A point is a limit point if it is an accumulation point of an orbit. For Fuchsian groups all limit points are on the unit circle; the set of limit points will be referred to as the *limit set* $\Lambda(G)$. Except for some trivial cases there are only two alternatives: either Λ is the whole unit circle, or it is a perfect nowhere dense subset. With an unimaginative, but classical, terminology Fuchsian groups are accordingly classified as being of the first kind or second kind.

If S is of finite type, then G is always of the first kind; what is more, G has a fundamental region with finite noneuclidean area. Consider a q.c. mapping $f : S_0 \to S$ with corresponding groups G_0 and G. Then f lifts to a mapping $f : U \to U$ (which we continue to denote by the same letter), and if $g_0 \in G_0$ there is a $g \in G$ such that $f \circ g_0 = g \circ f$. This defines an isomorphism $\theta : G_0 \to G$ which is uniquely determined, up to conjugation, by the homotopy class of f. Moreover, f

extends to a homeomorphism of the closed disks, and the boundary correspondence is again determined uniquely up to normalization. The Teichmüller problem becomes that of finding f with given boundary correspondence and smallest maximal dilatation. The extremal mapping has a Beltrami coefficient $\mu = k\bar{\varphi}|\varphi|$ where φ is an invariant quadratic differential with respect to G_0.

Incidentally, the problem of extremal q.c. mappings with given boundary values makes sense even when there is no group, but the solution need not be unique. The questions that arise in this connection have been very successfully treated by Hamilton, K. Strebel, and E. Reich.

For a more general situation, let $\mu d\bar{z}/dz$ be any Beltrami differential, defined in the whole plane and invariant under G_0 in the sense that $(\mu \circ g_0)\bar{g}_0'/g_0' = \mu$ a.e. for all $g_0 \in G$. Suppose f is a solution of the Beltrami equation $f_{\bar{z}} = \mu f_z$. It follows from the chain rule that $f \circ g_0$ is another solution of the same equation. Therefore $f \circ g_0 \circ f^{-1}$ is conformal everywhere, and hence a Möbius transformation g. In this way μ determines an isomorphic mapping of G_0 on another group G, but this time G will in general not leave U invariant. For this reason G is a Kleinian group rather than a Fuchsian group. It has two invariant regions $f(U)$ and $f(U^*)$, separated by a Jordan curve $f(\delta U)$. The surfaces $f(U)/G$ and $f(U^*)/G$ are in general not conformal mirror images.

The group $G = fG_0f^{-1}$ is said to be obtained from G_0 by q.c. deformation, and it is called a quasifuchsian group. Evidently, quasifuchsian groups have much the same structure as fuchsian groups, except for the lack of symmetry. The curve that separates the invariant Jordan regions is the image of the unit circle under a q.c. homeomorphism of the whole plane. Such curves are called quasicircles. It follows by a well-known property of q.c. mappings that every quasicircle has zero area, and consequently the limit set $\Lambda(G)$ has zero two-dimensional measure.

Strangely enough, quasicircles have a very simple geometric characterization: A Jordan curve is a quasicircle if and only if for any two points on the curve at least one of the subarcs between them has a diameter at most equal to a fixed multiple of the distance between the points. It means, among other things, that there are no cusps.

7. The Bers representation. There are two special cases of the construction that I have described: (1) If μ satisfies the symmetry condition $\mu(1/\bar{z})\bar{z}^2/z^2 = \bar{\mu}(z)$, then G is again a Fuchsian group and f preserves symmetry with respect to the unit circle. (2) If μ is identically zero in U and arbitrary in U^*, except for being invariant with respect to G_0, then f is conformal in U, and $f(U)/G$ is conformally equivalent to $S = U/G$, while $f(U^*)/G$ is a q.c. mirror image of S.

I shall refer to the second construction as the Bers mapping. Two Beltrami differentials μ_1 and μ_2 will lead to the same group G and to homotopic maps f_1, f_2 if and only if $f_1 = f_2$ on ∂U (up to normalizations). When that is the case we say that μ_1 and μ_2 are equivalent, and that they represent the same point in the Teichmüller space $T(G_0)$ based on the Fuchsian group G_0.

In other words the equivalence classes are determined by the values of f on the unit circle. These values obviously determine $f(U)$, and hence f, at least up to a normalization. One obtains strict uniqueness by passing to the Schwarzian derivative $\varphi = S_f$ defined in U (recall that $S_f = (f''/f')' - \frac{1}{2}(f''/f')^2$). From the properties of the Schwarzian it follows that $\varphi(gz)g'(z)^2 = \varphi(z)$ for all $g \in G_0$. Furthermore, by a theorem of Nehari $|\varphi(z)|(1-|z|^2)^2$ is bounded (actually $\leqslant 6$). Thus φ belongs to the Bers class $B(G_0)$ of bounded quadratic differentials with respect to the group G_0. The Bers map is an injection $T(G_0) \to B(G_0)$.

It is known that the image of $T(G_0)$ under the Bers map is *open*, and as a vector space $B(G_0)$ has a natural complex structure. The mapping identifies $T(G_0)$ with a certain open subset of $B(G_0)$ which in turn endows $T(G_0)$ with its own complex structure. If S is of type (p, m) the complex dimension is $3p - 3 + m$. The nature of the subset that represents $T(p, m)$ in C^{3p-3+m} is not well known. For instance, it seems to be an open problem whether $T(1, 1)$ is a Jordan region in C.

The case where $G = I$, the identity group, is of special interest because it is so closely connected with classical problems in function theory. An analytic function φ, defined on U, will belong to $T(I)$ if and only if it is the Schwarzian S_f of a schlicht (injective) function on U with a q.c. extension to the whole plane. The study of such functions has added new interest to the classical problems of schlicht functions.

To illustrate the point I would like to take a minute to tell about a recent beautiful result due to F. Gehring. Let $\boldsymbol{S}$ denote the space of all $\varphi = Sf$, f analytic and schlicht in U, with the norm $\|\varphi\| = \sup\,(1-|z|^2)^2|\varphi(z)|$, and let $\boldsymbol{T} = T(I)$ be the subset for which f has a q.c. extension. Gehring has shown (i) that $\boldsymbol{T} = \operatorname{Int} \boldsymbol{S}$, (ii) the closure of $\boldsymbol{T}$ is a proper subset of $\boldsymbol{S}$. To prove the second point, which gives a negative answer to a question raised by Bers maybe a dozen years ago, he constructs, quite explicitly, a region with the property that no small deformation, measured by the norm of the Schwarzian, changes it to a Jordan region, much less to one whose boundary is a quasicircle. I mention this particular result because it is recent and because it is typical for the way q.c. mappings are giving new impulses to the classical theory of conformal mappings.

In the finite dimensional case $T(p, m)$ has a compact boundary in $B(G_0)$. It is an interesting and difficult problem to find out what exactly happens when φ approaches the boundary. The pioneering research was carried out by Bers and Maskit. They showed, first of all, that when φ approaches a boundary point the holomorphic function f will tend to a limit which is still schlicht, and the groups G tend to a limit group which is Kleinian with a single, simply connected invariant region. Such groups were called B-groups (B stands either for Bers or for boundary) in the belief that any such group can be obtained in this manner. It can happen that the invariant simply connected region is the whole set of discontinuity; such groups are said to be *degenerate*. Classically, degenerate groups were not known, but Bers proved that they must exist, and more recently Jørgensen has been able to construct many explicit examples of such groups.

Intuitively, it is clear what should happen when φ goes to the boundary. We are interested to follow the q.c. images $f(U^*)$. In the degenerate case the image disappears completely. In the nondegenerate case the fact that one approaches the boundary must be visible in some way, and the obvious guess is that one or more of the closed geodesics on the surface is being pinched to a point. In the limit $f(S^*)$ would either be of lower genus or would disintegrate to several pieces, and one would end up with a more general configuration consisting of a "surface with nodes", each pinching giving rise to two nodes.

A lot of research has been going on with the intent of making all this completely rigorous, and if I am correctly informed these attempts have been successful, but much remains to be done. This is the general trend of much of the recent investigations of Bers, Maskit, Kra, Marden, Earle, Jørgensen, Abikoff and others; I hope they will understand that I cannot report in any detail on these theories which are still in *status nascendi*.

In a slightly different direction the theory of Teichmüller spaces has been extended to a study of the so-called universal Teichmüller curve, which for every type (p, m) is a fiber-space whose fibers are the Riemann surfaces of that type. A special problem is the existence, or rather non-existence, of holomorphic sections.

The Bers mapping is not concerned with extremal q.c. mappings, and it is rather curious that one again ends up with holomorphic quadratic differentials. The Bers model has a Kählerian structure obtained from an invariant metric, the Petersson–Weil metric, on the space of quadratic differentials. The relation between the Petersson–Weil metric and the Teichmüller metric has not been fully explored and is still rather mystifying.

8. Kleinian groups. I would have preferred to speak about Kleinian groups in a section all by itself, but they are so intimately tied up with Teichmüller spaces that I was forced to introduce Kleinian groups somewhat prematurely. I shall now go back and clear up some of the terminology.

It was Poincaré who made the distinction between Fuchsian and Kleinian groups and who also coined the names, much to the displeasure of Klein. He also pointed out that the action of any Möbius transformation extends to the upper half space, or, equivalently, to the unit ball in three-space. Any discrete group of Möbius transformations is discontinuous on the open ball. Limit points are defined as in the Fuchsian case; they are all on the unit sphere, and the limit set Λ may be regarded either as a set on the Riemann sphere or in the complex plane. The elementary groups with at most two limit points are usually excluded, and in modern terminology a Kleinian group is one whose limit set is nowhere dense and perfect. A Kleinian group may be looked upon as a Fuchsian group of the second kind in three dimensions. As such it cannot have a fundamental set with finite non-euclidean volume. Therefore, the relatively well developed methods of Lie group theory which require finite Haar measure are mostly not available for Kleinian groups. However, the important method of Poincaré series continues to make sense.

Let G be a Kleinian group, Λ its limit set, and Ω the set of discontinuity, that is to say the complement of Λ in the plane or on the sphere. The quotient manifold Ω/G inherits the complex structure of the plane and is thus a disjoint union of Riemann surfaces. It forms the boundary of a three-dimensional manifold $M(G) = B(1) \cup \Omega/G$.

What is the role of q.c. mappings of Kleinian groups? For one thing one would like to classify all Kleinian groups. It is evident that two groups that are conjugate to each other in the full group of Möbius transformations should be regarded as essentially the same. But as in the case of quasifuchsian groups two groups can also be conjugate in the sense of q.c. mappings, namely if $G' = fGf^{-1}$ for some q.c. mapping of the sphere. In that case G' is a q.c. deformation of G, and such groups should be in the same class.

But this is not enough to explain the sudden blossoming of the theory under the influence of q.c. mappings. As usual, linearization pays off, and it has turned out that infinitesimal q.c. mappings are relatively easy to handle. An infinitesimal q.c. mapping is a solution of $f_{\bar{z}} = \nu$ where the right-hand member is a function of class L^∞. This is a non-homogeneous Cauchy–Riemann equation, and it can be solved quite explicitly by the Pompeiu formula, which is nothing else than a generalized Cauchy integral formula. In order that f induce a deformation of the group ν must be a Beltrami differential, $\nu \in \text{Bel } G$, this time with arbitrary finite bound. There is a subclass N of trivial differentials that induces only a conformal conjugation of G, and the main theorem asserts that the dual space of Bel G/N can be identified with the space of quadratic differentials on $\Omega(G)/G$ which are of class L^1.

This technique is particularly successful if one looks only at finitely generated groups. In that case the deformation space is finite dimensional, so that there are only a finite number of linearly independent integrable quadratic differentials. This result led me to announce, somewhat prematurely, the so-called *finiteness theorem*: If G is finitely generated, then $S = \Omega(G)/G$ is a finite union of Riemann surface of finite type. I had overlooked that fact that a triply punctured square carries no quadratic differentials. Fortunately, the gap was later filled by L. Greenberg, and again by L. Bers who extended the original method to include differentials of higher order. With this method Bers obtained not only an upper bound for the number of surfaces in terms of the number of generators, but even a bound on the total Poincaré area of S.

It was not unreasonable to expect that finitely generated Kleinian groups would have other simple properties. For instance, since a finitely generated Fuchsian group has a fundamental polygon with a finite number of sides one could hope that every finitely generated Kleinian group would have a finite fundamental polyhedron. All such hopes were shattered when L. Greenberg proved that a degenerate group in the sense of Bers and Maskit can never have a finite fundamental polyhedron. Groups with a finite fundamental polyhedron are called *geometrically finite*, and it has been suggested that one should perhaps be content to study only geometrically finite groups. With his constructive methods that go back to Klein, Maskit has been

able to give a complete classification of all geometrically finite groups, and Marden has used three-dimensional topology to study the geometry of the three-manifold. These are very farreaching and complicated results, and it would be impossible for me to try to summarize them even if I had the competence to do so.

9. The zero area problem. An interesting problem that remains unsolved is the following: Is it true that every finitely generated Kleinian group has a limit set with two-dimensional measure zero?

The most immediate reason for raising the question is that it is easy to prove the corresponding property for Fuchsian groups of the second kind, two-dimensional measure being replaced by one-dimensional. How does one prove it? If the limit set of a Fuchsian group has positive measure one can use the Poisson integral to construct a harmonic function on the unit disk with boundary values 1 a.e. on the limit set and 0 elsewhere. If the group is finitely generated the surface must have a finitely generated fundamental group, and it is therefore of finite genus and connectivity. The ideal boundary components are then representable as points or curves. If they are all points the group would be of the first kind, and if there is at least one curve the existence of a nonconstant harmonic function which is zero on the boundary violates the maximum principle. Therefore the limit set must have zero linear measure. The proof is thus quite trivial, but it is trivial only because one has a complete classification of surfaces with finitely generated fundamental groups.

For Kleinian groups it is easy enough to imitate the construction of the harmonic function, which this time has to be harmonic with respect to the hyperbolic metric of the unit ball. If the group is geometrically finite this leads rather easily to a proof of measure zero. For the general case it seems that one would need a better topological classification of three-manifolds with constant negative curvature. It is therefore not suprising that the problem has come to the attention of the topologists, and I am happy to report that at least two leading topologists are actively engaged in research on this problem. I believe that this pooling of resources will be very fruitful, and it would of course not be the first time that analysis topology, and vice versa.

Some time ago W. Thurston became interested in a topological concerning foliations of surfaces, and he proved a theorem which is closely related to Teichmüller theory. I have not seen Thurston's work, but I have seen Bers' interpretation of it as a new extremal problem for self-mappings of a surface. It is fascinating, and I could and perhaps should have talked about it in connection with the Teichmüller extremal problem, but I am a little hesitant to speak about things that are not yet in print, and therefore not quite in the public domain. Nevertheless, since many exciting things have happened quite recently in this particular subject, I am taking upon myself to report very informally on some of the newest developments, including some where I have to rely on faith rather than proofs.

Thurston has now begun to apply his remarkable geometric and topological intuition and skill to the problem of zero measure. I certainly do not want to

preempt him in case he is planning to talk about it in his own lecture, and I have seen only glimpses of his reasoning, but it would seem that he can prove zero area for all groups that are limits, in one sense or another, of geometrically finite groups. This would be highly significant, for it would show that all groups on the boundary of Teichmüller space have limit sets with zero measure. It would neither prove nor disprove the original conjecture, but it would be a very big step. Personally, I feel that a definitive solution is almost imminent.

Very recently there was a highly specialized conference on Riemann surfaces in the United States, and there was an air of excitement caused not only by what Thurston had done and was doing, but also by the presence of D. Sullivan who had equally fascinating stories to tell. Sullivan, too, has worked hard on the area problem, and he has come up with a by-product that does not solve the problem, but is extremely interesting in itself. He applies the powerful tool of what has been called topological dynamics. If a transformation group acts on a measure space, the space splits into two parts, a dissipative part with a measurable fundamental set, and a recurrent part whose every measurable subset meets infinitely many of its images in a set of positive measure. This powerful theorem, which goes back to E. Hopf, does not seem to have been familiar to those who have approached Kleinian group from the point of view of q.c. mappings. The dissipative part of a finitely generated group is the set of discontinuity, and nothing more; this is a known theorem. The recurrent part is the limit set, and it is of interest only if it has positive measure. But even if the area conjecture is true Sullivan's work remains significant for groups whose limit set is the whole sphere.

Sullivan has several theorems, but the one that has captured my special interest because I understand it best asserts that there is no invariant vector field supported on the limit set. If the limit set is the whole sphere there is no invariant vector field, period. In an equivalent formulation, the limit set carries no Beltrami differential. It was known before that there are only a finite number of linearly independent Beltrami differentials on the limit set of a finitely generated Kleinian group, but that there are none was a surprise to me, and Sullivan's approach gives results even for groups that are not finitely generated. Sullivan's results, taken as a whole, give a new outlook on the ergodic theory of Kleinian groups. They are related to, but go beyond the results of E. Hopf which were already considered deep and difficult, and as a corollary Sullivan obtains a strenthening of Mostow's rigidity theorem. I cannot explain the proofs beyond saying that they are very clever and show that Sullivan is not only a leading topologist, but also a strong analyst.

10. Several dimensions. In the remaining time I shall speak briefly about the generalizations to more than two dimensions. There are two aspects: q.c. mappings *per se*, and Kleinian groups in several dimensions.

The foundations for q.c. mappings in space are essentially due to Gehring and J. Väisälä, but very important work has also been done in the Soviet Union and Romania. I have already mentioned, in passing, that correct definitions can be

based on modules of curve families, and the modules give the only known workable technique. Otherwise, the difficulties are enormous. It is reasonably clear that the Beltrami coefficient should be replaced by a matrix-valued function, but this function is subject to conditions that were already known to H. Weyl, but which are so complicated that nobody has been able to put them to any use. Very little is known about when a region in n-space is q.c. equivalent to a ball, and there is not even an educated guess what Teichmüller's theorem should be replaced by. On the positive side one knows a little bit about boundary correspondence.

In two dimensions there is not much use for mappings that are locally q.c. but not homeomorphic, for by passing to Riemann surfaces they can be replaced by homeomorphisms. In several dimensions the situation is quite different, and there has been rapid growth of the theory of so-called quasiregular mappings from one n-dimensional space to another. It has been developed mostly in the Soviet Union and Finland, and this is perhaps a good opportunity to congratulate the young Finnish mathematicians to their success in this area. In the spirit of Rolf Nevanlinna they have even been able to carry over parts of the value distribution theory to quasiregular functions. In fact, less than a month ago I learned that Rickman has succeeded in proving a generalization of Picard's theorem that I know they have been looking for for a long time. It is so simple that I cannot resist quoting the result: There exists $q = q(n,k)$ such that any K-q.c. mapping $f : R^n \to R^n$ — $\{a_1, \ldots, a_q\}$ is constant. (They believe that the theorem is true with $q = 2$.)

As for Kleinian groups, they generalize trivially to any number of dimensions, and the distinction between Fuchsian and Kleinian groups disappears. Some properties that depend purely on hyperbolic geometry will carry over, but they are not the ones that use q.c. mappings. However, infinitesimal q.c. mappings have an interesting counterpart for several variables. There is a linear differential operator that takes the place of $f_{\bar{z}}$, namely $Sf = \frac{1}{2}(Df + Df') - (1/n)\mathrm{tr}\ Df \cdot 1_n$ which is symmetric matrix with zero trace. It has the right invariance, and the conditions under which the Beltrami equation $Sf = \nu$ has a solution can be expressed as a linear integral equation. The formal theory is there, but it will take time before it leads to tangible results.

My survey ends here. I regret that there are so many topics that I could not even mention, and that my report has been so conspicuously insufficient as far as research in the Soviet Union is concerned. I know that I have not given a full picture, but I hope that I have given you an idea of the extent to which q.c. mappings have penetrated function theory.

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THE WORK OF L. SCHWARTZ

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At a meeting of the organizing committee of the International Congress held in 1924 at Toronto the resolution was adopted that at each international mathematical congress two gold medals should be awarded, and in a memorandum the donor of the fund for the founding of the medals, the late Professor J. C. Fields, expressed the wish that the awards should be open to the whole world and added that, while the awards should be a recognition of work already done, it was at the same time intended to be an encouragement for further mathematical achievements. The funds for the Fields' medals were finally accepted by the International Congress in Zürich in 1932, and two Fields medals were for the first time awarded at the Congress in Oslo 1936 to Professor Ahlfors and Professor Douglas. And now, after a long period of fourteen years, the mathematicians meet again at an international congress, here in Harvard.

In the fall of 1948 Professor Oswald Veblen, as nominee of the American Mathematical Society for the presidency of the Congress in Harvard, together with the chairman of the organizing committee, and with the secretary of the Congress, appointed an international committee to select the two recipients of the Fields medals to be awarded at the Congress in Harvard, the committee consisting of Professors Ahlfors, Borsuk, Fréchet, Hodge, Kolmogoroff, Kosambi, Morse, and myself. With the exception of Professor Kolmogoroff, whose valuable help we were sorry to miss, all the members of the committee have taken an active part in the discussions. As chairman of the committee I now have the honor to inform the Congress of our decisions and to present the gold medals together with an honorarium of $1,500 to each of the two mathematicians selected by the committee.

The members of the committee were, unanimously, of the opinion that the medals, as on the occasion of the first awards in Oslo, should be given to two really young mathematicians, without exactly specifying, however, the notion of being "young". But even with this principal limitation the task was not an easy one, and it was felt to be very encouraging for the expectations we may entertain of the future development of our science that we had to choose among so many young and very talented mathematicians, each of whom should certainly have been worthy of an official appreciation of his work. Our choice fell on Professor *Atle Selberg* and

Professor *Laurent Schwartz*, and I feel sure that all members of the Congress will agree with the committee that these two young mathematicians not only are most promising as to their future work but have already given contributions of the uttermost importance and originality to our science; indeed they have already written their names in the history of mathematics of our century.

I now turn to the work of the slightly older, of the two recipients, the French mathematician *Laurent Schwartz*. Having passed through the old celebrated institution École Normale Supérieure in Paris, he is now Professor at the University of Nancy. He belongs to the group of most promising and closely collaborating young French mathematicians who secure for French mathematics in the years to come a position worthy of its illustrious traditions. Like Selberg, Schwartz can look back on an extensive and varied production, but when comparing the work of these two young mathematicians one gets a strong impression of the richness and variety of the mathematical science and of its many different aspects. While Selberg's work dealt with clear cut problems concerning notions which, as the primes, are, so to say, given *a priori*, one of the greatest merits of Schwartz's work consists on the contrary in his creation of new and most fruitful notions adapted to the general problems the study of which he has undertaken. While these problems themselves are of classical nature, in fact dealing with the very foundation of the old calculus, his way of looking at the problem is intimately connected with the typical modern development of our science with its highly general and often very abstract character. Thus once more we see in Schwartz's work a confirmation of the words of Felix Klein that great progress in our science is often obtained when new methods are applied to old problems. In the short time at my disposal I think I may give the clearest impression of Schwartz's achievements by limiting myself to speak of the very central and most important part of his work, his theory of "distribution". The first publication of his new ideas was given in a paper in the Annales de ℓ'Université de Grenoble, 1948, with the title *Généralisation de la notion de fonction*, *de dérivation*, *de transformation de Fourier et applications mathématiques et physiques*, a paper which certainly will stand as one of the classical mathematical papers of our times. As the title indicates it deals with a generalization of the very notion of a function better adapted to the process of differentiation than the ordinary classical one. In trying to explain briefly the new notions of Schwartz and their importance I think I can do no better than start by considering the same example as that used by Schwartz himself, namely, the simple function $f(x)$ which is equal to 0 for $x \leqq 0$ and equal to 1 for $x > 0$. This function has a derivative $f'(x) = 0$ for every $x \neq 0$, but this fact evidently does not tell us anything about the magnitude of the jump of $f(x)$ at the point $x = 0$. In order to overcome this inconvenience the physicist and the technicians had accustomed themselves to say that the function $f(x)$ has as derivative the "Dirac-function" $f'(x) = \delta(x)$ which is 0 for $x \neq 0$ and equal to $+\infty$ for $x = 0$ and moreover has the property that its integral over any interval containing the point $x = 0$ shall be equal to 1. But this is of course not a legitimate way of speaking; from a mathematical point of view — using the idea of a Stieltjes' integral — we naturally

would think of the derivative of our function $f(x)$ not as a function but as a mass-distribution, in this case of the particularly simple type with the whole mass 1 placed at the origin $x = 0$. Now, according to a classical theorem of F. Riesz, there is a most intimate connection, in fact a one-to-one correspondence, between an arbitrary mass-distribution μ on the x-axis and a linear continuous functional $\mu(\phi)$ defined in the space of all continuous functions $\phi(x)$, vanishing outside a finite interval, where the topology of the ϕ-space is fixed by the simple claim that convergence of sequence ϕ_n shall mean that the functions $\phi_n(x)$ are all zero outside a fixed finite interval and that the sequence $\phi_n(x)$ shall be uniformly convergent. This correspondence between the mass-distributions μ and the functionals $\mu(\phi)$ is given simply by the relation

$$\mu(\phi) = \int_{-\infty}^{+\infty} \phi(x)\, d\mu\,.$$

In the Schwartz theory of distributions the new notion, generalizing, or rather replacing, that of a function is nothing else than just such a linear continuous functional, but of a kind essentially different from that above, the underlying ϕ-space and its topology being of a quite different nature. The new notion — once invented — is so easy to explain that I cannot resist the temptation, notwithstanding the general solemn nature of this opening meeting, to go into some detail. Let us consider, then, with Schwartz a quite arbitrary function $f(x)$, assumed only to be integrable in the sense of Lebesgue over any finite interval, and let us try to characterize the function $f(x)$, not as in the classical Dirichlet way by the values it takes for the different values of x, but by what we may call its effect when operating on an arbitrary auxiliary function $\phi(x)$ of which, for the moment, we suppose only, as above, that $\phi(x)$ is continuous and equal to zero outside a finite interval; by the effect of the given function $f(x)$ on the auxiliary function $\phi(x)$ we here mean simply the value of the integral

$$\int_{-\infty}^{+\infty} f(x)\phi(x)\, dx\,.$$

This integral, obviously, is a linear functional, associated with the function $f(x)$, and we shall denote it $f(\phi)$, using the same letter f, as we wish, so to speak, to identify it with the function $f(x)$ itself. In the special case where the given function $f(x)$ has a continuous derivative $f'(x)$ we may of course, starting with $f'(x)$ instead of with $f(x)$, in the same way build a functional associated with $f'(x)$, i.e., the fuctional

$$f'(\phi) = \int_{-\infty}^{+\infty} f'(x)\phi(x)\, dx\,.$$

If now — and this is an essential point — we assume also the auxiliary function $\phi(x)$ to have a continuous derivative $\phi'(x)$, we immediately find through partial integration

$$\int_{-\infty}^{+\infty} f'(x)\phi(x)\, dx = -\int_{-\infty}^{+\infty} f(x)\phi'(x)\, dx\,,$$

i.e., the simple relation

$$f'(\phi) = -f(\phi')\,.$$

In order that the derivative of any function $\phi(x)$ of our space shall also belong to the space, we must obviously assume, with Schwartz, that the auxiliary function $\phi(x)$ to be considered shall possess derivaties not only of the first order but of arbitrarily high order. In the space consisting of all such functions $\phi(x)$, i.e., of all functions $\phi(x)$ zero outside a finite interval and with derivatives of any order, and topologized by the definition that convergence of a sequence ϕ_n shall mean not only as above that the function $\phi_n(x)$ shall all be zero outside a fixed finite interval and that $\phi_n(x)$ shall converge uniformly, but moreover that all the derivated sequences $\phi_n'(x), \phi_n''(x)\ldots$ shall converge uniformly, Schwartz now takes into consideration all continuous linear functionals $J(\phi)$. These functionals $J(\phi)$ are just what Schwartz denotes as "distributions". Among them are in particular the distributions $f(\phi)$ derived in the manner above from an ordinary function $f(x)$, and more generally we have distributions, which we denote by $\mu(\phi)$, which are associated with a mass-distribution μ — evidently the word distribution has been chosen to remind us vaguely of these mass-distributions — but the whole class of Schwartz distributions is far from being exhausted by the special distributions of the type $f(\phi)$ or $\mu(\phi)$. Now — and this is the decisive point in the theory — Schwartz assigns to every one of his distributions $J(\phi)$ another distribution $J'(\phi)$ as the derivative of $J(\phi)$, namely, immediately suggested by the consideration above, the distribution defined by

$$J'(\phi) = -J(\phi')\,.$$

In the special case where the distribution $J(\phi)$ is of the type $f(\phi)$ and moreover is derived from a function $f(x)$ with a continuous derivative, the derivated distribution $J'(\phi) = -J(\phi')$ is, evidently, nothing else than the distribution $f'(\phi)$ associated with the function $f'(x)$. But generally, if $f(x)$ is an arbitrary function with no derivative, the corresponding functional $f(\phi)$ still has a derivative $f'(\phi)$ which, however, is no longer associated with any function, neither, generally, with any mass-distribution, but is just some Schwartz distribution.

And now, one will naturally ask, what has been gained by Schwartz's generalization of a function $f(x)$ to that of a distribution $J(\phi)$. Naturally, the aim of any such generalization of basic notions — as, for instance, the generalization of the notion of a real number to that of a complex number — is, in principle, the same and of a double kind; on the one hand, and this is the primary purpose, one aims at getting simplifications in the treatment of problems concerning the old notions through the greater freedom in carrying out operations, provided by the new notions, and on the other hand, one may hope to meet with new fruitful problems concerning these new notions themselves. In both these respects the theory of Schwartz may be said to be a great success. I think that every reader of his cited paper, like myself, will have left a considerable amount of pleasant excitement, on seeing the wonderful harmony of the whole structure of the calculus to which the theory leads and on understanding how essential an advance its application may mean to many parts

of higher analysis, such as spectral theory, potential theory, and indeed the whole theory of linear partial differential equations, where, for instance, the important notion of the "finite part" of a divergent integral, introduced by Hadamard, presents itself in a most natural way when the distributions and not the functions are taken as basic elements. And as to the harmony brought about I shall mention only one single, very simple, but most satisfactory result. Not only has, as we have seen, every distribution $J(\phi)$ a derivative $J'(\phi)$, and hence derivatives of every order, but conversely it also holds that every distribution possesses a primitive distribution, i.e., is the derivative of another distribution which is uniquely determined apart from an additive constant (i.e., of course, a distribution associated with a constant). The simplification obtained, and not least the easy justification of different "symbolic" operations often used in an illegitimate way by the technicians, is of such striking nature that it seems more than a utopian thought that elements of the theory of the Schwartz distributions may find their place even in the more elementary courses of the calculus in universities and technical schools.

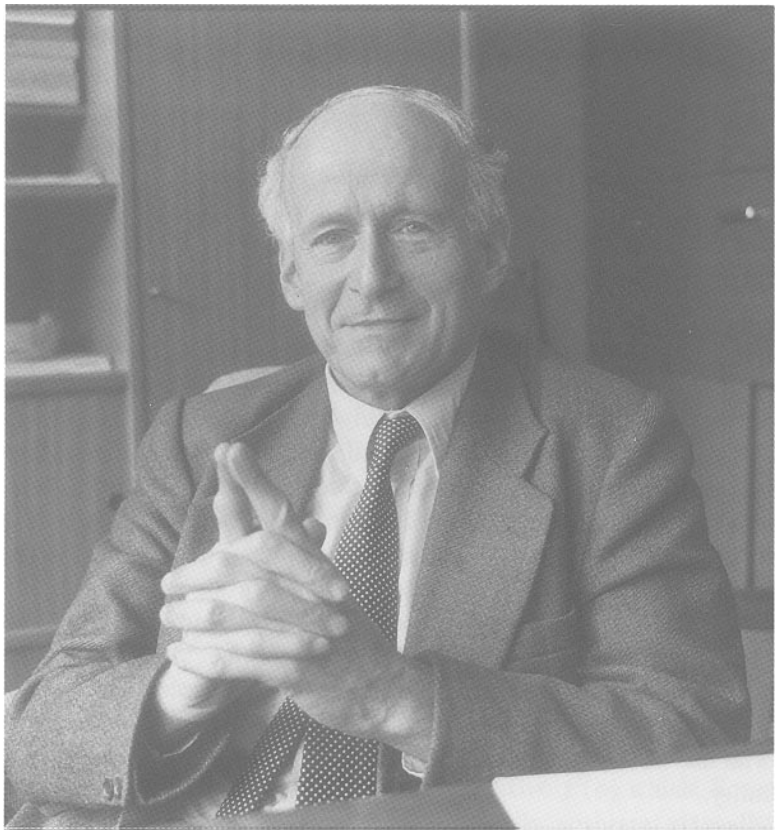
Schwartz is now preparing a larger general treatise on the theory of distributions, the first, very rich, volume of which has already appeared. In his introduction to this treatise he emphasizes the fact that ideas similar to those underlying his theory have earlier been applied by different mathematicians to various subjects — here only to mention the methods introduced by Bochner in his studies on Fourier integrals — and that the theory of distibutions is far from being a "nouveauté revolutionaire." Modestly he characterizes his theory as "une synthèse et une simplification". However, as in the case of earlier advances of a general kind — to take only one of the great historic examples, that of Descartes' development of the analytic geometry which, as is well-known, was preceded by several analytic treatments by other mathematicians of special geometric problems — the main merit is justly due to the man who has clearly seen, and been able to shape, the new ideas in their purity and generality.

No wonder that the work of Schwartz has met with very great interest in mathematical circles throughout the world, and that a number of younger mathematicians have taken up investigations in the wide field he has opened for new researches.

And now I have the honor to call upon Professor Selberg and Professor Schwartz to present to them the golden medals and the honorarium.

In the name of the committee, I think I dare say of the whole Congress, I congratulate you most heartily on the awards of the Fields medals. Repeating the wish of Fields himself I may finally express the hope that the great admiration of your achievements of which the medals are a token may also mean an encouragement to you in your future work.

Harvard University
Cambridge, Massachusetts
August 30, 1950



Laurent Schwartz

LAURENT SCHWARTZ

Etudes et diplômes

1934–1937	Ecole Normale Supérieure
1937	Agrégation de Mathématíques
1943	Doctorat ès-Sciences, Faculté des Sciences de Strasbourg réfugiée á Clermont Ferrand.

Carrière

1944–1945	Chargé de cours à la Faculté des Sciences de Grenoble
1945–1952	Maître de conférences, puis professeur à la Faculté des Sciences de Nancy
1953–1969	Maître de conférences, puis professeur à la Faculté des Sciences de Paris
1959–1980	Professeur à l'Ecole Polytechnique
1980–1983	Professeur á l'Université Paris VII
1983	Retraite

Prix

1950	Médaille Fields, Congrès international des mathématiciens, Cambridge (Mass.) U.S.A.
1955	Prix Carrière de l'Académie des Sciences, Paris, France
1964	Grand prix de Mathématiques et de Physique, Académie des Scíences, Paris, France
1972	Prix Cognac-Jay, Académie des Sciences, Paris, France (avec J. L. Lions et B. Malgrange).
1956	Professeur honoraire de l'Université d'Amérique, Bogota, Colombie, et professeur honoraire de l'Université de Buenos-Aires, Argentine
1957	Membre corespondant de la Société royale des Sciences de Liège, Belgique
1958	Membre honoraire de l'Union mathématique Argentine, académicien honoraire
1960	Docteur Honoris Causa de l'Université de Humboldt, Berlin, R.D.A.
1962	Docteur Honoris Causa de l'Université libre de Bruxelles, Belgique
1964	Membre correspondant de l'Académie des Sciences du Brésil
1965	Professeur honoraire de l'Université nationale des Ingénieurs du Pérou, Lima, Pérou
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1972	Membre correspondant, Académie des Sciences, Paris, France

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1981 Docteur Honoris Causa de l'Université de Lund (Suède)
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1988 Membre étranger de l'"Accademia Nazionale dei Lincei", Rome
1991 Membre associé de l'Académie Polonaise des Sciences
1993 Docteur honoris causa de l'Université d'Athènes (Grèce)
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CALCUL INFINITESIMAL STOCHASTIQUE

by

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Dédié à Jacques–Louis Lions pour son soixantième anniversaire

Cet article ne contient presque pas de résultats nouveaux, mais donne, à partir de rien, et sans démonstrations, de nombreux résultats personnels du calcul infinitésimal stochastique. Je suis très heureux de le dédier à Jacques–Louis Lions.

1. Processus[(1)]

La donnée initiale est toujours un ensemble Ω (ensemble des échantillons), une tribu $\mathcal{F}$ de parties de Ω, une probabilité $\mathcal{P}$ (mesure ≥ 0 de masse 1) sur $(\Omega, \mathcal{F})$. On se donne en outre une "filtration", $(\mathcal{F}_t)_{t\in\overline{\mathbb{R}}_+}$, de sous tribus de la tribu $\mathcal{P}$-mesurable, $\mathcal{P}$-complète, croissante, continue à droite ($\mathcal{F}_t = \cap_{t'>t}\mathcal{F}_{t'}$, pour $t < +\infty$). On prend ici pour ensemble des temps t l'ensemble $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$, compact; d'autres préfèrent prendre $\mathbb{R}_+$; c'est sans importance, puisque nous prendrons plus loin (chapitre 8) des sous-ensembles A de $\overline{\mathbb{R}}_+$ comme ensembles des temps. Nous ajouterons souvent un élément $\overline{+\infty} > +\infty$, isolé, $\mathcal{F}_{\overline{+\infty}} = \mathcal{F}_{+\infty}$.

Une variable aléatoire, par exemple à valeurs dans un espace vectoriel E de dimension finie N, est une application $\mathcal{P}$-mesurable $\Omega \to E$. Un processus X à valeurs dans E est une application $\overline{\mathbb{R}}_+ \times \Omega \to E$; $X_t : \omega \mapsto X(t,\omega)$, l'etat du processus à l'instant t, sera supposé $\mathcal{F}_t$-mesurable; $X(\omega) : t \mapsto X(t,\omega)$ est la trajectoire du processus correspondant à l'échantillon $\omega \in \Omega$. Un processus est donc une trajectoire aléatoire. Puisque X_s est $\mathcal{F}_t$-mesurable pour $s \leq t$, $\mathcal{F}_t$ apparaît comme la tribu du passé (présent inclus !) de l'instant t. Le processus est dit continu, continu à droite, à variation finie, ..., si, pour $\mathcal{P}$-presque tout ω (expression que nous ne répèterons plus), la trajectoire $X(\omega)$ est continue, continue à droite, à variation finie, ...

2. Processus à variation finie et martingales

Un processus V sera dit `*a variation finie* s'il est *continu* à variation finie. Une *martingale* M (voir Dellacherie–Meyer [1], volume 2, chapitre V) est un processus continu intégrable (pour tout t, M_t est intégrable), tel que, si $s \leq t$ et $A \in \mathcal{F}_s$:

(1) Il est impossible de donner ici une liste des livres d'initiation aux probabilités. Nous renverrons souvent à Dellacherie–Meyer [1], mais on pourrait aussi souvent renvoyer à Ikeda–Watanabe [1]

$$\int_A M_s d\mathcal{P} = \int_A M_t d\mathcal{P}\,. \tag{2.1}$$

En probabilités, les intégrales s'écrivent en général E (espérance); (2.1) s'écrit alors

$$E(1_A M_s) = E(1_A M_t)\,. \tag{2.2}$$

On écrit cela aussi en utilisant la notion très féconde d'espérance conditionnelle, mais le lecteur pourra s'en passer pour la suite. L'espérance conditionnelle d'une variable aléatoire intégrable Y, par rapport à une tribu $\mathcal{T}$, P-mesurable, notée $Y^{\mathcal{T}}$ ou $E(Y|\mathcal{T})$, est une variable aléatoire $\mathcal{T}$-mesurable, ayant mêmes intégrales que Y sur les ensembles de la tribu $\mathcal{T}$; elle n'est définie qu'à un ensemble $\mathcal{T}$-mesurable $\mathcal{P}$-négligeable près. Si $\mathcal{T}$ est toute la tribu $\mathcal{P}$-mesurable, $Y^{\mathcal{T}} = Y$; si $\mathcal{T}$ est la tribu triviale, $\mathcal{T} = \{\phi, \Omega\}$, $Y^{\mathcal{T}}$ est la constante égale à l'intégrale $\int_\Omega Y d\mathcal{P} = E(Y)$. Alors M est une martingale si elle est intégrable et si, pour $s \leq t$:

$$M_s = E(M_t|\mathcal{F}_s)\,. \tag{2.3}$$

2'. Martingales locales et temps d'arrêt[(2)]

On a besoin en pratique de processus un peu plus généraux, les *martingales locales*; local n'a pas ici le sens topologique habituel. Le processus M est une martingale locale s'il est continu, et s'il existe une suite croissante $(T_n)_{n\in\mathbb{N}}$ de temps aléatoires (T_n est une variable aléatoire $\Omega \to [0, \overline{+\infty}]$) tendant *stationnairement* vers $\overline{+\infty}$ (pour presque tout ω, il existe $N(\omega)$ tel que, pour $n \geq N(\omega)$, $T_n(\omega) = \overline{+\infty}$; on écrira $T_n \uparrow\uparrow \overline{+\infty}$) et une suite de martingales $(M_n)_{n\in\mathbb{N}}$, tels que, dans $[0, T_n[= \{(t,\omega);\, t < T_n(\omega)\}$, $M = M_n$; on ne met pas $[0, T_n]$, car cela imposerait que M_o soit intégrable, ce que l'on ne souhaite pas; M_t n'est peut-être intégrable pour aucun t. Mais, par continuité, $M = M_n$ dans $[0, T_n] \cap (\overline{\mathbb{R}}_+ \times \Omega) \cap \{T_n > 0\} = \{(t,\omega); T_n(\omega) > 0,\ 0 \leq t \leq T_n(\omega) \leq +\infty\}$.

Cela s'écrit aussi en utilisant les temps d'arrêt (voir Dellacherie–Meyer [1], volume 1, chapitre IV, 49, page 184). Les temps d'arrêt ont été introduits par Doob. Nous ne les développerons pas ici, malgré leur rôle fondamental, parce que nous ne les utiliserons pas plus loin. Bornons-nous à dire qu'un temps d'arrêt est une variable aléatorie $T : \Omega \to [0, \overline{+\infty}]$, telle que, pour tout $t \leq +\infty$, l'ensemble $\{T \leq t\} = \{\omega; T(\omega) \leq t\}$ soit $\mathcal{F}_t$-mesurable. Un temps d'arrêt sert à arrêter des processus; si X est un processus, le processus arrêté au temps T, noté X^T, est le processus pour lequel la trajectoire $X^T(\omega)$ coïncide avec $X(\omega)$ aux temps $\leq T(\omega)$, mais reste fixée à $X(T(\omega), \omega)$ aux temps $\geq T(\omega)$; on peut écrire $X_t^T = X_{T\wedge t}$ (voulant dire, avec l'habitude des probabilités de ne jamais écrire $\omega : X_t^T(\omega) = X_{T(\omega)\wedge t}(\omega)$). Un théorème de Doob dit qu'un processus arrêté d'une martingale est encore une martingale (voir Dellacherie–Meyer [1], volume 2, chapitre VI, théorème 10). Alors M

(2) Le lecteur pressé pourra passer ce chapitre

est une martingale locale s'il existe $(T_n) \uparrow\uparrow \overline{+\infty}$ tels que, pour tout n, $1_{\{T_n>0\}} M^{T_n}$ soit une martingale. *Dans la suite, nous abrègerons martingale locale par martingale.*

3. Semi-martingales

Une semi-martingale (voir Dellacherie–Meyer [1], volume 2, chapitre VII, 2) est un processus continu qui peut s'écrire

$$X = V + M\,, \tag{3.1}$$

V à variation finie, M martingale (sous-entendu locale). On normalise part M_o (valeur initiale) $= 0$, alors la décomposition est unique, à un ensemble $\mathcal{P}$-négligeable près (ce qui signifie qu'une martingale à variation finie est un processus constant, $M = M_o$). L'unicité de la décomposition est le théorème de Doob–Meyer (voir Dellacherie–Meyer [1], volume 2, chapitre VIII, 45). On écrit habituellement

$$X = \widetilde{X} + X^c\,; \tag{3.2}$$

$\widetilde{X}$ s'appelle la caractéristique locale de X, X^c sa compensée.

4. Calcul intégral stochastique d'Ito

La tribu optionnelle Opt sur $\overline{\mathbb{R}}_+ \times \Omega$ est (voir Dellacheric–Meyer [1], volume 1, chapitre IV, 61) la tribu engendrée par les processus réels continus à droite et les parties $\mathcal{P}$-négligeables (c.-à-d. dont la projection sur Ω est $\mathcal{P}$-négligeable). (Elle est aussi engendrée par les intervalles stochastiques $[S,T[= \{(t,\omega); S(\omega) \leq t < T(\omega)\}, S, T$ temps d'arrêt, ou simplement les $[S,+\infty] = [S,\overline{+\infty}[$, et les parties $\mathcal{P}$-négligeables).

Si H est une fonction réelle optionnelle bornée sur $\overline{\mathbb{R}}_+ \times \Omega$, on peut définir une intégrale $H.V$,

$$(H.V)_t = \int_{]0,t]} H_s\, dV_s\,, \quad (H.V)_o = 0\,, \tag{4.1}$$

où, comme toujours, ω n'est pas marqué; cela veut dire

$$(H.V)(t,\omega) = \int_{]0,t]} H(s,\omega) dV(s,\omega) \quad (d = d_s)\,. \tag{4.2}$$

Elle se calcule individuellement pour tout ω (intégrale de Stieltjes); $H.V$ est encore un processus à variation finie.

Mais Ito a montré que, si H est optionnelle bornée, et si M est une martingale, on peut encore définir une intégrale stochastique $H.M$ (voir Dellacherie–Meyer [1], volume 2, chapitre VIII); mais elle ne se calcule plus individuellement pour tout ω, elle n'a qu'un sens global, comme $\mathcal{P}$-classe de processus (modulo les processus $\mathcal{P}$-presque partout nuls, c.-à-d. nuls pour $\mathcal{P}$-presque tout ω, pour tout t) un peu comme

l'intégrale de Fourier d'une fonction de L^2 est seulement une classe de Lebesgue de fonctions de carré intégrable, sans valeur précise en chaque point; la méthode de définition est d'ailleurs la même, c'est un prolongement par continuité; c'est encore une martingale (voulant dire, comme indiqué à la fin du chapitre 2', une martingale locale; même si M est une martingale vraie, et si H est optionnelle bornée, $H.M$ n'est en général qu'une martingale locale).

Si alors X est une semi-martingale, et si H est optionnelle bornée, il y aura encore une intégrale stochastique $H.X$, nulle au temps 0; et $(H.X)^\sim = H.\widetilde{X}, (H.X)^c = H.X^c$. Inversement un théorème de Dellacherie (voir Dellacherie–Meyer [1], volume 2, chapitre VIII, 4) dit que les semi-martingales sont les seuls processus donnant lieu, dans un sens à préciser, à des intégrales stochastiques. De là, on passe facilement à des H optionnelles non seulement bornées, mais X-intégrables. On peut même passer à des H optionnelles qui ne sont plus intégrables, et définir des $H.X$ qui sont seulement des semi-martingales formelles (voir Schwartz [1]), au sens où la dérivée d'une fonction sans dérivée est une distribution, ou fonction formelle; d'ailleurs une semi-martingale formelle peut se définir comme une $H.X$, X semi-martingale, H optionnelle, non nécessairement X-intégrable. On pourra désormais écrire $H.X$, pour H optionnelle arbitraire, X semi-martingale formelle et ce sera encore une semi-martingale formelle. Mais on ne pourra plus parler de sa valeur $X(t,\omega)$, comme une distribution n'a pas de valeur en un point. On a aussi des processus à variation finie formels, et des martingales formelles, et on a toujours:

$$(H.X)^\sim = X.\widetilde{X}, (H.X)^c = H.X^c, H.(K.X) = HK.X\,. \tag{4.3}$$

Bien entendu, si X est à valeurs dans l'espace vectoriel E, H n'a pas besoin d'être réelle; si H est à valeurs dans $\mathcal{L}(E;F)$, alors $H.X$ est à valeurs dans F. Notons que, si S et T sont des temps d'arrêt, $1_{[S,T[}.X = X^T - X^S$. Nous ne nous occuperons pas des processus formels, mais nous négligerons les problémes d'intégrabilité.

5. Le crochet[3]

Si, dans (2.1), (2.2), (2.3), on remplace $=$ par $\leq$, on a la notion de sous-martingale; de là on passe aux sous-martingales locales, qu'on abrège encore par sous-martingales.

Si M est une martingale réelle, son carré M^2 est une sous-martingale ≥ 0; elle est alors une semi-martingale et sa composante à variation finie se note $[M,M]$, qui est un processus croissant. Si X est une semi-martingale quelconque, on pose

$$[X,X] = [X^c,X^c] + X_o^2\,. \tag{5.1}$$

On peut l'interpréter facilement. En effet, la trajectoire $X(\omega)$ n'est pas en général à variation finie (à cause de la composante martingale X^c), mais elle a une

[3] Voir Dellacherie–Meyer [1], volume 2, chapitre VII, 42

variation quadratique finie, et justement $[X,X]_t(\omega)$ est la variation quadratique de $X(\omega)$ de 0 à t (voir Dellacherie–Meyer [1], volume 2, chapitre VIII, théorème 20):

$$[X,X]_t(\omega) = \lim\,(X_o^2(\omega) + \sum_i (X_{t_{i+1}\wedge t} - X_{t_i\wedge t})^2(\omega))\,, \tag{5.2}$$

lorsque le pas de la subdivision tend vers 0 (le pas est mesuré par rapport à une distance sur $\overline{\mathbb{R}}_+$ compatible avec sa topologie compacte); lim. est une limite en probabilité, uniformément en t. Par polarisation, si X, Y sont deux semi-martingales, on définit leur crochet, à variation finie $[X,Y]$. Si X, Y sont à valeurs dans des espaces vectoriels $E, F, [X,Y]$ est à valeurs dans $E \otimes F$; pour $F = E$, on prend en général $[X,Y]$ à valeurs dans le produit tensoriel symétrique $E \odot E$, quotient de $E \otimes E$. On a toujours

$$[H.X, K.Y] = HK.[X,Y]\,. \tag{5.3}$$

Un théorème, dû à Girsanov (voir Dellacherie–Meyer [1], volume 2, chapitre VII, 45), dit que, si Q est une probabilité équivalente à $\mathcal{P}$ sur $(\Omega, \mathcal{F})$, un processus X est une Q-semi-martingale si et seulement s'il est une $\mathcal{P}$-semi-martingale. Les processus $\widetilde{X}, X^c$, ne sont pas les mêmes pour $\mathcal{P}$ et Q, mais $[X,X]$ est le même d'après (5.2).

$[X,Y]$ est l'unique processus à variation finie tel que $XY - [X,Y]$ soit une martingale nulle au temps 0.

6. Le mouvement brownien

La plus connue des martingales (vraie, pas seulement locale, mais sur $\mathbb{R}_+ \times \Omega$, $\mathbb{R}_+ = [0, +\infty[$) est le mouvement brownien. Dans un espace euclidien E de dimension N, on appelle mouvement brownien normal une martingale B vérifiant:

a) $B_o = 0$ $\mathcal{P}$-presque sûrement;
b) Pour $t > s$, $B_t - B_s$ est indépendante de la tribu $\mathcal{F}_s$, et sa loi dans E (la mesure image $(B_t - B_s)(\mathcal{P})$) est la loi de Gauss centrée de paramètre $\sqrt{t-s}$:

$$\left(\frac{1}{\sqrt{2\pi(t-s)}}\right)^N \exp(-|x|^2/2(t-s))dx$$

L'existence d'une telle martingale a été démontrée par N. Wiener; elle n'est évidemment pas unique ($\Omega, \mathcal{P}, \ldots$ peuvent varier), mais elle est unique "en loi". Les plus importants résultats sur le brownien ont été donnés par P. Lévy.

On montre qu'on peut aussi caractériser le brownien par le fait que c'est une martingale dont le crochet $[B,B]$, à valeur dans $E \odot E$. est

$$[B,B]_t = \sum_{k=1}^N (e_k \odot e_k)t\,, \quad \text{ou}\,[B^i, B^j]_t = \delta^{ij}t\,,$$

si (e_k) est une base orthonormée quelconque, δ le symbole de Kronecker; ou encore, en omettant toujours ω:

$$t \mapsto B_t \odot B_t - \sum_{k=1}^{N} (e_k \odot e_k)t \text{ est une martingale}.$$

(Voir Dellacherie–Meyer [1], volume 2, chapitre VIII, théorème 5.9). Ce théorème est dû à Paul Lévy.

7. Le calcul différentiel stochastique vectoriel; la formule de changement de variables d'Ito

Soient E, F des espaces vectoriels, U un ouvert de E, Φ une application C^2 : $U \to F, X : \overline{\mathbb{R}}_+ \times \Omega \to E$ une semi-martingale, à valeurs dans U. Alors $\Phi(X)$ est une semi-martingale à valeurs dans F, qui s'écrit (voir Dellacherie–Meyer [1], volume 2, chapitre VIII, théorème 27):

$$\Phi(X) = \Phi(X_o) + \Phi'(X).X + \frac{1}{2}\Phi''(X).[X, X] \tag{7.1}$$

Comme X n'est pas à variation finie, mais a une variation quadratique $[X, X]$, il n'est pas étonnant qu'apparaisse le deuxième terme de la formule de Taylor, avec Φ''. L'interprétation est immédiate: X est à valeurs dans $E, \Phi'(X)$ à valeurs dans $\mathcal{L}(E; F)$, donc $\Phi'(X).X$ à valeurs dans $F; [X, X]$ est à valeurs dans $E \odot E$, $\Phi''(X)$ à valeurs dans l'espace des applications bilinéaires symétriques de $E \times E$ dans F, donc linéaires de $E \odot E$ dans F, et $\Phi''(X).[X, X]$ est bien encore à valeurs dans F. On en déduit:

$$\widetilde{\Phi(X)} = \Phi(X_o) + \Phi'(X).\widetilde{X} + \frac{1}{2}\Phi''(X).[X, X] \tag{7.2}$$

$$(\Phi(X))^c = \Phi'(X).X^c \tag{7.3}$$

$$\frac{1}{2}[\Phi(X), \Phi(X)] = \frac{1}{2}\Phi(X_o) \odot \Phi(X_o) + \frac{1}{2}(\Phi'(X) \odot \Phi'(X)).[X, X]\,. \tag{7.4}$$

Dans la dernière formule, $\Phi'(X)$ est une application linéaire de E dans F, son carré tensoriel symétrique $\Phi'(X) \odot \Phi'(X)$ est linéaire de $E \odot E$ dans $F \odot F, [X, X]$ est à valeurs dans $E \odot E$, donc $(\Phi'(X) \odot \Phi'(X))[X, X]$ est bien à valeurs dans $F \odot F$.

8. Première extension: formules locales[(4)]

Soit $A \subset \overline{\mathbb{R}}_+ \times \Omega$, optionnel. On dira que A est ouvert si, pour tout ω, la section $A(\omega) = \{t; (t, \omega) \in A\}$ est un ouvert de $\overline{\mathbb{R}}_+$.

On dira qu'un processus X est équivalent à 0 sur A, $X \underset{A}{\sim} 0$, si pour $\mathcal{P}$-presque tout ω, $X(\omega)$ est localement constant sur $A(\omega)$ (localement, cette fois, au sens

(4) Voir Schwartz [2], chapitres 2 et 3, et Schwartz [1]

topologique habituel: $X(\omega)$ est constant sur tout intervalle de $A(\omega)$). On dira que $X \underset{A}{\sim} Y$ si $X - Y \underset{A}{\sim} 0$.

On montre que, pour X, Y, semi-martingales:

$$X \underset{A}{\sim} 0 \Longleftrightarrow \widetilde{X} \underset{A}{\sim} 0\,, \quad X^c \underset{A}{\sim} 0\,, \tag{8.1}$$

$$X \underset{A}{\sim} \Rightarrow [X, Y] \underset{A}{\sim} 0\,, \tag{8.2}$$

(8.3) Si H est optionnel borné, *nul* sur A, ou si $X \underset{A}{\sim} 0$, alors $H.X \underset{A}{\sim} 0$.

Alors

$$X \underset{A}{\sim} X'\,, \quad Y \underset{A}{\sim} Y' \Rightarrow [X, Y] \underset{A}{\sim} [X', Y'] \tag{8.4}$$

(Noter que ce n'est pas vrai pour le produit multiplicatif ordinaire: les équivalences de gauche n'entraînent pas $XY \underset{A}{\sim} X'Y'$).

$$H \underset{A}{=} H'\,, \quad X \underset{A}{\sim} X' \Rightarrow H.X \underset{A}{\sim} H'.X'\,. \tag{8.5}$$

On peut alors définir des semi-martingales sur des ouverts A de $\overline{\mathbb{R}}_+ \times \Omega$. Un processus $X : A \to E$ est une semi-martingale s'il existe une suite $(A_n)_{n\in\mathbb{N}}$ d'ouverts de réunion A, et de semi-martingales $(X_n)_{n\in\mathbb{N}}$ sur $\overline{\mathbb{R}}_+ \times \Omega$ tel que $X \underset{A_n}{\sim} X_n$; de même pour les processus à variation finie et martingales. Si $A = \overline{\mathbb{R}}_+ \times \Omega$, on retrouve les objets définis antérieurement sur $\overline{\mathbb{R}}_+ \times \Omega$ (à condition d'avoir introduit les martingales locales du chapitre 2'). Si X est une semi-martingale formelle sur A, on peut définir $\widetilde{X}, X^c, [X, X]$, et $H.X$ pour H processus optionnel vrai sur A (restriction à A d'un processus optionnel sur $\overline{\mathbb{R}}_+ \times \Omega$); mais ce ne sont que des processus formels sur A, même si X est une semi-martingale vraie sur A (une égalité $X = \widetilde{X} + X^c$, où X est vraie et $\widetilde{X}, X^c$ seulement formelles, est à comparer à une solution u de l'equation des ondes, où $\frac{\partial u}{\partial t}$ et Δu sont seulement des distributions, pour $\frac{\partial u}{\partial t} - \Delta u = 0$ fonction) et ils ne sont définis qu'à une équivalence près sur A; autrement dit ce sont des classes d'équivalence sur A de semi-martingales formelles; et toutes les formules antérieures subsistent dans ce cadre.

La formule (7.1) d'Ito subsiste: si X est une semi-martingale sur A, et si Φ est C^2, $\Phi(X)$ est une semi-martingale sur A, mais les termes du deuxième membre ne sont que des classes d'équivalence sur A de semi-martingales formelles, et (7.2), (7.3), (7.4) subsistent aussi, où même les premiers membres sont des classes d'équivalence sur A de semi-martingales formelles. On conviendra d'appeler *différentielle semi-martingale* une telle classe d'équivalence; si X est une semi-martingale, sa classe se notera dX. Alors $d(H.X) = H\ dX$; on a $H(K\,dX) = H(d(K.X)) = d(H.(K.X))$, et $(H\,K)dX = d(HK.X)$, donc l'égalité (4.3) donne $H(K\,dX) = (H\,K)dX$, ce qui est une notation cohérente; les différentielles semi-martingales sur A (mais aussi les différentielles à variation finie et les différentielles martingales) forment un module sur l'anneau des fonctions optionnelles sur A, pour l'intégration stochastique.

Par ailleurs il est aussi commode de remplacer $d[X,X]$ par $dX \odot dX$ ou même $dX\ dX$, puisque c'est une variation quadratique (chapitre 5), et alors les formules d'Ito (chapitre 7) si X est une semi-martingale sur A, s'écrivent sous forme différentielle:

$$d(\Phi(X)) = \Phi'(X)dX + \frac{1}{2}\Phi''(X)dX\ dX \tag{8.6}$$

$$d(\widetilde{\Phi(X)}) = \Phi'(X)\widetilde{dX} + \frac{1}{2}\Phi''(X)dX\ dX \tag{8.7}$$

$$d(\Phi(X)^c) = \Phi'(X)dX^c \tag{8.8}$$

$$\frac{1}{2}d(\Phi(X))d(\Phi(X)) = \frac{1}{2}(\Phi'(X) \odot \Phi'(X))dX\ dX\ . \tag{8.9}$$

9. Deuxième extension: semi-martingales sur des variétés[5] de classe C^2

Soit V une telle variété, $X : A \subset \overline{\mathbb{R}}_+ \times \Omega \to V$ un processus. On dira que c'est une V-semi-martingales si, pour toute fonction φ réelle C^2 sur V, $\varphi(X)$ est une semi-martingale réelle. Par Ito, si $V = E$ espace vectoriel, on retrouve la notion antérieure. Si $\Phi : V \to V'$ est une application C^2, $\Phi(X)$ est ene V'-semi-martingale. Si maintement W est une sous-variété (non nécessairement fermée) de V et X une V-semi-martingale prenant ses valeurs dans W, c'est une W-semi-martingale. Bien entendu, on peut parler de V-processus à variation finie, mais pas de V-martingale; $X^c, \widetilde{X}, [X,X]$ n'ont aucun sens, ni la notion d'équivalence sur A, car il n'y a pas d'addition sur V. Mais nous allons donner un sens aux différentielles, en utilisant les espaces vectoriels tangents à V.

D'après ce qui est dit ci-dessus, comme V peut (Whitney) être plongée dans un espace vectoriel E, un V-processus est une V-semi-martingale si et seulement si c'est une E-semi-martingale.

10. Différentielles semi-martingales sections d'un fibré vectoriel optionnel[6]

Soient $A \subset \overline{\mathbb{R}}_+ \times \Omega$ ouvert, et G_A un espace fibré optionnel au dessus de A, à fibres vectorielles de dimension finie, de fibre-type G; $G_{(t,\omega)}$ est la fibre au dessus de $(t,\omega) \in A$. Il y a des cartes produits, $G_{A'} \to A' \times G$; la formule de transition d'une carte à une autre est optionnelle, $((t,\omega),g) \mapsto ((t,\omega),\alpha(t,\omega)g)$, où α est optionnelle sur A' à valeurs dans $\mathcal{L}(G;G)$. Alors on appellera *différentielle semi-martingale section de* G_A la donnée, pour chaque carte $G_{A'} \to A' \times G$, d'une G-différentielle semi-martingale dX' sur A', avec la formule de transition $dX' \mapsto \alpha dX'$, pour la structure de module par intégration stochastique définie au chapitre 8.

[5] Voir Schwartz [2], [3]
[6] Voir Schwartz [4]

Seule existe ici la différentielle semi-martingale section de G_A, dX, par le système cohérent des dX'; X elle-même n'existe pas; d'abord les différentielles sont des classes d'équivalence de semi-martingales formelles; ensuite, si on peut écrire symboliquement $dX(t,\omega) \in G_{(t,\omega)}, dX(t,\omega)$ est un petit vecteur semi-martingale $\in G_{(t,\omega)}$ (exactement par le même abus de langage par lequel, si T est une section-distribution d'un fibré vectoriel G_V au-dessus d'une variété V, on se perment d'écrire, en l'absence d'étudiants, pour $v \in V$, $T(v) \in G_v$, alors que T n'a de valeur en aucun point), les fibres $G_{(t,\omega)}$ varient avec t, ω, et "l'intégrale" $X_t(\omega) = \int_{]o,t]} dX_s(\omega)$ n'est nulle part!

Mais la structure de module est très riche. Si dX est une G_A-différentielle semi-martingale au dessus de A, β une section optionnelle d'un fibré $\mathcal{L}(G_A; H_A)$ (de fibre $\mathcal{L}(G(t,\omega); H(t,\omega))$ au-dessus de (t,ω)), le produit βdX (intégration stochastique) est une H_A-différentielle semi-martingale. On a de même des différentielles à variation finie et martingales. Toute G_A-différentielle de semi-martingale dX a une décomposition unique $dX = d\widetilde{X} + dX^c, d\widetilde{X}$ G_A-différentielle à variation finie, dX^c G_A-différentielle martingale; et $d[X,X]$ ou $dX\ dX$ est une différentielle à variation finie section de $G_A \odot G_A$.

11. Différents espaces tangents à une variété C^2 (7)

L'espace m-tangent $T^m(V;v)$ à une variété V de classe C^m, en un point v, est l'espace vectoriel des formes linéaires sur $C^m(V;\mathbb{R})$ (C^m = espace vectoriel des fonctions réelles de classe C^m) qui s'annulent sur les fonctions m-plates en v (une fonction est m-plate en v si ses dérivées d'ordre $1, 2, \ldots, m$, sont nulles en v; ceci a un sens par des cartes). Un élément de $T^m(V;v)$ est la trace au point v d'un opérateur différentiel d'ordre $\leq m$ sur V, sans terme d'ordre 0; et un tel opérateur différentiel d'ordre $\leq m$ n'est autre qu'un champ de vecteurs m-tangents, ou une section du fibré $T^m(V)$ m-tangent. Nous aurons besoin de $m = 1, 2$; $T^1(V;v) \subset T^2(V;v)$. Le quotient $T^2(V;v)/T^1(V;v)$ est canoniquement isomorphe au produit tensoriel symétrique $T^1(V;v) \odot T^1(V;v)$. Soient $\xi, \eta \in T^1(V;v)$. Ils se prolongent en $\bar{\xi}, \bar{\eta}$, opérateurs différentiels d'ordre 1 sur V; le composé $\bar{\xi}\bar{\eta}$ est un opérateur différentiel d'ordre 2; sa trace $(\bar{\xi}, \bar{\eta})_v$ en v est un élément de $T^2(V;v)$. Il dépend, bien entendu, des prolongements choisis; mais, modulo $T^1(V;v)$, il n'en dépend pas. On définit ainsi une application bilinéaire de $T^1(V;v) \times T^1(V;v)$ dans $T^2(V;v)/T^1(V;v)$. Elle est symétrique, car $[\bar{\xi}, \bar{\eta}]$ (crochet de Lie) est un opérateur différentiel d'ordre 1. Donc elle définit une application linéaire de $T^1(V;v) \odot T^1(V;v)$ dans $T^2(V;v)/T^1(V;v)$, qu'une carte montre être bijective. Si D est un opérateur différentiel d'ordre ≤ 2 sans terme d'ordre 0, c.-à-d. une section du fibré $T^2(V)$, son symbole principal, en théorie des équations aux dérivées partielles, au point v, est son image dans $T^2(V;v)/T^1(V;v)$, donc un élément de $T^1(V;v) \odot T^1(V;v)$, ou un polynôme homogène de degré 2 sur le fibré cotangent $T^{*1}(V;v)$.

(7) Voir Schwartz [3], chapitres 1,2

Dans une carte de V sur un ouvert d'un vectoriel $E, T^1(V;v) \simeq E, T^2(V;v) \simeq E \oplus (E \odot E)$, le quotient devient facteur direct. On peut d'aileurs identifier $T^m(V;v)$ à l'espace des opérateurs différentiels `*a coefficients constants* d'ordre $\leq m$, sans terme d'ordre 0. Si (e_k), $k = 1, 2, \ldots, N$, est une base de E, on peut identifier e_k à la dérivée partielle ∂_k, un élément de $T^1(V;v) \simeq E$ est $\sum_{k=1}^N b^k \partial_k$, et un élément de $T^2(V;v) \simeq E \oplus (E \odot E)$ est

$$\sum_{k=1}^N b^k \partial_k + \frac{1}{2} \sum_{i,j=1}^N a^{i,j} \partial_i \partial_j \,, \qquad a^{i,j} = a^{j,i} \,;$$

$T^1(V;v)$ est de dimension $N, T^2(V;v)$ de dimension $N + N(N+1)/2$.

Si on passe d'une carte à une autre, par une application Φ de classe C^2 d'un ouvert de E dans F, et si on représente les vecteure 2-tangents par la différentielle d'ordre 2 de Φ,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} \Phi'(v)\alpha + \Phi''(v)\beta \\ \Phi'(v) \odot \Phi'(v)\beta \end{pmatrix} \in \begin{pmatrix} F \\ \oplus \\ F \odot F \end{pmatrix} , \tag{11.1}$$

ou

$$\begin{pmatrix} \Phi'(v) & \Phi''(v) \\ 0 & \Phi'(v) \odot \Phi'(v) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} .$$

12. Formule d'Ito et espaces tangents

Soit X une V-semi-martingale. Dans une carte sur un ouvert U d'un vectoriel E, où nous l'écrirons encore X, introduisons la semi-martingale complète associée, $\underline{X}$, $(E \oplus (E \odot E))$-semi-martingale qui n'a de sens que dans cette carte:

$$\underline{X} = \begin{pmatrix} X \\ \frac{1}{2}[X, X] \end{pmatrix} , \qquad \underline{dX} = \begin{pmatrix} dX \\ \frac{1}{2} dX \; dX \end{pmatrix} . \tag{12.1}$$

Dans le changement de cartes $E \to F$, les formules d'Ito (8.6), (8.7), (8.8), (8.9) s'écrivent, si $Y = \Phi(X)$:

$$\underline{dY} = \begin{pmatrix} \Phi'(v) & \Phi''(v) \\ 0 & \Phi'(v) \odot \Phi'(v) \end{pmatrix} \underline{dX} \,, \tag{12.2}$$

$$\widetilde{\underline{dY}} = \begin{pmatrix} \Phi'(v) & \Phi''(v) \\ & \Phi(v) \odot \Phi'(v) \end{pmatrix} \widetilde{\underline{dX}} \,, \tag{12.3}$$

$$\underline{dY^c} \text{ qu'on écrit } dY^c = \Phi'(X) dX^c \,, \tag{12.4}$$

$$\frac{1}{2} dY dY = \Phi'(X) \odot \Phi'(X) \frac{1}{2} dX dX \,. \tag{12.5}$$

En comparant (12.2) et (11.1), on voit que l'on peut considérer $\underline{dX}_t$ comme un petit vecteur semi-martingale $\in T^2(V;X_t)$, $\underline{dX}(t,\omega) \in T^2(V;X(t,\omega))$. Les définitions du chapitre 10 montrent que, si X est une V-semi-martingale sur $A \subset \overline{\mathbb{R}}_+ \times \Omega$, on peut considérer sa différentielle $\underline{dX}$ (voir Schwartz [4], proposition (2.7), p. 710) comme une différentielle semi-martingale section du fibré $T^2(V)$ au dessus de V, le long de X, c.-à-d. une différentielle semi-martingale section du fibré G_A, où G_A est le fibré image réciproque du fibré $T^2(V)$ par l'application $X : A \mapsto V$; la fibre $G_{(t,\omega)}$ est $T^2(V;X(t,\omega))$. De même sa composante à variation finie, $\widetilde{\underline{dX}}$, est une différentielle section du même fibré. Mais la composante martingale dX^c, par la formule (12.4), est une différentielle martingale section du sous-fibré $T^1(V)$ le long de X, tandis que l'image de $\underline{dX}$ dans le quotient $T^2(V)/T^1(V)$ est une différentielle à variation finie section du quotient $T^1(V) \odot T^1(V)$; c'est $\frac{1}{2}dX\ dX$. Ainsi il n'y a pas de décomposition $X = \widetilde{X} + X^c$, ni de crochet $[X,X]$, mais il y a une décomposition $\underline{dX} = \widetilde{\underline{dX}} + dX^c$, $\underline{dX}(t,\omega) \in T^2(V;X(t,\omega))$, $\widetilde{\underline{dX}}(t,\omega) \in T^2(V;X(t,\omega))$, $dX^c(t,\omega) \in T^1(V;X(t,\omega))$, et un produit $\frac{1}{2}dX\ dX(t,\omega) \in T^1(V;X(t,\omega)) \odot T^1(V;X(t,\omega))$, $\frac{1}{2}dX\ dX$ image de $\underline{dX}$ et de $\widetilde{\underline{dX}}$ dans $T^2(V)/T^1(V)$.

13. Equations différentielles stochastiques (EDS)

Ici $A = \overline{\mathbb{R}}_+ \times \Omega$. Une EDS sur un ouvert d'un espace vectoriel E de dimension finie N est une équation de la forme

$$dX = \sum_{k=1}^{m} H_k(X) dZ^k\,, \tag{13.1}$$

où H_k est un champ de vecteurs, Z^k une semi-martingale réelle. Il y a existence et unicité de la solution, pour des H_k localement lipschitziens (avec explosion ou temps de mort), si on se donne la valeur initiale X_o, variable aléatoire $\mathcal{F}_o$-mesurable. Si on fait un changement de cartes, la formule d'Ito montre que les crochets $[Z^i, Z^j]$ interviendront; autant vaut les mettre tout de suite. Nous considérerons donc plutôt une EDS de la forme:

$$dX_t = \sum_{k=1}^{m} H_k(X_t) dZ_t^k + \frac{1}{2} \sum_{i,j=1}^{m} H_{i,j}(X_t) d[Z^i, Z^j]_t\,, \tag{13.2}$$

où les H_k et les $H_{i,j}$ sont des champs de vecteurs, $H_{j,i} = H_{i,j}$.

Si, dans une carte, les $H_{i,j}$ sont nuls, c'est un accident, ils ne le seront pas dans une autre. En passant à la semi-martingale complète associée, on trouve:

$$\frac{1}{2} dX_t dX_t = \frac{1}{2} \sum_{i,j=1}^{m} H_i(X_t) \odot H_j(X_t) d[Z^i, Z^j]_t \tag{13.3}$$

d'où

$$\underline{dX}_t = \sum_{k=1}^{m} H_k(X_t) dZ_t^k + \frac{1}{2} \sum_{i,j=1}^{m} \underline{H_{i,j}(X_t)} d[Z^i, Z^j]_t\,, \tag{13.4}$$

où les H_k sont des champs de E-vecteurs, les $H_{i,j}$ des champs de $(E \oplus (E \odot E))$-vecteurs, $\underline{H_{j,i}} = \underline{H_{i,j}}$; la composante de $\underline{H_{i,j}}$ dans $E \odot E$ étant $H_i \odot H_j$.

Mais alors ceci devient indépendant de toute carte. Une EDS sur une variété V (voir Schwartz [3], chapitre 8) sera une équation de la forme (13.4); les Z^k sont des champs de vecteurs 1-tangents donnés; les $\underline{H_{i,j}}$, avec $\underline{H_{j,i}} = \underline{H_{i,j}}$, sont des champs de vecteurs 2-tangents donnés; ces champs doivent être localement lipschitziens, donc V doit être C^2-lipschitz; et on a la condition de compatibilité suivante: la projection de $\underline{H_{i,j}}$ dans le quotient $T^2/T^1 = T^1 \odot T^1$ doit être $H_i \odot H_j$ qui exprime que l'image du deuxième membre dans le quotient $T^2(V; X_t)/T^1(V; X_t) = T^1(V; X_t) \odot T^1(V; X_t)$ est $\frac{1}{2} dX_t dX_t$. Une solution est une V-semi-martingale X, dont la différentielle $\underline{dX}$ au sens du chapitre 12 doit être égale au deuxième membre, différentielle semi-martingale section de $T^2(V)$ le long de X.

On peut donc écrire globalement une EDS sur une variété. Et, pas plus difficilement sur une variété que sur un espace vectoriel, on montre l'existence et l'unicité de la solution pour une condition initiable donnée X_o, $\mathcal{F}_o$-mesurable, avec explosion ou temps de mort.

14. Diffusion brownienne sur une variété

Ici on est sur $\mathbb{R}_+ \times \Omega$, $\mathbb{R}_+ = [0, +\infty[$. Le mouvement brownien sur un espace euclidien E vérifie l'EDS $dX = dB$, donc, compte tenu du chapitre 6:

$$\underline{dX}_t = \begin{pmatrix} dB_t \\ \frac{1}{2}\Sigma_{k=1}^N e_k \odot e_k dt \end{pmatrix} ;$$

on en déduit

$$\underline{\widetilde{dX}}_t = \begin{pmatrix} 0 \\ \frac{1}{2}\Sigma_{k=1}^N e_k \odot e_k \end{pmatrix} dt \,. \tag{14.1}$$

L'opérateur différentiel du second ordre correspondant au champ de vecteurs constant $\frac{1}{2}\Sigma_{k=1}^N e_k \odot e_k$ est $\frac{1}{2}\Delta$, qu'on peut écrire comme champ 2-tangent: $v \mapsto \frac{1}{2}\Delta = \frac{1}{2}\Delta(v)$. C'est cela qui traduit que son semi-groupe, de générateur infinitésimal $\frac{1}{2}\Delta$, est le semi-groupe de la chaleur, et que la loi de $X_t = B_t$ est la gaussienne de paramètre $\sqrt{t}$.

Si alors L est un opérateur différentiel du second ordre semi-elliptique ≥ 0 sans terme d'ordre 0, sur une variété V, on appellera *diffusion L-brownienne* une solution de l'EDS (voir Schwartz [5]):

$$\underline{\widetilde{dX}}_t = L(X_t)dt \,. \tag{14.2}$$

On voit bien que L est un champ de vecteurs 2-tangents, sa valeur en X_t est $L(X_t) \in T^2(V; X_t)$, et $\underline{\widetilde{dX}}_t$ est un petit vecteur à variation finie $\in T^2(V; X_t)$, au sens du chapitre 12, donc (14.2) est bien cohérent. Cette EDS n'est pas mise sous la forme canonique (13.4), assurant l'existence et l'unicité de la solution, pour une condition initiale donnée; on ne donne ici que $\underline{\widetilde{dX}}_t$, il manque la composante martingale dX_t^c,

et la condition de compatibilité requise n'est donc pas là non plus. Mais, si L est de range constant (cas strictement elliptique par exemple), la connaissance d'un $\widetilde{\underline{dX}}_t$ proportionnel à dt permet (moyennant un élargissement de $\Omega, \mathcal{P}$), de trouver le terme dX_t^c; ou démontre qu il est nécessairement de la forme

$$dX_t^c = \sum_{l=1}^{m} \sigma_l(X_t) dB_t^l \,, \tag{14.3}$$

où $(B^l)_{l=1,2,\ldots,m}$ est un brownien normal d'un espace $\mathbb{R}^m$, et où, pour $v \in V$, $\sigma_l(v) \in T^1(V;v)$, et

$$\frac{1}{2}\sum_{l=1}^{m} \sigma_l(v) \odot \sigma_l(v) = \text{image de } L(v) \text{ dans } T^2(V;v)/T^1(V;v) = T^1(V;v) \odot T^1(V;v)\,.$$

Alors

$$\underline{dX}_t = \sum_{l=1}^{m} \sigma_l(X_t) dB_t^l + L(X_t)dt \,, \tag{14.4}$$

donc $\widetilde{\underline{dX}}_t = L(X_t)dt$; c'est une EDS sous la forme canonique (13.4), $\sigma_l(v) \in T^1(V;v)$, $L(v) \in T^2(V;v)$. La condition de compatibilité est vérifiée; l'image dans le quotient $T^2(V;v)/T^1(V;v)$ de $L(v)$ est $\frac{1}{2}\Sigma_{l=1}^m \sigma_l(v) \odot \sigma_l(v)$, et il y a bien une solution unique, pour une condition initiale donnée. Si V est riemannienne, son brownien associé est son L-brownien, $L = \frac{1}{2}\Delta$. Par des méthodes difficiles, on peut montrer, pour L strictement elliptique ≥ 0, l'existence et l'unicité lorsque L a ses coefficients d'ordre 2 continus, et d'ordre 1 boréliens localement bornés: c'est le théorème de Stroock et Varadhan, voir Meyer–Priour–Spitzer [1], article de Priouret, chapitre VI.

15. Mouvement brownien et problème de Dirichlet

Montrons succinctement le rôle que joue le L-brownien pour la résolution d'un problème de Dirichlet pour L. Soit U un ouvert relativement compact de V, de frontière ∂U. Soit X le L-brownien issu de $a \in U$, $X_o = a$. La formule $\widetilde{\underline{dX}}_t = L(X_t)dt$ est une égalité entre petits vecteurs 2-tangents à V (en réalité entre différentielles semi-martingales sections du fibré $T^2(V)$ le long de X). Si φ est une fonction réelle C^2 à support compact sur V, on peut calculer les valeurs de ces petits vecteurs 2-tangents sur φ; la valeur du premier est $d\widetilde{\varphi(X)}_t$, la valeur du deuxième est $L\varphi(X_t)dt$. Donc:

$$d\widetilde{\varphi(X)}_t = L\varphi(X_t)dt\,. \tag{15.1}$$

Désignons part T le temps d'entrée dans le complémentaire de U (ou dans ∂U):

$$T = \inf\{t \geq 0; X_t \notin U\}\,;$$

c'est un temps d'arrêt (voir chapitre 2'); si on suppose V connexe non compacte, on voit que $T < +\infty$ et que les intégrales qui vont suivre ont un sens. Calculons le $E\int_o^T$ des deux membres de (14.1), en se souvenant que $X_o = a$:

$$E(\varphi(X))_{\widetilde{T}} - \varphi(a) = E\int_o^T (L\varphi)(X_t)dt\,. \tag{15.2}$$

(si $Y = (\varphi(X))^\sim$ est un processus, Y_T est sa valeur en T, $Y_T(\omega) = Y(T(\omega), \omega)$, c'est une variable aléatoire, à ne pas confondre avec le processus arrêté Y^T).

Mais $\varphi(X)$ et $(L\varphi)(X)$ sont bornés, donc $(\varphi(X))^\sim$ aussi aux temps bornés, donc $(\varphi(X))^c$ aussi, donc c'est une vraie martingale pas seulement locale, et nulle au temps 0; $E(\varphi(X))^c_T = 0$, et $E(\varphi(X))^\sim_t = E(\varphi(X))_T = E\varphi(X_T)$. Donc

$$\varphi(a) = E(\varphi(X_T)) - E\int_o^T (L\varphi)(X_t)dt\,. \tag{15.3}$$

On définit ainsi deux mesures ≥ 0:

$$\mu_a, \mu_a(\varphi) = E(\varphi(X_T))\,, \quad \text{de masse } +1\,, \text{portée par } \partial U\,;$$

$$\Gamma_a, \Gamma_a(\psi) = E\int_o^T \psi(X_t)dt\,, \quad \text{portée par } U\,.$$

La deuxième est absolument continue par rapport aux mesures de Lebesgue de V. Si en effet A est un ensemble Lebesgue-négligeable,

$$E\int_o^T 1_A(X_t)dt \leq E\int_o^{+\infty} 1_A(X_t)dt = \int_o^{+\infty} \mathcal{P}\{X_t \in A\}dt\,;$$

mais on démontre que la loi de X_t, loi image $X_t(\mathcal{P})$ de $\mathcal{P}$ par X_t, est toujours absolument continue par rapport aux mesures de Lebesgue, donc $\mathcal{P}\{x_t \in A\} = 0$. Donc A Lebesgue-négligeable est Γ_a-négligeable. Donc $\Gamma_a = G(a,x)dx$, si dx est une mesure de Lebesgue sur V. Alors:

$$\varphi(a) = \mu_a(\varphi) - \int_U G(a,x)L\varphi(x)dx\,: \tag{15.4}$$

μ_a est la mesure L-harmonique sur ∂U relative à a $\in U$, et $G(a,x)dx$ est le L-noyau de Green. (La formule (15.4) n'est démontrée que si la fonction φ sur $\bar{U}$ est restriction d'une fonction C^2 sur V; il faut que ∂U soit assez régulier pour qu'on puisse l'etendre à des données de Dirichlet plus générales).

16. Le flot d'une EDS (13.4)

Si les coefficients d'une EDS sont C^∞ et en nous bornant au cas où il n'y a pas à considére de temps de mort (cas dit complètement conservatif), on peut appeler $\Phi_t(\omega; x)$ la valeur en (t, ω) de la solution correspondant à la valeur initiale x. Alors un théorème très remarquable dit que l'on peut choisir Φ tel que,

pour $\mathcal{P}$-presque tout $\omega, \Phi(\omega;.)$ soit C^∞ en x, à dérivées continues en t, x; Φ est le flot. (Ce qui est remarquable, c'est qu'on n'ait pas seulement: pour tout x, pour presque tout $\omega, \Phi(\omega; x)$ est continue en t, mais: pour presque tout $\omega, \ldots$). Pour une donnée initiale X_o $\mathcal{F}_o$-mesurable, la solution est $\Phi(., X_o)$. Alors $\Phi_t(\omega;.)$ est un C^∞-difféomorphisme de V sur un ouvert de V, et sur V elle-même moyennant des hypothèses plus restrictives, par exemple si V est compacte, ou si $V = \mathbb{R}^N$ et si les champs H_k de (13.1) sont globalement lipschitziens.

On peut aller plus loin, et résoudre l'équation sans probabilité (voir Schwartz [6], chapitre 4). L'ensemble $(Z^1(\omega), Z^2(\omega), \ldots, Z^m(\omega))$ est une trajectoire $uu \in C([0, +\infty]; \mathbb{R}^m])$; $(H_1(v), H_2(v), \ldots, H_m(v))$ est une application linéaire $H(v)$ de $\mathbb{R}^m$ dans $T^1(V; v)$, $\underline{(H_{i,j}(v))}_{i,j=1,2,\ldots,m}$ une application linéaire de $\mathbb{R}^m \odot \mathbb{R}^m$ dans $T^2(V; v)$. On peut écrire l'équation avec des notations à une variable:

$$\underline{dX}_t = H(X_t)dZ_t + \frac{1}{2}\underline{H}(X_t)d[Z, Z]_t\,, \tag{16.1}$$

l'image de $\underline{H}(v) \in \mathcal{L}(\mathbb{R}^m \odot \mathbb{R}^m; T^2(V; v))$ dans $\mathcal{L}(\mathbb{R}^m \odot \mathbb{R}^m;\ T^2(V; v)/T^1(V; v))$ étant $H(v) \odot H(v) \in \mathcal{L}(\mathbb{R}^m \odot \mathbb{R}^m; T^1(V; v) \odot T^1(V; v))$.

Ecrivon l'EDS en termes de trajectoires:

$$\begin{aligned} &\underline{d\xi}_t = H(\xi_t)dw_t + \frac{1}{2}\underline{H}(\xi_t)d[w, w]_t\,, \\ &w \in C(\overline{\mathbb{R}}_+; \mathbb{R}^m) \text{ donnée}, \xi \in C(\overline{\mathbb{R}}_+; V) \text{ inconnue}. \end{aligned} \tag{16.2}$$

Cette équation a un sens si w est à variation finie, avec $d[w, w] = dwdw = 0$; sinon, pour un w donné, elle n'a pas de sens (pas seulement parce que $[w, w]$ n'a pas de sens, il peut en avoir si w a une variation quadratique, mais parce que $H(\xi_t)dw_t$ n'en a pas). Mais il existe, indépendamment de toute probabilité, une application de $C(\overline{\mathbb{R}}_+; \mathbb{R}^m) \times V$ dans $C(\overline{\mathbb{R}}_+; V)$, appelée flot Φ, ayant la propriété suivante: si Z est une $\mathbb{R}^m$-semi-martingale sur un $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathcal{P}), X_o$ une condition initiale $\mathcal{F}_o$-mesurable, la solution de (15.1) avec la condition initiale X_o est donnée par $X_t(\omega) = \Phi_t(Z(\omega), X_o(\omega))$. L'application unique Φ résout toutes les EDS pour les champs $H, \underline{H}$ donnés, sans probabilité. En particulier, pour $\Omega, \mathcal{F}, (\mathcal{F}_t), Z, X_o$ donnés, la solution est la même pour toutes les probabilités $\mathcal{P}$ qui font de Z une semi-martingale. En outre, l'ensemble $\mathcal{W}$ des $w \in C(\overline{\mathbb{R}}_+; \mathbb{R}^m)$ pour lesquels $\Phi(w, .)$ est une fonction C^∞ en x, à dérivées continues en t, x, et pour lesquels $\Phi_t(w, .)$ est, pour tout t, un C^∞-difféomorphisme de V sur un ouvert de V, est "universellement presque sûr": pour tout $\mathcal{P}$ qui fait de Z une semi-martingale, $Z(\omega) \in \mathcal{W}$ $\mathcal{P}$-presque sûrement.

17. Relèvement d'une semi-martingale par une connexion[(8)]

Soit G_V un fibré sur une variété V, de fibre G_v en $v \in V$. Soit σ une connexion; si $\hat{v} \in G_v, \sigma(\hat{v}) \in \mathcal{L}(T(V; v); T(G_V; \hat{v}))$, $\pi\sigma =$ identité, si π est la projection de G_V

[(8)] Pour ce chapitre 17, voir Schwartz [3], chapitre 13

sur V. On démontre que cette connexion définit automatiquement des connexions d'ordre supérieur, $\sigma^m, \sigma^m(\hat{v}) \in \mathcal{L}(T^m(V;v); T^m(G_V;\hat{v}))$; σ^2 respecte les structures de sous-espace et quotient du chapitre 11, c.-à-d. σ^2 induit $\sigma^1 = \sigma$ sur $T^1(V;v)$, et définit $\sigma^1 \odot \sigma^1$ sur $T^2(V;v)/T^1(V;v) = T^1(V;v) \odot T^1(V;v)$.

On sait que les connexions définissent des transports parallèles le long de courbes C^1 de V, ou relèvements horizontaux de ces courbes. On peut prolonger ces relèvements aux semi-martingales. Soit X une V-semi-martingale; ses relevées $\hat{X}$ sont des G_V-semi-martingales horizontales, vérifiant la relation différentielle

$$\begin{aligned} &\underline{d\hat{X}}_t = \sigma^2(\hat{X}_t)\underline{dX}_t\,, \quad dX_t = \pi(\hat{X}_t)\,; \\ &\underline{d\hat{X}}_t \in T^2(V;\hat{X}_t)\,, \quad \sigma^2(\hat{X}_t) \in \mathcal{L}(T^2(V;X_t);\, T^2(G_V;\hat{X}_t))\,. \end{aligned} \tag{17.1}$$

Le relèvement s'obtient comme dans le cas d'une courbe C^1. On montre d'abord que toute semi-martingale X sur V est solution globale d'une EDS (13.4) sur V. Mais le relèvement d'une EDS est évident; on posera $\hat{H}(\hat{v}) = \sigma(\hat{v})H(v)$, $\underline{\hat{H}}(\hat{v}) = \sigma^2(\hat{v})\underline{H}(v)$; l'EDS relevée est

$$\underline{d\hat{X}}_t = \hat{H}(\hat{X}_t)dZ_t + \frac{1}{2}\underline{\hat{H}}(\hat{X}_t)d[Z,Z]_t\,. \tag{17.2}$$

Si l'EDS de X satisfait à la condition de compatibilité, celle de $\hat{X}$ aussi; elle a donc une solution unique (avec temps de mort), pour un rèvement initial donné $\hat{X}_o$, au dessus de X_o, $\mathcal{F}_o$-mesurable, et c'est le relèvement de X.

Il y a même un flot, comme au chapitre 16, indépendant de toute probabilité. Ceci, à ma connaissance, n'est démontré nulle part, mais c'est une conséquence facile du chapitre 16. Il existe une application $(w,\hat{x}) \mapsto \hat{w} = \Phi(w,\hat{x})$, de $C(\overline{\mathbb{R}}_+;V) \times G_V$ dans $C(\overline{\mathbb{R}}_+;G_V)$, définie seulement pour $\pi\hat{x} = w_o$, telle que, si X est une $(\Omega,\mathcal{F},(\mathcal{F}_t),\mathcal{P})$-$V$-semi-martingale, son relèvement, de valeur initiale $\hat{X}_o$ au-dessus de X_o, soit la semi-martingale $\hat{X} = \Phi(X,\hat{X}_o)$, $\hat{X}_t(\omega) = \Phi_t(X(\omega),\hat{X}_o(\omega))$. En particulier, si X est un V-processus sur $(\Omega,\mathcal{F},(\mathcal{F}_t)_t)$, et si $\hat{X}_o$ est un relèvement $\mathcal{F}_o$-mesurable de X_o, $\hat{X} = \Phi(X,\hat{X}_o)$ est son relèvement correspondant pour toutes les probabilités $\mathcal{P}$ sur Ω qui font de X une semi-martingale.

Supposons que X soit une L-diffusion sur V, suivant le chapitre 14; L est un opérateur différentiel du second ordre ou champ de vecteurs 2-tangents, donc il a un relèvement $\sigma^2 L = \hat{L}$, champe de vecteurs 2-tangents sur G_V ou opérateur différentiel du second ordre, $\hat{L}(\hat{v}) = \sigma^2(\hat{v})L(v)$. Ensuite X a un relèvement horizontal $\hat{X}$ (pour une condition initiale donnée $\hat{X}_o$). De (16.1) on déduit:

$$\widetilde{\underline{d\hat{X}}}_t = \sigma^2(\hat{X}_t)\widetilde{\underline{dX}}_t = \sigma^2(\hat{X}_t)L(X_t)dt = \hat{L}(\hat{X}_t)dt\ : \tag{17.3}$$

$\hat{X}$ est une $\hat{L}$-diffusion sur G_V. On peut donc construire $\hat{X}$ comme relèvement de X ou comme $\hat{L}$-diffusion, ou encore construire les $\hat{L}$-diffusions directement ou en relevant les L-diffusions. On peut aussi construire une L-diffusion sur V en projetant sur V une $\hat{L}$-diffusion sur G_V.

Il se trouve que, si V est riemannienne, si $L = \frac{1}{2}\Delta$, et si G_V est le fibré des repères orthonormés (fibré principal), muni de la connexion de Levi–Civita, $\hat{L} = \frac{1}{2}\hat{\Delta}$ est facile à construire, parce que le fibré tangent de G_V est trivial! Si $(e_k)_{k=1,2,\ldots,N}$ est la base canonique de $\mathbb{R}^N$, un point $\hat{v}$ de G_v au-dessus de v est un repère $\mathbb{R}^m \to T(V;v); \hat{v}(e_k) \in T(V;v)$ a un relèvement horizontal $\sigma(\hat{v})(\hat{v}(e_k)) \in T(G_V;\hat{v})$, soit $\eta_k(\hat{v})$. Alors chaque η_k est un champ de vecteur 1-tangent ou opérateur différentiel d'ordre 1 sur G_V, appelé champ basique associé à e_k. On peut construire les opérateurs différentiels du deuxième ordre $\eta_k^2 = \eta_k \circ \eta_k$ sur G_V, ou champ de vecteurs 2-tangents. Alors le relèvement $\hat{\Delta}$ de Δ est $\Sigma_{k=1}^N \eta_k^2$. On peut résoudre su G_V l'EDS.

$$d\hat{X}_t = \sum_{k=1}^{m} \eta_k(\hat{X}_t)dB_t^k + \frac{1}{2}\hat{\Delta}(\hat{X}_t)dt\,, \tag{17.4}$$

qui admet une solution unique pour $\hat{X}_o$ donné; c'est une $\frac{1}{2}\hat{\Delta}$-diffusion; si $X_o = \pi\hat{X}_o$, alors $X = \pi\hat{X}$ sera la $\frac{1}{2}\Delta$-diffusion ou mouvement brownien de valeur initiale X_o d'où une construction du mouvement brownien sur V à partir de son relevé sur le fibré des repères. Avec des notations différentes, et même une conception assez différente, c'est la méthode de Eels–Elworthy (voir Ikeda-Watanabe [1], chapitre V, paragraphe 4). Ce sont ces jeux de relévements par connexion de mouvements browniens qui ont permis à J.M. Bismut de donner une nouvelle démonstration du théorème de l'indice d'Atiyah–Singer pour l'opérateur de Dirac.

18. Semi-martingales à valeurs dans des espaces de dimension infinie[(9)]

Soit $\mathcal{H}$ un Banach. On pourrait appeler $\mathcal{H}$ semi-martingale un processus somme d'un processus à variation finie et d'une martingale. I1 y a cependant deux difficultés. L'une est qu'il y a d'autres processus que les semi-martingales qui donnent lieu à des intégrales stochastiques suivant le chapitre 4, et cela quel que soit $\mathcal{H}$ de dimension infinie. Ce n'est pas trop gênant, mais ce qui l'est plus c'est qu'en général, une semi-martingale ne donne pas lieu à des intégrales stochastiques. C'est toutefois vrai si $\mathcal{H}$ est hilbertien séparable. C'est pourquoi, si G est un espace vectoriel topologique localement convexe séparé, on dira que X, processus à valeurs dans G, est une G-semi-martingale, s'il existe une suite de temps d'arrêt (voir chapitre 2') $(T_n)_{n\in\mathbb{N}} \uparrow\uparrow \overline{+\infty}$, et une suite de sous-espaces hilbertiens séparables $\mathcal{H}_n$ de G, tels que $1_{\{T_n>0\}}X^{T_n}$ soit une $\mathcal{H}_n$-semi-martingale. Un cas particulièrement intéressant est celui des espace G de Fréchet nucléaires (Ustunel, voir Schwartz [7]); si X est un G-processus, et si, pour tout $\xi \in G', <\xi, X>$ est une semi-martingale, X est une G-semi-martingale; et même il existe un même $\mathcal{H}$, sous-espace hilbertien séparable de G, tel que X soit une $\mathcal{H}$-semi-martingale. Un cas intéressant est celui de $G = C^\infty(E;F)$, où E et F sont des espaces vectoriels de dimension finie,

(9) Voir Schwartz [7]

ou plus généralement $C^\infty(U;F)$, où U est une variété de dimension finie; c'est un Fréchet nucléaire. Si maintenant V est une autre variété, $C^\infty(U;V)$ n'est plus un espace vectoriel, ni même une variété. On peut cependant dire que Φ, $C^\infty(U;V)$-processus, est une $C^\infty(U;V)$-semi-martingale, si, pour toute fonction φ réelle C^∞ sur V, $\varphi \circ \Phi$ est une $C^\infty(U;\mathbb{R})$-semi-martingale. On a alors les trois théorèmes suivants, si U, V, W sont des variétés de dimension finie, Φ une $C^\infty(U;V)$-semi-martingale, Ψ une $C^\infty(V;W)$-semi-martingale, X une U-semi-martingale:

(1) $\Phi(X)$ est une V-semi-martingale;
(2) $\Psi \circ \Phi$ est une $C^\infty(U;W)$-semi-martingale;
(3) si, pour presque tout ω, pour tout t, $\Phi(t,\omega)$ est un difféomorphisme de U sur V, d'inverse $\Phi^{-1}(t,\omega), \Phi^{-1}$ est une $C^\infty(V;U)$-semi-martingale.

Par exemple, si Φ est le flot d'une EDS sur une variété V (chapitre 16), dans le cas complètement conservatif (par exemple V compacte), Φ est une $C^\infty(V;V)$-semi-martingale.

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THE WORK OF K. F. ROTH

by
H. DAVENPORT

On the three previous occasions on which Fields Medals have been presented, the addresses on the achievements of the recipients have been given either by the Chairman or by a member of the awarding Committee. On this occasion, Professor Siegel was to have spoken about the work of Dr Roth, but as he is unfortunately unable to be present the duty has devolved on me. It is a pleasant duty, in that it requires me to pay tribute to the work of a colleague and friend.

Dr Roth's greatest achievement is by now well known to mathematicians generally; it is his solution, in 1955, of the principal problem concerning approximation to algebraic numbers by rational numbers.

If α is any irrational number, whether algebraic or not, there are infinitely many rational numbers p/q such that

$$\left|\frac{p}{q}-\alpha\right| < \frac{1}{q^2}\,;$$

for example the convergents to the continued fraction for α. It is therefore natural to attempt to characterize irrational numbers in terms of the exponents μ for which there are infinitely many approximations satisfying

$$\left|\frac{p}{q}-\alpha\right| < \frac{1}{q^\mu}\,.$$

For convenience, I denote by $\bar{\mu} = \bar{\mu}(\alpha)$ the upper bound of such exponents μ. Obviously $\overline{\mu}(\alpha) \geqslant 2$.

The problem is: what can be said about the value of $\overline{\mu}(\alpha)$ when α is *algebraic*? In 1844 Liouville showed, in a very simple manner, that $\overline{\mu}(\alpha) \leqslant n$ if α is an algebraic number of degree n. In fact, if α is a root of the irreducible equation $f(x) = 0$, where $f(x)$ has integral coefficients (not all 0), then on the one hand

$$\left|f\left(\frac{p}{q}\right)\right| \geqslant \frac{1}{q^n}\,,$$

and on the other hand it is easily seen that

$$\left|f\left(\frac{p}{q}\right)\right| = \left|f\left(\frac{p}{q}-\alpha+\alpha\right)\right| < c\left|\frac{p}{q}-\alpha\right|\,,$$

where c depends only on α. Comparison of these inequalities leads to the result. If $n = 2$ we get $\bar{\mu}(\alpha) = 2$; thus quadratic irrationals are about as badly approximable as any irrational number can be.

There are simple considerations which suggest that Liouville's result is far from being best possible. But it was not until 1908 that this was proved; in that year the Norwegian mathematician Axel Thue showed that $\overline{\mu}(\alpha) \leqslant \frac{1}{2}n + 1$. In 1921 Siegel made further very substantial progress, and obtained $\bar{\mu}(\alpha) < 2\sqrt{n}$ approximately, the precise result being a little better than this. In 1947 Dyson improved Siegel's inequality to $\bar{\mu}(\alpha) \leqslant \sqrt{(2n)}$.

In all this work, extending over a period of 40 years, the basic idea was the use of polynomials in two variables. Suppose $f(x_1, x_2)$ is a polynomial with integral coefficients, of degree r_1 in x_1 and r_2 in x_2, and suppose p_1/q_1 and p_2/q_2 are two rational approximations to α. Then

$$\left| f\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right) \right| \geqslant \frac{1}{q_1^{r_1} q_2^{r_2}},$$

provided of course that

$$f\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right) \neq 0.$$

Suppose further that $f(\alpha, \alpha) = 0$ and that the Taylor expansion of $f(x_1, x_2)$ in powers of $x_1 - \alpha$ and $x_2 - \alpha$ has all its 'early' coefficients zero, a condition which can be made precise in various ways. Then one can obtain an upper bound for

$$\left| f\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right) \right|$$

in terms of

$$\left| \frac{p_1}{q_1} - \alpha \right| \text{ and } \left| \frac{p_2}{q_2} - \alpha \right|,$$

and the principle is to combine this with the previous lower bound in such a way as to establish that p_1/q_1 and p_2/q_2 cannot both be very good approximations to α. Finally, p_1/q_1 and p_2/q_2 are chosen in a suitable way from the infinite sequence of approximations.

The proof of the existence of a polynomial $f(x_1, x_2)$ with all the desired properties is a difficult matter, and the condition that

$$f\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right) \neq 0$$

is particularly troublesome. No explicit construction for such a polynomial has yet been found. During the course of their work, the four mathematicians I have mentioned developed methods of great subtlety and interest, and other important ideas which are relevant to the problem were contributed by Gelfond, Mahler and Schneider.

In 1955 Roth finally solved the problem: he proved that $\bar{\mu}(\alpha) = 2$ for any algebraic number α. The achievement is one that speaks for itself; it closes a chapter, and a new chapter will now be opened. Roth's theorem settles a question which is both of a fundamental nature and of extreme difficulty. It will stand as a landmark in mathematics for as long as mathematics is cultivated.

It is not my intention to describe or analyse Dr Roth's proof, particularly as he will be speaking about it himself. My own impression of his proof is that it is a structure, inevitably of some complexity, every part of which fits into its proper place and carries its proper share of the total load. As you have probably anticipated from my description of previous work, it uses polynomials in an arbitrarily large number of variables, instead of in two variables. It had indeed long been realized that this would be necessary, but the difficulties in the way had appeared to be quite insuperable.

I turn now to another achievement of Dr Roth, which seems to me to be also of the first magnitude, though the problem to which it relates is perhaps of less universal interest. Let

$$n_1,\, n_2,\, n_3,\, \ldots$$

be a sequence of natural numbers, and suppose that no three of the numbers are in arithmetic progression; in other words,

$$n_i + n_j \neq 2n_k$$

unless $i = j = k$. It was conjectured by Erdős and Turán in 1935 (though the conjecture is believed to be older) that such a sequence must have zero density, that is, the number $N(x)$ of terms not exceeding x must satisfy

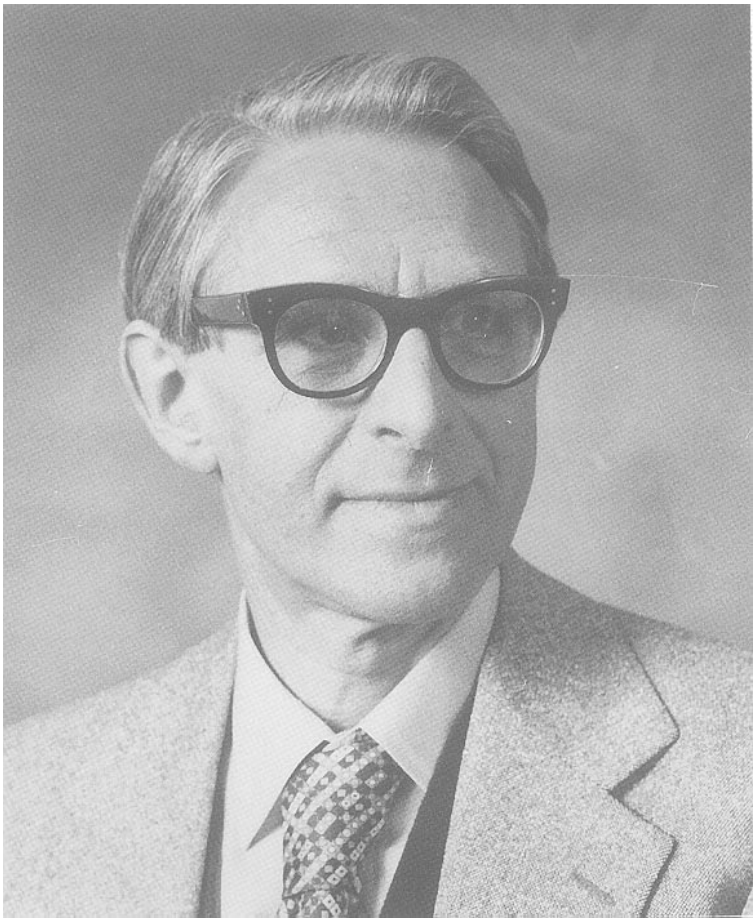
$$\frac{N(x)}{x} \to 0 \quad \text{as} \quad x \to \infty\,.$$

This problem was the subject of several interesting and ingenious papers, but it resisted all attempts at solution for a long time. The conjecture was proved by Dr Roth in 1952, and his proof is one of great interest and originality. He first considers a set of numbers $n_1, n_2, \ldots, n_\tau \leqslant x$ with the property in question, for which τ is a maximum, and proves that such a set, if dense, would have to have considerable regularity of distribution. This regularity is of two kinds: regularity of distribution in position and regularity of distribution among the residue-classes to any modulus. Then he applies the analytic method developed by Hardy and Littlewood for problems of an additive character, and the features of regularity prove to be just sufficient to give the estimates necessary for the method to succeed. The final conclusion is that $N(x) < cx/(\log\log x)$, where c is an absolute constant. I can recall no other instance, of comparable importance, in which the Hardy–Littlewood method has been used to elucidate the additive properties of an unknown sequence, instead of a special sequence such as the kth powers or the primes.

There are other achievements of Dr Roth which stand out by their originality and novelty, and it is with reluctance that I pass over them. Those I have outlined

already will, I am sure, satisfy you that the recognition which has come to him is well merited.

The Duchess, in *Alice in Wonderland*, said that there is a moral in everything if only you can find it. It is not difficult to find a moral in Dr Roth's work. It is that the great unsolved problems of mathematics may still yield to direct attack, however difficult and forbidding they appear to be, and however much effort has already been spent on them.



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RATIONAL APPROXIMATIONS TO ALGEBRAIC NUMBERS

by
K. F. ROTH

1. Let α be any algebraic irrational number and suppose there are infinitely many rational approximations h/q to α such that

$$\left|\alpha - \frac{h}{q}\right| < \frac{1}{q^\kappa}\,. \tag{1}$$

I proved in 1955 that this implies $\kappa \leqslant 2$. In this talk I shall try to outline the proof, and to say a few words about some of the possible extensions, and about the limitations of the method.

It is easily seen that there is no loss of generality in supposing that α is an algebraic integer. Accordingly, we shall suppose that α is a root of the polynomial

$$f(x) = x^n + a_1 x^{n-1} + \cdots + a_n \tag{2}$$

with integral coefficients and highest coefficient 1.

2. Previous work on the problem entailed the use of polynomials in two variables. It has long been recognized that further progress would demand the use of polynomials in more than two variables, and that polynomials in a large number of variables would have to be used to obtain the full result. It is not difficult to formulate properties which a polynomial should have in order to be useful for our purpose.

Suppose that $h_1/q_1, \ldots, h_m/q_m$ are rational approximations to α, all satisfying (1). Let $Q(x_1, \ldots, x_m)$ denote a polynomial with integral coefficients, of degree at most r_j in x_j for each j. Then

$$\left|Q\left(\frac{h_1}{q_1}, \ldots, \frac{h_m}{q_m}\right)\right| \geqslant \frac{1}{P}\,, \tag{3}$$

where $P = q_1^{r_1} \ldots q_m^{r_m}$, provided, of course, that

$$Q\left(\frac{h_1}{q_1}, \ldots, \frac{h_m}{q_m}\right) \neq 0\,.$$

Let the Taylor expansion of $Q(h_1/q_1, \ldots, h_m/q_m)$ in powers of $(h_1/q_1) - \alpha, \ldots, (h_m/q_m) - \alpha$ be

$$\sum_{i_1} \cdots \sum_{i_m} Q_{i_1,\ldots,i_m} \left(\frac{h_1}{q_1} - \alpha\right)^{i_1} \cdots \left(\frac{h_m}{q_m} - \alpha\right)^{i_m} .$$

Suppose now that Q has the properties:

A: $$\sum_{i_1} \cdots \sum_{i_m} |Q_{i_1}, \ldots, i_m| < P^{\Delta} ,$$

where Δ is small;

B: $$Q_{i_1,\ldots,i_m} = 0 \text{ for all } i_1, \ldots, i_m \text{ satisfying}$$

$$q_1^{i_1} \cdots q_m^{i_m} \leqslant P^{\phi} \quad (\phi > 0) .$$

Then each term in this Taylor expansion with a non-zero coefficient has

$$\left|\frac{h_1}{q_1} - \alpha\right|^{i_1} \cdots \left|\frac{h_m}{q_m} - \alpha\right|^{i_m} < \frac{1}{(q_1^{i_1} \cdots q_m^{i_m})^{\kappa}} < \frac{1}{P^{\kappa\phi}} .$$

Hence $$\left|Q\left(\tfrac{h_1}{q_1}, \ldots, \tfrac{h_m}{q_m}\right)\right| < \tfrac{1}{P^{\kappa\phi - \Delta}} . \tag{4}$$

Comparison of (3) and (4) yields

$$\kappa < \frac{1 + \Delta}{\phi} . \tag{5}$$

We must bear in mind that to obtain (5) we used, in addition to A and B, the condition

C: $$Q\left(\tfrac{h_1}{q_1}, \ldots, \tfrac{h_m}{q_m}\right) \neq 0 .$$

To prove our theorem we shall establish the existence of a polynomial Q satisfying the conditions A, B, C with ϕ near to $\frac{1}{2}$. For this purpose it will be necessary to take m large and to choose the approximations $h_1/q_1, \ldots, h_m/q_m$ suitably. Only after choosing these approximations do we choose the polynomial Q, and the latter choice depends on the former.

[With $m = 2$ it is only possible to satisfy condition B with ϕ of the order of $n^{-\frac{1}{2}}$ (where n is the degree of α), and this leads to an estimate of the type $\kappa < cn^{\frac{1}{2}}$.]

3. The logical structure of the proof is as follows. We suppose $\kappa > 2$; m is chosen sufficiently large and is fixed throughout. A small positive number $\delta (< 1/m)$ will be fixed until the end of the proof, when we let $\delta \to 0$. We denote by Δ any function of δ and m such that $\Delta \to 0$ as $\delta \to 0$ for fixed m.

We begin by choosing $h_1/q_1, \ldots, h_m/q_m$ [with $(h_j, q_j) = 1$] from the assumed infinite sequence of approximations to α (satisfying (1)) by first taking q_1 sufficiently

large (in terms of m, δ, α), then taking q_2 sufficiently large in terms of q_1, and so on. It will in fact suffice if

$$\frac{\log q_j}{\log q_{j-1}} > \delta^{-1} \quad (j = 2, \ldots, m)\,.$$

Then we choose integers $r_1, \ldots, r_m$ which are sufficiently large in relation to $q_1, \ldots, q_m$ and which satisfy

$$q_1^{r_1} \leqslant q_j^{r_j} < q_1^{r_1(1+\frac{1}{10}\delta)}\,. \tag{6}$$

This presents no difficulty. We note that (6) implies

$$q_1^{mr_1} \leqslant P < q_1^{mr_1(1+\Delta)}\,. \tag{7}$$

Condition B now takes the form that the Taylor coefficients $Q_{i_1,\ldots,i_m}$ vanish for all $i_1, \ldots, i_m$ satisfying

$$\frac{i_1}{r_1} + \cdots + \frac{i_m}{r_m} < m\phi + \Delta\,. \tag{8}$$

4. We shall first outline a proof of the existence of a polynomial Q^* satisfying conditions A and B only (with ϕ near to $\frac{1}{2}$). This proof, due essentially to Siegel, is based on the use of Dirichlet's compartment principle. The question of satisfying C as well, which gives rise to the principal difficulty, is deferred until later.

We put $B_1 = q_1^{\delta r_1}$ and consider all polynomials $W(x_1, \ldots, x_m)$ of degree at most r_j in x_j, having positive integral coefficients, each less than B_1. We try to find two such polynomials W', W'' such that their derivatives of order $i_1, \ldots, i_m$ are equal when $x_1 = \ldots = x_m = \alpha$, for all $i_1, \ldots, i_m$ satisfying (8). Since any such derivative is of the form

$$A_0 + A_1\alpha + \cdots + A_{n-1}\alpha^{n-1}\,,$$

where $A_0, \ldots, A_{n-1}$ are integers, one can estimate the number of possibilities for a derivative for given $i_1, \ldots, i_m$. This number can be shown to be less than $B_1^{n(1+3\delta)}$. The number of polynomials W is about B_1^r, where $r = (r_1 + 1)\ldots(r_m + 1)$. Thus the number of polynomials W will exceed the number of possible distinct sets of derivatives provided that the number of sets of $i_1, \ldots, i_m$ satisfying (8), and with no i exceeding the corresponding r, is less than about $r/\{n(1 + 3\delta)\}$. The number of integer points $(i_1, \ldots, i_m)$ in the region defined by the above conditions can be shown to be less than $\frac{2}{3}r/n$ if ϕ is chosen so that

$$m\phi + \Delta = \tfrac{1}{2}m - 3nm^{\frac{1}{2}}\,. \tag{9}$$

The polynomial $Q^* = W' - W''$ satisfies B, by its definition, and can be shown by a process of simple estimation to satisfy A. Furthermore, on letting $\delta \to 0$ we would obtain

$$\phi = \tfrac{1}{2} - 3nm^{-\frac{1}{2}}\,,$$

so that ϕ could be assumed to be sufficiently near to $\frac{1}{2}$, since m is large. [It is in order to be able to choose ϕ near to $\frac{1}{2}$, that we must work with polynomials in many variables.]

5. To find a polynomial Q which satisfies A, B and C as well, we seek a derivative

$$Q = \frac{1}{j_1!}\left(\frac{\partial}{\partial x_1}\right)^{j_1} \cdots \frac{1}{j_m!}\left(\frac{\partial}{\partial x_m}\right)^{j_m} Q^* ,$$

of not too high an order, of the polynomial Q^* just considered. We want Q not to vanish at $(h_1/q_1, \ldots, h_m/q_m)$. The 'order' is measured by $j_1/r_1 + \cdots + j_m/r_m$. The replacement of Q^* by Q will involve a weakening of condition B, but provided the order in question is a Δ, this will make no difference on letting $\delta \to 0$. There will also be an effect on condition A, but this turns out to be insignificant. Condition C is the essential requirement now.

The existence of such a derivative, whose order is a Δ, is not easy to establish. One would in fact expect this to cause difficulty, as the choice of Q^* was designed to make Q^* very small at $(h_1/q_1, \ldots, h_m/q_m)$.

At this stage it is convenient to introduce the notion of the index of a polynomial at a point. We define the index of a polynomial at the point $(\alpha_1, \ldots, \alpha_m)$ relative to positive parameters $r_1, \ldots, r_m$ to be the minimal order of derivative (measured as above) which does not vanish at the point $(\alpha_1, \ldots, \alpha_m)$. In this language, we need to show that the index of Q^* at $(h_1/q_1, \ldots, h_m/q_m)$ is a Δ.

For polynomials in two variables, two quite different lines of reasoning have been used to obtain upper bounds for the index of Q^* at $(h_1/q_1, \ldots, h_m/q_m)$. The first, due to Siegel, is algebraic in nature. It is based on the principle that, under certain conditions, the sum of the indices of a polynomial at a finite number of points (not restricted to be rational) is bounded in terms of its degrees in the various variables. Since Q^* satisfies condition B (with an appropriate ϕ), it has an almost maximal index (in a certain sense) at the point $(\alpha, \ldots, \alpha)$ and at the points obtained by replacing α by its conjugates; and it can be deduced that the index of Q^* is small at any other point. I have been unable to extend this method to polynomials in more than 2 variables.

The second method, due to Schneider, is arithmetic in nature. It is based on the principle that, under certain conditions, the index of a polynomial at a rational point is bounded in terms of the magnitude of its coefficients. Since the coefficients of Q^* are not too large, this leads to a result of the desired kind.

My treatment is based on Schneider's approach and enables me to prove the following lemma.

Principal lemma. Let $0 < \delta < m^{-1}$, and let $r_1, \ldots, r_m$ be positive integers satisfying

$$r_m > 10\delta^{-1}, \quad \frac{r_{j-1}}{r_j} > \delta^{-1} \quad (j = 2, \ldots, m) .$$

Let $q_1, \ldots, q_m$ be positive integers satisfying

$$q_1 > c = c(m, \delta), \quad q_j^{r_j} \geqslant q_1^{r_1} .$$

Consider any polynomial R, not identically zero, with integral coefficients of absolute value at most $q_1^{\delta r_1}$ and of degree at most r_j in x_j. Then

$$\text{index } R < 10^m \delta^{(\frac{1}{2})^m} ,$$

where the index is taken at a point $(h_1/q_1, \ldots, h_m/q_m)$ relative to $r_1, \ldots, r_m$; the h's being integers relatively prime to the corresponding q's.

This suffices for the purpose of finding Q; the hypotheses of the lemma are satisfied when $R = Q^*$, and the lemma shows that the index of Q^* at $(h_1/q_1, \ldots, h_m/q_m)$ is a Δ, as required.

6. The proof of the lemma is self-contained, as indeed it must be, for it uses induction on the number of variables, whereas in the main proof the number m is fixed. Furthermore, the lemma has to be generalized before the induction can be set up.

We consider the class of all polynomials $R(x_1, \ldots, x_m)$ with integral coefficients, each coefficient being numerically at most B, say, and of degree at most r_j in x_j. We obtain, under certain conditions, an upper bound for the indices of polynomials of this class at a point $(h_1/q_1, \ldots, h_m/q_m)$ relative to $r_1, \ldots, r_m$. During the course of the proof, which is by induction on m, it is necessary to consider various different sets of values of the parameters involved. The final estimate is of the type required to establish the lemma.

The case $m = 1$ is simple. Suppose the coefficients of a polynomial $R(x_1)$ are numerically less than B. If θ_1 is the index of R at h_1/q_1 relative to r_1, the polynomial $R(x_1)$ is divisible by

$$\left(x_1 - \frac{h_1}{q_1}\right)^{\theta_1 r_1} .$$

It follows from Gauss's theorem on the factorization of polynomials with integral coefficients into polynomials with rational coefficients, that

$$R(x_1) = (q_1 x_1 - h_1)^{\theta_1 r_1} R^*(x_1) ,$$

where $R^*(x_1)$ is a polynomial with integral coefficients. Hence the coefficient of the highest term in R^* is an integral multiple of $q_1^{\theta_1 r_1}$, so that

$$q_1^{\theta_1 r_1} \leqslant B, \quad \theta_1 \leqslant \frac{\log B}{r_1 \log q_1} .$$

This gives an upper bound of the required type for $m = 1$.

Now suppose that upper bounds of this kind have been obtained for $m = 1, 2, \ldots, p-1$, where $p \geqslant 2$. We wish to deduce an upper bound for the indices for classes of polynomials in p variables.

For any given polynomial $R(x_1, \ldots, x_p)$ we consider all representations of the form

$$R = \phi_0(x_p)\psi_0(x_1, \ldots, x_{p-1}) + \cdots + \phi_{l-1}(x_p)\psi_{l-1}(x_1, \ldots, x_{p-1}) , \tag{10}$$

where the ϕ_ν and ψ_ν are polynomials with rational coefficients, subject to the condition that the ϕ_ν and ψ_ν are of degree at most r_j in x_j. Such a representation is possible, e.g. with $l-1=r_p$ and $\phi_\nu(x_p)=x_p^\nu$. From all such representations we select one for which l is least.

In this representation the polynomials ϕ form a linearly independent set, and so do the ψ's. Thus the Wronskian $W(x_p)$ of the ϕ's is not identically zero, and the same is true of a certain generalized Wronskian $G(x_1,\ldots,x_{p-1})$ of the ψ's. From (10) and the rule for multiplication of determinants by rows, it follows that

$$G(x_1,\ldots,x_{p-1})\ W(x_p)=F(x_1,\ldots,x_p) \tag{11}$$

is a certain determinant whose elements are all of the form

$$R_{j_1,\ldots,j_p}(x_1,\ldots,x_p)\,.$$

Since G and W have rational coefficients, there is an equivalent factorization of F in the form

$$F(x_1,\ldots,x_p)=U(x_1,\ldots,x_{p-1})\ V(x_p)\,, \tag{12}$$

where U and V have integral coefficients.

If the coefficients of R are assumed to be numerically less than B, this will imply an upper bound for the coefficients of F; and this, in turn, will imply upper bounds for the coefficients of U and V. The induction hypothesis then gives us upper bounds for the indices of U and V at the points $(h_1/q_1,\ldots,h_{p-1}/q_{p-1})$ and h_p/q_p respectively; and by a multiplicative property of indices, (12) then yields an upper bound for the index of F at $(h_1/q_1,\ldots,h_p/q_p)$.

On the other hand, F is obtained from R by the operations of differentiation, addition and multiplication; and by using some simple properties of indices relating to these operations, one obtains a lower bound for the index of F in terms of the index of R. Thus the upper bound for the index of F leads to an upper bound for the index of R.

In this way it is possible to set up the induction on m, although the details are somewhat more complicated than they are made to appear above.

This concludes the outline of the proof of our theorem. We note that the proof of the existence of a polynomial Q satisfying conditions A, B, C of § 2 is very indirect, and it would be of considerable interest if such a polynomial could be obtained by a direct construction.

7. The theorem can be generalized and extended in various ways. For example, instead of considering rational approximations to the algebraic number α, we may consider approximations to α by algebraic numbers β (a) lying in a fixed algebraic field, or (b) of fixed degree. In each case the accuracy of the approximation is measured in terms of $H(\beta)$, the maximum absolute value of the rational integral coefficients in the primitive irreducible equation satisfied by β.

The results already found by Siegel can be improved in both cases. In case (a) the best possible result has been obtained. † In case (b), Siegel's result is significant only if the degree of β is not too large compared to the degree of α, and I do not know how to obtain an improvement which does not suffer from a similar limitation.

The theorem can also be extended to p-adic and g-adic number fields, and this has been done by Ridout and Mahler respectively.

Various deductions can be made from the theorem. For example, for a given α and $\kappa > 2$, it is possible to estimate the number of solutions of (1), as has been done by Davenport and myself. This leads to estimates for the number of solutions of certain Diophantine equations.

The method is subject to a severe limitation, however, due to the role played by the selected approximations $h_1/q_1, \ldots, h_m/q_m$. One cannot answer questions of the following type:

(i) Can one give, in terms of α and κ (if $\kappa > 2$), an upper bound for the greatest denominator q among the finite number of solutions h/q of (1)?

(ii) Can one prove that

$$\left|\alpha - \frac{h}{q}\right| < q^{-(2+f(q))}$$

has only a finite number of solutions h/q for some explicit function $f(q)$ such that $f(q) \to 0$ as $q \to \infty$?

Liouville's result

$$\left|\alpha - \frac{h}{q}\right| > c(\alpha)q^{-n}$$

remains the only known result of its type for which an explicit value of the constant can be given.

Our method can only throw light on such questions if some assumption is made concerning the 'gaps' between the convergents to α. It would appear that a completely new idea is needed to obtain any information concerning problems of the above type.

One outstanding problem is to obtain a theorem analogous to ours concerning simultaneous approximations to two or more algebraic numbers by rationals of the same denominator. In the case of such simultaneous approximation to two algebraic numbers α_1, α_2 (subject to a suitable independence condition), one would expect the inequalities

$$\left|\alpha_1 - \frac{h_1}{q}\right| < q^{-\kappa}, \quad \left|\alpha_2 - \frac{h_2}{q}\right| < q^{-\kappa}$$

to have at most a finite number of solutions for any $\kappa > \frac{3}{2}$. But practically nothing is known in this connection.

A complete solution of the problem of simultaneous approximations could lead to the complete solution of many others, such as, for example, case (b) of the first problem mentioned in this section.

†See W. J. LeVeque, *Topics in Number Theory*, Addison Wesley, 1956.

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THE WORK OF R. THOM

by
H. HOPF[1]

René Thom was born in Montbéliard in 1923. He studied at the Ecole Normale Supérieure in Paris from 1943 to 1946 and then went to the University of Strasbourg where he is now "Professeur sans Chaire". In Strasbourg, he prepared his thesis "Espaces fibrés en sphères et carrés de Steenrod". He presented it in Paris in 1951 in order to get the degree of Docteur ès Sciences. In 1954 his paper "Quelques propriétés globales des variétés différentiables" appeared in Comm. Math. Helvet. 28. It was prepared from his thesis. There, one finds the foundations of the theory of cobordism. Today, four years later, one can say that, for a long time, only few events have so strongly influenced Topology and, through topology, other branches of mathematics as the advent of this work.

I would like to try to briefly describe here the basic idea and the grounds of the theory of cobordism.

One considers k-dimensional compact, oriented manifolds without boundary $A, B, \ldots$; it is not necessarily supposed that they are connected (a 1-dimensional A is a finite union of mutually disjoint closed lines, a 2-dimensional A is a finite union of closed surfaces which do not intersect, etc,...) $A + B$ denotes the disjoint union of copies of A and B, $-A$ denotes the manifold which is homeomorphic to A but has the opposite orientation, $A - B$ denotes the sum of A and $(-B)$. $A.B$ is, for manifolds of arbitrary dimensions, the Cartesian product. Besides, one considers, for each $k, (k+1)$-dimensional compact oriented manifolds U, V with boundary whose boundaries are k-dimensional manifolds. If, for a Ω k-dimensional A, there exists a $(k+1)$-dimensional U whose boundary is a copy of A, one says "A is bordant" and one writes: $A \sim 0$; when $A - B$ is bordant, then one says "A and B are cobordant" and one writes: $A \sim B$. The relation $\sim$ defines the "cobordism classes" of the manifolds $A, B, \ldots$ This classification is compatible with addition; the classes of each dimension k form an abelian group Ω with respect to addition; the 0-Element is the class of the bordant A (i.e. which are boundaries). This classification is also compatible with the Cartesian multiplication; so the groups Ω^k

[1]English translation from German by Stefan Weinzierl, with revisions by C. Kopper and V. Rivasseau.

with $k = 0, 1, 2, \ldots$ are fused to become the Ring

$$\Omega = \Omega^0 + \Omega^1 + \cdots + \Omega^k + \cdots$$

This is the Thom algebra.

Here one may ask: "Is all this not totally trivial? Can such a primitive definition be the origin of new and interesting insights?" But the same question was raised in 1955 when Hurewicz defined the Homotopy groups, and when the consequences of that primitive definition were realized many a mathematician had to admit "that is something that I could not have discovered. It would have been too easy for me". Indeed it required an ingenious mathematician like Hurewicz, not to be afraid of the apparently simple, but to see how deeply these simple concepts intrude upon the heart of the problem. I think Hurewicz was one of the greatest geometers of our time — we feel painfully for his early death during this conference — and personally I can hardly make a greater compliment to a mathematician than to place him near Hurewicz.

Then, Thom's discovery of cobordism and the algebra Ω reminds me again and again — in spite of the different contexts — of the discovery of the Homotopy groups by Hurewicz.

One of the, by no means trivial, insights which Thom had obviously from the beginning was that the notion of cobordism is particularly suited for the study of differentiable manifolds. We will restrict from now on to those. A differentiable manifolds can always be equipped with a Riemannian Metric, that is a metric that is euclidean at the infinitesimal level; this is how the orthogonal groups enter the game. Thom connects the orthogonal groups $\mathrm{SO}(n)$ with certain topological complexes $M(\mathrm{SO}(n))$; with the help of some highly non-trivial tools, developed only in recent years (Eilenberg, Maclane, Cartan, Serre) he shows that the Groups Ω^k are isomorphic to certain homotopy Groups of the spaces $M(\mathrm{SO}(n))$ and he succeeds in many cases to calculate these Groups or at least to make rather precise statements about them. The main results are the following:

$$\Omega^o = \mathbb{Z},\ \Omega^1 = \Omega^2 = \Omega^3 = 0,\ \Omega^4 = \mathbb{Z},\ \Omega^5 = \mathbb{Z}_2,\ \Omega^6 = \Omega^7 = 0\,,$$

where $\mathbb{Z}$ denote the infinite cyclic group, $\mathbb{Z}_2$ denote the Group of order 2. For all k, which are $\not\equiv 0 \bmod 4$, Ω^k is finite. For $k = 4m$, Ω^k is a direct sum of $\pi(n)$ Groups $\mathbb{Z}$, where $\pi(m)$ is the number of partitions (decompositions into positive summands) of m, and of a finite Group. The finite Groups just mentioned are very hard to determine; however one can get rid of these groups by replacing Ω by the "weak" Thom algebra Ω', which is obtained from Ω by admitting rational numbers as coefficients of the elements of Ω (in such a way that Ω' is the tensor product of Ω with the field of rational numbers); now one can describe exactly not only the additive but also the multiplicative structure of the algebra: Ω' is the algebra of the polynomials with rational coefficients in the variables $P^0, P^2, \ldots, P^{2m}$, where P^{2m} is represented by the Complex Projective Space of $2m$ complex dimensions (that means $4m$ real dimensions). It follows that for each manifold M there exists an

entire multiple nM, which is cobordant with well-defined integer linear combination of cartesian products of complex projective spaces of even dimension.

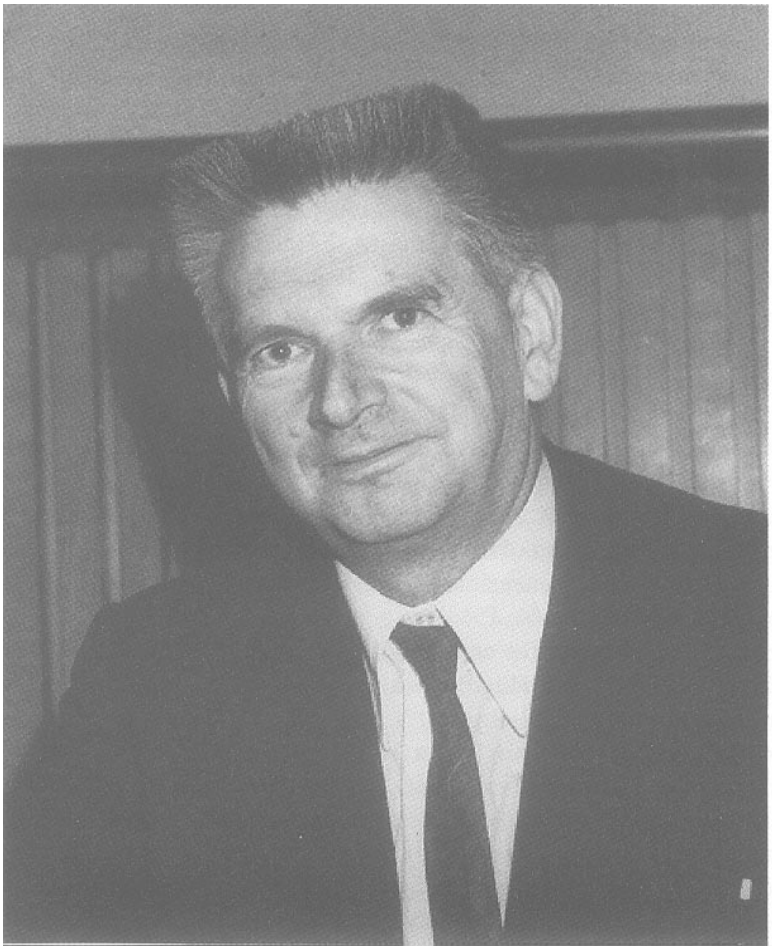
All this serves primarily to study those properties of manifolds which are invariants of the cobordism classes. In the first place, these are the properties which are connected with the "Pontrjagine numbers" of the manifold whose study is *a priori* very difficult. The theorem which was just mentioned shows that it is essential, on the one hand, to represent the considered manifold in the algebra Ω and, on the other hand, to determine the Pontrjagine numbers of the spaces P^m; the rule, which these spaces, stemming from Classical Algebraic Geometry, are playing for the topology of arbitrary (differentiable) manifolds, is very remarkable.

With this framework, we find the so far most important consequence of the cobordism theory; it constitutes one of the pillars on which F. Hirzebruch's theory of the Riemann–Roch theorem is founded, a theory which in turn is considered to be one of the most important mathematical achievements in recent years (F. Hirzebruch, Neue topologische Methoden in der algebraischen Geometrie, Springer–Verlag, 1956).

A different application of the theory of cobordism was given by J. Milnor; he gave a proof of the spectacular fact that the sphere S^7 can have several distinct differentiable structures; this shows for the first time that there are two differentiable manifolds which can be mapped onto each other homeomorphically and differentiably; on the top of Milnor's proof, there is the theorem of Thom, that $\Omega^7 = 0$, i.e. that every 7-dimensional (closed, orientable) manifold is a boundary (bordant) (Ann. Math. (2), 64, 1956).

I have no time here to present more of the many interesting results of Thom, neither from the papers mentioned above, nor from those I have not touched. But I have one remark of principal nature.

Topology is today, like many other branches of mathematics, in a process of powerful and consequent algebraization; this process has had remarkable success in clarification simplification and unification, and it has led to some unexpected new results; it is not simply that algebra only serves as a tool for the treatment of topological problems; rather one finds that most of the problems have an aspect of particularly algebraic character themselves. But the great successes of this development give rise, in my opinion, to a certain danger, the danger of a perturbation of the mathematical balance: there is a tendency to neglect totally the geometrical content of the topological problems and situations; but this neglect would imply an impoverishment of mathematics. Just in view of this danger, I feel that the achievements of Thom have something extraordinarily encouraging and enjoyable: of course Thom also masters and uses modern algebraic methods and sees the algebraic aspects of his problems, but his basic ideas, the grand simplicity of which I already talked of are of a very geometric and intuitive nature. These ideas have significantly enriched mathematics, and everything seems to indicate that the impact of Thom's ideas — whether they find their expression in the already known or in forthcoming works — is not exhausted by far.



René Thom

AUTOBIOGRAPHY OF RENÉ THOM

When I was asked what I had gained from receiving the Fields medal (in Edinburgh 1958), I thought it would be simpler to come up with a short autobiography, adding a few personal comments whenever they might seem necessary.

Childhood: 1923–1940

I was born on September 2nd 1923 in Montbéliard (Doubs). In those days, Montbéliard looked like a medieval town. It lies on an affluent part of the Doubs, the Allan, which, flowing from the Swiss Jura, bathes it on the southern side, whilst to the north, the river is fed by another affluent part from the southern Vosges, the Luzine. At that time the Luzine was not covered over; nor were the two little watercourses coming from the east. The canals drained water from the Savoureuse and were built by the princes of Montbéliard in order to strengthen the fortifications of the medieval castle, a very well preserved castle set upon a rock betwixt Allan and Luzine. Our house was at Number 21, rue de Belfort, right at the southern foot of the castle rock, from which it was separated by a space 10 or 11 meters wide. What remained of the old rampart walk which, after the fall of the princely house, had been "privatised" into little gardens. This detail played an important part in my childhood, for I often went to play in this space called "the garden". For years we grew flowers and vegetables and kept a chicken-run. Our house was very old indeed; to reach the upper floors we had to climb a spiral stone staircase (called a "yorbe" in the local patois). The date 1631 could still be seen on a stone doorway at the foot of the stairs. I had a relatively happy childhood. My father and mother got on well. They had a small grocery-cum-drugstore at the same address, 21 rue de Belfort. My father was a tall, well-built man, with a good secondary education. He enjoyed reciting in Latin and, though he was a fervent agnostic, singing the Latin mass. My mother was a loving if sometimes rather forceful woman. I had an elder brother, (Robert) with whom I (René) didn't hit it off too well, because, at that age, he used to take unfair advantage of the strength of his three extra years. I must mention how well I remember my grandmothers. I had two of them, Papa's mother, whose maiden name was Blazer, and Mama's, born a Ramel. Both lived with us. I even had (up to the age of four) a great grandmother, née Beucler. These old ladies took a great interest in the children of the family and made up the framework of my life up to the time when I was sent to the Primary School, Market Building, in 1931. I did well in school from the start and passed the scholarship exam (the first notable event leading to my university cursus). It must be said that although they were shopkeepers, my parents were not particularly good at commerce and, financially speaking, the situation was shaky. It stayed much the same, with no major incidents, until 1940, the year war broke out.

1940 – The War

At the start of the war, our parents insisted on our leaving for the south, as so many people were doing at the time. They themselves stayed at home, but my brother Robert and I took to our bikes and the southbound roads. On the plateau where the southern Doubs makes a loop, near Maiche, we came upon the last skirmishes of a routed army, forced to cross the border into Switzerland. We joined their convoy and so passed into Switzerland. The surprising warmth with which we were welcomed there, all those people offering food and drink at the roadside, still fills me with emotion. Thus we made our way to Bienne, where we were given a place to sleep in the straw. Later we were allowed to stay for some time in Switzerland, haymaking in the Gruyère valley near Romont. But towards the end of September a convoy was organized to bring us back to France by train, to Lyons in the free zone. Luckily my mother had a friend in Lyons, related to one of our uncles, who was able to take care of us. I had already passed the scientific *Baccalaur´eat* (*Math. Elem.*) in 1940 in Besançon, so I now made the most of our stay in Lyons and passed the *Baccalaur´eat* in Philosophy in June 1941. But we couldn't stay there forever and started thinking of going home, which we did at the end of June 1941. We took the train to a village near the Loue, a frontier river flowing into the Doubs. Guides helped us cross the river by night with our bikes. From there we cycled to Besançon where we boarded a train to Montbéliard. Needless to say, our parents were overjoyed to have us back.

This period was short-lived, for very soon it was time to think of my future studies. Thanks to the good advice of G. Becker, my teacher in *Premi`ere* (a man of letters!) I was directed towards the *Lyc´ee Saint-Louis* in Paris. After a first interview, I was accepted there without further discussion. I tried for the *´Ecole Normale Sup´erieure* (Paris) in 1942 but failed and had to try again in 1943. This time I was successful (though not brilliantly so!). All this brings us to October 1943, when I entered the ENS, rue d'Ulm.

The periodic trips home to see my parents had become difficult, for Montbéliard was now in a "forbidden zone". Nevertheless, thanks to the complicity of railway workers on the eastern network, it was still possible to cross from one zone to another by stratagem, and I did so several times. The commitment and resourcefulness of railway workers was admirable (for example, we could use the dog compartment of the wagon and cross over by Vitrey). Unfortunately my brother Robert had to leave Montbéliard for the STO. He was working in a factory at Mödling, southern suburb of Vienna, right up to the end of the war three years later. In Montbéliard itself the food situation was pretty bleak. I was able to go back there for good in 1945, when the French were drawing near from the south. Of my studies at the Lycée St Louis, I have retained a special admiration for the teachers we had, both in *Hypotaupe* and in *Taupe*. I entered ENS in October 1943 (with a modest rank) and started on my "higher" education.

ENS, 1943–1946

I remember those years at the ENS under the Occupation as a horribly tense period. It is true that the school authorities did all they could to make our stay easier. The political antagonism between the Director Carcopino and the Deputy Director Bruhat obviously did not make the task of administration any easier. But I have to say that, until G. Bruhat was arrested, there were no major disturbances (at least in so far as I was concerned). The yellow star, of course, soon made its appearance to set its mark on the Jews. Only the last year, after the "victory", was a year of opening, bringing with it the impression of once more living life to the full. Of this rebirth I recall a sensation of freedom that I found it hard to control. But the evenings spent in students' rooms with friends like Duvert, Pechmajou, Pecker, and others, will stay forever in my memory.

Henri Cartan was our master there from day one, teaching us the rudiments of Bourbakism, spheres and balls in all dimensions. Later I had a glimpse of Malraux speechifying in a corridor. It was too late to plunge further into literary country, although I had a feeling that a whole universe lay there to explore. But the evident superiority of the arts students was so crushing that we did not contest it.

From the mathematical point of view this was the time when we made the acquaintance of Bourbaki. And at the end of 1946 I decided to go to Strasbourg, to follow Henri Cartan.

Strasbourg. Entering the CNRS

Towards November 1946 I moved into a rented room in Strasbourg; at first I was way off on the road to Lyons near Graffenstaden, but later found a room in town at the Krutenau. It was then that I got to know mathematicians of my own age, or near enough, some of whom had already given proof of their talent. I should mention Koszul who drew my attention to the new Steenrod squares, Wu Wen Tsün, a discreet and kindly Chinese whose calculations were legendary in their reliability, Reeb, creator of the theory of foliation, and Charles Ehresmann, another expert in Topology, who was to stay there a few years, and whose theories struck me profoundly. For all that, the formation of fundamental notions in Algebraic Topology was not self-evident and a great deal was said (in Paris, not in Strasbourg) about the difference between a "shell" theory and a "blanket" theory. The years 1940–1950, though they came in with a mighty war, were astonishingly fertile for the foundations of Algebraic Topology. They were happy years for me. I met the girl who was to become my wife and bear me three children. I came to know some amazing minds, some very general theories, including the "categories" developed by Ehresmann. My first piece of work in Mathematics was a C.R.A.S. Note on Morse theory, seen from the then novel point of view of a cellular subdivision of the manifold, a point of view which later helped considerably in Smale's resolution of Milnor's conjecture. Then, in 1951, I defended my thesis under the direction of Henri Cartan: "Fibre Spaces in Spheres and Steenrod Squares".

It was also in 1951 that I obtained a fellowship for the United States. So there I went, leaving the children, Françoise and her little sister Elisabeth, with my wife and mother-in-law. This gave me the opportunity to learn some English and to see such famous people as Einstein and Hermann Weyl (whose last lectures I listened to). In Princeton, I met N. Steenrod, sadly already suffering from the spinal disorder of which he was later to die. The English I learned during that stay has stood me in good stead, for it is that little part that has not since been forgotten. I was also able to attend the seminars of Calabi and Kodaira.

Back in France in 1953, I was reunited with my family and was given a chair in Strasbourg made vacant by the departure of Chabauty for Grenoble. This period in Strasbourg was very fertile for me, although complicated for the first few months by the necessity of commuting between Strasbourg and Grenoble. It was around that time that I began to be interested in the theory of differentiable maps. A certain conversation comes to mind in which de Rham drew my attention to *Sard's Theorem*, according to which the set of singular values $F(\Sigma)$ of the critical set Σ of a smooth map $F : X \to M$ is nil in M. This theorem was to contribute in an essential way to the rise of differential topology by restriction to generic situations. It was the basic tool for my work on smooth manifolds.

Thus I obtained results that were to be rewarded by the Fields Medal in 1958: the construction of the theory of cobordism. It must be said that Serre and Milnor played an essential part in this progress. Milnor, through his discovery of exotic spheres, showed that intuition is not always enough. . . : there exist spheres which are not the boundaries of balls (in C^∞)!

After 1958

I have little to say about the attribution of the Fields Medal in itself. It was an impressive ceremony, at which I bowed respectfully before the Lord Mayor of Edinburgh. But at bottom this success left me with a slightly bitter taste, for I have the impression that work was done just a very little while later that was greater in depth and sagacity than mine and whose authors were quite as deserving, if not more so, of the medal (such as my co-medallist Roth). I am thinking too of Barry Mazur's demonstration of the *Sch¨onfliess conjecture*: Every shpere S^{n-1} in R^n with regular boundary is the boundary of an n-ball. Not to mention the discovery by Milnor of exotic shperes. In this sense, the Fields medal meant for me a certain fragility which the future was to make even more visible. If there was any consequence of my receiving the Fields medal, it was the invitation to come to the newly created Institut des Hautes Études Scientifiques (IHES). Finally, three or four years later, I took it up and moved with my family to Bures-sur-Yvette. I had been invited by the founding director, Léon Motchane, with whom I had many friendly conversations. Relations with my colleague Grothendieck were less agreeable for me. His technical superiority was crushing. His seminar attracted the whole of Parisian mathematics, whereas I had nothing new to offer. That made

me leave the strictly mathematical world and tackle more general notions, like the theory of morphogenesis, a subject which interested me more and led me towards a very general form of "philosophical" biology. The director, L. Motchane, had no objection (or if he did he kept it to himself). And so I ran what was known as a "crazy" seminar, that lasted the best part of my first year at the Institute. Three or four years passed, during which I returned to Aristotle and classical Greek. My first book, "Stabilité structurelle et Morphogenèse" (Structural Stability and Morphogenesis) had difficulty in getting to press with my editor (Pergamon Press). It was only with the help of Christopher Zeeman (then professor at the University of Warwick) that it was eventually published, and that the notion of "catastrophe" appeared in the literature (and in the press).

Catastrophe theory has as earlier origin. From the Strasbourg days I had been interested in the (classical) theory of envelopes; I looked for the structure of singular sets of envelopes. It was then that the notion of "generic singularity" came to me, starting with the case of envelopes of curves in the plane. I recalled H. Whitney's result (October 1947) on the generic singularities of plane to plane mappings with the notion of "cusp".

From there on I began to look at the structure of smooth maps of Euclidean spaces (and/or manifolds) with respect to each other. In fact, in Strasbourg, I had persuaded a physicist colleague (Ph. Pluvinage) to set up an optical apparatus allowing me to study the structure of generic caustics. It was the fruit of an old curiosity which had led me to write a paper "On envelope theory".

I have no clear memory of exactly when I launched "catastrophe theory". It was already there in 1972 when the first edition of "Stabilité structurelle et Morphogenèse" came out. But it was the fruit of a long semi-conscious genesis. Catastrophe terminology has often been held against me. It was certainly not introduced just to draw attention. It was meant to mark the presence of a pregnant discrete detail in the midst of an undifferentiated homogeneous continuum, and that is the fundamental characteristic of the *eidos* within the *genos*, the most evident manifestation of phenomenal existence.

It is a fact that catastrophe theory (CT) is dead. But one could say that it died of its own success. It was brought down by the extension from analytical (or algebraic) models to models that were only smooth. For as soon as it became clear that the theory did not permit quantitative prediction, all good minds, following Rutherford's phrase, decided that it was of no value. When it comes down to it, this extension resulted from B. Malgrange's extension of the preparation theorem to the class C^∞. Thus CT fell victim to its own progress. As for understanding the value of strictly qualitative theory, that takes a certain breadth of vision. Zeeman's models required that intelligence, and perhaps it is an intuitive quality vital to the global perception of a conflict.

The sociological history of CT is not without interest. It was marked by enormous popular success, appearing after the IMU congress of Vancouver (1972), where E. C. Zeeman was the perfect propagandist. The theory was world renowned

until it was ruined by the success of B. Malgrange. Some notions remain, introduced by algebraists, such as the concept of versal unfolding of a singularity, now undoubtedly classic.

However that may be, the wish to escape from the evident superiority of excellent mathematicians, led me to move into other areas of research, including those connected with biology. I can claim no success there, at least not in society. Gangway for molecules! even if it takes millions of computer operations to calculate the interaction between two macromolecules. My retreat towards the global perception has led me to a few (perhaps obvious) discoveries in linguistics and biology. Maybe they will be resurrected in days to come...

To conclude, I would like to say just this. The Fields medal brought me the freedom to choose what research I wanted to do, and that is essential. When we were driving from France to Scotland, Suzanne, my wife, felt a little sick during the long journey, a "specific" sickness that lasted right up to the ceremony in Edinburgh. Then she discovered her pregnant state. Back in Strasbourg, a son, Christian, our third child, was born to us on April 17th 1959. This fact anticipates by some twenty years the theory of "pregnance" expounded later in "Esquisse d'une Sémiophysique". One might say — and it is hardly surprising — that the Fields medal confers on its recipient a certain "pregnance", both in the realm of mathematics and elsewhere. It remains up to him to stay worthy of it when the inevitable falling-off of old age sidles in.

René Thom
(trad. Vendla Meyer)

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HÖRMANDER'S WORK ON LINEAR DIFFERENTIAL OPERATORS

by
L. GÅRDING

Lars Hörmander was born 1931, studied at Lund University and is now professor of mathematics at Stockholm University. He has been given a Fields Medal for his outstanding work in the theory of partial differential equations. Before trying to describe some of his results I shall have to lead you through a somewhat lengthy introduction.

Let me first acquaint you with a few notations. Coordinates and derivatives in real n-space will be denoted by x_k and $D_k = i^{-1}\partial/\partial x_k$ respectively. Higher derivatives will be written as powers $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$; we put $|\alpha| = \alpha_1 + \dots + \alpha_n$. A linear differential operator of order m is a sum

$$P = P(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$$

with principal part

$$P_m(x, D) = \sum_{|\alpha| = m} a_\alpha(x) D^\alpha \,.$$

We assume that the coefficients are smooth complex functions. Substituting complex numbers ξ_j for the derivatives we get the characteristic polynomials

$$P(x, \xi) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha$$

and $P_m(x, \xi)$. The Fourier transform will convert a differential operator with constant coefficients $P(D)$ into multiplication by $P(\xi)$. The adjoint and the conjugate of P are defined by

$$P^* = \sum D^\alpha \bar{a}_\alpha(x)$$

and

$$\overline{P} = \sum \bar{a}_\alpha(x) D^\alpha \,.$$

Linear differential equations

$$Pu = v \tag{1}$$

and systems of such equations occur in many mathematical models of physics, e.g. hydrodynamics, elasticity, thermodynamics, electricity and magnetism, and magnetohydrodynamics. We can think of u as a function describing some aspect of the state of the physical system and v as some kind of exterior force. The manifold of solutions u gives us the possible states under the action v. Additional restraints mostly given as boundary conditions give us a solution describing a particular situation. In practice it is the boundary conditions that account for the variety while P is given by the model itself.

It was pointed out very emphatically by Hadamard that it is not natural to consider only analytic solutions and source functions even if P has analytic coefficients. This reduces the interest of the Cauchy–Kowalevski theorem which says that (1) has locally analytic solutions if P and v are analytic. The Cauchy–Kowalevski theorem does not distinguish between classes of differential operators which have, in fact, very different properties.

If

$$P = D_1^2 + \cdots + D_n^2$$

is Laplace's operator, we are dealing with potential theory and if

$$P = D_1^2 - D_2^2 - \cdots - D_n^2$$

is the wave operator, we are studying wave propagation. The solutions of the homogeneous equation $Pu = 0$ behave quite differently. In the first case they are harmonic and hence real analytic, in the second case the example $u =$ arbitrary function of $x_1 \pm x_2$ shows that they may have very complicated singularities.

A natural reaction to all this is the question: what is it in P that makes this difference? More generally we can ask for properties of P or rather its characteristic polynomial which are intrinsic in the sense that they are more or less equivalent to properties of the solutions. Questions of this nature have no physical background but a very solid motivation: mathematical curiosity. They lead Hadamard to the fruitful notion of correctly set boundary problems. The first complete results, however, are due to Petrovsky who proved among other things that

$$Pu = 0 \Rightarrow u \text{ analytic} \tag{2}$$

and

$$\xi \text{ real } \neq 0 \Rightarrow P_m(\xi) \neq 0 \tag{3}$$

are equivalent properties for operators with constant coefficients. The second condition has an immediate extension to operators with variable coefficients. They are said to be elliptic. A simple sample is Laplace's operator. Actually Petrovsky's main achievement on that occasion is a proof that all solutions of suitably defined elliptic nonlinear analytic systems are analytic. The problem goes back to one of Hilbert's problems from the Paris congress in 1900.

Petrovsky felt that his results were just a beginning and in a lecture in 1945 he explicitly asked for a general theory of linear differential operators including

those which do not appear in the mathematical models of physics. At the same time the theory of distributions appeared as a new tool in analysis. In his book on distributions Laurent Schwartz pointed out that the equivalence of (2) and (3) holds if the first u in (2) is assumed to be a distribution. He also stated a number of problems about differential operators. Since then a rather comprehensive theory has been worked out. Many people have contributed but the deepest and most significant results are due to Hörmander. I can describe only a few of them.

One of Schwartz's questions was the following one: what becomes of (3) if we replace (2) by the weaker statement

$$Pu = 0 \Rightarrow u \in C^\infty? \tag{4}$$

The answer, namely

$$\text{Im } \xi \text{ bounded }, \quad \xi \to \infty \Rightarrow P(\xi) \to \infty\,,$$

was obtained by Hörmander in his thesis as a byproduct of a study of operators with constant coefficients, which I shall not go into. The corresponding operators are called hypoelliptic. A characteristic point of the proof is that the theorem of the closed graph is used to replace (4) by an equivalent inequality which establishes the problem in a suitable analytical form. The last part of the thesis deals with variable coefficients and establishes an important inequality which can be described as follows.

Let Ω be an open part of R^n and let H_0^k be the closure of $C_0 = C_0^\infty(\Omega)$ in the norm

$$|u|_k^2 = \int \sum_{|\alpha| \le k} |D^\alpha u(x)|^2 dx\,.$$

It is a Hilbert space and we can identify its dual H^{-k} with the space of distributions u in Ω for which

$$|u|_{-k} = \sup |(u,v)|/|v|_k < \infty\,,$$

where $(u,v) = \int u(x)\overline{v(x)}dx$ and v runs over C_0. We observe that $H^{-0} = H_0^0$ and that H^{-k} increases with k. A simple result from the theory of linear operators tells us that if A and B are Banach spaces and L is a densely defined linear operator such that

$$A \xrightarrow{L} B \text{ has a continuous inverse}\,, \tag{5}$$

then the adjoint map

$$B^* \xrightarrow{L^*} A^*$$

is onto. By a piece of ingenious and powerful analysis Hörmander established (5) in the form of an inequality

$$|u|_{m-1} \le C|Pu|_0 \quad (u \in C_0)$$

and hence also the solvability of

$$P^*u = v \in H^{1-m}\,. \tag{6}$$

It is assumed that

$$\Omega \text{ is sufficiently small,} \tag{7}$$

$$P_m \text{ is real,} \tag{8}$$

$$\xi \text{ real } \neq 0 \Rightarrow \sum_1^n \left| \frac{\partial P_m(x,\xi)}{\partial \xi_k} \right| > 0\,. \tag{9}$$

The solvability of (6) was a great breakthrough in the theory and the conditions (7) and (9) but perhaps not (8) of reasonable generality. There were, however, surprises to come. Let us weaken (6) to

$$Pu = v \text{ has a solution (possibly a distribution) for every } v \in C_0\,. \tag{10}$$

By an ingenious but very special reasoning Hans Lewy proved in 1957 that the operator

$$D_1 + iD_2 + i(x_1 + ix_2)D_3$$

does not have this property. An almost complete investigation of the situation was given by Hörmander in 1960. Rephrasing (10) in the form of an inequality he got the following result. Put

$$C(x,D) = (P(x,D)\overline{P}(x,D) - \overline{P}(x,D)P(x,D))_{2m-1}\,.$$

This operator only depends on P_m and vanishes if P_m is real. Then one has

$$(10) \;\Rightarrow (P_m(x,\xi) = 0\,, \quad \xi \text{ real } \Rightarrow C(x,\xi) = 0)\,. \tag{11}$$

Furthermore, this statement almost has a converse. We say that P is principally normal if there exists a polynomial $Q_{m-1}(x,\xi)$ of degree $m-1$ in ξ such that

$$C(x,\xi) = P_m(x,\xi)\bar{Q}_{m-1}(x,\xi) + \overline{P}_m(x,\xi)Q_{m-1}(x,\xi)\,.$$

This property implies the right side of (11) and Hörmander proved that

$$P \text{ principally normal} \tag{12}$$

together with (7) and (9) implies the solvability of (6). The discovery of these results and their proofs is a first-class achievement. Recently he has given a global version of the theory.

I will finish by touching upon the problem of unique continuation which is the following. Consider a regular surface $S : s(x) = 0$ in R^n and a solution u of $Pu = 0$

which vanishes on one side of S. Does it vanish on the other side? If the coefficients of P are analytic the answer is yes provided S in non-characteristic for P, i.e.

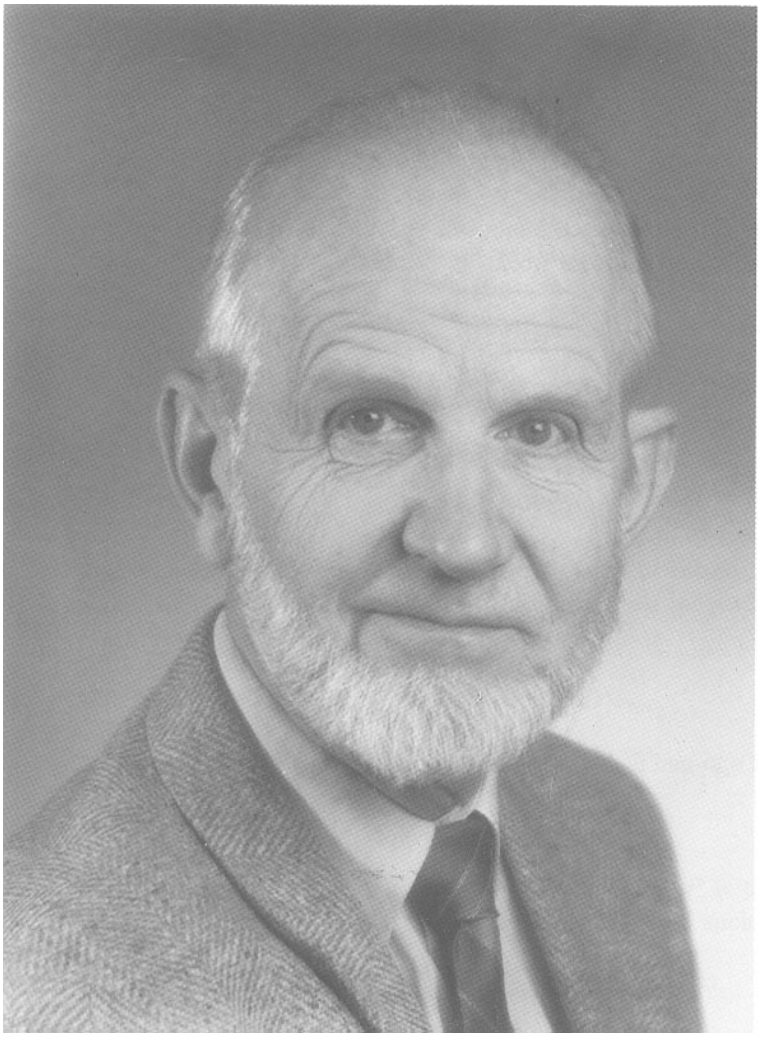
$$P_m(x, \text{grad } s(x)) \neq 0\,. \tag{13}$$

This is an old result by Holmgren. The same problem for non-analytic coefficients is much harder and there are counterexamples to show that (13) is not enough. The first positive results are due to Carleman for two variables and to Calderón in the general case. Hörmander has clarified the situation considerably by inventing a sufficient convexity condition bearing on S and P which also comes close to being necessary.

I hope that this sketch has given you an idea of the power and the drive that characterizes Hörmander's work.

Reference

L. Hörmander, *Linear Partial Differential Operators*, Springer 1963.



Lars Hörmander

AUTOBIOGRAPHY OF LARS HÖRMANDER

I was born on January 24, 1931, in a small fishing village on the southern coast of Sweden where my father was a teacher. After elementary school there and "realskola" in a nearby town which could be reached daily by train I went to Lund to attend "gymnasium", as my older brothers and sisters had done before me. I was more fortunate than they, for the principal was just starting an experiment which meant that three years were decreased to two with only three hours daily in school. This meant that I could mainly work on my own, with much greater freedom than the universities in Sweden offer today, and that suited me very well. I was also lucky to get an excellent and enthusiastic mathematics teacher who was a docent at the University of Lund. He encouraged me to start reading mathematics at the university level, and it was natural to follow his advice and go on to study mathematics at the University of Lund when I finished "gymnasium" in 1948.

In 1950 I got a masters degree and started as a graduate student. Marcel Riesz was my advisor, as he had been for my "gymnasium" mathematics teacher. Riesz was close to his retirement in 1952, and his lectures which I had actually attended since 1948 were not devoted to partial differential equations where he had recently made major contributions but rather to his earlier interests in classical function theory and harmonic analysis. My first mathematical attempts were therefore in that area. Although they did not amount to much this turned out to be an excellent preparation for working in the theory of partial differential equations. That became natural when Marcel Riesz retired and left for the United States while the two new professors Lars Gårding and Åke Pleijel appointed in Lund were both working on partial differential equations.

After a year's absence for military service 1953–1954, spent largely in defense research which gave ample opportunity to read mathematics, I finished my thesis in 1955 on the theory of linear partial differential operators. It was to a large extent inspired by the thesis of B. Malgrange which was announced in 1954, combined with techniques developed for hyperbolic differential operators by J. Leray and L. Gårding. Soon after that I was ready for my first visit to the United States, where in 1956 I spent the Winter and Spring quarters at the University of Chicago, the summer at the Universities of Kansas and Minnesota, and the fall in New York at what is now called the Courant Institute. (At the time R. Courant was still the director and it was called the Institute of Mathematical Sciences.) In Chicago there was no activity at all in my field, but the Zygmund seminar, conducted in his absence by E. M. Stein and G. Weiss, gave a useful addition to my background in harmonic analysis. At the other places I visited there was much to learn in my proper field.

At the end of this stay I was appointed to a full professorship at the University of Stockholm (called Stockholms Högskola then), which I had applied for before leaving for the United States. I took up my duties there in January 1957 and remained as professor until 1964. However, already during the academic year 1960–61 I was back in the United States as a member of the Institute for Advanced Study. During the summers of 1960 and 1961 I lectured at Stanford University and wrote a major part of my first book on partial differential equations. It was published by Springer Verlag in the Grundlehren Series in 1963 after the manuscript had been completed and polished back in Stockholm during the academic year 1961–62.

The 1962 International Congress of Mathematicians was held in Stockholm. In view of the small number of professors in Sweden at the time it was inevitable that I should be rather heavily involved in the preparations but it came as a complete surprise to me when I was informed that I would receive one of the Fields medals at the congress.

Some time after the two summers at Stanford I received an offer of a part time appointment as professor at Stanford University. I had declared that I did not want to leave Sweden, so the idea was that I should spend the Spring and Summer quarters at Stanford but remain in Stockholm most of the academic year there, from September through March. A corresponding partial leave of absence was granted by the ministry of education in Sweden and the arrangement became effective in 1963. However, I had barely arrived at Stanford when I received an offer to come to the Institute for Advanced Study as permanent member and professor. Although I had previously been determined not to leave Sweden, the opportunity to do research full time in a mathematically very active environment was hard to resist. After an attempt to create a research professorship for me in Sweden had failed, I finally decided in the fall of 1963 to accept the offer from the Institute and resign from the universities of Stockholm and Stanford to take up the new position in Princeton in the fall of 1964.

At that time the focus of interest in Princeton was definitely not in analysis which was felt both as a challenge and as a great opportunity to broaden my mathematical outlook. However, it turned out that I found it hard to stand the demands on excellence which inevitably accompany the privilage of being an Institute professor. After two years of very hard work I felt that my results were not up to the level which could be expected. Doubting that I would be able to stand a lifetime of this pressure I started to toy with the idea of returning to Sweden when a regular professorship became vacant. An opportunity arose in 1967, and I decided to take it and return as professor in Lund from the fall term 1968. After the decision had been taken I felt much more relaxed, and my best work at the Institute was done during the remaining year.

So in 1968 I had completed a full circle and was back in Lund where I had started as an undergraduate in 1948. I have remained there since then, with interruptions for some visits mainly to the United States. During the Fall term of 1970 I was visiting professor at the Courant Institute in New York, during the Spring term

1971 I was a member of the Institute for Advanced Study, and during the Summer quarter 1971 I was back at Stanford University, where I also lectured during the Summer quarters of 1977 and 1982. During the academic year 1977–1978 I was again a member of the Institute for Advanced Study which had a special year in microlocal analysis then, and during the Winter quarter 1990 I was visiting professor at the University of California in San Diego.

After five years devoted to writing a four volume monograph on linear partial differential operators I spent the academic years 1984–1986 as director of the Mittag–Leffler Institute in Stockholm. I had only accepted a two year appointment with a leave of absence from Lund since I suspected that the many administrative duties there would not agree very well with me. The hunch was right, and since 1986 I have been in Lund where I became professor emeritus in January 1996.

In my contribution to this volume I have summed up my work in partial differential equations and microlocal analysis which can be considered as the continuation of the work which gave me a Fields medal. Another major interest has been the theory of functions of several complex variables and its applications to the theory of partial differential equations. Lecture notes on this subject written at Stanford during the summer of 1964 became a book published in 1966, and extended editions were published in 1973 and 1990. A book on convexity theory published in 1994 is also to a large extent devoted to this field.

LOOKING FORWARD FROM ICM 1962

by
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1. Introduction

When I received the invitation to contribute to a volume entitled *Fields Medallists' Lectures*, intended to be similar to *Nobel Lectures in Physics*, my first reaction was quite negative. The Nobel prizes and the Fields medals are so very different in character; while Nobel prizes are supposed to be given for work of already recognized importance, often the work of a lifetime, the Fields medals are given "in recognition of work already done, and *as an encouragement for further achievement on the part of the recipients"*. However, since this may imply an obligation to account for the expected further achievements, I have decided to contribute to this volume a brief survey of the later development of the work for which I assume that I received a Fields medal in 1962. It will not be a complete survey even of the development of these topics in the theory of linear partial differential equations since 1962, for I shall concentrate on my own work and only mention work by others interacting with it.

At the ICM in 1962 I gave a half hour lecture [H1] with the following table of contents:

1. Notations
2. Equations without solutions
3. Existence theorems
4. Hypoelliptic operators
5. Holmgren's uniqueness theorem
6. Carleman estimates
7. Uniqueness of the Cauchy problem
8. Unique continuation of singularities

The paper [H1] was written just after the manuscript of my first book [H2] had been completed, and all the topics 2–8 are presented in detail there. When discussing the later development I shall often compare it with the new version [H3] published in four volumes about 20 years later.

2. Pseudodifferential Operators and The Wave Front Set

In [H2] the main tool for proving *a priori* estimates for differential operators was the Fourier transformation. When the coefficients were variable their arguments

were first "frozen" at a point, sometimes after a preliminary integration by parts as in [H2, Chap. VIII]. For the success of the procedure it was of course necessary that the error committed in freezing the coefficients was in some sense small compared to the quantities to be estimated. This worked well also for the proof of uniqueness theorems originally proved by Calderón [Ca] using singular integral operators; in fact, it was possible to reduce his regularity assumptions. Singular integral operators were not included in [H2], for they appeared to have many drawbacks: they required an artificial reduction to operators of order 0, only principal symbols were handled, and the role of the Fourier transformation was so suppressed that the calculations involved seemed artificial. These objections were removed by the introduction of pseudodifferential operators by Kohn and Nirenberg [KN]. They considered operators of the form

$$\begin{aligned} a(x,D)u(x) &= (2\pi)^{-n}\int a(x,\xi)e^{i\langle x,\xi\rangle}\hat{u}(\xi)d\xi\,, \\ \hat{u}(\xi) &= \int e^{-i\langle x,\xi\rangle}u(x)dx\,, \quad u\in C_0^\infty(\mathbf{R}^n)\,, \end{aligned} \tag{2.1}$$

where a is asymptotically a sum of terms which are homogeneous of integer order, which means that this algebra of operators is essentially generated by singular integral operators, differential operators and standard potential operators. However, the important point was that the calculus formulas for the symbols a were essentially the same as the familiar formulas for differential operators, not only on a principal symbol level. It was then easy and natural to introduce more general symbols which were useful in the further development of the theory of linear differential operators. The properties of parametrices of hypoelliptic operators with constant coefficients led me to introduce in [H4] the symbol class $S^m_{\varrho,\delta}$ of C^∞ functions in $\mathbf{R}^n\times\mathbf{R}^n$ such that

$$\left|\partial_x^\beta\partial_\xi^\alpha a(x,\xi)\right| \le C_{\alpha,\beta}(1+|\xi|)^{m-\varrho|\alpha|+\delta|\beta|}\,. \tag{2.2}$$

When $0\le\delta<\varrho\le 1$ operators with such symbols give a calculus with very good properties, and if $\delta\ge 1-\varrho$, hence $\varrho>\frac{1}{2}$, it is invariant under changes of variables so it can be transplanted to smooth manifolds. (Such operators had already been used in [H5] to construct left parametrices of hypoelliptic differential operators, but I had not understood that an algebra of operators could be obtained in this way.)

Pseudodifferential operators were used at once to localize the study of singularities of solutions of partial differential equations. However, it took a few years before this was codified in the notion of wave front set: if $u\in\mathcal{D}'(\mathbf{R}^n)$ then the wave front set $WF(u)\subset T^*(\mathbf{R}^n)$ was defined in [H6] as the intersection of the characteristic sets of all pseudodifferential operators A such that $Au\in C^\infty(\mathbf{R}^n)$. By standard elliptic regularity theory it follows that the projection of $WF(u)$ in $\mathbf{R}^n$ is equal to the singular support of u, and the definition guarantees that $WF(Au)\subset WF(u)$ if A is a pseudodifferential operator. The wave front set describes both the location of the singularities and the directions of the frequencies which cause the singularities.

A similar resolution of analytic singularities was given independently by Sato [Sa] but is technically more difficult to explain. For the original definitions we refer to [SKK], and various alternative definitions can be found in [H3, Chap. VIII, IX].

The proof of estimates is often reduced to positivity of an operator in L^2. The classical Gårding inequality states that if $a(x,D)$ is a differential operator with $a \in S^m_{1,0}$, $m > 0$, and $a(x,\xi) \geq c(1+|\xi|)^m$ for some $m > 0$ and $c > 0$, then

$$\mathrm{Re}\ (a(x,D)u,u) \geq -C(u,u)\,, \quad u \in C_0^\infty(\mathbf{R}^n)\,, \tag{2.3}$$

where $(\cdot\,,\cdot)$ is the scalar product in L^2. This remains true for pseudodifferential operators with symbol in $S^m_{\varrho,\delta}$, where $0 \leq \delta < \varrho \leq 1$. The proof is quite trivial in the pseudodifferential framework since $a(x,D) + a(x,D)^* = b(x,D)^* b(x,D) + c(x,D)$ for some $b \in S^{m/2}_{\varrho,\delta}$ and $c \in S^0_{\varrho,\delta}$. A stronger result, called the "sharp Gårding inequality" was proved in [H7]; an extended version given in [H3, Theorem 18.6.7] states that if $a \in S^{\varrho-\delta}_{\varrho,\delta}$ and $a \geq 0$ then

$$\mathrm{Re}\ (a(x,D)u,u) \geq -C(u,u)\,, \quad u \in C_0^\infty(\mathbf{R}^n)\,. \tag{2.4}$$

An important refinement of the right-hand side was given by Melin [Mel] (see also [H3, Theorem 22.3.3]), and Fefferman–Phong [FP] (see also [H3, Theorem 18.6.8]) proved that (2.4) remains valid when $a \in S^{2(\varrho-\delta)}_{\varrho,\delta}$ and $a \geq 0$. This is often a very significant technical improvement.

The study of solvability of (pseudo)differential equations led Beals–Fefferman [BF1] to introduce much more general symbol classes than the classes $S^m_{\varrho,\delta}$ above, which could be tailored to the equation being studied. A further extension was made in [H8] (see [H3, Chap. XVIII]) using a variant of the definition (2.1) proposed already by Weyl [W] in his work on quantum mechnanics. It is the symplectic invariance of the approach of Weyl which allows a greater generality. Even more general symbols have been studied by Bony and Lerner [BL].

3. Fourier Intergral Operators

Since pseudodifferential operators cannot increase the wave front set, a (pseudo)differential operator cannot have a pseudodifferential fundamental solution (or parametrix) unless it is hypoelliptic. Parametrices of a different kind were constructed already in 1957 by Lax [La] for certain hyperbolic differential operators, as linear combinations of what is now called Fourier integral operators. In the simplest case these are of the form (2.1) with a modified exponent,

$$\begin{aligned} Au(x) &= (2\pi)^{-n} \int a(x,\eta) e^{i\varphi(x,\eta)} \hat{u}(\eta)\, d\eta\,, \\ \hat{u}(\eta) &= \int e^{-i\langle y,\eta\rangle} u(y)\, dy\,, \quad u \in C_0^\infty(\mathbf{R}^n)\,, \end{aligned} \tag{3.1}$$

where $\varphi(x,\eta)$ is positively homogeneous of degree 1 in η and $\det \partial^2\varphi/\partial x \partial\eta \neq 0$. (A reference to this construction in [La] was given in [H2, p. 230] but it could not be

studied with the techniques used in [H2].) With an operator of the form (3.1) there is associated a canonical transformation χ with the generating function φ,

$$\chi : (\partial\varphi(x,\eta)/\partial\eta, \eta) \mapsto (x, \partial\varphi(x,\eta)/\partial x)\,; \tag{3.2}$$

it has the important property that

$$WF(Au) \subset \chi WF(u)\,, \quad u \in \mathcal{D}'\,. \tag{3.3}$$

A systematic study of such operators was initiated in [H9] in connection with a study of spectral asymptotics (see Section 9 below), and a thorough, more general and global theory was presented in [H10]. Fourier integral operators permit simplifications of (pseudo)differential operators, for as observed by Egorov [Eg1] conjugation by an invertible Fourier integral operator changes the principal symbol by composition with the canonical transformation. This was exploited in [DH], a sequel to [H10] written jointly with J. J. Duistermaat. A somewhat different exposition of these matters is given in [H3, Chap. XXV]. Some early papers in the area have been collected and provided with an introduction and bibliography by Brüning and Guillemin [BG].

4. Hypoellipticity

A (pseudo)differential operator P in an open set $X \subset \mathbf{R}^n$ (or a manifold X) is called hypoelliptic if

$$\text{sing supp } u = \text{sing supp } Pu\,, \quad u \in \mathcal{D}'(X)\,. \tag{4.1}$$

Hypoelliptic differential operators with constant coefficients in $\mathbf{R}^n$ were characterised in my thesis (see [H2, Chap. IV]). The simplest class of hypoelliptic differential operators with variable coefficients is the class of elliptic operators: A differential operator P of order m is elliptic if the principal symbol $p(x,\xi)$, which is homogeneous of degree m, never vanishes when $\xi \in \mathbf{R}^n \setminus \{0\}$. If $H^{\text{loc}}_{(m)} = \{u; D^\alpha u \in L^2_{\text{loc}}, |\alpha| \le m\}$, then P is elliptic if and only if $Pu \in H^{\text{loc}}_{(0)}$ implies $u \in H^{\text{loc}}_{(0)}$. The definition of the Sobolev spaces $H^{\text{loc}}_{(s)}$ can be extended to all $s \in \mathbf{R}$ in a unique way such that for all elliptic pseudodifferential operators of order $m \in \mathbf{R}$ we have $Pu \in H^{\text{loc}}_{(0)} \Longleftrightarrow u \in H^{\text{loc}}_{(m)}$, and then we have more generally $Pu \in H^{\text{loc}}_{(s)} \Longleftrightarrow u \in H^{\text{loc}}_{(s+m)}$.

Elliptic operators (with smooth coefficients) have been understood for a very long time, for they can easily be treated as mild perturbations of elliptic operators with constant coefficients. A class of hypoelliptic differential operators which can be studied similarly starting from hypoelliptic differential operators with constant coefficients was investigated by B. Malgrange and myself; the results were included in [H2, Chap. VII]. At the time they seemed quite general, but hypoellptic operators not covered by these results were soon found by Treves [T1]. In [H5] his ideas were developed as a very primitive and incomplete version of pseudodifferential operator

theory already mentioned. In the more mature form of pseudodifferential operators in [H4] it was a simple consequence of the calculus, that if P is a pseudodifferential operator with symbol $p \in S^m_{\varrho,\delta}$, where $0 \le \delta < \varrho \le 1$, and if

$$\begin{aligned} |p(x,\xi)^{-1}\partial_x^\beta \partial_\xi^\alpha p(x,\xi)| &\le C_{\alpha,\beta}|\xi|^{-\varrho|\alpha|+\delta|\beta|}\,, \quad && |\xi| > C\,, \\ |p(x,\xi)^{-1}| &\le C|\xi|^{m'}\,, && |\xi| > C\,, \end{aligned}$$

then P is hypoelliptic. (In [H4, Section 4] there is actually a more general result for systems.) When $1-\varrho \le \delta$ this gives a class of hypoelliptic operators which is invariant under changes of variables. However, that requires $\varrho > \frac{1}{2}$, and it is easy to see that second order differential operators satisfying this condition must in fact be elliptic. This led to the detailed study in [H11] of second order hypoelliptic operators.

A classical model equation, the Kolmogorov equation, was known in the theory of Brownian motion but was more familiar to probabilists than to experts in partial differential equations. The Kolmogorov operator is

$$P = (\partial/\partial x)^2 + x\partial/\partial y - \partial/\partial t$$

in $\mathbf{R}^3$ with coordinates (x,y,t). Kolmogorov [Ko] constructed an explicit fundamental solution, singular only on the diagonal, which implies hypoellipticity. Freezing the coefficients would give an operator with constant coefficients acting only along a two dimensional subspace, so the operator is obviously not covered by the results mentioned so far. However, the vector fields $\partial/\partial x$ and $x\partial/\partial y - \partial/\partial t$ do not satisfy the Frobenius integrability condition so the operator does not act only along submanifolds. The importance of this fact was established in [H11] where it was proved more generally that if

$$P = \sum_1^r X_j^2 + X_0 + c\,, \tag{4.2}$$

where $X_0, \dots, X_r$ are smooth real vector fields in a manifold M of dimension n, and if among the operators $X_j, [X_j, X_k], [X_j, [X_k, X_l]], \dots$ obtained by taking repeated commutators it is possible to find a basis for the tangent space at any point in M, the P is hypoelliptic. The condition on the vector fields is necessary in the sense that if the rank is smaller than n in an open subset then the operator only acts in less than n local coordinates, if they are suitably chosen, so it cannot be hypoelliptic. However, the condition can be relaxed at smaller subsets. (See e.g. [OR], [KS], [BM].) Simplified proofs of these results (with slightly less precise regularity) due to J. J. Kohn can be found in [H3, Section 22.2]. There is now a very extensive literature on operators of the form (4.2) (see e.g. [RS], [HM], [NSW], [SC]), partly motivated by the importance in probability theory.

If P is of the form (4.2), then the principal symbol of P is ≤ 0. Many of the results concerning operators of the form (4.2) have been extended to general pseudodifferential operators such that the principal symbol (microlocally) takes its

values in an angle $\subset \mathbf{C}$ with opening $< \pi$. We refer to [H3, Chap. XXII] for such results and references to their origins.

A pseudodifferential operator of order m and type 1, 0 is called *subelliptic* with loss of δ derivaties, if $0 < \delta < 1$ and

$$u \in \mathcal{D}'(\Omega)\,, \quad Pu \in H^{\mathrm{loc}}_{(s)} \Longrightarrow u \in H^{\mathrm{loc}}_{(s+m-\delta)}\,. \tag{4.3}$$

For $\delta = 0$ this condition would be equivalent to ellipticity, and the assumption $\delta < 1$ implies that (4.3) depends only on the principal symbol $p(x,\xi)$ of P, which we assume to be homogeneous. In [H7] it was proved that $\delta \geq \frac{1}{2}$ when (4.3) is valid and P is not elliptic, and that (4.3) is valid for $\delta = \frac{1}{2}$ if and only if

$$\{\mathrm{Re}\, p(x,\xi), \mathrm{Im}\, p(x,\xi)\} > 0\,, \quad \text{when } p(x,\xi) = 0\,. \tag{4.4}$$

(The term subellipticity was introducted in [H3] just for the case $\delta = \frac{1}{2}$. There are pseudodifferential operators with this property but no differential operators since the Poisson bracket is then an odd function of ξ.) In [H7] a very implicit condition for subellipticity with loss of $\frac{1}{2}$ derivatives was also given for systems. The condition was made somewhat more explicit in [H12] but there are no really satisfactory results on subellipticity for systems with loss of δ derivatives even for $\delta = \frac{1}{2}$, and very little is known for $\delta \in (\frac{1}{2}, 1)$. However, for the scalar case Egorov [Eg2] found necessary and sufficient conditions for subellipticity with loss of δ derivatives; the proof of sufficiency was completed in [H13]. The results prove that the best δ is always of the form $k/(k+1)$ where k is a positive integer, and the conditions replacing (4.4) then involve the Poisson brackets of $\mathrm{Re}\, p$ and $\mathrm{Im}\, p$ of order $\leq k$, at the characteristics. A slight modification of the presentation in [H13] is given in [H3, Chap. XXVII], but it is still very complicated technically. Another approach which also covers systems operating on scalars has been given by Nourrigat [No] (see also the book [HN] by Helffer and Nourrigat), but it is also far from simple so the study of subelliptic operators may not yet be in a final form.

The hypoelliptic operators discussed here are all microlocally hypoelliptic in the sense that (4.1) can be strengthened to

$$WF(u) = WF(Pu)\,, \quad u \in \mathcal{D}'(X)\,. \tag{4.1$'$}$$

This remains valid in an open conic subset of the cotangent bundle if the conditions above are only satisfied there.

5. Solvability and Propagation of Singularities

The discovery by Lewy [Lew] that the equation

$$\partial u/\partial x_1 + i\partial u/\partial x_2 + 2i(x_1 + ix_2)\partial u/\partial x_3 = f$$

for most $f \in C^\infty(\mathbf{R}^3)$ (in the sense of category) has no solution in any open subset of $\mathbf{R}^3$ led to a systematic study of local solvability of differential equations in [H14],

[H15] and [H2, Chap. VI, VIII]. It was proved there that if $p(x,\xi)$ is the principal symbol of a differential operator P of order m in $\mathbf{R}^n$ then the equation $Pu = f$ has no distribution solution in any neighborhood of x^0 for most $f \in C^\infty$ unless

$$p(x,\xi) = 0 \Longrightarrow \{\operatorname{Re} p(x,\xi), \operatorname{Im} p(x,\xi)\} = 0\,, \tag{5.1}$$

when x is in a neighborhood of x^0. On the other hand, if

$$\{\operatorname{Re} p(x,\xi), \operatorname{Im} p(x,\xi)\} = \operatorname{Re}(p(x,\xi)a(x,\xi))\,, \tag{5.2}$$

in a neighborhood of x^0 for some polynomial a in ξ with C^1 coefficients, and

$$p'_\xi(x,\xi) \neq 0 \ \text{ for } \ \xi \in \mathbf{R}^n \setminus \{0\}\,, \tag{5.3}$$

it was proved that the equation $Pu = f$ has a solution $u \in H^{\mathrm{loc}}_{(s+m-1)}$ in a neighborhood of x^0 for every $f \in H^{\mathrm{loc}}_{(s)}$. There is of course a substantial gap between the conditions (5.1) and (5.2). For first order differential equations it was filled to a large extent by Nirenberg–Treves [NT1]. The study of boundary problems for elliptic operators led to the extension of the solvability problem from differential to pseudodifferential operators in [H7]. There it was proved that the results of [H14], [H15] remain valid for pseudodifferential operators if (5.1), (5.2) are modified to

$$p(x,\xi) = 0 \Longrightarrow \{\operatorname{Re} p(x,\xi), \operatorname{Im} p(x,\xi)\} \leq 0\,, \tag{5.1$'$}$$

$$\{\operatorname{Re} p(x,\xi), \operatorname{Im} p(x,\xi)\} \leq \operatorname{Re}(p(x,\xi)a(x,\xi))\,. \tag{5.2$'$}$$

Thus (5.1)$'$ forbids $\operatorname{Im} p$ to change sign from $-$ to $+$ at a simple zero in the forward direction on a bicharacteristic of $\operatorname{Re} p$. Nirenberg and Treves [NT2] proved that such sign changes must not occur at any zeros of finite order either, and later work based on an idea of Moyer [Mo] has shown that local solvability implies that there are no such sign changes at all. (See [H3, Theorem 26.4.7].) This is called the condition (Ψ). For differential operators we have $p(x,-\xi) = (-1)^m p(x,\xi)$ and it follows that (Ψ) implies that there are no sign changes at all, which is called condition (P). It was also proved in [NT2] that local solvability follows from condition (P) and (5.3) provided that p is real analytic, a condition which was later removed by Beals and Fefferman [BF2]. However, it is still not known whether there is local solvability under condition (Ψ); there are many positive and negative results by Lerner [Ler1], [Ler2], [Ler3] and others. For these matters we refer to a recent survey paper [H16] and to [H3, Chap. XXVI].

The existence theorems just mentioned, first established in [H17], are not only local. They are valid for arbitrary compact sets which do not trap any complete bicharacteristics for the operator (see [H3, Theorem 26.11.1]), and they also give C^∞ solutions for C^∞ data. The key to such semiglobal results is the study of propagation of singularities of solutions of pseudodifferential equations. In [H1] there was just a very primitive result of this kind, Theorem 8.1, stating that if

the principal symbol p of a differential operator P is real and satisfies (5.3), then a distribution u such that $Pu \in C^\infty$ is in C^∞ in a neighborhood of a point x^0 if there is a C^2 hypersurface through x^0 which has positive curvature at x^0 with respect to tangential bicharacteristics such that $u \in C^\infty$ outside the surface. A first step toward more precise results was taken by Grushin [Gr] who proved in the constant coefficient case that if $x^0 \in$ sing supp u then sing supp u contains a bicharacteristic line through x^0. An extension of this result to operators with variable coefficients was announced in [H18, p. 39]. However, the conclusion was only local which was a great weakness, for if one has concluded that an interval from x^0 to x^1 on a bicharacteristic Γ_0 is contained in sing supp u, it follows that there is an interval around x^1 on a bicharacteristic Γ_1 contained in sing supp u, but there is no guarantee that Γ_1 should be a continuation of Γ_0. The proof announced in [H18] depended on an early local version of the theory of Fourier integral operators. The global theory presented in [H10] was developed precisely to remove this flaw so that it could be proved directly that a complete bicharacteristic must be contained in sing supp u. However, just as this machinery was completed the idea of the wave front set presented in [H6] was conceived. The right statement on the propagation of singularities is that if $Pu \in C^\infty$ and $(x^0, \xi^0) \in WF(u)$, then the bicharacteristic strip starting at (x^0, ξ^0) is contained in $WF(u)$, provided that the principal symbol p is real valued. (If one removes the hypothesis $Pu \in C^\infty$ the conclusion is that the bicharacteristic strip remains in $WF(u)$ until it encounters $WF(Pu)$.) A bicharacteristic curve is the base projection of a bicharacteristic strip, so it is not determined by its starting point whereas a bicharacteristic strip is. This means that for the improved microlocal version of the theorem on propagation of singularities the local result implies the global one. A simple proof was outlined in [H6]. In [H19] the propagation theorem was extended to symbols with Im $p \geq 0$, and in [DH] an analogue for the case of characteristics where d Re p and d Im p are linearly independent was established under condition (P). The main point in [H17] was a fairly complete discussion of propagation of singularities for arbitrary pseudodifferential operators satisfying condition (P). The results on propagation of singularities were completed by Dencker [D], replacing for some bicharacteristic a weaker result in [H17] which gave the same existence theorems though. Some of the results on propagation of singularities remain valid for operators satisfying only condition (Ψ), but there are no satisfactory general results in that case.

6. Holmgren's Uniqueness Theorem

The classical uniqueness theorem of Holmgren states that a classical solution of a linear differential equation $P(D)u = 0$ vanishing on one side of a C^1 surface must vanish in a neighborhood of every noncharacteristic point. This was extended to distribution solutions in [H2], and by a simple geometric argument it was concluded (see [H1, Theorem 5.1]) that if the surface is in C^2 the assertion remains valid at

characteristic points where the curvature of the surface is positive with respect to the corresponding tangential bicharacteristic.

It had been observed already by John [Jo] that the proof of Holmgren's uniqueness theorem could be modified to proving analyticity theorems such as the analyticity of solutions of elliptic differential equations with analytic coefficients. This observation could be reversed when the analytic singularities had been microlocalized to a set similar to the wave front set. The definition of Sato [Sa] (see also [SKK]) mentioned above works for hyperfunctions, whereas a definition of such a set $WF_A(u)$ modelled on the definition of $WF(u)$ introduced in [H20] only works for distributions. Since the equivalence with the definition of Sato in this case has been verified by Bony [Bo] we shall use the notation $WF_A(u)$ for arbitrary hyperfunctions u. (A proof of the equivalence of these definitions and another definition due to Bros and Iagolnitzer [BI] in the case of distributions is also given in [H3, Chap. VIII, IX].) The base projection of $WF_A(u)$ is of course the analytic singular support sing $\operatorname{supp}_A u$, the complement of the largest open set where u is a real analytic function.

The connection with Holmgren's uniqueness theorem is given by two facts:

(i) If P is a differential operator with real analytic coefficients and $Pu = 0$, then $WF_A(u)$ is contained in the characteristic set of P.

(ii) If u is a hyperfunction vanishing on one side of a C^2 surface passing through $x^0 \in \operatorname{supp} u$, then $WF_A(u)$ contains (x^0, ν) if ν is conormal to the surface at x^0. Part (i) is a microlocal version of the standard analytic regularity theorem for elliptic differential equations, and part (ii) follows from a part of the arguments used originally to prove Holmgren's uniqueness theorem. If the principal part p of P is real valued then a theorem on propagation of analytic singularities combined with (i) and (ii) proves that if $Pu = 0$ and u vanishes on one side of a C^1 hypersurface through $x^0 \in \operatorname{supp} u$ then the surface is characteristic at x^0 and the full bicharacteristic strip through a conormal must remain in $WF_A(u)$ so the base projection stays in supp u. This is a great improvement of [H1, Theorem 5.1] first obtained independently by Kawai in [Ka] and myself in [H20]. Many other improvements of Holmgren's uniqueness theorem have been obtained through a deeper understanding of (i) and (ii); a recent survey is given in [H21].

7. Analytic Hypoellipticity and Propagation of Analytic Singularities

Having made no contributions to this area beyond the first steps taken in [H20] we shall content ourselves here with some references to results which are relevant in connection with the improvements of Holmgren's uniqueness theorem discussed in the preceding section. A differential operator with real analytic coefficients is called analytic hypoelliptic if

$$\operatorname{sing\,supp}_A\, u = \operatorname{sing\,supp}_A\, Pu\,, \quad u \in \mathcal{D}'(X)\,, \tag{7.1}$$

and P is said to be microlocally analytically hypoelliptic if

$$WF_A(u) = WF_A(Pu)\,, \quad u \in \mathcal{D}'(X)\,. \tag{7.2}$$

(One sometimes insists on these equalities for all hyperfunctions.) If P is subelliptic in the C^∞ sense then it was proved by Treves [T2] that P is analytically hypoellptic, and by Trépreau [Tre] in even greater generality that P is also microlocally analytically hypoelliptic.

However, operators of the form (4.2) with analytic coefficients satisfying the commutator condition which implies hypoellipticity are not always analytically hypoelliptic even if $X_0 = 0$. (Lower order terms are not expected to affect analytic hypoellipticity.) Simple examples are given in e.g. Christ [Ch]. However, if the characteristic set is a symplectic manifold then microlocal analytic hypoellipticity has been proved, also for some classes of operators with higher order multiplicities for the characteristics (see [Tar], [T3], [Met], [Sj1], and also [DT] for recent results and additional references).

Concerning propagation of analytic singularities we shall content ourselves with referring to the very general results by Kawai and Kashiwara [KK] and Grigis, Schapira and Sjöstrand (see [Sj2]), although these are in no way the last words on the subject.

8. Carleman Estimates

Section 6 in [H1] and Chap. VIII in [H2] were devoted to Carleman estimates of the form

$$\begin{aligned} \tau \int |D^{m-1}u|^2 e^{2\tau\varphi} dx \le C_1 \int |Pu|^2 e^{2\tau\varphi}\, dx \\ + C_2 \sum_0^{m-2} \tau^{2(m-j)-1} \int |D^j u|^2 e^{2\tau\varphi}\, dx\,, \quad u \in C_0^\infty(K) \end{aligned} \tag{8.1}$$

where P is a differential operator of order m in a neighborhood of the compact set $K \subset \mathbf{R}^n$. Such estimates with $C_2 \neq 0$ were applied to the proof of existence theorems and some very weak results on propagation of singularities, and these results have been made obsolete by those discussed in Section 5. The estimates with $C_2 = 0$,

$$\tau \int |D^{m-1}u|^2 e^{2\tau\varphi}\, dx \le C_1 \int |Pu|^2 e^{2\tau\varphi}\, dx, \quad u \in C_0^\infty(K), \tag{8.1$'$}$$

are still of interest though. It was proved in [H2] that (8.1)$'$ implies that if p is the principal symbol of P then

$$|\xi + i\tau\varphi'(x)|^{2(m-1)} \le C_1 \{\overline{p(x, \xi + i\tau\varphi'(x)}, p(x, \xi + i\tau\varphi'(x))\}/2i\tau\,, \tag{8.2}$$

$$\text{if } \ \xi \in \mathbf{R}^n,\ \tau > 0, p(x, \xi + i\tau\varphi'(x)) = 0\,. \tag{8.3}$$

When (5.2) is fulfilled it is easy to conclude as a limiting case when$\tau \to 0$ that

$$|\xi|^{2(m-1)} \le C_1 \operatorname{Re}\{\overline{p(x,\xi)}, \{p(x,\xi), \varphi(x)\}\}\,, \tag{8.2$'$}$$

$$\text{if } \ \xi \in \mathbf{R}^n,\ p(x,\xi) = 0\,. \tag{8.3$'$}$$

It was proved in [H2] that conversely the conditions (8.2),(8.3) imply (8.1)$'$ (with a larger constant C_1) when(5.2) is valid with $a(x,\xi)$ in C^1 and polynomial in ξ. It was remarked in [H1, p. 343] that (8.2), (8.3) imply (5.1) but that it might be possible to eliminate the extraassumption (5.2) to a large extent. This was done in [H3, Chap. XXVIII] where (5.2) was weakened to

$$|\{\overline{p(x,\xi)}, p(x,\xi)\}| \le C_3 |p(x,\xi)||\xi|^{m-1}\,, \tag{8.4}$$

by means of the powerful lower bound for pseudodifferential operators established by Fefferman and Phong [FP].

In the application of the estimate (8.1)$'$ to the proof ofuniqueness theorems only the level sets of φ are important. Replacing φ by an increasing convex function of φsuch as $e^{\lambda\varphi}$ with some large positive λwill add a positive term in the right-hand side of (8.2), (8.2)$'$unless

$$\langle p'_\xi(x, \xi + i\tau\varphi'(x)), \varphi'(x)\rangle = \{p(x, \xi + i\tau\varphi'(x)), \varphi(x)\} = 0\,. \tag{8.5}$$

If (8.2), (8.2)$'$ are only assumed to be valid when (8.5) is addedto the hypotheses (8.3), (8.3)$'$ (with $\tau = 0$ in (8.5) in thesecond case), one can modify φ in this way without changing the level sets so that (8.2), (8.2)$'$ become valid with another constant C_1 under the hypotheses (8.3), (8.3)$'$ only. By the standard Carleman argument it follows then that if $Pu = 0$, $u \in H^{\mathrm{loc}}_{(m-1)}$ in a neighborhood of x^0 and $u = 0$when $\varphi(x) > \varphi(x^0)$, then $u = 0$ in a neighborhood ofx^0. (The proof uses also that the convexity conditions (8.2),(8.2)$'$ are stable under small perturbations.)

Already in [H2, Section 8.9] some examples based on constructionsby A. Pliš and P. Cohen were given which proved that the convexity assumption in this uniqueness result could not in general be relaxed. A more systematic study of such examples wasgiven in [H22]. However, these constructions relied onperturbations of P by terms of lower but positive order. Much better examples were constructed by Alinhac [Al] who proved that uniqueness fails after addition to P of a suitable C^∞ term of order 0 if (5.1) is not valid, or p has real coefficients and $\varphi(x) \le \varphi(x^0)$ on a bicharacteristic curvethrough x^0, or the right-hand side of (8.2) vanishes for a suitable family of zeros satisfying (8.3) and (8.5). For first order differential operators it was proved by Strauss and Treves [ST] that there is uniqueness for all non-characteristic surfaces if condition (P) is fulfilled, but recently Colombini and Del Santo [CD] proved that the condition (8.4) cannot be replaced by condition (P) in the uniqueness theorem above; in their example there are no non-trivial solutions of (8.3), (8.3)$'$ satisfying (8.5).

A surprising new discovery concerning the uniqueness of the Cauchy problem was made a few years ago by Robbiano [Ro] who proved that for a differential operator in $\mathbf{R}^{n+1}$ of the form

$$P = D_t^2 - A(x, D_x)$$

which is hyperbolic with respect to t, there is uniqueness for the Cauchy problem on every surface which is cylindrical in the t direction. This would be false in general for lower order terms with coefficients depending on t. A quantitatively more precise form of this result given in [H23] states that there is unique continuation of solutions of the equation $Pu = 0$ across a timelike surface with conormal (τ, ξ) at a point (t, x) provided that

$$27\tau^2/23 - a(x, \xi) < 0\,;$$

it is of course classical that there is unique continuation acrossspacelike surfaces, that is, surfaces with $\tau^2 - a(x, \xi) > 0$. Very recently Tataru [Tat] has proved that there is uniquecontinuation across all non-characteristic surfaces. This is aspecial case of a general result stating that if the principal symbolis translation invariant along a linear subspace $\mathcal{A}$ and all coefficients are real analytic in the direction of $\mathcal{A}$, then the uniqueness theorem above is valid if the convexity conditions (8.2), $(8.2)'$ are satisfied when to the conditions (8.3), $(8.3)'$ and (8.5) is added that ξ is a conormal of $\mathcal{A}$. In [H24] it is proved that it suffices to assume that the restriction of the principal symbol to the conormal bundle of $\mathcal{A}$ and its parallel spaces is translation invariant, and there are similar improvements of the other results of [Tat] as well.

9. Spectral Asymptotics

When I was a graduate student in Lund the two professors, Lars Gårding and Åke Pleijel, were both working on asymptotic properties of eigenvalues and eigenfunctions of elliptic differential operators P, and so were most of the graduate students. I chose a different direction for my thesis but became of course aware of the state of the field.

Since the work of Carleman in the 1930's the main approach to suchquestions has been to study the kernel of some function of P, suchas the resolvent or the Laplace transform. When pseudodifferentialoperators had appeared and been recognized as a powerful tool for theconstruction of parametrices, it was natural to try their strength inthis field.

Let $P(x, D)$ be an elliptic differential operator of order m with C^∞ coefficients in an open set $\Omega \subset \mathbf{R}^n$ with a self-adjoint extension $\mathcal{P}$ in $L^2(\Omega)$ which is bounded below. Let (E_λ) be the spectral resolution of $\mathcal{P}$ and let $e(x, y, \lambda)$ be the kernel of E_λ. Gårding [Gå1] proved using the simplest asymptotic properties of the resolvent of $\mathcal{P}$ that

$$R(x, \lambda) = \lambda^{-n/m} e(x, x, \lambda) - (2\pi)^{-n} \int_{p(x,\xi)<1} d\xi \tag{9.1}$$

converges to 0 as $\lambda \to +\infty$. Here p is the principalsymbol of P. For operators with constant coefficients he proved in [Gå2] that $R(x,\lambda) = O(\lambda^{-1/m})$. For second order operators with variable coefficients this wasproved by Avakumovič [Av] and in part by Lewitan [Lewi]. By afairly straightforward application of pseudodifferential calculus tothe construction of the resolvent $(\mathcal{P} - z)^{-1}$ it was proved in [H25] that $R(x,\lambda) = O(\lambda^{-\theta/m})$ for every $\theta < \frac{1}{2}$ (every $\theta < 1$ if the coefficients of the principal part are constant). The same result was obtained independently with different methods by Agmon and Kannai [AK]; in fact, Agmon had been the first to prove such bounds with a positive θ. It was also proved in [H25] that the value of θ has decisive importance for the summability properties of the eigenfunction expansion with respect to $\mathcal{P}$. The reason why Avakumovič obtained the optimal value $\theta = 1$ was that he used the Hadamard construction of parametrices for second order differential equations which takes advantage of geodesic coordinate systems. In [H9] it was proved that $R(x,\lambda) = O(\lambda^{-1/m})$ in general by applying Fourier integral operators to construct a parametrix for the hyperbolic pseudodifferential operator

$$i\partial/\partial t - \mathcal{P}^{1/m}$$

after a reduction to a compact manifold and a positive operator $\mathcal{P}$ which makes $\mathcal{P}^{1/m}$ a well defined pseudodifferential operator. This construction, which takes into account the geometrical optics description of propagation of singularities, was the starting point for the work on Fourier integral operators described in Section 3. What is required is only an understanding of the operator $e^{-it\mathcal{P}^{1/m}}$ for small values of $|t|$. Later work by many mathematicians where this unitary group is studied also for large values of t has led to much deeper understanding of the eigenvalues of elliptic differential operators, in particular the connection between clustering of eigenvalues and closed bicharacteristics. Some of this work is covered by [H3, Chap. XXIX]. (In the first edition there is an error in Theorem 29.1.4 corrected in the second edition.) However, it would carry too far to give a survey of this work here.

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L'OEUVRE DE MICHAEL F. ATIYAH

by
HENRI CARTAN

Je parlerai très brièvement des travaux d'Atiyah dans trois domaines, d'ailleurs ètroitement reliés entre eux: la K-théorie, la formule de l'indice, et la ≪ formule de Lefschetz ≫. Je laisserai de côté d'autres contributions, fort intéressantes d'ailleurs, à la Géométrie algébrique ou à la théorie du cobordisme; et je passerai aussi sous silence les résultats tout récents, encore inédits, dont l'auteur parlera lui-même dans sa conférence pendant ce Congrès.

1. La *K*-théorie. La plupart des travaux d'Atiyah en K-théorie ont été faits en collaboration avec F. Hirzebruch. C'est en 1956 que paraissait l'ouvrage fondamental de Hirzebruch (≪ Neue topologische Methoden in der algebraischen Geometrie ≫) dont le but ultime était le théorème fameux qui porte aujourd'hui le nom de ≪ théorème de Riemann–Roch–Hirzebruch ≫. Il s'agissait de géométrie algébrique sur le corps complexe. Peu après, Grothendieck cherchait et obtenait une démonstration purement algébrique (valable sur tout corps de base algébriquement clos, de caractéristique quelconque) d'un théorème plus général [1], puisqú'au lieu de considérer une variété algébrique X il étudiait un morphisme $X \to Y$ (le cas traité par Hirzebruch étant celui où la variété algébrique Y est réduite à un point). C'est à cette occasion que Grothendieck introduisit un foncteur contravariant qui, à chaque variété algébrique X, associe un *anneau* construit à l'aide des classes d'isomorphie de fibrés vectoriels algébriques de base X. Atiyah et Hirzebruch [2] eurent l'idée de faire de même pour un espace topologique *compact* X et pour les classes de fibrés vectoriels *complexes* de base X (il s'agit de fibrés topologiques, localement triviaux). On définit ainsi un anneau $K(X)$ pour tout espace compact X, d'où le nom de K-théorie. Il y a aussi une KO-théorie pour les fibrés vectoriels *r´eels*, et une KSp-théorie pour les fibrés vectoriels quaternioniens.

Bornons-nous, pour simplifier, à la K-théorie. On définit des groupes relatifs $K(X,Y)$ (pour Y sous-espace fermé de X), puis, par suspension, des groupes $K^n(X,Y)$ pour n entier $\leqslant 0$, avec $K^0(X,Y) = K(X,Y)$. On a alors une suite exacte

$$\cdots \longrightarrow K^n(X,Y) \longrightarrow K^n(X) \longrightarrow K^n(Y) \longrightarrow K^{n+1}(X,Y) \longrightarrow \cdots$$

analogue à la suite exacte de cohomologie. D'autre part, Atiyah observe que le célèbre théorème de périodicité de Bott [qui concerne les groupes d'homotopie du groupe unitaire infini $U = \varinjlim_m U(m)$] peut s'exprimer par un isomorphisme explicite

$$K^n(X) \approx K^{n+2}(X)\,,$$

ce qui permet de définir le foncteur K^n aussi pour n entier > 0. De cette facon on obtient une ≪ théorie cohomologique ≫ au sens d'Eilenberg-Steenrod, à cela près qu'un des axiomes d'Eilenberg-Steenrod (l'axiome ≪ de dimension ≫) n'est pas vérifié. Cette théorie fut d'abord baptisée ≪ cohomologie extraordinaire ≫.

Si on veut comparer la cohomologie extraordinaire à la cohomologie ordinaire, on peut dire en gros ceci: au lieu de considérer, comme en cohomologie ordinaire, les classes d'homotopie d'applications d'un espace X dans les espaces d'Eilenberg-MacLane $K(\pi, n)$, on envisage, dans la K-théorie, le groupe unitaire infini U (ou, ce qui revient au même, le groupe linéaire complexe infini), et son espace classifiant BU; ce sont eux qui servent d'espaces de comparaison. Les relations existant entre les deux théories cohomologiques (ordinaire et extraordinaire) s'expriment par une suite spectrale, et le ≪ caractère de Chern ≫

$$\mathrm{ch} : K^*(X, Y) \longrightarrow H^*(X, Y; \mathbf{Q})$$

est un homomorphisme multiplicatif d'une théorie dans l'autre.

L'importance de la ≪ cohomologie extraordinaire ≫ fut vite mise en évidence par les applications qu'Atiyah et Hirzebruch en firent, en Topologie algébrique et ailleurs [3]. Citons quelques exemples qui illustrent ces applications de la K-théorie:

— un théorème du type ≪ Riemann–Roch–Grothendieck ≫, valable cette fois pour les variétés différentiables [4];
— le calcul de $K(X)$ pour certains espaces homogènes, et le lien de cette question avec la théorie des représentations des groupes de Lie compacts [2];
— des théorèmes de non-plongement [5]: par exemple, l'espace projectif complexe $P_n(\mathbf{C})$ ne peut pas être différentiablement plongé dans l'espace numérique $\mathbf{R}^{4n-2\alpha(n)}$, où $\alpha(n)$ désigne le nombre des chiffres 1 du développement dyadique de l'entier n;
— des critères permettant de reconnaître si une classe de cohomologie d'une variété analytique complexe compacte peut être représentée par une sous-variété analytique [6].

Toutes ces applications sont dues à la collaboration d'Atiyah avec Hirzebruch. Il y en a d'autres; par exemple, c'est grâce à la K-théorie et à l'introduction de certains foncteurs $K \to K$ (dont l'idée revient essentiellement à Grothendieck) que J. F. Adams [7] a pu résoudre complètement un problème classique, resté longtemps

sans réponse: celui de la détermination exacte, en fonction de l'entier n, du nombre maximum de champs de vecteurs linéairement indépendants sur la sphère S^n (voir la conférence d'Adams au Congrès de Stockholm en 1962).

2. Le théorème de l'indice. Mais la plus belle application de la K-théorie devait être faite par Atiyah lui-même: je veux parler du *théorème de l'indice* (1963), démontré en collaboration avec I. Singer [8].

Soit D un opérateur elliptique sur une variété différentiable compacte X (supposée sans bord), opérant de l'espace vectoriel $\Gamma(E)$ des sections différentiables d'un fibré vectoriel complexe E dans l'espace $\Gamma(F)$ des sections différentiables d'un fibré vectoriel complexe F. On sait que le noyau et le conoyau de l'application linéaire $D : \Gamma(E) \to \Gamma(F)$ sont de dimension finie; l'*indice* $i(D)$ est l'entier défini par

$$i(D) = \dim\,(\mathrm{Ker}\ D) - \dim\,(\mathrm{Coker}\ D)\,.$$

Les travaux de plusieurs mathématiciens soviétiques avaient mis en évidence le fait que $i(D)$ ne change pas quand D varie d'une facon continue, et I. M. Gelfand, en 1960 [9], avait conjecturé que $i(D)$ devait donc pouvoir être calculé au moyen d'invariants purement topologiques liés à la donnée de X et de D. C'est ce problème qu'Atiyah et Singer ont complètement résolu. Les termes homogènes de plus haut degré de l'opérateur D définissent un $\ll$ symbole $\gg$ $\sigma(D)$ qui permet d'abord de définir l'ellipticité de D, puis, par l'intervention de la K-théorie, du caractère de Chern, et de la classe de Todd du fibré cotangent à X (lequel a une structure presque complexe), de définir finalement une classe de cohomologie, élément de $H^*(X;\mathbf{Q})$. Sa composante de degré $n = \dim\ X$ est un élément de $H^n(X\ ;\ \mathbf{Q}) \approx \mathbf{Q}$ (on suppose X orientable, pour simplifier). D'où un *nombre rationnel* $i_t(D)$ attaché à D (et à X), et défini au signe près; on peut l'appeler l'$\ll$ indice topologique $\gg$ de D. Le théorème d'Atiyah–Singer dit alors que *l'indice topologique* $i_t(D)$ *est égal à l'indice* $i\ (D)$ (moyennant des conventions convenables d'orientation). Ce théorème établit ainsi un pont entre deux vastes domaines des mathématiques: l'analyse des équations aux dérivées partielles d'une part, la topologie algébrique d'autre part.

Observons que, par définition, $i(D)$ est un entier. Il s'ensuit que le nombre rationnel $i_t(D)$ fourni par la Topologie algébrique est, en fait, un entier. On obtient par ce moyen, d'une facon naturelle et par le choix d'opérateurs elliptiques appropriés, tous les $\ll$ théorèmes d'intégralité $\gg$ relatifs aux classes caractéristiques des variétés (intégralité du L-genre, du genre de Todd, du $\hat{A}$-genre). Inversement, toute information fournie par la Topologie algébrique donne un résultat qui intéresse l'Analyse; par exemple, on voit facilement que l'indice topologique $i_t(D)$ est nul si la variété X est de dimension impaire.

La démonstration du théorème $i(D) = i_t(D)$ est laborieuse, mais fort intéressante, car elle conduit à introduire des opérateurs plus généraux que les opérateurs différentiels, à savoir les opérateurs intégraux singuliers de Calderon-Zygmund et de Seeley. La démonstration repose également sur une théorie du cobordisme qui

constitue une généralisation (relativement facile) de celle due à Thom. En fait il existe une nouvelle démonstration, plus récente, de la formule de l'indice, qui évite le recours au cobordisme.

Au lieu de considérer un seul opérateur elliptique, on peut envisager une suite d'opérateurs différentiels

$$\text{(D)} \qquad \Gamma(E_0) \longrightarrow \Gamma(E_1) \longrightarrow \cdots \longrightarrow \Gamma(E_k)$$

formant un ≪ complexe ≫ (i.e: le composé de deux opérateurs consécutifs est zéro). On définit l' ≪ ellipticité ≫ d'un tel complexe. A chaque complexe elliptique on attache encore un nombre rationnel $i_t(D)$. D'autre part les groupes d'homologie du complexe elliptique (D) sont des espaces vectoriels de dimension finite; soit $\chi(D)$ la somme alternée de leurs dimensions (c'est une sorte d'invariant d'Euler-Poincaré). Alors on a le théorème:

$$\chi(D) = i_t(D)\,.$$

Cette forme plus générale du théorème de l'indice est fort utile dans les applications. Par exemple, si on l'applique à une variété analytique complexe compacte X, et au ≪ complexe ≫ défini par l'opérateur difféfentiel d'' (noté aussi $\overline{\partial}$) des formes différentielles, on retrouve exactement l'énoncé du théorème de Riemann–Roch-Hirzebruch. Ce dernier n'était démontré auparavant que pour les variétés algébriques sans singularité; il est désormais valable pour toute variété analytique compacte.

Je laisse de côté le théorème de l'indice pour les variétés à bord [10]; il nécessite une nouvelle définition de l'ellipticité qui tienne compte des ≪ conditions aux limites ≫. La question a été entièrement résolue par Atiyah en collaboration avec Bott et Singer.

3. Formules de points fixes. Le théorème de l'indice n'est, en réalité, qu'un cas extrême d'une situation dont un autre cas extrême est, lorsque le complexe elliptique est celui défini par l'opérateur de différentiation extérieure des formes différentielles, la formule de Lefschetz relative aux points fixes (supposés isolés) d'une transformation d'une variété compacte X en elle-même. Il y a de nombreux cas intermédiaires, dont l'étude est en cours. Les résultats déjà obtenus sont dûs à la collaboration d'Atiyah et de Bott [11]. Expliquons sur un exemple de quoi il s'agit: soit X une variété analytique complexe, compacte, et soit $\mathfrak{f} : X \to X$ une application holomorphe; on sait que les espaces vectoriels de cohomologie $H^q(X, \mathcal{O})$ à coefficients dans le faisceau $\mathcal{O}$ des fonctions holomorphes sont de dimension finie; soit $L(\mathfrak{f})$ la somme alternée des traces

$$(-1)^q \ \mathrm{Tr}\ \left(\mathfrak{f}|_{H^q(X,\mathcal{O})}\right)\,.$$

C'est un entier; dans le cas où $\mathfrak{f}$ est l'identité, cet entier n'est autre que le premier membre de l'égalité de Riemann–Roch-Hirzebruch. Dans le cas général, on se propose d'exprimer cet entier à l'aide des propriétés topologiques de $\mathfrak{f}$ au voisinage de

l'ensemble des points fixes de $\mathfrak{f}$. Si $\mathfrak{f}$ est l'identité, on peut considérer que la formule de Hirzebruch (démontrée par Atiyah-Singer) résout le problème. Supposons au contraire que $\mathfrak{f}$ n'ait qu'un nombre fini de points fixes P, et que la différentielle $d\mathfrak{f}_P$ n'admette pas la valeur propre 1 (c'est notamment le cas lorsque $\mathfrak{f}$ est une transformation d'ordre fini). Alors le déterminant

$$\det_C(1 - d\mathfrak{f}_P)$$

est un nombre complexe $\neq 0$; le théorème prouvé par Atiyah et Bott affirme que, sous ces hypothèses, l'entier $L(\mathfrak{f})$ est égal à la somme des inverses de ces nombres complexes.

Nous bornant à cet exemple, nous ajouterons simplement que les résultats déjà obtenus fournissent une démonstration $\ll$ sans calculs $\gg$ de la formule de H. Weyl donnant le caractère d'une représentation d'un groupe semi-simple, et qu'ils permettent aussi de résoudre des problèmes de Conner-Floyd sur les variétés compactes où opère un groupe fini. Signalons aussi que, d'après Hirzebruch [12], on peut en déduire une formule de Langlands donnant la dimension des espaces vectoriels de formes automorphes pour un groupe discret à quotient compact.

En conclusion, l'on doit à Michael Atiyah plusieurs contributions majeures qui mettent en relation étroite la Topologie et l'Analyse. Chacune d'elles a été réalisée en collaboration; sans diminuer en rien la part qui revient à des collaborateurs aussi éminents que Hirzebruch, Singer ou Bott, il ne fait aucun doute que dans chaque cas l'intervention personnelle d'Atiyah a été décisive. Il nous donne l'exemple d'un mathématicien chez qui la clarté des conceptions et la vision d'ensemble des phénomènes s'allie harmonieusement à l'imagination créatrice, et aussi à la persévérance qui conduit aux grands achèvements.

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Michael F. Atiyah

MICHAEL F. ATIYAH

Michael F. Atiyah was born in 1929. His father was a distinguished Lebanese and his mother came from a Scottish background. He was educated at Victoria College, Cairo, and Manchester Grammar School. After National Service, he went to Trinity College, Cambridge, where he obtained his BA and PhD degrees and continued with further research, finally as a University lecturer and Fellow of Pembroke College. In 1961 he moved to Oxford, initially appointed to a Readership, and later to the Savilian Professorship of Geometry. From 1969 he was Professor of Mathematics at the Institute for Advanced Study in Princeton, USA (where he had held a Commonwealth Fund Fellowship in 1955) until 1972 when he returned to Oxford as a Royal Society Research Professor and Fellow of St Catherine's College. He held this post until 1990 when he became Master of Trinity College, Cambridge, and Director of the new Isaac Newton Institute for Mathematical Sciences.

Michael Atiyah has contributed to a wide range of topics in Mathematics centering around the interaction between geometry and analysis. His first major contribution (in collaboration with F. Hirzebruch) was the development of a new and powerful technique in topology (K-theory) which led to the solution of many outstanding difficult problems. Subsequently (in collaboration with I. M. Singer) he established an important theorem dealing with the number of solutions of elliptic differential equations. This "index theorem" had antecedents in algebraic geometry and led to important new links between differential geometry, topology and analysis. Combined with considerations of symmetry it led (jointly with R. Bott) to a new and refined "fixed-point theorem" with wide applicability.

All these ideas were subsequently found to be directly relevant to gauge theories of elementary particle physics. The index theorem could be interpreted in terms of quantum theory and has proved a useful tool for theoretical physicists.

Beyond these linear problems, gauge theories involved deep and interesting nonlinear differential equations. In particular, the Yang–Mills equations have turned out to be particularly fruitful for mathematicians. Atiyah initiated much of the early work in this field and his student Simon Donaldson went on to make spectacular use of these ideas in 4-dimensional geometry.

Most recently Atiyah has been influential in stressing the role of topology in quantum field theory and in bringing the work of theoretical physicists, notably E. Witten, to the attention of the mathematical community.

In the past few years, he has taken on major responsibilities in the educational and scientific arena. As President of the Royal Society from 1990–95 he was much involved with science policy, both nationally and internationally. In Cambridge, as Master of Trinity College and first Director of the Isaac Newton Institute for Mathematical Sciences, he has a substantial involvement both in education and in research.

He has received numerous honours, including the Fields Medal awarded to him in Moscow in 1966. He was knighted in 1983 and made a member of the Order of Merit in 1992.

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THE INDEX OF ELLIPTIC OPERATORS

by
MICHAEL F. ATIYAH

Introduction

The index theorem is an outgrowth of the Riemann–Roch theorem in algebraic geometry and in these lectures I shall follow its historical development, starting from the theory of algebraic curves and gradually leading up to the modern developments. Since the Riemann–Roch theorem has been a central theorem in algebraic geometry the history of the theorem is to a great extent a history of algebraic geometry. My purpose therefore is really to use the theorem as a focus for a general historical survey. For convenience I shall divide up the four lectures roughly according to the following periods:

(1) The classical era pre 1939, from Riemann to Hodge.
(2) The post war period up to 1954, culminating in Hirzebruch's proof of the Riemann–Roch theorem.
(3) 1955–62, notable for the work of Grothendieck and the introduction of K-theory.
(4) 1963–: the move from algebraic geometry to general elliptic operators.

I should emphasize that this is not meant to be a balanced account of the whole subject either in terms of time-scale or in terms of content. It is simply a personal point of view emphasizing those aspects which have been my own particular specialty. In particular my interest lies mainly in the transcendental theory connecting algebraic geometry (over the complex field) with holomorphic functions of several complex variables and the theory of harmonic functions. I am concerned with the Laplace operator and other operators of the same type, namely the elliptic ones. I shall therefore essentially ignore the purely algebraic aspects of the subject which have of course been the center of great interest in recent years. However, most important ideas straddle the border between the algebraic and transcendental areas and it is impossible to make a strict separation.

Finally, I should like to make it clear that these notes are only an approximate indication of the lectures: they are a complement and not a substitute for the spoken word.

1. The Classical Period

1.1. *Algebraic curves*

Classical algebraic geometry starts from the study of algebraic curves given by a polynomial equation $f(x,y) = 0$ or, to include points "at infinity," a homogeneous equation $f(x,y,z) = 0$ where (x,y,z) are homogeneous coordinates for a point in the projective plane. The coefficients of f and the values of (x,y,z) are complex numbers. Assuming that f is irreducible and non-singular (i.e., the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$ do not vanish simultaneously for $(x,y,z) \neq 0$) the curve is a compact subspace of the projective plane which can be locally parametrized by one complex variable (one of the six ratios x/y, y/z ... will do). In other words the curve is a compact Riemann surface or in modern parlance a compact one-dimensional complex analytic manifold. In fact, the theories of

(a) algebraic curves
(b) compact Riemann surfaces
(c) algebraic function fields of one variable

essentially coincide, being simply the geometric, analytic and algebraic aspects of the same mathematical entity. One also gets different points of view according as to how much emphasis one places on the choice of coordinates in the plane. In the early days one considered the (inhomogeneous) equation $f(x,y) = 0$ as defining y as a (multi-valued) algebraic function of x. Then as x wandered over the complex line (including ∞) y described a multiple-sheeted covering with branch points. The Riemann surface of the curve was thus viewed as always sitting over the 2-sphere, and it was not until much later that the idea of an abstract Riemann surface evolved.

On a compact Riemann surface the basic objects of study are naturally the holomorphic and meromorphic functions. Globally any holomorphic function must be a constant (Liouville's theorem) and, more generally, any meromorphic function is a rational function of (x,y) (when we consider the algebraic point of view). This is the basis for the use of global transcendental methods in the study of algebraic functions. From a quantitative point of view the simplest question we can ask is: how many linearly independent global meromorphic functions are there with a prescribed set of poles (of given multiplicities)? Thus if $P_1, \ldots, P_r$ are distinct points on the Riemann surface and $n_1, \ldots, n_r$ are positive integers we consider all meromorphic functions φ having at each P_i a pole of order $\leq n_i$ (and no other poles). Putting formally $D = \sum_{i=1}^{r} n_i P_i$ and denoting by $H(D)$ the space of all φ as above we get, by Liouville's theorem

$$h(D) = \dim\ H(D) \leq 1 + d$$

where $d = \deg\ D = \sum_{i=1}^{r} n_i$. If the Riemann surface is the 2-sphere, so that we are dealing just with rational functions of one variable, then we have equality. But if

the Riemann surface has *genus* $g \geq 1$ we get instead the *Riemann–Roch formula* $h(D) = 1 + d - (g - i(D))$. Here $i(D)$, the "index of specialty" of D, denotes the number of (independent) holomorphic differentials which vanish on D (i.e., have zeros at P_i of order $\geq n_i$): thus $0 \leq i(D) \leq g$. Now the total number of zeros of a holomorphic differential is always $2g - 2$ and so $i(D) = 0$ if deg $D > 2g - 2$. Thus if the number of poles is large the Riemann–Roch formula gives $h(D) = 1 + d - g$. Note also that we can allow the integers n_i to be negative provided we interpret a pole of negative order as a zero in the obvious way.

Remark. For a meromorphic differential ω Cauchy's theorem implies that $\sum_P \mathrm{Res}_P \omega = 0$ (since the Riemann surface is closed and so has no boundary). This gives restrictions on the principal parts of φ and Riemann–Roch asserts that these are the only restrictions.

The only numerical invariant of a Riemann surface is its genus and it appears in many guises. Thus

(1) g = number of "handles" = $\frac{1}{2}$ first Betti number
(2) g = number of independent holomorphic differentials
(3) $2g - 2$ = number of zeros - poles of a meromorphic differential
(4) $1 - g$ = constant term in the (linear) polynomial in m given by $h(mD)$ for m large: this is called the *arithmetic genus*.

1.2. *Algebraic surfaces*

For a non-singular algebraic surface we have more invariants, and the genus of a curve generalizes in several different directions. Thus we have

(1) Betti numbers: $B_1 = B_3$ and B_2
(2) Number of independent holomorphic 1-forms: g_1
Number of independent holomorphic 2-forms: g_2
(3) The divisor class of zeros - poles of a meromorphic 2-form: $-C_1$.

Note that $-C_1$ is a class of curves and so, by taking its self-intersection, we obtain a numerical invariant denoted by C_1^2. More generally, if D is a given curve we can define an intersection number $C_1 \cdot D$.

Suppose now $D = \sum n_i P_i$ is a divisor, i.e., a formal linear combination of irreducible curves P_i, and denote by $H(D)$ the space of meromorphic functions φ with poles along P_i of order $\leq n_i$. Then the Italian algebraic geometers proved a weak form of Riemann–Roch, namely

$$h(D) = \dim\ H(D) \geq \frac{D^2}{2} + \frac{C_1 D}{2} + \frac{C_1^2 + C_2}{12} - i(D)$$

where D^2 is the self-intersection number of D, $C_2 = 2 - 2B_1 + B_2$ is the Euler–Poincaré characteristic, and the index of specialty $i(D)$ is the number of holomorphic

2-forms vanishing on D. The difference between the left and right hand sides of this inequality was termed the *super-abundance* of D and remained a mystery for a long time. Moreover, if D was say a plane section (assuming the surface in P_3) then $i(mD)$ and the super-abundance of mD vanished for large m, $h(mD)$ was then a polynomial (of degree 2) in m with constant term $\frac{C_1^2+C_2}{12}$. Generalizing property (4) of the genus of a curve it was known that this "arithmetic genus" was given by

$$1 - g_1 + g_2 = \frac{C_1^2 + C_2}{12}\,.$$

1.3. *Higher dimensions*

For algebraic varieties of higher dimension the first problem is to provide the appropriate generalization of the known invariants for curves and surfaces. One main step was taken by J. A. Todd [12] in the mid-thirties (and independently by Eger) by showing that one could define for each i a canonical subvariety C_i of (complex) co-dimension i (unique up to an appropriate equivalence) generalizing C_1 and C_2 for surfaces. It was claimed by Severi that the number $\sum(-1)^i g_i$ should be expressible in terms of the C_i and Todd found explicitly the polynomials that were needed for the first few dimensions. Thus

$$\sum_{i=0}^{n}(-1)^i g_i = T_n(C_1, \ldots, C_n)$$

where T_n was a polynomial of weight n in the C_i. These polynomials now known as the Todd polynomials begin as follows:

$$T_1 = \tfrac{1}{2}C_1 \qquad T_2 = \tfrac{C_1^2+C_2}{12}\,, \qquad T_3 = \tfrac{C_1C_2}{24}$$
$$T_4 = \tfrac{1}{720}(-C_4 + C_3C_1 + 3C_2^2 + 4C_2C_1^2 - C_1^4)\,.$$

Moreover, it was also conjectured by Severi that $\sum(-1)^i g_i$ should again coincide with the arithmetic genus, i.e., the constant term in $h(mD)$ for m large and D a hyperplane section.

The other major step in the thirties was the development by Hodge [11] of the theory of harmonic forms. In particular Hodge introduced numerical invariants $h^{p,q}$ which refined the Betti numbers and were related to holomorphic differentials. Precisely

$$B_i = \sum_{p+q=i} h^{p,q} \qquad g_p = h^{p,0}$$

and $h^{p,q} = h^{q,p} = h^{n-p,\,n-q}$. An important formula due to Hodge was the *signature formula*. If the complex dimension n is even (e.g. for a surface) the middle homology group has a quadratic form given by intersection: the signature τ is defined as $p-q$ where p, q are the number of $+$ and $-$ signs in a diagonalization of the quadratic form. Hodge's signature theorem asserts that

$$\tau = \sum_{p,q}(-1)^q h^{p,q}\,.$$

The formal similarity between this alternating sum, that for the arithmetic genus and the Euler characteristic

$$C_n = \sum_{p,q} (-1)^{p+q} h^{p,q}$$

is very striking. The full implications of this similarity were not however understood until much later.

So much for the new numerical invariants in higher dimensions. The Riemann–Roch theorem however did not seem to generalize even as an inequality for dimensions ≥ 3, although one still knew (from the work of Hilbert) that $h(mD)$ was a polynomial in m of degree n provided D was a hyperplane section and m was large.

2. Sheaf Theory

In the post war years powerful new methods were introduced into algebraic geometry. The theory of sheaves developed by J. Leray and applied by H. Cartan and J -P. Serre to complex analysis led to rapid developments, notably the work of Kodaira–Spencer and of Hirzebruch [10]. In addition to sheaf theory many other techniques and ideas were taken over from algebraic topology. One of the simplest and most useful concepts was that of a *vector bundle*. A vector bundle over a space X is roughly speaking a continuous family of vector spaces parametrized by the points of X. If X is a differentiable manifold we can consider differentiable bundles, where the family varies differentiably while if X is a complex manifold we can consider holomorphic bundles where the family varies holomorphically. In differential geometry vector bundles arise naturally by taking for example the family of tangent spaces or more generally the tensor spaces of various types. A section S of a vector bundle E is a function $x \longmapsto s(x) \in E_x$ which is continuous, differentiable or holomorphic as the case may be. For the tangent vector bundle a section is a tangent vector field, for the bundle of skew-symmetric covariant p-tensors it is an exterior differential p-form.

Chern showed that a complex vector bundle E over a space X had certain invariantly associated cohomology classes $C_i(E) \in H^{2i}(X)$. If X is an algebraic variety and E its tangent bundle these are dual to the classes defined by Todd. These classes are now called the *Chern classes* of E. For real vector bundles similar invariants had earlier been introduced by Whitney and Pontrjagin.

For a holomorphic vector bundle E over a compact complex manifold X the space $H^0(X, E)$ of holomorphic sections is *finite-dimensional*. For example, if E is the bundle Ω^p of p-forms then dim $H^0(X, \Omega^p) = g_p$ in our previous notation.

Vector bundles of dimension one – also called line-bundles – turn up in algebraic geometry in quite another direction. Over the projective space $P_{n-1}(\mathbb{C})$ we have a natural line-bundle obtained by associating to each point $x \in P_{n-1}(\mathbb{C})$ the line L_x in $\mathbb{C}^n$ defined by x. In fact, it is better to work with the dual L_x^* and these give a bundle H over $P_{n-1}(\mathbb{C})$. The important point is that the holomorphic sections of H are precisely the linear forms on $\mathbb{C}^n$. More generally the homogeneous

polynomials on $\mathbb{C}^n$ of degree k are the holomorphic sections of the line-bundle $H^k = H \otimes H \otimes \cdots \otimes H$ (k times). If now $X \subset P_{n-1}$ is an algebraic subvariety we get a line-bundle H_X by restriction and if D is a hyperplane section then

$$H(D) \cong H^0(X, H_X)\,.$$

More generally, given any divisor D we can associate to it a line-bundle L so that

$$H(D) \cong H^0(X, L)$$

and moreover $C_1(L)$ is the class dual to the $(2n-2)$-cycle defined by D.

We are now ready to import sheaf cohomology. Not only can we define the groups $H^0(X, L)$ but there are also *cohomology groups* $H^q(X, L)$ for $q = 0, 1, \ldots, n$, and the same holds for holomorphic vector bundles of any dimension. For example, taking the bundle Ω^p of p-forms it turns out that the Hodge numbers can be identified as

$$h^{p,q} = \dim\ H^q(X, \Omega^p)\,.$$

In view of the symmetry $h^{p,q} = h^{q,p}$ we see that

$$\sum_{i=0}^{n} (-1)^i g_i = \sum_{i=0}^{n} (-1)^i \dim\ H^i(X, \Omega^0) = \chi(X, \Omega^0)$$

is a holomorphic Euler characteristic (Ω^0 denotes here the trivial line-bundle — whose sections are just functions on X). Euler characteristics are well-known to have better properties than dimensions of individual cohomology groups so it is quite reasonable to expect a formula for $\chi(X, L)$ in terms of $C_1(L)$ and the Chern classes† $C_i(X)$. For a curve X it is clear from the classical Riemann–Roch that we have such a formula provided

$$\dim\ H^1(X, L) = i(D) = \dim\ H^0(X, \Omega^1 \otimes L^*)\,.$$

This is, in fact, a special case of the Serre duality theorem which asserts that

$$H^q(X, E) \text{ and } H^{n-q}(X, \Omega^n \otimes E^*)$$

are canonically dual for any holomorphic vector bundle E.

For a surface we then see that $H^1(X, L)$ should be the mysterious superabundance. For dimension ≥ 3 it is also clear why even an inequality is not available in classical terms because both H^1 and H^2 enter and have opposite signs.

With these preliminaries we can now formulate the Hirzebruch form of the Riemann–Roch theorem for line-bundles:

$$\chi(X, L) = \frac{C_1(L)^n}{n!} + \frac{T_1 \cdot C_1(L)^{n-1}}{(n-1)!} + \frac{T_2 \cdot C_1(L)^{n-2}}{(n-2)!} + \cdots + T_n$$

†The standard convention is to write $C_i(X)$ for the Chern classes of the tangent bundle of X.

where the T_i are the Todd polynomials in the Chern classes of X. Moreover, Hirzebruch gave a generating function for the Todd polynomials as follows: we put

$$\prod_{i=1}^{n} \frac{x_i}{1-e^{-x_i}} = 1 + T_1 + T_2 + \cdots + T_n$$

where the T_i are polynomials in the elementary symmetric functions C_j of $x_1, \ldots, x_n$.

Not only does this give a complete generalization of the classical Riemann–Roch but it also disposes of the Severi–Todd conjectures concerning the arithmetic genus. In fact, taking L to be the trivial bundle, $C_1(L) = 0$ and we get

$$\sum (-1)^i g_i = \chi(X, \Omega^0) = T_n\,.$$

Moreover, if $L = H_X^m$ then Kodaira (and also Cartan–Serre) proved that $H^q(X, L) = 0$ for $q \geq 1$ and m large. On the other hand

$$\chi(X, H_X^m) = \sum_{j=0}^{n} \frac{T_j C^{n-j}}{j!}\,.\,m^{n-j}$$

where $C = C_1(H_X)$ is the class of a hyperplane section. This identifies the Hilbert polynomial in m and shows that the constant term is in fact T_n.

More generally still Hirzebruch [10] established a formula for $\chi(X, E)$ for any holomorphic vector bundle E. This may be written

$$\chi(X, E) = \sum_{j=0}^{n} \frac{\mathrm{ch}_{n-j}(E) \cdot T_j}{j!} \tag{HRR}$$

where $\mathrm{ch}\, E = \sum \mathrm{ch}_k(E)$ (the Chern character) is a polynomial in the Chern classes of E defined by the identities

$$\mathrm{ch}\, E = \sum_{i=1}^{q} e^{x_i} \qquad \sum_{j=0}^{q} C_j(E) = \prod_{i=1}^{q} (1 + x_i)\,.$$

Applying this formula to the vector bundles Ω^p, summing and using the Hodge signature theorem we end up with a formula for the signature (when $n = 2k$)

$$\tau = L_k(p_1, \ldots, p_k)$$

where the p_i are related to the Chern classes C_j of X by

$$\sum p_i = \prod (1 + x_j^2) \qquad \sum C_i = \prod (1 + x_j)$$

and the polynomials L_k are given by

$$\prod_{i=1}^{k} \frac{x_i}{\tanh x_i} = 1 + L_1 + \cdots + L_k$$

(that is, we express the L_i in terms of the elementary symmetric functions p_i of the x_j^2).

This Hirzebruch signature formula now makes sense for any oriented differentiable manifold of dimension $4k$ – not necessarily a complex manifold. In fact, Hirzebruch established this formula first using Thom's cobordism theory and used it as a step to the Riemann–Roch theorem.

One of the interesting consequences of HRR is that the Todd polynomial $T_n(C_1, \ldots, C_n)$ gives an integer even though it has large denominators. For example when $n = 2$, $T_2 = \frac{C_1^2+C_2}{12}$ and so we deduce that $C_1^2 + C_2 \equiv 0 \bmod 12$. Many interesting integrality theorems of this kind can be deduced from HRR and these have been very important in differential topology. For example, the exotic differentiable structures on spheres discovered by Milnor were detected using such integrality theorems and the relationship is quite deep. Although much mystery remains the topological significance of HRR became clearer later on as we shall see.

3. K-Theory

It might seem that HRR was the last word on the subject, but around 1957 Grothendieck introduced some revolutionary new ideas (see [7]). Part of Grothendieck's aim was to give a purely algebraic proof of HRR without transcendental methods. However his ideas go far beyond this and have had a very significant impact in the transcendental and topological domain.

Grothendieck starts by observing that $\chi(X, E)$ is an additive invariant of E. That is,

$$\chi(X, E) = \chi(X, E') + \chi(X, E'')$$

whenever E' is a sub-vector bundle of E and E'' is the quotient bundle. The right-hand side of HRR is also such an additive invariant. Grothendieck therefore conceived the idea of studying all such additive invariants, by introducing an abelian group $K(X)$ with one generator $[E]$ for every vector bundle on X and one relation $[E] = [E'] + [E'']$ for every exact sequence $0 \to E' \to E \to E'' \to 0$ (i.e., E' a subbundle and $E'' \cong E/E'$). Clearly the additive invariants of bundles with values in some abelian group A are just given by homomorphisms $K(X) \to A$. For example the Chern character ch E defined earlier is additive and so defines a homomorphism $\mathrm{ch}: \ K(X) \to H^*(X, \mathbf{Q})$. In fact, the tensor product of bundles turns $K(X)$ into a commutative ring and ch is even a ring homomorphism. Moreover, Grothendieck showed that $K(X)$ had many of the formal properties of cohomology and he was able to compute it for certain important spaces such as projective space.

For a (holomorphic) map $f : Y \to X$ of non-singular algebraic varieties Grothendieck was able to define a homomorphism $f_! : K(Y) \to K(X)$ which generalized the Euler characteristic χ. More precisely, if X is a point then $K(X) \cong \mathbb{Z}$ and $f_![E] = \chi(Y, E)$. His version of RR was then that the diagram

$$\begin{array}{ccc} K(Y) & \xrightarrow{\text{ch}} & H^*(Y,\mathbf{Q}) \\ \Big\downarrow f_! & & \Big\downarrow f_* \\ K(X) & \xrightarrow{\text{ch}} & H^*(X,\mathbf{Q}) \end{array}$$

commutes up to multipliers $\mathcal{T}_Y, \mathcal{T}_X$ (the total Todd classes $1+T_1+T_2+\cdots$ of Y and X respectively), where f_* denotes the map on cohomology given by using Poincaré duality. Thus for a vector bundle E on Y we have

$$\mathcal{T}_X \cdot \text{ch } f_![E] = f_*\{\mathcal{T}_Y \cdot \text{ch } E\}\,. \tag{GRR}$$

Clearly this reduces to HRR when X is a point.

The greater generality of GRR is not only more appealing but it also leads to a simpler and more natural proof. Thus to prove (HRR) we factor the map $Y \to$ point into $Y \xrightarrow{i} P \xrightarrow{\pi}$ point where P is projective space and i is some embedding: we then prove GRR for i and π separately and this implies it for the composition $\pi \circ i$. For π it is easy because $K(P)$ is completely known so the main step in the proof is the proof for the embedding i.

Almost at the same time there were startling new developments on the topological front. Bott discovered his famous periodicity theorems concerning the stable homotopy groups of the classical groups.† For $\mathrm{GL}(N,C)$ these assert

$$\pi_n(\mathrm{GL}(N,C)) \cong \pi_{n+2}(\mathrm{GL}(N,C)) \text{ for large } N$$

from which one deduces that $\pi_n \cong \mathbb{Z}$ for n odd and 0 for n even. Since a map $S^{2n-1} \to \mathrm{GL}(N,C)$ defines a complex N-dimensional vector bundle over S^{2n} Bott's result classifies vector bundles of large dimension over even spheres.

Shortly after all this I became interested in a topological problem‡ concerning complex projective spaces and I found it convenient to combine the work of Grothendieck and Bott. The results were so successful that it soon became apparent that a powerful topological tool was in the making. Hirzebruch and I therefore systematically transposed the formalism of Grothendieck to the topological context (see[2]). Thus we defined a new $K(X)$ using continuous vector bundles over any (compact) space X. It turned out that Bott's periodicity theorem was the basic building stone of the new theory and a substitute for the corresponding theorems of Grothendieck. For example, in both contexts, a basic result asserts that

$$K(X \times P) \cong K(X) \otimes K(P)$$

where P is a complex projective space.

Topological K-theory has by now justified itself by numerous important applications. Interestingly enough one of the most subtle and striking was the solution by J. F. Adams [1] of the vector-field problem on spheres. This is in essence the

†For a historical discussion see Bott′s Colloquium Lectures published in [8].
‡See[4].

real counterpart of the problem which started the whole thing off – though I must point out that the real case is considerably more delicate and used much more of the formal structure of K-theory.

In view of its mixed parentage it should surprise no one to learn that the integrality theorems derived from HRR were now explained in satisfactory terms by K-theory. Moreover, similar results, following from GRR, could now be proved in the topological context. All the same there was still more to follow as we shall see in the next lecture.

As a preparation for later on it is useful here to recall how the generator of the group $\pi_{2n-1}(\mathrm{GL}(N,C))$ can be constructed. Assume that we have $2n$ complex $N \times N$ matrices $A_1, \ldots, A_{2n}$ satisfying the Clifford identities

$$A_i^2 = -1, \quad A_i A_j = -A_j A_i \text{ for } i \neq j$$

so that $(\sum x_i A_i)^2 = -\sum x_1^2$. Thus for a unit vector $x = (x_1, \ldots, x_{2n})$ the matrix $A(x) = \sum x_i A_i$ is invertible and so $x \longmapsto A(x)$ defines a map $S^{2n-1} \to \mathrm{GL}(N,C)$. The Bott generator is given by taking N as small as possible: in fact, this value is 2^{n-1} and the matrices A_i are essentially unique. For $n = 1$, putting $J = A_1^{-1} A_2$ we recognize the usual map $x \longmapsto x_1 + Jx_2$ sending S^1 into the non-zero complex numbers. For $n = 2$ we get the unit quaternions mapped in $\mathrm{GL}(2,C)$.

Note that the Clifford matrices arose in the work of Dirac in which he sought to express the Laplace operator† as the square of a first order system:

$$-\sum \frac{\partial^2}{\partial x_i^2} = \left(\sum A_i \frac{\partial}{\partial x_i}\right)^2 .$$

4. The Index Theorem

In attempting to understand further one of the integrality theorems of Hirzebruch, Singer and I were led (in 1961) to rediscover the Dirac operator and its curved analogue on a Riemannian manifold. We then conjectured that the Hirzebruch integer in question should be expressible in terms of *harmonic spinors* in much the same way as the arithmetic genus was expressed in terms of holomorphic forms. While pondering this question our attention was drawn to some recent papers of Gel′fand [9] and his associates. In these papers the index problem was posed for general elliptic differential operators and some first steps were taken towards its solution.

A partial differential operator† $P = \sum_{|\alpha| \leq k} a_\alpha(x) \frac{\partial}{\partial x^\alpha}$ (with smooth coefficients $a_\alpha(x)$) is said to be elliptic (or order k) if $\sum_{|\alpha|=k} a_\alpha(x) \xi^\alpha \neq 0$ for $0 \neq \xi \in \mathbb{R}^n$. If

†Of course Dirac was concerned with the indefinite Lorentz metric but the algebra is the same.
†We use the usual abbreviated notation $\alpha = (\alpha_1, \ldots, \alpha_n), |\alpha| = \sum \alpha_i, \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots (\partial x_n)^{\alpha_n}$ and $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$.

P is acting on vector-valued functions so that the $a_\alpha(x)$ are square matrices the condition is that the symbol

$$\sigma_P(x,\xi) = \sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha$$

has non-zero determinant for $\xi \neq 0$. These definitions make sense on a manifold X and when vector-valued functions are generalized to sections of vector bundles. Thus in general P is a linear operator $C^\infty(X,E) \to C^\infty(X,F)$ (where C^∞ denotes the C^∞ sections). Elliptic operators share the main qualitative properties of the Laplace operator. In particular if X is compact the space of solutions of $Pu = 0$ has finite-dimension (say α) and the equation $Put = v$ can be solved provided v satisfies a number (say β) of linear relations. The index of P is the number $\alpha - \beta$. By introducing metrics and passing to the adjoint P^* this can be expressed

$$\text{index } P = \dim\,(\text{Ker } P) - \dim\,(\text{Ker } P^*)\,.$$

The most significant property of this index is that it is stable under perturbation and hence in particular it depends only on the highest order terms. It is therefore reasonable to expect a topological formula for it in terms of the geometrical data given by the symbol $\sigma_P(x,\xi)$. To interpret σ_P geometrically we must view ξ as a vector in the cotangent space T_x^*, and then $\sigma_P(x,\xi)$ is a linear map $E_x \to F_x$. Since σ is an isomorphism for $\xi \neq 0$ we can construct a vector bundle $V(\sigma)$ on the double $\sum X$ of the unit ball bundle BX of T^*X (using a Riemannian metric). Namely we lift E to one copy of BX, F to the other and identify E with F along the boundary SX using σ. The index of P depends only on $V(\sigma)$ and in fact the index extends to give a homomorphism

$$K(\Sigma X) \longrightarrow \mathbb{Z}\,.$$

This shows that K-theory is just the right tool to study the general index problem.

It is not hard to reformulate Riemann–Roch so that it appears as an index problem. For Riemann surfaces this is quite clear, we simply take the $\overline{\partial}$-operator. Its kernel consists of the holomorphic functions (or more generally of holomorphic sections) and the kernel of its adjoint can be identified with holomorphic differentials. Thus $h(D) - i(D)$ is just the index of the $\overline{\partial}$-operator of the line-bundle corresponding to the divisor D. In higher dimensions we have to replace $\overline{\partial}$ by $\overline{\partial} + \overline{\partial}^*$ (where $\overline{\partial}^*$ is the adjoint of $\overline{\partial}$) acting from forms of type (0, even) to forms of type (0, odd).

The Hirzebruch signature formula for an oriented (real) $4k$-manifold can also be viewed as an index formula for a suitable first-order system. Finally the Dirac operator leads to an index problem and this was precisely what Singer and I were studying.

We see, therefore, that there are numerous classical examples of first-order elliptic systems which give interesting index problems. Now it might seem that the general index problem for higher-order operators would be vastly more difficult

than for the classical first-order ones. In fact, this is not so: *a solution of the index problem for all classical operators implies a solution of all index problems*. This surprising and striking fact is essentially a consequence of Bott's periodicity theorem which shows that the generating bundle on S^{2n} is in fact given by the symbol of the Dirac operator as we saw in the last lecture. More precisely one shows that the group $K\left(\sum X\right)$ is (essentially) generated by the symbols of classical operators.†

With the whole machinery of K-theory at our disposal and with the formulae of Hirzebruch for various classical operators it was not difficult for Singer and me to guess what the general index formula ought to be. It can be written as follows:

$$\text{index } P = \sum_{k=0}^{n} T_k(X)\mathrm{ch}_{n-k}(V(\sigma_P))$$

where $T_k(X)$ denote the Todd polynomials of the complexified tangent bundle of X and n is the (real) dimension of X.

As I have explained it is only necessary to establish this formula for all classical operators. In our first proof [5] we followed Hirzebruch's procedure of using cobordism. The main point was to show there was some formula for index P involving characteristic classes of X and $V(\sigma)$ – the precise formula could then be found by computing various special cases. Very recently an alternative approach has been found [3] in which Riemannian geometry replaces cobordism.

There is also another quite different proof [6] which involves embedding X in R^n and transferring the given index problem on X to one on R^N (or rather on the N-sphere $R^N \cup \infty$). This is in the spirit of the proof of GRR, though of

Thus K-theory which was introduced to study a very special index problem, namely Riemann–Roch, turned out to be precisely the right tool for the general case. As I have mentioned this depends on Bott's periodicity theorem, itself closely related to Grothendieck's work on K-theory. In trying to unravel this story Bott and I were led to various new proofs of the periodicity theorem (see [8]) in which the index of certain operators played a key role. In fact, the deeper one digs the more one finds that K-theory and index theory are one and the same subject!

The various proofs of the index theorem have different merits and lead to generalizations in different directions. I cannot pursue these here but suffice it to say that there is plenty of life in the subject yet.

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This is a very small and somewhat arbitrary selection from the vast literature. For more references consult [10].

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†For X even-dimensional and orientable.

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Postscript (1996)

Developments related to index theory since 1973 are very extensive. A short list of (inter-linked) major topics includes:

1. C^*-algebras and associated K-theory as developed by Kasparov and others.
2. Non-commutative differential geometry developed by Connes.
3. Local versions in the context of Riemannian Geometry, developed in particular by Bismut.
4. Super-symmetry formalism, borrowed from theoretical physics, by Quillen and others.
5. Infinite-dimensional versions as exploited by physicists, chiefly Witten.
6. Elliptic cohomology and modular forms.

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SUR LES TRAVAUX DE STEPHEN SMALE

by
RENÉ THOM

Le premier travail scientifique de S. Smale est sa thèse de Ph.D. soutenue en 1956 à l'Université de Michigan (Ann Arbor). Faite sous la direction de Raoul Bott, elle témoigne déjà d'une éclatante maîtrise. Le résultat essentiel, maintenant bien connu, est le théorème du relèvement des homotopies des immersions d'une variété modulo une sous-variété. Etabli par des constructions géométriques raffinées, ce résultat témoignait chez son auteur de capacités d'intuition de tout premier ordre. Grâce à lui, on pouvait établir une conjecture — vieille alors d'une dizaine d'années — de C. Ehresmann sur la classification des immersions d'une variété dans une autre; il en résultait qu'il était possible, par une déformation régulière (c'est-à-dire sans sortir des immersions) de transformer le plongement canonique de la $\ll$ 2-sphère $\gg$ dans l'espace euclidien R^3 à trois dimensions en un plongement antipodique; ce résultat n'alla pas sans soulever la curiosité des topologues, dont beaucoup s'ingénièrent à préciser cette déformation. Mais, la thèse de Smale donnait plus que cette curiosité, elle ouvrait une voie d'attaque dans tout un domaine de questions jusqu'alors inabordables, et tout un chapitre de Topologie Différentielle, l'étude des immersions et plongements d'une variété différentiable dans une autre, marqué par les travaux de M. Hirsh, Haefliger, etc., en est plus ou moins directement sorti.

Avec les grands travaux de 1960 sur la conjecture de Poincaré, nous abordons la partie la plus connue de l'œuvre de Smale, celle qui, sans doute, nous vaut sa présence ici. On savait déjà, par suite de la théorie de Morse, que toute variété compacte se divise en cellules de gradient, et que, si l'on se donne à chaque niveau critique l'attachement de la cellule de gradient correspondante, on est en mesure de reconstituer la variété; on avait déjà commencé, — à la suite des travaux de Kervaire, Milnor, Wallace — à pratiquer la $\ll$ chirurgie $\gg$ des variétés, c'est-à-dire la technique qui consiste à transformer une variété plus simple par résection d'un couple d'anses duales.

L'énorme mérite de Smale, en cette question, est d'avoir osé entreprendre ce que tout autre mathématicien du temps aurait considéré comme sans espoir: étant donnée une fonction de Morse sur une sphère d'homotopie, simplifier la présentation de cette variété en éliminant par chirurgie les couples d'index k, $k+1$ de points

critiques excédentaires. Que cela fût possible, on connaissait trop les difficultés dans les petites dimensions (trois ou quatre), pour l'espérer; Smale osa, et réussit. I1 comprit que les difficultés entrevues étaient un phénomène spécial aux petites dimensions: en se bornant aux dimensions supérieures à cinq, on se trouvait plus à l'aise pour travailler, et la chirurgie s'effectuait plus aisément; par des constructions très ingénieuses, Smale vint alors à bout des dernières difficultés: il élimine sur une sphère d'homotopie tous les couples de points critiques d'indice k différent de zéro et n; il obtient ainsi une variété sur laquelle il existe une fonction ne présentant que deux points critiques (minimum et maximum). D'après un théorème de Reeb, cette variété est une sphère topologique (mais non nécessairement difféomorphe à la sphère usuelle, comme l'ont montré les examples dus à Milnor).

Ce résultat extrêmement brillant s'est trouvé complété peu après par le théorème dit du h-cobordisme, qui le généralise. Si deux variétés compactes M_1, et M_2 forment le bord d'une même variété à bord W, dont elles sont rétractées par déformation et si elles sont simplement connexes, alors M_1 et M_2 sont difféomorphes. Ce théorème perment, à l'aide d'un résultat ultérieur de Novikov et Browder, de ramener le problème de la classification des variétés différentiables à un pur problème d'homotopie, en fait à un problème d'algèbre (il est vrai, difficile). Les techniques usées dans la démonstration du théorème du h-cobordisme n'ont probablement pas donné tout leur fruit, et des travaux plus récents, comme ceux de J. Cerf, en ont élargi le champ d'application.

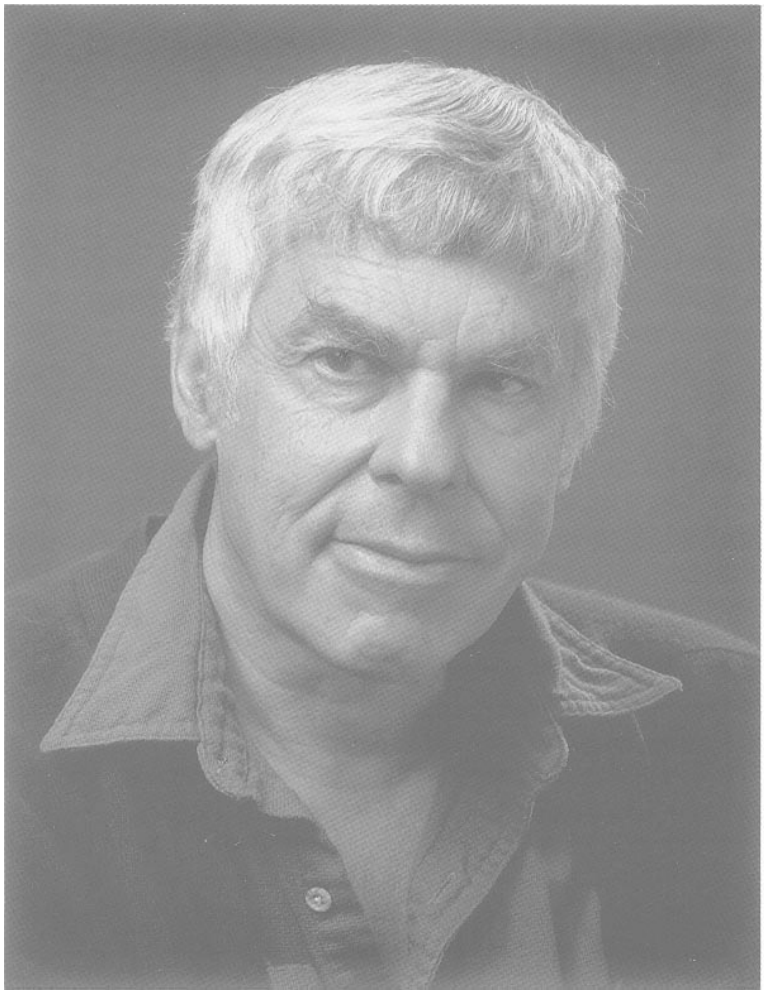
Avec le résultat contemporain de B. Mazur sur la conjecture de Schönflies, les travaux de Smale tournent une page en Topologie algébrique. On peut dire que la topologie des $\ll$ espaces $\gg$, des variétés différentiables est désormais quasi-achevée. I1 subsiste certes beaucoup de questions non résolues: les structures algébriques définies par la classification ne sont pas élucidées — en général —; mais le seront-elles un jour? I1 ne reste guère que la théorie — qui a fait d'ailleurs récemment de beaux progrès — de ces êtres malgré tout quelque peu pathologiques que sont les variétés semi-linéaires et les variétés purement topologiques. Dans ces conditions, si la Topologie veut se renouveler, et ne pas se cantonner en des problèmes ardus d'une vaine technicité, elle doit se préoccuper de renouveler ses matériaux et aborder des problèmes neufs. Avec les objets géométriques associés aux structures différentiables: formed différentielles, tenseurs, structures feuilletées, opérateurs différentiels, un champ immense est ouvert au topologue. On a vu d'ailleurs qu'un de nos lauréats s'est vu récompenser pour un résultat dans cette voie.

A côté de l'Analyse classique, essentiellement linéaire, il y a le domaine pratiquement inexploré de l'analyse non linéaire; là, le topologue peut espérer encore mieux utiliser ses méthodes, et peut-être sa qualité essentielle, à savoir la vision intrinsèque des choses. C'est ce que comprendra très vite Smale; dans un article en collaboration avec R. Palais, il définira les meilleures conditions possibles d'application de la théories de Morse au Calcul des variations; il en déduira ensuite des théorèmes relatifs à l'existence des solutions de problèmes elliptiques non

linéaires. Mais, très tôt, Smale se tourne vers une théorie — alors bien délaissée — : la théorie qualitative des systèmes différentiels sur une variété. Quasiment seul, Smale lit Poincaré et Birkhoff; devant l'inextricable du problème, il comprend très vite l'intérêt d'une notion essentielle, celle de $\ll$ stabilité structurelle introduite par Andronov et Pontrjagin, cette notion vise à caractériser, parmi les champs de vecteurs sur une variété, ceux qui jouissent d'une propriété de stabilité qualitative, au sens suivant: tout champ (Z) assez voisin du champ donné (X) (avec la C^1-topologie) donne naissance à un champ de trajectoires homéomorphe au champ défini par (X). Le problème central est alors le problème de l'approximation: tout champ de vecteurs peut-il être approché par un champ structurellement stable? Ce problème, résolu positivement par Peixoto pour les variétés compactes de dimension inférieure à deux, était posé pour les dimensions supérieures. Smale construit alors une variété compacte M de dimension quatre, et un champ (X) sur M, tel qu'aucun champ (Z) assez voisin de (X) ne soit structurellement stable. Le problème général de la stabilité des systèmes différentiels est ainsi résolu par la négative. Cependent, la notion même de stabilité structurelle est loin d'avoir perdu tout son intérêt: d'abord, parce qu'il existe, dans l'espace fonctionnel des champs de vecteurs d'une variété M, un ouvert $\ll$ relativement important $\gg$ de champs structurellement stables: celui formé par les champs de vecteurs de type gradient génériques (sans récurrence) et, probablement, une classe de champs définis par Smale (les champs dits de Morse-Smale), qui présentent de la récurrence (avec des trajectoires fermées) mais sous une forme bénigne et sévèrement contrôlée. Mais, par l'étude des configurations de trajectoires associées aux points homocliniques de Poincaré, Smale se convainc bien vite que d'autres champs, à toplogie complexe et rigide, sont structurellement stables. I1 revenait aux brillants travaux de l'école soviétique (avec Sinai, Arnold, Anosov) d'établir l'existence d'une classe étendue de champs structurellement stables, du type du flot géodésique sur une variété riemannienne à courbure négative. Ces travaux ont exercé sur Smale une grande influence, et ont infléchi ses recherches dans la direction actuelle, à savoir la mise en évidence d'une $\ll$ stabilité structurelle par morceaux $\gg$, chaque $\ll$ morceau $\gg$ étant lié à une configuration rigide de trajectoires récurrentes (non-wandering) au sens de Birkhoff. Ces recherches sont en cours et semblent fort prometteuses.

Je m'en voudrais de ne pas insister sur un dernier point: si les œuvres de Smale ne possèdent peut-être pas la perfection formelle du travail définitif, c'est que Smale est un pionnier qui prend ses risques avec un courage tranquille; dans un domaine complètement inexploré, dans une jungle géométrique d'une inextricable richesse, il est le premier à avoir tracé la route et posé les premiers jalons. Et l'on peut prévoir que son œuvre revêtira à l'avenir une importance fondamentale, comparable à celle des grands précurseurs, Poincaré et Birkhoff. Après tout bien des problèmes classiques, tels le problème de Fermat ou la conjectures de Riemann, peuvent attendre leur solution encore quelques années; mais si la science veut user de l'outil différentiel pour décrire les phénomènes naturels, elle ne peut se permettre d'ignorer encore longtemps la structure topologique des attracteurs d'un système

dynamique structurellement stable, car tout ≪ état physique ≫ présentant une certaine stabilité, une certaine permanence, est nécessairement représenté par un tel attracteur. Selon certaines vues de Smale, ces attracteurs seraient des espaces homogènes de groupes de Lie d'un type spécial. Si ces vues pouvaient s'étendre aux systèmes hamiltoniens, on pourrait peut-être s'expliquer l'apparition — jusqu'ici si incomprise — des groupes de Lie dans la Physique des particules élémentaires. En ce sens, le problème de Smale est — à mes yeux — d'une importance épistomologique essentielle.



Stephen Smale

STEPHEN SMALE

1930	Born July 15 at Flint, Michigan
1952	B.S., University of Michigan
1953	M.S., University of Michigan
1957	Ph.D., University of Michigan
1956–68	Instructor, University of Chicago
1958–60	Member, Institute for Advanced Study, Princeton
1962	Visiting Professor, College de France, Paris (Spring)
1960–61	Associate Professor of Mathematics, University of California, Berkeley
1961–64	Professor of Mathematics, Columbia University
1964–present	Professor of Mathematics, University of California, Berkeley
1960–62	Alfred P. Sloan Research Fellow
1966	Member, Institute for Advanced Study, Princeton (Fall)
1967–68	Research Professor, Miller Institute for Basic Research in Science, Berkeley
1969–70	Visiting Member, Institut des Hautes Études Scientifiques (Fall)
1972–73	Visiting Member, Institut des Hautes Études Scientifiques (Fall)
1972–73	Visiting Professor, University of Paris, Orsay (Fall)
1974	Visiting Professor, Yale University (Fall)
1976	Visiting Professor, Instituto de Matematica Pura e Aplicada, Rio de Janeiro (April)
1976	Visiting Professor, Institut des Hautes Études Scientifiques (May, June)
1976–present	Professor without stipend, Department of Economics, University of California, Berkeley
1979–80	Research Professor, Miller Institute for Basic Research in Science, Berkeley
1987	Visiting Scientist, IBM Corporation, Yorktown Heights (Fall)
1987	Visiting Professor, Columbia University (Fall)

Honors

1960–62	Alfred P. Sloan Research Fellowship
1964–present	Foreign Membership, Brazilian Academy of Science
1965	Veblen Prize for Geometry, American Mathematical Society
1966	Fields Medal, International Mathematical Union
1967	University of Michigan Sesquicentennial Award

1967–present	Membership, American Academy of Arts and Sciences
1967–68	Research Professorship, Miller Institute for Basic Research in Science, Berkeley
1970–present	Membership, National Academy of Sciences
1972	Colloquium Lecturership, American Mathematical Society
1974	Honorary D.Sc. University of Warwick, England
1979–80	Research Professorship, Miller Institute for Basic Research in Science, Berkeley
1983–present	Election as Fellow, The Econometric Society
1983	Faculty Research Lecturership, University of California, Berkeley
1987	Honorary D.Sc., Queens University, Kingston, Ontario
1988	Chauvenet Prize, Mathematical Association of America
1989	Von Neumann Award, Society for Industrial and Applied Mathematics
1990	Honorary Membership in Instituto de Mathematica, Pura e Aplicada (IMPA), Rio de Janeiro
1990	Research Professorship, Miller Institute for basic Research in Science, Berkeley
1991–92	Trinity Mathematical Society, Dublin, Honorary Member Bishop Berkeley Lecture, Trinity College Dublin Distinguished Lecture, Fields Institute

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A SURVEY OF SOME RECENT DEVELOPMENTS IN DIFFERENTIAL TOPOLOGY

by
S. SMALE

1. We consider differential topology to be the study of differentiable manifolds and differentiable maps. Then, naturally, manifolds are considered equivalent if they are diffeomorphic, i.e., there exists a differentiable map from one to the other with a differentiable inverse. For convenience, differentiable means C^∞; in the problems we consider, C' would serve as well.

The notions of differentiable manifold and diffeomorphism go back to Poincaré at least. In his well-known paper, *Analysis situs* [**27**] (see pp. 196–198), topology or analysis situs for Poincaré was the study of differentiable manifolds under the equivalence relation of diffeomorphism. Poincaré used the word homeomorphism to mean what is called today a diffeomorphism (of class C'). Thus differential topology is just topology as Poincaré originally understood it.

Of course, the subject has developed considerably since Poincaré; Whitney and Pontrjagin making some of the major contributions prior to the last decade.

Slightly after Poincaré's definition of differentiable manifold, the study of manifolds from the combinatorial point of view was also initiated by Poincaré, and again this subject has been developing up to the present. Contributions here were made by Newman, Alexander, Lefschetz and J. H. C. Whitehead, among others.

What started these subjects? First, it is clear that differential geometry, analysis and physics prompted the early development of differential topology (it is this that explains our admitted bias toward differential topology, that it lies close to the main stream of mathematics). On the other hand, the combinatorial approach to manifolds was started because it was believed that these means would afford a useful attack on the differentiable case. For example, Lefschetz wrote [**13**, p. 361], that Poincaré tried to develop the subject on strictly "analytical" lines and after his *Analysis situs*, turned to combinatorial methods because this approach failed for example in his duality theorem.

Naturally enough, mathematicians have been trying to relate these two viewpoints that have developed side by side. S. S. Cairns is an example of such.

An address delivered before the Stillwater meeting of the Society on August 31, 1961, by invitation of the Committee to Select Hour Speakers for Western Sectional Meetings; received by the editors November 28, 1962.

In the last decade, the three domains, differential topology, combinatorial study of manifolds, and the relations between the two, have all advanced enormously. Of course, these developments are not isolated from each other. However, we would like to make the following important point.

It has turned out that the main theorems in differential topology did not depend on developments in combinatorial topology. In fact, the contrary is the case; the main theorems in differential topology inspired corresponding ones in combinatorial topology, or else have no combinatorial counterpart as yet (but there are also combinatorial theorems whose differentiable analogues are false).

Certainly, the problems of combinatorial manifolds and the relationships between combinatorial and differentiable manifolds are legitimate problems in their own right. An example is the question of existence and uniqueness of differentiable structures on a combinatorial manifold. However, we don't believe such problems are the goal of differential topology itself. This view seems justified by the fact that today one can substantially develop differential topology most simply without any reference to the combinatorial manifolds.

We have not mentioned the large branch of topology called homotopy theory until now. Homotopy theory originated as an attack on the homeomorphism or diffeomorphism problem, witness the "Poincaré Conjecture" that the homotopy groups characterize the homeomorphism type of the 3-sphere, and the Hurewicz conjecture that the homeomorphism type of a closed manifold is determined by the homotopy type. One could attack the homotopy problem more easily than the homeomorphism one and, for many years, most of the progress in topology centered around the homotopy problem.

The Hurewicz conjecture turned out not to be true, but amazingly enough, as we shall see, the last few years have brought about a reduction of a large part of differential topology to homotopy theory. These problems do not belong so much to the realm of pure homotopy theory as to a special kind of homotopy theory connected with vector space bundles and the like, as exemplified by work around the Bott periodicity theorems.

Of course, there are a number of important problems left in differential topology that do not reduce in any sense to homotopy theory and topologists can never rest until these are settled. But, on the other hand, it seems that differential topology has reached such a satisfactory stage that, for it to continue its exciting pace, it must look toward the problems of analysis, the sources that led Poincaré to its early development.

We here survey some developments of the last decade in differential topology itself. Certainly, we make no claims for completeness. A notable omission is the work of Thom, on cobordism, and the study of differentiable maps. The reader is referred to expositions of Milnor [**21**] and H. Levine [**14**] for accounts of part of this work.

2. We now discuss what must be considered a fundamental problem of differential topology, namely, the diffeomorphism classification of manifolds.

The classification of closed orientable 2-manifolds goes back to Riemann's time. The next progress on this problem was the development of numerical and algebraic invariants which were able to distinguish many nondiffeomorphic manifolds. These invariants include, among others, the dimension, betti numbers, homology and homotopy groups and characteristic classes.

For dimension greater than two, there was still (at the beginning of 1960) no known case where the existing numerical and algebraic invariants determined the diffeomorphism class of the manifold. The simplest case of this problem (or so it appeared) was that posed by Poincaré: Is a 3-manifold which is closed and simply connected, homeomorphic (equivalently diffeomorphic) to the 3-sphere? This has never been answered.

The surprising thing is, however, that without resolving this problem, the author showed that in many cases, the known numerical and algebraic invariants were sufficient to characterize the diffeomorphism class of a manifold. Generally speaking in fact, considerable information on the structure of manifolds was found. We will now give an account of this.

To see how manifolds can be constructed, one defines the notion of *attaching a handle*. Let M^n be a compact manifold with boundary ∂M (we remind the reader that everything is considered from the C^∞ point of view, manifolds, imbeddings, etc.) and let D^s be the s-disk (i.e., the unit disk of Euclidean s-space E^0). Suppose $f : (\partial D^s) \times D^{n-s} \to M$ is an imbedding. Then $X(M; f; s)$, "M with a handle attached by f_0^n is defined by identifying points under f and imposing a differentiable structure on $M \cup_f D^s \times D^{n-s}$ by a process called "straightening the angle." Similarly if $f_i : (\partial D_1^s) \times D_1^{n-s} \to M$, $i = 1, \ldots, k$, are imbeddings with disjoint images one can define $X(M; f_1, \ldots, f_k; s)$. If M itself is a disk, then $X(M; f_1, \ldots, f_k; s)$ is called a *handlebody*.

(2.1) Theorem. *Let f be a C^∞ function on a closed (i.e. compact with empty boundary) manifold with no critical points on $f^{-1}[-\epsilon, \epsilon]$ except k nondegenerate ones on $f^{-1}(0)$, all of index s. Then $f^{-1}[-\infty, \epsilon]$ is diffeomorphic to $X(f^{-1}[-\infty, -\epsilon]; f_1, \cdots, f_k; s)$ (for suitable f_i).*

(For a reference to the notion of nondegenerate critical point and its index, see e.g. [**1**].) This might well be regarded as the basic theorem of finite dimensional Morse theory. Morse [**23**] was concerned with the homology version of this theorem, Bott [**1**], the homotopy version of 2.1. The proof of 2.1 itself is based on the ideas of the proofs of the weaker statements.

(2.2) Theorem. (Morse–Thom). *On every closed manifold W, there exists a C^∞ function with nondegenerate critical points.*

For a proof see [**41**].

By combining 2.1 and 2.2 we see that every manifold can be obtained by attaching handles successively to a disk (we have been restricting ourselves to the compact, empty boundary case only for simplicity).

The main idea of the following theory is to remove superfluous handles (or equivalently critical points) without changing the diffeomorphism type of the manifold. For this one starts (after 2.2) with a "nice function," a function on M given by 2.2 with the additional property that the handles are attached in order according to their dimension (the s in $D^s \times D^{n-s}$).

(2.3) Theorem. *On every closed manifold there exists a function f with nondegenerate critical points such that at each critical point, the value of f is the index.*

This was proved by A. Wallace [**44**] and the author [**38**] independently by different methods. (For a general references to this section see [**34**; **37**].)

The actual removing of the extra handles is the main part and for this one needs extra assumptions. The next is the central theorem (or its generalization "2.4′" to include manifold with boundary).

(2.4) Theorem. *Let M be a simply connected closed manifold of dimension > 5. Then on M there is a nondegenerate (and nice) function with the minimal number of critical points consistent with the homology structure of M.*

We make more explicit the conclusion of 2.4. Let $\sigma_{i1}, \cdots, \sigma_{ip(i)}, \tau_{i1}, \cdots, \tau_{ig(i)}$, $0 \leqq i \leqq n$, be a set of generators for a direct sum decomposition of $H_i(M)$, σ_{ij} free, τ_{ij} of finite order. Then one can obtain the function of 2.4 with type numbers M_i (the number of critical points of index i) satisfying $M_i = p(i) + g(i) + g(i-1)$.

The first special case of 2.4 is

(2.5) Theorem. *Let M be a simply connected closed manifold of dimension greater than 5 with no torsion in the homology of M. Then there exists a nondegenerate function on M with type numbers equal to the betti numbers of M.*

We should emphasize that 2.4 and 2.5 should be interpreted from the point of view of 2.1. One may apply 2.5 to the case of a "homotopy sphere" (using 2.1 of course).

(2.6) Theorem. *Let M^n be a simply connected closed manifold with the homology groups of a sphere, $n > 5$. Then M can be obtained by "gluing" two copies of the n-disk by a diffeomorphism from the boundary of one to the boundary of the other.*

For $n = 5$, the theorem is true and can be proved by an additional argument.

Theorem 2.6 implies the weaker statement ("the generalized Poincaré conjecture in higher dimensions") that a homotopy sphere of dimension $\geqq 5$ is homeomorphic to S^n, see [**33**]. Subsequently, Stallings [**39**] and Zeeman [**50**] found a proof of this last statement.

Theorem 2.4 was developed through the papers [**34; 35**] and appears in the above form in [**37**]. Rather than try to give an idea of the proof of 2.4, we refer the reader to these papers. One may also refer to [**2; 3**] and [**15**].

The analogue of 2.4, say $2.4'$ is also proved for manifolds with boundary [**37**] and this analogue implies the h-cobordism theorem stated below. The simplest case of $2.4'$ is

(2.7) Disk Theorem. *Let M^n be a contractible compact manifold, $n > 5$, ∂M connected and simply connected. Then M^n is diffeomorphic to the disk D^n.*

We now discuss another aspect of the preceding theory, the relationship between diffeomorphism and h-cobordism. Two oriented, closed manifolds M_1, M_2 are *cobordant* if there exists a compact oriented manifold W with $\partial W = M_1 - M_2$ (taking into account orientations). Thom in [**40**] reduced the cobordism classification of manifolds to a problem in homotopy theory which has since been solved.

Two closed oriented manifolds M_1, M_2 are h-cobordant (following Milnor [**18**] if one can choose W as above so that the inclusions $M_i \to W$, $i = 1, 2$ are homotopy equivalences. The idea of h-cobordism (homotopy-cobordism) is to replace diffeomorphism by a notion of equivalence, *a priori* weaker than diffeomorphism and amenable to study using homotopy and cobordism theory.

The following theorem was prove in [**37**], with special cases in [**34; 35**]. It is also a consequence of $2.4'$.

(2.8) The h-Cobordism Theorem. *Let M_1^n, M_2^n be closed oriented simply connected manifolds, $n > 4$, which are h-cobordant. The M_1 and M_2 are diffeomorphic.*

Milnor [**20**] has shown that 2.7 is false for nonsimply connected manifolds. On the other hand Mazur [**16**] has generalized 2.7 to a theorem which includes the nonsimply connected manifolds.

Theorem 2.8 reduces the diffeomorphism problem for a large class of manifolds to the h-cobordism problem. This h-cobordism problem has been put into quite good shape for homotopy spheres by Milnor [**19**], Kervaire and Milnor [**12**], and recently Novikov [**24**] has found a general theorem.

Kervaire and Milnor show that homotopy spheres of dimension n, with equivalence defined by h-cobordism form an abelian group $\mathcal{H}^n$. Their main theorem is the following.

(2.9) Theorem. *$\mathcal{H}^n$ is finite, $n \neq 3$.*

Kervaire and Milnor go on to find much information about the structure of this finite abelian group, in particular, to find its order for $4 \leqq n \leqq 18$. The main technique in the proof of 2.9 is what is called spherical modification or surgery, see [**44; 22**].

Putting together 2.8 and 2.9 one obtains a classification of the simplest type of closed manifolds, the homotopy spheres, of dimension n for $5 \leqq n \leqq 18$ and for general n the finiteness theorem. This also gives theorems on differentiable structures on spheres. See [**35**] or [**12**].

Most recently, Novikov has found a very general theorem on the h-cobordism structure of manifolds (and hence by 2.8, the diffeomorphism structure). We refer the reader to [**24**] for a brief account of this.

Lastly we mention some specific results on the manifold classifaction problem [**36; 37**].

(2.10) Theorem. (a) *There is a* **1–1** *correspondence between simply-connected closed* 5*-manifolds with vanishing* 2*nd Stiefel–Whitney class and finitely generated abelian groups, the correspondence given by $M \to$ Free part $H_2(M)+\frac{1}{2}$ Torsion part $H_2(M)$.*

(b) *Every* 2*-connected* 6*-manifold is diffeomorphic to S^6 or a sum* (*in the sense of* [**18**]) *of r copies of $S^3 \times S^3$, r a positive integer.*

C. T. C. Wall has very general theorems which extend the above results to $(n-1)$-connected $2n$-manifolds [**43**].

3. We give a substantial account of immersion theory because the main problem here has been completely reduced to homotopy theory.

An immersion of one mainifold M^k in a second X^n is a C^∞ map $f : M \to X$ with the property that for each $p \in M$; in some coordinate systems (and hence all) about p and $f(p)$, the Jacobian matrix of f has rank k. A regular homotopy is a homotopy $f_t : M \to X$, $0 \leqq t \leqq 1$ which for each t is an immersion and which has the additional property that the induced map $F_t : T_M \to T_X$ (the derivative) on the tangent spaces is continuous (on $T_M \times I$). One could obtain an equivalent theory by requiring in place of the last property that f_t be a differentiable map of $M \times I$ into X.

The fundamental problem of immersion theory is: given manifolds M and X, find the equivalence classes of immersions of M in X, equivalent under regular homotopy. This includes in particular the problem of whether M can be immersed in X at all. This general problem is in good shape. The complete answer has been given recently in terms of homotopy theory as we shall see.

The first theorem of this type was based on "general position" arguments and proved by Whitney [**46**] in 1936.

(3.1) Theorem. *Given manifolds M^k, X^n, any two immersions $f, g: M \to X$ which are homotopic are regularly homotopic if $n \geqq 2k+2$. If $n \geqq 2k$ there exists an immersion of M in X.*

Recent proofs of the second part of this theorem can be found in [**17**] and [**28**].

The first statement of 3.1 is equally true with $2k+2$ replaced by $2k+1$. Most of the theorems in this survey on the existence of immersions and imbeddings can be strengthened with an approximation property of some sort. Although these are important, for simplicity we omit them.

The first immersion theorem for which arguments transcending general position are needed was the Whitney–Graustein theorem [**45**] (proved in a paper by Whitney who gives much credit to Graustein). For an immersion $f: S^1 \to E^2$, S^1, E^2 oriented, the induced map on the tangent vectors yields a map of S^1 into S^1; the degree (an integer) of this map times 2π is called by Whitney the rotation number.

(3.2) Theorem (Whitney–Graustein). *Two immersions of S^1 in E^2 are regularly homotopic if and only if they have the same rotation number. There exists an immersion of S^1 in E^2 with prescribed rotation number of the form an integer times 2π.*

The next step in the theory of immersions was again taken by Whitney [**48**] in 1944 with the following theorem.

(3.3) Theorem. *Every k-manifold can be immersed in E^{2k-1} for $k > 1$.*

The proof of this is quite difficult and involved a careful analysis of the critical points of a differentiable mapping of M^k into E^{2k-1}. In this dimension, these critical points are isolated in a suitable approximation of a given map, but have to be removed to obtain an immersion. The difficulties in the study of singularities of differentiable maps have limited this method, although, very recently, Haefliger [**4**] has used effectively Whitney's ideas in the above proof, both in studying imbeddings and immersions. We shall say more about this later.

Some of the recent progress in imbeddings and immersions can be measured by Whitney's statement in the above paper. "It is a highly difficult problem to see if the imbedding and immersion theorems of the preceding and the present one can be improved upon." He goes on to ask if every open or orientable M^4 may be imbedded in E^7 and immersed in E^6. The complex projective plane cannot be immersed in E^6, but every open M^4 can be imbedded in E^7. See Hirsch [**8; 9**]. Whitney finally asks if every M^3 can be imbedded in E^5. Hirsch has proved this is so if M^3 is orientable [**10**].

The next progress in the subject of immersions occured in papers [**30; 31**] and [**32**] of the author in 1957–1959. The first generalized the Whitney–Graustein theorem for circles immersed in the plane to circles immersed in an arbitrary manifold,

and here methods were introduced which soon led to the solution of the general problem mentioned previously.

Consider "based" immersions of $S^1 = \{0 \leqq \theta \leqq 2\pi\}$ in a manifold X, that is those which map $\theta = 0$ into a fixed point x_0 of X and the positive unit tangent at $\theta = 0$ into a fixed tangent vector of X at x_0. To each based immersion of S^1 in X, the differential of the immersion associates an element of the fundamental group of T'_X, the unit tangent bundle of X.

(3.4) Theorem. *The above is a* **1–1** *correspondence between* (*based*) *regular homotopy classes of based immersions of* S^1 *in* X *and* $\pi_1(T'_X)$.

In the next paper [**31**] corresponding theorems are proved with S^1 replaced by S^2. A noteworthy special case of the theorem proved there is that any two immersions of S^2 in E^3 are regularly homotopic. It is a good mental exercise to check this for a reflection through a plane of S^2 in E^3 and the standard S^2 in E^3. This check has been carried through independently by A. Shapiro and N. H. Kuiper, unpublished.

In [**32**], the classification, under regular homotopy, of immersions of S^k in E^n is given (any k, n). This can be stated as follows.

If $f, g : S^k \to E^n$ are based immersions (i.e., at a fixed point x_0 of S^k, $f(x_0) = g(x_0)$ is prescribed and the derivatives of f and g at x_0 are prescribed and equal) one can define an invariant $\Omega(f, g) \in \pi_k(V_{n,k})$ where $\pi_k(V_{n,k})$ is the kth homotopy group of the Stiefel manifold of k-frames in E^n.

(3.5) Theorem. *Based immersions* $f, g : S^k \to E^n$ *are* (*based*) *regularly homotopic if and only if* $\Omega(f, g) = 0$. *Furthermore, given a based immersion* $f : S^k \to E^n$ *and* $\Omega_0 \in \pi_k(V_{n,k})$, *there is a based immersion* $g : S^k \to E^n$ *such that* $\Omega(f, g) = \Omega_0$.

The content of Theorem 3.2 is that the homotopy group $\pi_k(V_{n,k})$ classifies immersions of S^k in E^n. Information on the groups can be found in [**25**]. An application of this theorem is that immersions of S^k in E^{2k} are classified by the integers if k is even, the correspondence given by the intersection number.

R. Thom in [**42**] has given a rough exposition of the proof of the previous theorem, which contributes to the theory of conceptualizing part of the proof.

M. Hirsch in his thesis [**8**], using the results of [**32**], has generalized 3.5 to the case of immersions of an arbitrary manifold in an arbitrary manifold. If M^k and X^n are manifolds T_M, T_X their tangent bundles, a monomorphism $\phi : T_M \to T_X$ is fiber preserving map which is a vector space monomorphism on each fiber. For each immersion $f : M \to X$ the derivative is a monomorphism $\phi_f : T_M \to T_X$.

(3.6) Theorem. *If* $n > k$, *the map* $f \to \phi_f$ *induces a* **1–1** *correspondence between regular homotopy classes of immersions of* M *in* X *and* (*monomorphism*) *homotopy classes of monomorphisms of* T_M *into* T_X.

In this theorem one can replace homotopy classes of monomorphisms of T_M into T_X by equivariant homotopy classes (equivariant with respect to the action of $GL(k)$) of the associated k-frame bundles of T_M and T_X respectively. Still another interpretation is that the regular homotopy classes of immersions of M in X are in a **1–1** correspondence with homotopy classes of cross-sections of the bundle associated to the bundle of k-frames of M whose fiber is the bundle of k-frames of X. Recently Hirsch (unpublished) has established the theorem for the case $n = k$ provided M is not closed.

Theorem 3.6 includes as special cases all the previous theorems mentioned here on immersions and has the following consequences as well.

(3.7) Theorem. *If $n > k$ and M^k is immersible in E^{n+r} with a normal r-field, then it is immersible in E^n. Conversely, if M^k is immersible in E^n then (trivially) it is immersible in E^{n+r} with a normal r-field.*

(3.8) Theorem. *If M^k is parallelizable (admits k independent continuous tangent vector fields), it can be immersed in E^{k+1}. Every closed 3-manifold can be immersed in E^4; every closed 5-manifold can be immersed in E^8.*

Theorem 3.6 is the fundamental theorem of immersion theory. It reduces all questions pertaining to the existence or classification of immersions to a homotopy problem. The homotopy problem, though far from being solved, has been studied enough to yield much information on immersions through Theorem 3.6 as can be seen for example in Theorem 3.8. Most further work on the existence and classification of immersions would thus seem to lie outside of differential topology proper and in the corresponding homotopy problems.

We note that Haefliger [**4**] has very recently given another very different proof of Theorem 3.6 under the additional assumption $n > 3(k+1)/2$. See also [**7**].

We return now to discuss very briefly some of the methods used to prove the theorems of the previous section. The first step is to introduce function spaces of immersions. If M^k, X^n are manifolds, let $\mathrm{Im}^r(M, X)$ be the space of all immersions of M in X endowed with the C^r topology, $1 \leqq r \leqq \infty$. This means roughly that two immersions are close if they are pointwise close and their first r derivatives are close. Of course $\mathrm{Im}^r(M, X)$ might be empty! A point in $\mathrm{Im}^r(M, X)$ is an immersion of M in X and an arc in $\mathrm{Im}^1(M, X)$ is a regular homotopy, so the main problem amounts to finding the arc-components of $\mathrm{Im}^1(M, X)$ or $\pi_0(\mathrm{Im}^1(M, X))$. One now generalizes the problem to finding not only $\pi_0(\mathrm{Im}^1(M, X))$, but all the homotopy groups of $\mathrm{Im}^1(M, X)$. The homotopy groups of $\mathrm{Im}^r(M, X)$ do not depend on r and we sometimes omit it. To find these homotopy groups one uses the exact homotopy sequence of a fiber space and one of the main problems becomes, to show certain maps are fiber maps.

The following in fact is perhaps the most difficult part of [**32**].

(3.9) Theorem. *Define a map* $\pi : \mathrm{Im}^2(D^k, E^n) \to \mathrm{Im}^2(\partial D^k, E^n)$ *by restricting an immersion of* D^k *to the boundary. If* $n < k+1$, π *has the covering homotopy property.*

Actually, one uses an extension of this theorem to the case where boundary conditions involving first order derivates are incorporated into the range space of π. Since it was first proved, Theorem 3.9 has been generalized and strengthened. The general version, due to Hirsch and Palais [**11**] is as follows.

(3.10) Theorem. *Let* V *be a submanifold of a manifold* M, X *another manifold and* $\pi : \mathrm{Im}(M,\ X) \to \mathrm{Im}(V,\ X)$ *defined by restriction. Then* π *is a fiber map in the sense of Hurewicz* (*and hence has the covering homotopy property*).

A version of 3.10 is also proved with the boundary conditions mentioned above.

An idea not present in the author's original proof of 3.9, but introduced by Thom in [**42**], was to prove theorems of types 3.9 and 3.10 by first explicitly proving the coresponding theorem for spaces of imbeddings, this theorem being much easier and quite useful itself. The final version of this intermediate result is due to Palais [**26**].

(3.11) Theorem. *Let* M *be a compact manifold,* V *a submanifold, and* X *any manifold. Let* $\mathcal{E}(M,\ X)$, $\mathcal{E}(V,\ X)$ *be the respective spaces of imbeddings with the* C^r *topology,* $1 \leqq r \leqq \infty$, *and* $\pi : \mathcal{E}(M,\ X) \to \mathcal{E}(V,\ X)$ *defined by restriction. Then* π *is a locally trivial fiber map.*

Using Theorems 3.9, 3.10 and an induction basically derived from the fact that the dimension of the boundary of a manifold is one less than the manifold itself, one obtains weak homotopy equivalence theorems. The most general one is due to Hirsch and Palais [**11**]. Given manifolds M, X let $K(M,\ X)$ be the space of monomorphisms of T_M into T_X with the compact open topology. Then as described in Sec. 2, there is a map

$$\alpha :\ \mathrm{Im}(M, X) \to K(M, X)\,.$$

(3.12) Theorem. *The map* α *induces an isomorphism on all the homotopy groups* (*is a weak homotopy equivalence*) *if* $\dim X > \dim M$.

Theorem 3.12 applied to the zeroth homotopy groups or arc-components of $\mathrm{Im}(M,\ X)$ and $K(M,\ X)$ yields Theorem 3.6. Theorem 3.12 was first proved for $\mathrm{Im}(S^k, E^n)$ in [**32**].

4. An imbedding (or differentiable imbedding) is an immersion which is also a homeomorphism onto its image. A regular (or differentiable) isotopy is a regular homotopy which at each stage is an imbedding. The fundamental problem of imbedding theory is; given manifolds M^k, X^n, classify the imbeddings of M in X

under equivalence by regular isotopy. This includes the problem: does there exist an imbedding of M in X? Our discussion of imbedding theory is limited to work on this problem. The difficulty of the general problem is indicated by the special case of imbeddings of S^1 in E^3. This problem of classifying "classical" knots is far from being settled (and of course we omit any discussion of this special case although it could well be considered within the scope of differential topology).

Again the first theorems are due to Whitney in 1936 and are proved by general position arguments [**46**].

(4.1) Theorem. *A manifold M^k can always be imbedded in E^{2k+1}. Any two homotopic imbeddings of M in X^{2k+3} are regularly isotopic.*

One can replace X^{2k+3} by X^{2k+2} here.

See [**17**] or [**28**] for recent proofs of the first statement of 6.1.

In 1944, Whitney proved the much harder theorem [**47**].

(4.2) Theorem. *Every k-manifold can be imbedded in E^{2k}.*

The methods used in this paper have been important in subsequent developments in imbedding theory. A. Shapiro, in fact, has considerably developed Whitney's ideas in the framework of obstruction theory. Only the first stage of Shapiro's work is in print [**29**]. Besides being mostly unpublished, the theory has the further disadvantage from our point of view that it is a theory of imbedding for complexes and does not directly apply to give imbeddings (differentiable) of manifolds. On the other hand, Shapiro's work has in part inspired the important theorems of Haefliger that we will come to shortly.

Wu Wen Tsun in a number of papers, see e.g. [**49**], has a theory of imbedding and isotopy of complexes which overlaps with Shapiro's work. Shapiro's (unpublished) theorems on the existence of imbeddings of complexes in Euclidean space seem much stronger than those of Wu Wen Tsun. On the other hand, Shapiro works with spaces derived from the two-fold product of a space, while Wu studies stronger invariants derived from the p-fold products. Also Wu Wen Tsun not only considers existence of imbeddings but isotopy problems as well, including the following one for the differentiable case [**49**]. The proof is based on Whitney's paper [**47**].

(4.3) Theorem. *Any two imbeddings of a connected manifold M^k in E^{2k+1} are regularly isotopic.*

Haefliger [**4**] has taken a big step forward in the theory of imbeddings with the following theorems, proved by strong extensions of the work of Whitney, Shapiro and Wu Wen Tsun. Haefliger's main theorem can be expressed as follows.

As imbedding $f : M \to E^n$ induces a map $\phi_f : M \times M - M \to S^{n-1}$($M \times M - M$ is the product with the diagonal deleted) by $\phi_f(x, y) = (f(x) - f(y)/\|f(y) - f(x)\|$.

Then clearly ϕ_f is equivariant with respect to the involution on $M \times M - M$ which interchanges factors and the antipodal map of S^{n-1}.

(4.4) Theorem. *If $n > 3(k+1)/2$, the map $f \to \phi_f$ induces a* **1–1** *correspondence between regular isotopy classes of M in E^n and equivariant homotopy classes of $M \times M - M$ into S^{n-1}.*

The equivariant homotopy classes are in a **1–1** correspondence with homotopy classes of cross-sections of the following bundle E. Let M^* be the quotient space of $M \times M - M$ under the above involution. The two involutions described above define an action of the cyclic group of order two on $(M \times M - M) \times S^{n-1}$. The orbit space of this action is our bundle E with base M^* and fiber S^{n-1}.

Haefliger actually proves 4.4 with E^n replaced by an arbitrary manifold X^n.

Another of Haefliger's theorems is the following.

(4.5) Theorem. *If M^k and X^n are manifolds which are respectively $(r-1)$-connected, r-connected and $n \geqq 2k - r + 1$ then*

(a) *if $2r < n$, any continuous map of M in X is homotopic to an imbedding;*
(b) *if $2r < n + 1$, two homotopic imbeddings of M in X are regularly isotopic. Thus if $n > 3(k+1)/2$, any two imbeddings of S^k in E^n are regularly isotopic.*

Hirsch has proved some theorems on the existences of imbeddings of manifolds in Euclidean space. Perhaps the most interesting is the following [**10**].

(4.6) Theorem. *Every orientable 3-manifold can be imbedded in E^5.*

We do not discuss here, in general, the highly unstable problem of imbeddings of M^k in X^n where $n \leqq k+2$ except to mention that 2.7 is relevant to the differentiable Schonflies problem.

Since this section was first written Haefliger has obtained several further important results on imbeddings; see [**5**].

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THE WORK OF ALAN BAKER

by
PAUL TURÁN

The theory of transcendental numbers, initiated by Liouville in 1844, has been enriched greatly in recent years. Among the relevant profound contributions are those of A. Baker, W. M. Schmidt and V. A. Sprindzuk. Their work moves in important directions which contrast with the traditional concentration on the deep problem of finding significant classes of functions assuming transcendental values for all non-zero algebraic values of the independent variable. Among these, Baker's have had the heaviest impact on other problems of mathematics. Perhaps the most significant of these impacts has been the application to diophantine equations. This theory, carrying a history of more than thousand years, was, until the early years of this century, little more than a collection of isolated problems subjected to ingenious *ad hoc* methods. It was A. Thue who made the breakthrough to general results by proving in 1909 that all diophantine equations of the form $f(x, y) = m$, where m is an integer and f is an irreducible homogeneous binary form of degree at least three, with integer coefficients, have at most finitely many solutions in integers. This theorem was extended by C. L. Siegel and K. F. Roth (himself a Fields medallist) to much more general classes of algebraic diophantine equations in two variables of degree at least three. They even succeeded in establishing general upper bounds on the number of such solutions. A *complete* resolution of such problems however, requiring a knowledge of *all* solutions, is basically beyond the reach of these methods, which are what are called "ineffective". Here Baker made a brilliant advance. Considering the equation $f(x, y) = m$, where m is a positive integer, $f(x, y)$ an irreducible binary form of degree $n \geqq 3$, with integer coefficients, he succeeded in determining an *effective* bound B, depending only on n and on the coefficients of f, so that

$$\max(|x_0|, |y_0|) \leqq B$$

for any solution (x_0, y_0). Thus, although B is rather large in most cases, Baker has provided, in principle at least, and for the first time, the possibility of determining *all* the solutions explicitly (or the nonexistence of solutions) for a large *class* of equations. This is an essential step towards the positive aspects of Hilbert's tenth problem the interest of which is largely increased by the recent negative solution of the general problem by Ju. V. Matyaszevics. The significance of his theorem is

also enhanced by the fact that the so-called elliptic and hyperelliptic equations fall, after appropriate transformation, under its scope and again he gave explicit upper bounds on the totality of their solutions.

Joint work of Baker with J. Coates made effective for curves of genus 1 Siegel's classical theorem. Elaborating these methods and results Coates found among others the first *explicit* lower bound tending to infinity with n for the maximal primefactor of $|f(n)|$ where $f(x)$ stands for an arbitrary polynomial with integer coefficients apart from a trivial exception. The more fact that the maximal primefactor of $|f(n)|$ tends to infinity with n (conjectured for polynomials of second degree by Gauss) was established by K. Mahler several decades ago as well as an *explicit* lower bound for $n = 2$ by him and S. Chowla.

In collaboration with H. Davenport, Baker has shown by some examples how the upper bounds thus obtained permit actually the determination of *all* solutions.

As another consequence of his results he gave an *effective* lower bound for the approximability of algebraic numbers by rationals, the first one which is better than Liouville's.

As mentioned before, these results are all consequences of his main results on transcendental numbers. As is well known, the seventh problem of Hilbert asking whether or not α^β is transcendental whenever α and β are algebraic, certain obvious cases aside, was solved independently by A. O. Gelfond and T. Schneider in 1934. Shortly afterwards Gelfond found a stronger result by obtaining an *explicit* lower bound for $|\beta_1 \log \alpha_1 + \beta_2 \log \alpha_2|$ in terms of $\alpha'_\nu s$'s and of the degrees and heights of the $\beta'_\nu s$'s when the $\log \alpha'_\nu s$ are linearly independent. After Gelfond realised in 1948, in collaboration with Ju. V. Linnik, the significance of an *effective* lower bound for the three-term sum, he and N. I. Feldman soon discovered an *ineffective* lower bound for it. The transition from this important first step to *effective* bound for the three-term sum, and more generally for the k-term sum, resisted all efforts until Baker's success in 1966. This success enabled Baker to obtain a vast generalization of Gelfond–Schneider's theorem by showing that if $\alpha_1, \alpha_2, \ldots, \alpha_k$ $(\neq 0, 1)$ are algebraic, $\beta_1, \beta_2, \ldots, \beta_k$ linearly independent, algebraic and irrational, then $\alpha_1^{\beta_1}\alpha_2^{\beta_2}\cdots\alpha_k^{\beta_k}$ is transcendental. Some further appreciation of the depth of this result can be gained by recalling Hilbert's prediction that the Riemann conjecture would be settled long before the transcendentality of α^β. The analytic prowess displayed by Baker could hardly receive a higher testimonial. On the other hand, his brilliant achievement shows, after Gelfond–Schneider once more, that mathematics offers no scope for a doctrine of papal infallibility concerning its future. Among his other results generalizing transcendentality theorems of Siegel and Schneider I shall mention only one special case, in itself sufficiently remarkable, according to which the sum of the circumferences of two ellipses, whose axes have algebraic lengths, is transcendental.

His pathbreaking role is not diminished but perhaps even emphasized by the fact that in 1968 Feldman found another important lower estimate for the k-term sum which is stronger in its dependence upon the maximal height of the β_ν coefficients;

it is weaker in its dependence upon the maximal height of the α's which is relevant in most applications at present. It is reasonable to expect also new applications depending more on the former.

The 1948 discovery of Gelfond and Linnik, mentioned above, revealed an unexpected connection between such lower bounds for the three-term sum and a classical class-number problem. This has as its goal the determination of all algebraic extensions $R(\theta)$ of the rational field with class number 1. In its full generality this seems hopelessly out of reach at present. Restricting themselves to the imaginary quadratic case $R(\sqrt{-d})$, $d > 0$, H. Heilbronn and E. Linfoot showed in 1934 that at most ten such "good" fields can exist. Nine of these were found explicitly. Concerning the tenth it was known that its d would have to exceed $\exp(10^7)$. Hence, if it can be shown that there exists an upper bound $d_0 < \exp(10^7)$ for all "good" d's then the tenth possible field cannot exist. Now the Gelfond–Linnik discovery was that the afore mentioned *effective* lower bound for the three-term sum could furnish such an *effective* d_0. Baker found that one of his general results implies an upper bound $d_0 = 10^{500}$ enough by far for this purpose. This outcome provides a striking new example, illustrating once more how effectivity can play a *decisive* role in essential problems. Again, the value of this approach is of course not diminished by H. M. Stark's outstanding achievement in showing the non-existence of the tenth field, simultaneously and independently, by quite different methods.

To illustrate further the many-sided applicability of Baker's work, I mention that it could be employed to make effective some ineffective results of Linnik on the coefficients of a complete reduced set of binary quadratic forms belonging to a fixed negative discriminant (Linnik had used ideas from ergodic theory).

As one can guess, obtaining such long-sought solutions was a very complicated task. It is very difficult to attempt even a sketch of the underlying ideas in the short time at my disposal beyond the remark that they are of hard-analysis type. Fortunately, you will have the opportunity of hearing about them in some detail from Baker himself in his address to this Congress. To conclude, I remark that his work exemplifies two things very convincingly. Firstly, that beside the worthy tendency to start a theory in order to solve a problem it pays also to attack specific difficult problems directly. Particularly is this the case with such problems where rather singular circumstances do not make it probable that a solution would fall out as an easy consequence of a general theory. Secondly, it shows that a direct solution of a deep problem develops itself quite naturally into a healthy theory and gets into early and fruitful contact with other significant problems of mathematics. So, let the two different ways of doing mathematics live in peaceful coexistence for the benefit of our science.

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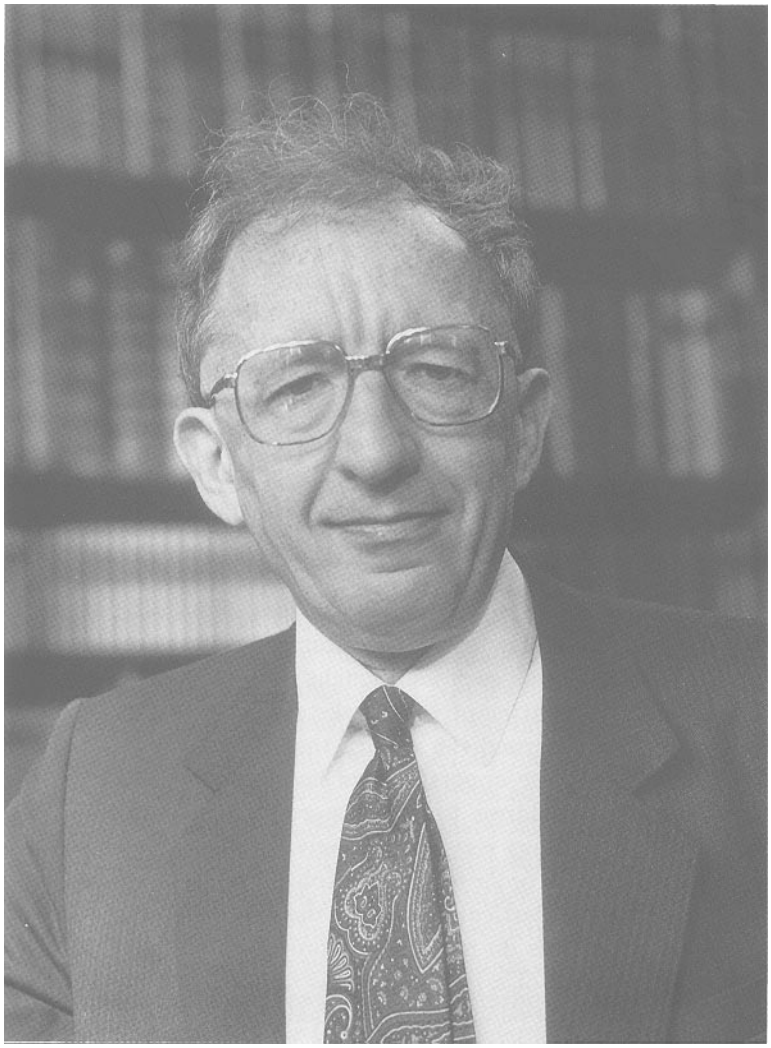

Alan Baker

ALAN BAKER

Professor Alan Baker obtained his first degree from University College, University of London and M.A., Ph.D. from Trinity College, Cambridge University. He has been Professor of Pure Mathematics at Cambridge University since 1974.

He was awarded a Fields Medal in 1970 at the International Congress of Mathematicians. In addition, he has a distinguished career with many awarded honours which include Fellow of Trinity College, Cambridge (1964), Adams Prize, Cambridge (1972), Fellow of the Royal Society (1973), First Turán Lecturer, Hungary (1978), Fellow, UCL (1979), Foreign Honorary Fellow of the Indian National Science Academy (1980).

He was Director of Studies in Mathematics, Trinity College, Cambridge (1968–74), Member of the Institute for Advanced Study, Princeton (1970), co-chairman of a program at MSRI, Berkeley (1993), and he has held visiting professorships at many universities including Stanford University (1974), University of Hong Kong (1988) and ETH, Zürich (1989).

Among his many important publications are "Transcendental Number Theory" (1975), "Transcendence Theory: Advances and Applications" (1977), "A Concise Introduction to the Theory of Numbers" (1984) (ed jtly), and "New Advances in Transcendence Theory" (1988), (ed).

Professor Alan Baker is well known internationally for his work in several areas of Number Theory. In particular, he succeeded in obtaining a vast generalization of the Gelfond–Schneider Theorem which is the solution to Hilbert's seventh problem. From this work he generated a large category of transcendental numbers not previously identified and showed how the underlying theory could be used to solve a wide range of Diophantine problems.

Because of his profound and significant contributions to Number Theory, Professor Alan Baker was awarded the Fields Medal. This is awarded every four years and is the most highly regarded international medal for outstanding discoveries in mathematics. Indeed, the Fields Medal is generally considered the equivalent of the Nobel Prize. In the Ceremony for the Award at the International Congress of Mathematicians, Professor P. Turán reported on Baker's work and said

> "... Some further appreciation of the depth of this (Baker's) result can be gained by recalling Hilbert's prediction that the Riemann conjecture would be settled long before the transcendentality of α^β. The analytic prowess displayed by Baker could hardly receive a higher testimonial ..."

This biographical sketch was prepared by the University of Hong Kong in connection with a Public Lecture given on 31 March 1995.

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EFFECTIVE METHODS IN THE THEORY OF NUMBERS

by

A. BAKER

1. Problems concerning the determination of the totality of integers possessing certain prescribed properties such as, for instance, solutions of systems of Diophantine equations or inequalities, have captured man's imagination since antiquity, and a wide variety of different techniques have been employed through the centuries to resolve a diverse multitude of problems in this field. Most of the early work tended to be of an *ad hoc* character, the arguments involved being specifically related to the particular numerical example under consideration, but gradually the emphasis has altered and the trend in recent times has been increasingly towards the development of general coherent theories. Two particular advances stand out in this connexion. First, investigations of Thue [39] in 1909 and Siegel [33] in 1929 led to the discovery of a simple necessary and sufficient condition for any Diophantine equation $F(x, y) = 0$, where F denotes a polynomial with integer coefficients, to possess only a finite number of solutions in integers; this occurs, namely, if and (reading "ganzartige" for "integer") only if the curve has genus at least 1 or genus 0 and at least three infinite valuations. The proof depends upon, amongst other things, Weil's well-known generalization [40] of Mordell's finite basis theorem and the earlier pioneering work of Thue and Siegel [32] concerning rational approximations to algebraic numbers. Secondly, in answer to a question raised by Gauss in his famous Disquisitiones Arithmeticae, Hecke, Mordell, Deuring and Heilbronn [29] showed in 1934 that there could exist only finitely many imaginary quadratic fields with any given class number, a result later to be incorporated in the celebrated Siegel–Brauer formula. These theorems and all their many ramifications, though of major importance in the evolution of much of modern number theory, nevertheless suffer from one basic limitation that of their non-effectiveness. The arguments depend on an assumption, made at the outset, that the relevant aggregates possess one or more elements that are, in a certain sense, large, and they provide no way of deciding whether or not these hypothetical elements exist. Thus the work leads merely to an estimate for the number of elements in question and throws no light on the fundamental problem of determining their totality.

Some special effective results in the context of the Thue–Siegel theory were obtained in 1964 by means of certain properties peculiar to Gauss' hypergeometric

function, in particular, the classic fact, certainly known to Padé, that quotients of such functions serve to represent the convergents to rational powers of $1-x$ (see [1, 2, 3]), but the first effective results applicable in a general context came in 1966 from a completely different source. One of Hilbert's famous list of problems raised at the International Congress held in Paris in 1900 asked whether an irrational quotient of logarithms of algebraic numbers is transcendental. An affirmative answer was obtained independently by Gelfond [26] and Schneider [30] in 1934, and shortly afterwards Gelfond established an important refinement giving a positive lower bound for a linear form in two logarithms (cf. [27]). It was natural to conjecture that an analogous result would hold for linear forms in arbitrarily many logarithms of algebraic numbers and a theorem of this nature was proved in 1994 [4]. The techniques devised for the demonstration form the basis of the principal effective methods in number theory known to data. I shall first describe briefly the main arguments and shall then proceed to discuss some of their applications(*).

2. The key result, which serves to illustrate most of the principal ideas, states that if $\alpha_1,\dots,\alpha_n$ are non-zero algebraic numbers such that $\log\alpha_1,\dots,\log\alpha_n$ are linearly independent over the rationals then $1,\log\alpha_1,\dots,\log\alpha_n$ are linearly independent over the field of all algebraic numbers. This implies, in particular, that $e^{\beta_0}\alpha_1^{\beta_1}\cdots\alpha_n^{\beta_n}$ is transcendental for all non-zero algebraic numbers $\alpha_1,\dots,\alpha_n$, $\beta_0,\dots,\beta_n$. It will suffice to sketch here the proof of a somewhat weaker result namely, if $\alpha_1,\dots,\alpha_n$, $\beta_1,\dots,\beta_{n-1}$ are non-zero algebraic numbers such that $\alpha_1,\dots,\alpha_n$ are multiplicatively independent, then the equation $\alpha_1^{\beta_1}\cdots\alpha_{n-1}^{\beta_{n-1}}=\alpha_n$ is untenable; it is under these conditions that our arguments assume their simplest form. We suppose the opposite and derive a contradiction. The proof depends on the construction of an auxiliary function of several complex variables which generalizes the function of a single variable employed originally by Gelfond. Functions of many variables were utilized by Schneider [31] in his studies concerning Abelian integrals but, for reasons that will shortly be explained, there seemed to be severe limitations to their serviceability in wider settings. The function that proved to be decisive in the present context is given by

$$\Phi(z_1,\dots,z_{n-1})=\sum_{\lambda_1=0}^{L}\cdots\sum_{\lambda_n=0}^{L}p(\lambda_1,\dots,\lambda_n)\alpha_1^{(\lambda_1+\lambda_n\beta_1)z_1}\cdots\alpha_{n-1}^{(\lambda_{n-1}+\lambda_n\beta_{n-1})z_{n-1}},$$

where L is a large parameter and the $p(\lambda_1,\dots,\lambda_n)$ denote rational integers not all 0. By virtue of the initial assumption we see at once that

$$\Phi(z,\dots,z)=\sum_{\lambda_1=0}^{L}\cdots\sum_{\lambda_n=0}^{L}p(\lambda_1,\dots,\lambda_n)\alpha_1^{\lambda_1 z}\cdots\alpha_n^{\lambda_n z}$$

and so, for any positive integer l, the value of Φ at $z_1=\cdots=z_{n-1}=l$ is an algebraic number in a fixed field. Moreover, apart from a multiplicative factor

(*)For a fuller survey of the applications see [13].

given by products of powers of the logarithms of the α's, the same holds for any derivative

$$\Phi_{m_1,\ldots,m_{n-1}} = (\partial/\partial_{z_1})^{m_1} \cdots (\partial/\partial z_{n-1})^{m_{n-1}} \Phi\,.$$

It follows from a well-known lemma on linear equations that, for any integers h, k, with hk^{n-1} a little less than L^n, one can choose the $p(\lambda_1, \ldots, \lambda_n)$ such that

$$\Phi_{m_1,\ldots,m_{n-1}}(l, \ldots, l) = 0 \quad (1 \leqslant l \leqslant h, m_1 + \cdots + m_{n-1} \leqslant k)$$

and, furthermore, an explicit bound for $|p(\lambda_1, \ldots, \lambda_n)|$ can be given in terms of h, k and L.

The real essence of the argument is an extrapolation procedure which shows that the above equation remains valid over a much longer range of values for l, provided that one admits a small diminution in the range of values for $m_1 + \cdots + m_{n-1}$. Although interpolation arguments have long been a familiar feature of transcendental number theory, work in this connexion has hitherto always involved an extension in the order of the derivatives while leaving the points of interpolation fixed; when dealing with functions of many variables, however, this type of argument requires that the points in question admit a representation as a Cartesian product and, as far as I can see, the condition can be satisfied only with respect to special multiply-periodic functions. Our algorithm proceeds by induction and it will suffice to illustrate the first step. We suppose that $m_1 + \cdots + m_{n-1} \leqslant \frac{1}{2}k$ and we prove that then

$$f(z) = \Phi_{m_1,\ldots,m_{n-1}}(z, \ldots, z)$$

vanishes at $z = l$, where $1 \leqslant l \leqslant h^2$. Now the condition $hk^{n-1} \leqslant L^n$ allows one to take $L \leqslant k^{1-\varepsilon}$ for some $\varepsilon > 0$ and h about $k^{\frac{1}{4}\varepsilon}$. This "saving" by an amount ε is crucial for it leads to a sharp bound for $|f(z)|$ on a circle centre the origin and radius slightly larger than h^2, thus including all the points l as above. Further, apart from a trivial multiplicative factor, $f(l)$ represents an algebraic integer in a fixed field and a similar bound obtains for each of the conjugates. But, by construction, we have

$$f_m(r) = 0 \quad \left(0 \leqslant m \leqslant \frac{1}{2}k, 1 \leqslant r \leqslant h\right),$$

and the maximum-modulus principle applied to the function $f(z)/F(z)$, where

$$F(z) = \{(z-1)\cdots(z-h)\}^{[\frac{1}{2}k]}\,,$$

now shows that $|f(l)|$ is sufficiently small to ensure that the norm of the algebraic integer is less than 1. Hence $f(l) = 0$ as required. The argument is repeated inductively and after a finite number of steps we conclude that

$$\Phi(l, \ldots, l) = 0 \quad (1 \leqslant l \leqslant (L+1)^n)\,.$$

But these represent linear equations in the $p(\lambda_1, \ldots, \lambda_n)$. The determinant of coefficients is of Vandermonde type and since, by hypothesis, $\alpha_1, \ldots, \alpha_n$ are multiplicatively independent, it does not vanish. The contradiction establishes our result.

3. The argument just described is capable of considerable refinement and generalization. In particular several other auxiliary functions can be taken in place of Φ, the points of extrapolation can be varied and greater use can be made in the latter part of the exposition of our information regarding the partial derivatives. Thus, for instance, results in the context of elliptic functions have been derived and, in particular, the transcendence has been established of any non-vanishing linear combination with algebraic coefficients of periods and quasi-periods associated with a Weierstrass p-function with algebraic invariants [10, 11, 12]. More relevant to the main theme of this talk, however, are refinements giving quantitative lower bounds for linear forms in logarithms. The main change in the preceding discussion required to obtain results of this nature is the replacement of the maximum-modulus principle by the Hermite interpolation formula. With this device one can show that

$$|\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n| > Ce^{-(\log H)^\kappa},$$

where $\alpha_1, \ldots, \alpha_n$ denote non-zero algebraic numbers such that $\log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over the rationals, $\beta_0, \ldots, \beta_n$ denote algebraic numbers, not all 0, with degrees and heights at most d and H respectively, $\kappa > n+1$ and $C > 0$ depends only on n, $\log \alpha_1, \ldots, \log \alpha_n$, κ and d; by the height of an algebraic number we mean the maximum of the absolute values of the relatively prime integer coefficients in its minimal defining polynomial [5]. With more complicated adaptations the number on the right can be strengthened to $CH^{-\kappa}$, where $\kappa > 0$ is specified like C above; this was shown by Feldman [22, 23]. In applications it frequently suffices to have simply a lower bound of the form $e^{-\delta H}$, valid for any $\delta > 0$ and all $H > C$, where C now depends on δ, and interest then attaches to the exact expression for C. Some explicit forms have been calculated (cf. [5, 6, 23, 24]) but there is certainly scope for improvement here and, indeed, the general efficacy of our methods seems to be closely linked to our progress in this connexion.

4. We now discuss some applications of our results in the theory of Diophantine equations. To begin with, they can be utilized to obtain a complete resolution of the equation originally considered by Thue, namely $f(x, y) = m$, where f denotes an irreducible binary form with integer coefficients and degree at least 3 [6]. Indeed our arguments enable us to find more generally all algebraic integers x, y in a given field K satisfying any equation $\beta_1 \cdots \beta_n = m$ where $\beta_j = x - \alpha_j y$, $n \geqslant 3$ and $\alpha_1, \ldots, \alpha_n$, m denote algebraic integers in K subject only to the condition that the α's are all distinct (cf. [15]). For denoting by $\theta^{(1)}, \ldots, \theta^{(d)}$ the field conjugates of any element θ of K and by $\eta_1, \ldots, \eta_r$ a fundamental system of units in K, it is readily seen that an associate

$$\gamma_i = \beta_i \eta_1^{b_{i1}} \cdots \eta_r^{b_{ir}}$$

of β_i can be determined such that

$$|\log |\gamma_i^{(j)}|| \leqslant C_1 \quad (1 \leqslant j \leqslant d),$$

where $C_1, C_2, \ldots,$ can be effectively computed in terms of f and m. Writing

$$H_i = \max |b_{ij}| \quad \text{and} \quad H_l = \max\ H_i$$

we have $|\beta_l^{(h)}| \leqslant C_2 e^{-H_l/C_3}$ for some h; and without loss of generality we can suppose that $\beta_l^{(h)} = \beta_l$. From the initial equation we see that $|\beta_k| \geqslant C_4^{-1}$ for some $k \neq l$ and if now j is any suffix other than k or l, the identity

$$(\alpha_k - \alpha_l)\beta_j - (\alpha_j - \alpha_l)\beta_k = (\alpha_k - \alpha_j)\beta_l$$

gives

$$\eta_1^{b_1} \cdots \eta_r^{b_r} - \alpha_{r+1} = \omega\,,$$

where

$$b_s = b_{ks} - b_{js}\,, \quad 0 < |\omega| < C_5 e^{-H_l/C_6}$$

and α_{r+1} is an element of K with degree and height $\leqslant C_7$. Now $|b_s| \leqslant 2H_l$ and hence the work of §3 can be applied to obtain a bound for H_l, whence also for all the conjugates of the β's and, finally, for the conjugates of x and y.

The last result enables one to solve many other Diophantine equations in two unknowns. In particular, one can now effectively determine all rational integers x, y satisfying $y^m = f(x)$, where m is any integer $\geqslant 2$ and f is a polynomial with integer coefficients possessing at least three simple zeros [8]. This includes the celebrated Mordell equation $y^2 = x^3 + k$, the hyperelliptic equation and the Catalan equation $x^n - y^m = 1$ with prescribed m, n. The demonstration involves ideal factorizations in algebraic number fields similar to those appearing in the first part of the proof of the Mordell–Weil theorem; in special cases one has readier arguments and, in particular, the elliptic equation has been efficiently treated by means of Hermite's classical theory of the reduction of binary quartic forms [7]. There is, moreover, little difficulty in carrying out the work more generally when the coefficients and variables represent algebraic integers in a fixed field, and, indeed, Coates and I have used this extension to give a new and effective proof Siegel's theorem on $F(x, y) = 0$ (see §1) in the case of curves of genus 1 [15, 21]. Here the equation of the curve is reduced to canonical form by means of a birational transformation similar to that described by Chevalley, the rational functions defining the transformation being constructed to possess poles only at infinity and thus be integral over a polynomial ring. Explicit upper bounds have been established in each instance for the size of all the solutions [6, 7, 8, 15]. The bounds tend to be large, with repeated exponentials, and current research in this field is centred on techniques for reducing their magnitude. In particular, Siegel [34] has recently given some improved estimates for units in algebraic number fields which should prove useful for this purpose, and, furthermore, devices have been obtained which, for a wide range of numerical examples, would seem to render the problem of determining the complete list of solutions in question accessible to practical computation (cf. [16]).

5. Finally we mention some further results that have been obtained as a consequence of these researches. One of the first applications was to establish an effective algorithm for resolving the old conjecture that there are only nine imaginary quadratic fields with class number 1 [4, 18]. The connexion between this problem and inequalities involving the logarithms of algebraic numbers was demonstrated by Gelfond and Linnik [28] in 1949 by way of an expression for a product of L-functions analogous to the well-known Kronecker limit formula. By a remarkable coincidence, Stark [38] established the conjecture at about the same time by an entirely different method with its origins in a paper of Heegner. Attention has subsequently focussed on the problem of determining all imaginary quadratic fields with class-number 2, and I am happy to report that an algorithm for this purpose was obtained very recently by means of a new result relating to linear forms in three logarithms [9, 14](*). It seems likely that this latest development will lead to advances in other spheres.

Among the original motivations of our studies was the search for an effective improvement on Liouville's inequality of 1844 relating to the approximation of algebraic numbers by rationals; from the work described in §4 we have now

$$|\alpha - p/q| > cq^{-n}e^{(\log\ q)^{1/\kappa}}$$

for all algebraic numbers α with degree $n \geqslant 3$ and all rationals $p/q (q > 0)$ where $\kappa > n$ and $c = c(\alpha, \kappa) > 0$ is effectively computable [6, 25]. For some particular α, such as the cube roots of 2 and 17, sharper results in this direction have been obtained from the work on the hypergeometric function mentioned in §1. Further, in the special case when p, q are comprised solely of powers of fixed sets of primes, a much stronger result can be obtained directly from the inequalities referred to in §3; indeed we have then

$$|\alpha - p/q| > c(\log\ q)^{-\kappa}$$

where $c > 0$, $\kappa > 0$ are effectively computable in terms of the primes and α, and this in fact furnishes an improvement on Ridout's generalization of Roth's theorem.

Analogues of the arguments of §3 and §4 in the p-adic realm have been given by Coates [19, 20]; his work leads, in particular, to an effective determination of all rational solutions of the equations discussed earlier with denominators comprised solely of powers of fixed sets of primes and so, more especially, provides a means for finding all elliptic curves with a given conductor (see also [35, 36, 37]). Furthermore, Brumer obtained in 1967 a natural p-adic analogue of the main theorem on logarithms which, in conjunction with work of Ax, resolved a well-known problem of Leopoldt on the non-vanishing of the p-adic regulator of an Abelian number field [17].

6. And now I must conclude my survey. It will be appreciated that I have been able to touch upon only a few of the diverse results that have been established with

(*)See also Stark's address to this Congress.

the aid of the new techniques, and, certainly, many avenues of investigation await to be explored. The work has demonstrated, in particular, a surprising connexion between the apparently unrelated seventh and tenth problems of Hilbert, as well as throwing an effective light on both of the fundamental topics referred to at the beginning concerning Diophantine equations and class numbers. Though the strength of this illumination has been steadily growing, and indeed the respective regions of shadow in these contexts have been receding at a remarkably similar rate, it would appear nevertheless that several further ideas will be required before our theories can be regarded as, in any sense, complete. The main feature to emerge is, I think, that the principal passage to effective methods in number theory lies, at present, deep in the domain of transcendence, and it is to be hoped that the territory so far gained in this connexion will be much extended in the coming years.

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EFFECTIVE METHODS IN DIOPHANTINE PROBLEMS

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1. Introduction

These notes are intended to serve as a guide to the various effective results in the theory of numbers that have been obtained as a consequence of recent researches. Most of the theorems derive in some way from the author's papers on the logarithms of algebraic numbers and we shall begin with an account of this work. The reader will be referred to the original memoirs for proofs.

2. On the Logarithms of Algebraic Numbers

At the International Congress of mathematicians held in Paris 1900, Hilbert raised, as the seventh of his famous list of 23 problems, the question whether an irrational logarithm of an algebraic number to an algebraic base is transcendental. The question is capable of various alternative formulations; thus one can ask whether an irrational quotient of natural logarithms of algebraic numbers is transcendental, or whether α^β is transcendental for any algebraic number $\alpha \neq 0, 1$ and any algebraic irrational β. A special case relating to logarithms of rational numbers had been posed by Euler more than a century before but no apparent progress had been made towards its solution. Indeed Hilbert expressed the opinion that the resolution of the problem lay farther in the future than a proof of the Riemann hypothesis or Fermat's last theorem.

The first significant advance was made by Gelfond in 1929. Employing interpolation techniques of the kind that he had utilized previously in researches on integral integer-valued functions, Gelfond showed that the logarithm of an algebraic number to an algebraic base cannot be an imaginary quadratic irrational, that is, α^β is transcendental for any algebraic number $\alpha \neq 0, 1$ and any imaginary quadratic irrational β; in particular, one sees that $e^\pi = (-1)^{-i}$ is transcendental. The result was extended to real quadratic irrationals β by Kuzmin in 1930. But it was clear that direct appeal to an interpolation series for $e^{\beta z}$, on which the Gelfond–Kuzmin work was based, was not appropriate for more general β, and further progress awaited a new idea. The search for the latter was concluded successfully by Gelfond and Schneider, independently, in 1934. The arguments they discovered were applicable

for any irrational β and, though differing in detail, both depended on the construction of an auxiliary function that vanished at certain selected points. A similar technique had been used a few years earlier by Siegel in the course of his investigations on the Bessel functions. Herewith, Hilbert's seventh problem was finally solved.

The Gelfond–Schneider theorem shows that for any nonzero algebraic numbers α_1, α_2, β_1, β_2 with $\log \alpha_1$, $\log \alpha_2$ linearly independent over the rationals we have

$$\beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 \neq 0\,.$$

It was natural to conjecture that an analogous theorem would hold for arbitrarily many logarithms of algebraic numbers and moreover it was soon realized that generalizations of this kind would have important consequences in number theory. But, for some thirty years, the problem of extension seemed resistant to attack. It was finally settled in 1966 [6], and the techniques devised for its solution have been the main instruments in establishing the various results described herein. The original theorem has been extended slightly to include also the case in which an additional nonzero algebraic number is present on the left and now reads as follows:

Theorem 1 [7]. *If $\alpha_1, \ldots, \alpha_n$ denote nonzero algebraic numbers such that $\log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over the rationals then $1, \log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over the field of all algebraic numbers.*

Here $\log \alpha_1, \ldots, \log \alpha_n$ are any fixed determinations of the logarithms. The proof of the theorem depends on the construction of an auxiliary function of several complex variables which generalizes the original function of a single variable employed by Gelfond. The subsequent arguments, however, involve an extrapolation procedure that is special to the present context and for which there is no precise earlier counterpart. Quantitative extensions of Theorem 1 will be discussed in the next section and applications of the results to various branches of number theory will be the theme of §§4 to 6.

We record now a few immediate corollaries of Theorem 1.

Theorem 2. *Any nonvanishing linear combination of logarithms of algebraic numbers with algebraic coefficients is transcendental.*

Theorem 3. *$e^{\beta_0}\alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}$ is transcendental for any nonzero algebraic numbers $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$.*

Theorem 4. *$\alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}$ is transcendental for any algebraic numbers $\alpha_1, \ldots, \alpha_n$, other than 0 or 1, and any algebraic number $\beta_1, \ldots, \beta_n$ with 1, $\beta_1, \ldots, \beta_n$ linearly independent over the rationals.*

Particular cases of the above theorems show that $\pi + \log \alpha$ is transcendental for any algebraic number $\alpha \neq 0$, and $e^{\alpha\pi+\beta}$ is transcendental for any algebraic numbers α, β with $\beta \neq 0$. One might also mention an analogy with Lindemann's classical theorem; this asserts that

$$\beta_1 \exp \alpha_1 + \cdots + \beta_n \exp \alpha_n \neq 0$$

for any distinct algebraic numbers $\alpha_1, \ldots, \alpha_n$ and any nonzero algebraic numbers $\beta_1, \ldots, \beta_n$; Theorem 1 shows that the same holds with "exp" replaced by "log" provided that the logarithms are linearly independent over the rationals.

3. Lower Bounds for Linear Forms

By the *height* of an algebraic number we shall mean the maximum of the absolute values of the relatively prime integer coefficients in its minimal defining polynomial. Soon after obtaining his solution to the seventh problem of Hilbert, Gelfond established an important refinement expressing a positive lower bound for a linear form in two logarithms; he proved that for any nonzero algebraic numbers α_1, α_2, β_1, β_2 with $\log \alpha_1$, $\log \alpha_2$ linearly independent over the rationals and any $\kappa > 5$ we have

$$|\beta_1 \log \alpha_1 + \beta_2 \log \alpha_2| > Ce^{-(\log H)^\kappa} ,$$

where H denotes the maximum of the heights of β_1, β_2, and $C > 0$ denotes a computable number depending only on $\log \alpha_1$, $\log \alpha_2$, and the degrees of β_1, β_2. Gelfond later improved the condition $\kappa > 5$ to $\kappa > 2$. Also he showed that, as a corollary to the Thue-Siegel theorem, about which we shall speak in §5, an inequality of the form

$$|b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n| > Ce^{-\delta H}$$

holds, where $\delta > 0$ and $b_1, \ldots, b_n$ are rational integers, not all 0, with absolute values at most H; but here $C > 0$ could not be effectively computed.

After the demonstration of Theorem 1 it proved relatively easy to obtain extensions of Gelfond's inequalities relating to arbitrarily many logarithms of algebraic numbers, and indeed the following theorem was established [7]

Theorem 5. *Let $\alpha_1, \ldots, \alpha_n$ denote nonzero algebraic numbers with $\log \alpha_1, \ldots, \log \alpha_n$ linearly independent over the rationals, and let $\beta_0, \ldots, \beta_n$ denote algebraic numbers, not all 0, with degrees and heights at most d and H respectively. Then, for any $\kappa > n + 1$, we have*

$$|\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n| > Ce^{-(\log H)^\kappa}$$

where $C > 0$ denotes an effectively computable number depending only on n, $\log \alpha_1, \ldots, \log \alpha_n$, and d.

Very recently the number on the right of the above inequality has been improved by Feldman to $C^{-1}H^{-\kappa}$, where $C > 0$, $\kappa > 0$ depend only on n, $\log \alpha_1, \ldots, \log \alpha_n$ and d [23], [24]. Estimates for C and κ have been explicitly calculated, but their values are large; the estimate for C takes the form $C' \exp[(\log A)^{\kappa'}]$ where A denotes the maximum of the heights of $\alpha_1, \ldots, \alpha_n$, κ' depends only on n, C' depends only on n, d and the degrees of $\alpha_1, \ldots, \alpha_n$. The value of C and, in particular, its dependence on A is of importance in applications; a more special result, but giving sharper

estimates with respect to the parameters other than H, was recently established by the author:

Theorem 6 [7]. *Suppose that $\alpha_1, \ldots, \alpha_n$ are $n \geq 2$ nonzero algebraic numbers and that the heights and degrees of $\alpha_1, \ldots, \alpha_n$ do not exceed integers A, d respectively, where $A \geq 4$, $d \geq 4$. Suppose further that $0 < \delta \leq 1$ and let $\log \alpha_1, \ldots, \log \alpha_n$ denote the principal values of the logarithms. If rational integers $b_1, \ldots, b_n$ exist with absolute values at most H such that*

$$0 < |b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n| < e^{-\delta H}$$

then

$$H < (4^{n^2} \delta^{-1} d^{2n} \log A)^{(2n+1)^2} .$$

Theorem 6 will be the only transcendence result to which we shall refer directly in the sequel. It is useful for application to a wide class of Diophantine problems and yields estimates that will be found, in many cases, to be accessible to practical computation [17].

4. On Imaginary Quadratic Fields with Class Number 1

In 1966 Stark [32] and the author [6], [12] showed independently how one could resolve the long-standing conjecture that the only imaginary quadratic fields $Q(\sqrt{-d})$ with class number 1 are those given by $d = 1, 2, 3, 7, 11, 19, 43, 67, 163$. Heilbronn and Linfoot had proved in 1934 that there could be at most ten such fields, and calculations had shown that the tenth field, if it existed, would satisfy $d > \exp(2 \cdot 2 \times 10^7)$. The work of Stark was motivated by an earlier paper of Heegner, and the work of the author was based on an idea of Gelfond and Linnik. We shall sketch briefly the latter method.

Let $-d < 0$ and $k > 0$ denote the discriminants of the quadratic fields $Q(\sqrt{-d})$ and $Q(\sqrt{k})$ respectively, and suppose that the corresponding class numbers are given by $h(d)$ and $h(k)$. Suppose also that $(d, k) = 1$ and let

$$\chi(n) = (k/n) , \qquad \chi'(n) = (d/n)$$

denote the usual Kronecker symbols. Further let

$$f = f(x, y) = ax^2 + bxy + cy^2$$

run through a complete set of inequivalent quadratic forms with discriminant $-d$. From a well-known formula for $L(s, \chi)L(s, \chi\chi')$ with $s > 1$ we obtain, on taking limits as $s \to 1$ and applying classical results of Dirichlet,

$$h(k)h(kd) \log \epsilon = \frac{1}{12} \pi k \sqrt{d} \left(\sum_f \chi(a) a^{-1} \right) \prod_{p|k} (1 - p^{-2})$$

$$+ B_0 + \sum_f \sum_{r=-\infty; r \neq 0}^{\infty} B_r e^{\pi i r b/(ka)} ,$$

where ϵ denotes the fundamental unit in the field $Q(\sqrt{k})$, $h(kd)$ denotes the class number of the field $Q(\sqrt{(-kd)})$, $B_0 = -\log p \sum_f \chi(a)$ if k is the power of a prime p, $B_0 = 0$ otherwise and, for $r \neq 0$, we have

$$|B_r| \leq k|r|e^{-\pi|r|\sqrt{d}/(ak)} .$$

On choosing $k = 21$ or 33, so that $h(k) = 1$ and the units ϵ are given by $\alpha_1 = \frac{1}{2}(5+\sqrt{21})$ and $\alpha_2 = 23 + 4\sqrt{33}$ respectively, we see that, if $h(d) = 1$,

$$|h(21d)\log\alpha_1 - \frac{32}{21}\pi\sqrt{d}| < e^{-\pi\sqrt{d}/100}, \quad |h(33d)\log\alpha_2 - \frac{80}{33}\pi\sqrt{d}| < e^{-\pi\sqrt{d}/100} .$$

Since, in particular, $h(21d) < 4\sqrt{d}$, $h(33d) < 4\sqrt{d}$, it follows, on writing

$$\delta^{-1} = 14 \times 10^3 , \quad H = 140\sqrt{d} , \quad b_1 = 35h(21d) , \quad b_2 = -22h(33d) ,$$

that the hypotheses of Theorem 6 are satisfied. We conclude that $H < 10^{250}$ whence $d < 10^{500}$. Note that, instead of Theorem 6, it would in principle have sufficed to have appealed to the earlier work of Gelfond and moreover, since $\pi = -2i\log i$, to have referred to the above formulae for just one value of k. A similar argument leads to the bound $d < 10^{500}$ for all imaginary quadratic fields $Q(\sqrt{-d})$ with class number 2, where d denotes a square-free positive integer with $d \not\equiv 3 \pmod 8$ [12].[1]

5. On the Representation of Integers by Binary Forms

We come now to the fundamental work begun by Thue in 1909 and subsequently developed by Siegel, Roth and others. Thue obtained a nontrivial inequality expressing a limit to the accuracy with which any algebraic number (not itself rational) can be approximated by rationals and thereby showed, in particular, that the Diophantine equation $f(x, y) = m$, where f denotes an irreducible binary form with integer coefficients and degree at least 3, possesses only a finite number of solutions in integers x, y. The work was much extended by Siegel first in 1921 when he strengthened the basic approximation inequality and then in 1929 when he applied his result, together with the Mordell–Weil theorem, to give a simple necessary and sufficient condition for any equation of the form $f(x, y) = 0$, where f denotes a polynomial with integer coefficients, to possess only a finite number of integer solutions. Siegel's work gave rise to many further developments; in particular, Mahler obtained far-reaching p-adic generalizations of the original theorems, and Roth succeeded in further improving the approximation inequality, establishing a result that is essentially best possible.

All the work which I have just described, however, suffers from one basic limitation, that of its noneffectiveness. The proofs depend on an assumption, made

[1] For another proof of the class number 1 result see P. Bundschuh and A. Hock [19]. An interesting application of Theorem 5 in a related context has been given by E. A. Anfert'eva and N. G. Čudakov [2].

at the outset, that the Diophantine inequalities, or the corresponding equations, possess at least one solution in integers with large absolute values and the arguments provide no way of deciding whether or not such a solution exists. Thus although the Thue-Siegel theory supplies information on the number of solutions of the equation $f(x,y)=0$, it does not enable one to determine whether or not a particular equation of this type is soluble; of course, such a determination would amount to a solution of Hilbert's tenth problem for polynomials in two unknowns. For the special equation $f(x,y)=m$, where $f(x,1)$ has at least one complex zero, another proof of the finiteness of the number of solutions was given by Skolem in 1935 by means of a p-adic argument very different from the original, but here the work depends on the compactness property of the p-adic integers and so is again noneffective.

As a consequence of Theorem 6 one can now give a new and effective proof of Thue's result on the representation of integers by binary forms.

Theorem 7 [8]. *If f denotes an irreducible binary form with degree $n \geq 3$ and with integer coefficients then, for any positive integer m, the equation $f(x,y)=m$ has only a finite number of solutions in integers x, y, and these can be effectively determined.*

The method of proof leads in fact to an explicit bound for the size of all the solutions; assuming that $\mathcal{H}$ is some number exceeding the maximum of the absolute values of the coefficients of f we have

$$\max\,(|x|,|y|) < \exp\left\{n\mathcal{H})^{(10n)^5} + (\log m)^{2n+2}\right\}.$$

In view of the mean-value theorem, this corresponds to an effective result on the approximation of algebraic numbers by rationals; indeed one can show that for any algebraic number α with degree $n \geq 3$ and for any $\kappa > n$ there exists an effectively computable number $c = c(\alpha,\kappa) > 0$ such that

$$|\alpha - p/q| > cq^{-n}\exp\left[(\log q)^{1/\kappa}\right]$$

for all integers p, q $(q > 0)$. Some slightly stronger quantitative results in this direction have been established for certain fractional powers of rationals [3], [4], [5] but here the work depends on particular properties of Gauss' hypergeometric function and is therefore of a special nature.

As regards the proof of Theorem 7, we assume, without loss of generality, that the coefficient of x^n in $f(x,y)$ is ± 1 and we denote the zeros of $f(x,1)$ by $\alpha_1,\ldots,\alpha_n$, where it is assumed that $\alpha^{(1)},\ldots,\alpha^{(s)}$ only are real and $\alpha^{(s+1)},\ldots,\alpha^{(s+t)}$ are the complex conjugates of $\alpha^{(s+t+1)},\ldots,\alpha^{(n)}$; thus it is implied that $n = s+2t$. The algebraic number field generated by $\alpha = \alpha^{(1)}$ over the rationals will be denoted by K, and $\theta^{(1)},\ldots,\theta^{(n)}$ will represent the conjugates of any elements θ of K corresponding to the conjugates $\alpha^{(1)},\ldots,\alpha^{(n)}$ of α. C_1, $C_2,\ldots$ will denote numbers

greater than 1 which can be specified explicitly in terms of m, n and the coefficients of f. Finally we denote by $\eta_1, \ldots, \eta_r$ a set of $r = s + t - 1$ units in K such that

$$\big|\log|\eta_i^{(j)}|\big| < C_1 \quad (1 \le i, j \le r)$$

and such that also the determinant Δ of order r with $\log|\eta_i^{(j)}|$ in the ith row and jth column satisfies $|\Delta| > C_2^{-1}$.

Suppose now that x, y are rational integers satisfying $f(x, y) = m$ and put $\beta = x - \alpha y$. Clearly β is an algebraic integer in K, and we have $|\beta^{(1)} \cdots \beta^{(n)}| = m$. Further it is easily seen that an associate γ of β can be determined such that

$$\big|\log|\gamma^{(j)}|\big| < C_3 \qquad (1 \le j \le n)\,.$$

Writing $\gamma = \beta\eta_1^{b_1} \cdots \eta_r^{b_r}$ and $H = \max|b_j|$ we deduce from the equations

$$\log|\gamma^{(j)}/\beta^{(j)}| = b_1 \log|\eta_1^{(j)}| + \cdots + b_r \log|\eta_r^{(j)}| \quad (1 \le j \le r)$$

that the maximum of the absolute values of the numbers on the left must exceed $C_4^{-1}H$, whence

$$\log|\beta^{(l)}| \le -(C_4^{-1}H - C_3)/(n-1)$$

for some l. In particular we have $|\beta^{(l)}| \le C_5$ and so $|\beta^{(k)}| \ge C_6^{-1}$ for some $k \ne l$. Since $n \ge 3$, there exists a superscript $j \ne k$, l, and we have the identity

$$(\alpha^{(k)} - \alpha^{(l)})\beta^{(j)} - (\alpha^{(j)} - \alpha^{(l)})\beta^{(k)} = (\alpha^{(k)} - \alpha^{(j)})\beta^{(l)}\,.$$

This gives $\alpha_1^{b_1} \cdots \alpha_r^{b_r} - \alpha_{r+1} = \omega$, where

$$\alpha_s = \eta_s^{(k)}/\eta_s^{(j)} \qquad (1 \le s \le r)\,,$$

$$\alpha_{r+1} = \frac{(\alpha^{(j)} - \alpha^{(l)})\gamma^{(k)}}{(\alpha^{(k)} - \alpha^{(l)})\gamma^{(j)}} \quad \text{and} \quad \omega = \frac{(\alpha^{(k)} - \alpha^{(j)})\beta^{(l)}\gamma^{(k)}}{(\alpha^{(k)} - \alpha^{(l)})\beta^{(k)}\gamma^{(j)}}\,.$$

Now by the choice of k and l we see that $0 < |\omega| < C_7 \exp(-H/C_8)$. Further, the degrees and heights of $\alpha_1, \ldots, \alpha_{r+1}$ are bounded above by numbers depending only on f and m. Noting that $|e^z - 1| < \frac{1}{4}$ implies that $|z - ik\pi| < 4|e^z - 1|$ for some rational integer k, we easily obtain an inequality of the type considered in Theorem 6. Hence we conclude that $H < C_9$; this gives $|\beta^{(j)}| < C_{10}$ for each j and thus

$$\max(|x|, |y|) < C_{11} \max(|\beta^{(1)}|, |\beta^{(2)}|) < C_{12}\,.$$

6. The Elliptic and Hyperelliptic Equations

Theorem 7 and its natural generalization to algebraic number fields can be used to solve effectively many other Diophantine equations in two unknowns. In particular it enables one to treat $y^2 = x^3 + k$ for any $k \ne 0$, an equation with

a long and famous history in the theory of numbers [9], [27]. The work rests on the classical theory of the reduction of binary cubic forms, due mainly to Hermite, and the techniques used by Mordell in establishing the finiteness of the number of solutions of the equation. More generally, by means of the theory of the reduction of binary quartic forms one can prove:

Theorem 8 [10]. *Let a ($\neq 0$), b, c, d denote rational integers with absolute values at most $\mathcal{H}$, and suppose that the cubic on the right of the equation*

$$y^2 = ax^3 + bx^2 + cx + d$$

has distinct zeros. Then all solutions in integers x, y satisfy

$$\max\,(|x|, |y|) < \exp\,\{10^6 \mathcal{H}^{10^6}\}\,.$$

Still more generally one can give bounds for all the solutions in integers x, y of equations of the form $y^m = f(x)$, where $m \geq 2$ and f denotes a polynomial with integer coefficients [11]. The work here, however, is based on a paper of Siegel and involves the theory of factorization in algebraic number fields; the bounds are therefore much larger than that specified in Theorem 8. As immediate consequences of the results one obtains inequalities of the type

$$|x^m - y^n| > c\,(\log\log x)^{1/n^2} \qquad (m, n \geq 3)\,,$$

where $c = c(m, n) > 0$; in particular one can effectively solve the Catalan equation $x^m - y^n = 1$ for any given m, n.

We referred earlier to the celebrated Theorem of Siegel on the equation $f(x, y) = 0$. By means of appropriate extensions of the results described above, a new and effective proof of Siegel's theorem in the case of curves of genus 1 has recently been obtained.

Theorem 9 [16]. *Let $F(x, y)$ be an absolutely irreducible polynomial with degree n and with integer coefficients having absolute values at most $\mathcal{H}$ such that the curve $F(x, y) = 0$ has genus 1. Then all integer solutions x, y of $F(x, y) = 0$ satisfy*

$$\max\,(|x|, |y|) < \exp\exp\exp\left\{(2\mathcal{H})^{10^{n^{10}}}\right\}.$$

The proof of the theorem involves a combination of some work of J. Coates [22] on the construction of rational functions on curves with prescribed poles, together with the techniques just mentioned for treating the equation $Y^2 = f(X)$. More precisely it is shown by means of the Riemann–Roch theorem that the integer solutions of $F(x, y) = 0$ can be related by a birational transformation to the solutions of an equation $Y^2 = f(X)$ as above, where now f denotes a cubic in X and where the coefficients and variables denote algebraic integers in a fixed field. From bounds for X, Y and their conjugates we immediately obtain the desired bounds for x, y.

In a recent series of papers [20], [21], Coates has generalized many of the theorems described above by employing analysis in the p-adic domain. In particular he has obtained explicit upper bounds for all integer solutions x, y, $j_1, \ldots, j_s$ of equations of the type

$$f(x,y) = mp_1^{j_1} \cdots p_s^{j_s} \quad \text{and} \quad y^2 = x^3 + kp_1^{j_1} \cdots p_s^{j_s} ,$$

where $p_1, \ldots, p_s$ denote fixed primes. The work involves, amongst other things, utilization of the Schnirelman line integral and the theory of S-units in algebraic number fields. As particular applications of his results, one can now give explicit lower bounds of the type $c(\log\log x)^{1/4}$ for the greatest prime factor of a binary form $f(x,y)$, and one can determine effectively all elliptic curves with a given conductor.

Several other extensions, applications and refinements of the theorems discussed here have been obtained by N. I. Feldman [25], [26], V. G. Sprindžuk [29], [30] and A. I. Vinogradov [31]. Recently Siegel [28] established some improved estimates for units in algebraic number fields which are likely to be of value in reducing the size of bounds. And, in 1967, Brumer [18] derived a natural p-adic analogue of Theorem 1 which, in conjunction with work Ax [1], resolved a well-known problem of Leopoldt on the nonvanishing of the p-adic regulator of an abelian number field.

7. On the Weierstrass Elliptic Functions

Let $\mathcal{P}(z)$ denote a Weierstrass $\mathcal{P}$-function, let g_2, g_3 denote the usual invariants occurring in the equation

$$(\mathcal{P}'(z))^2 = 4(\mathcal{P}(z))^3 - g_2\mathcal{P}(z) - g_3$$

and let ω, ω' denote any pair of fundamental periods of $\mathcal{P}(z)$. Seigel proved in 1932 that if g_2, g_3 are algebraic then at least one of ω, ω' is transcendental; hence both are transcendental if $\mathcal{P}(z)$ admits complex multiplication. Seigel's work was much improved by Schneider in 1937; Schneider showed that if g_2, g_3 are algebraic then any period of $\mathcal{P}(z)$ is transcendental and moreover, the quotient ω/ω' is transcendental except in the case of complex multiplication. Furthermore Schneider proved that if $\zeta(z)$ is the corresponding Weierstrass ζ-function, given by $\mathcal{P}(z) = -\zeta'(z)$, and if $\eta = 2\zeta\big(\frac{1}{2}\omega\big)$ then any linear combination of ω, η with algebraic coefficients, not both 0, is transcendental.

By means of techniques similar to those used in the proof of Theorem 1 these results can now be generalized as follows. Let $\mathcal{P}_1(z)$, $\mathcal{P}_2(z)$ be Weierstrass $\mathcal{P}$-functions (possibly with $\mathcal{P}_1 = \mathcal{P}_2$) for which the invariants g_2, g_3 are algebraic and let $\zeta_1(z)$, $\zeta_2(z)$ be the associated Weierstrass ζ-functions. Further let ω_1, ω_1' and ω_2, ω_2' be any pairs of fundamental periods of $\mathcal{P}_1(z)$, $\mathcal{P}_2(z)$ respectively, and put $\eta_1 = 2\zeta\big(\frac{1}{2}\omega_1\big)$, $\eta_2 = 2\zeta\big(\frac{1}{2}\omega_2\big)$. We have

Theorem 10 [13], [14]. *Any nonvanishing linear combination of* ω_1, ω_2, η_1, η_2 *with algebraic coefficients is transcendental.*

It will be recalled that ω_1, ω_2 and η_1, η_2 can be expressed as elliptic integrals of the first and second kinds respectively and so one sees, for instance, that the theorem establishes the transcendence of the sum of the circumferences of two ellipses with algebraic axes-lengths. Also, by an appropriate refinement of Theorem 10, one can obtain an upper estimate for the values assumed by a $\mathcal{P}$-function with algebraic invariants at an algebraic point. In particular, for any positive integer n, we have $|\mathcal{P}(n)| < C \exp[\log n)^{\kappa}]$ for some absolute constant $\kappa > 0$ and some $C > 0$ depending only on g_2, g_3.[2] The proof of Theorem 10 utilizes results on the division values of the elliptic functions.

8. Concluding Remarks

The three main problems left open by the work discussed here are

(i) To determine effectively all imaginary quadratic fields with a given class number ≥ 2.

(ii) To find an effective algorithm for determining all the integer points on any curve of genus ≥ 2.

(iii) To establish, under suitable conditions, the algebraic independence of the logarithms of algebraic numbers.

The resolution of these problems would represent a considerable advance in our knowledge.

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EFFECTIVE METHODS IN DIOPHANTINE PROBLEMS. II

by

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COMMENTS

1. Introduction

Three years ago, at a conference held in Stony Brook, I surveyed the theories which had then recently been developed for the effective resolution of a diverse collection of Diophantine problems [1] (see also [2]). Since that time, several of the topics have been considerably expanded and I should like to use the opportunity provided by the present Symposium to bring the account up to date.

2. Lower Bounds for Linear Forms

One of the most active fields of research has been concerned with improved bounds for linear forms in the logarithms of algebraic numbers. In particular, much study has been made of the special situation, of considerable importance in applications, when one of the algebraic numbers has a large height relative to the remainder. The primary result obtained in this connexion reads as follows.

Theorem 1. *Let $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_n$ be nonzero algebraic numbers with degrees at most d, let $\alpha_1, \ldots, \alpha_{n-1}$ have heights at most A' and let α_n and $\beta_1, \ldots, \beta_n$ have heights at most A and B respectively. If $\varepsilon > 0, \delta > 0$ and*

$$0 < |\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n| < e^{-\delta H}$$

for some $H > \exp((\log B)^{1/2})$, then $H < C(\log A)^{1+\varepsilon}$, where $C = C(n, d, \varepsilon, \delta, A')$ is effectively computable.

Special cases of the theorem were proved by Stark [21] and myself [3] in connexion with certain class number problems (see §4) and the full result was obtained by a combination of our methods [9]. Previous work, as described in [1], had led to a similar theorem but with $1 + \varepsilon$ replaced by a number greater than $n - 1$. The condition $H > \exp((\log B)^{1/2})$ can be relaxed to $H > (\log B)^{cn^2/\varepsilon}$ for a sufficiently

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large absolute constant c, provided that $\varepsilon < 1$, and, furthermore, one can replace $(\log A)^{\varepsilon}$ by some power of $\log\log A$, though this power is usually large. Very recently, by further developments of the arguments, it has been shown that, in the case when $\beta_1, \ldots, \beta_{n-1}$ are rational integers and $\beta_n = -1$, conditions frequently satisfied in applications, the exponent $1+\varepsilon$ can be replaced by 1, which is best possible.

Theorem 2 [5], [6]. *Let $\alpha_1, \ldots, \alpha_n$ be nonzero algebraic numbers with degrees at most d and let the heights of $\alpha_1, \ldots, \alpha_{n-1}$ and α_n be at most A' and A ($\geqq 2$) respectively. If, for some $\varepsilon > 0$, there exist rational integers $b_1, \ldots, b_{n-1}$ with absolute values at most B such that*

$$0 < |b_1 \log \alpha_1 + \cdots + b_{n-1} \log \alpha_{n-1} - \log \alpha_n| < e^{-\varepsilon B},$$

then $B < C \log A$ for some effectively computable number C depending only on n, d, A' and ε.

Theorem 2 is, in fact, an immediate consequence of another theorem [6] to the effect that there exists $C = C(n, d, A')$ such that, for any δ with $0 < \delta < \frac{1}{2}$, the inequalities

$$0 < |b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n| < (\delta/B')^{C \log A} e^{-\delta B}$$

have no solution in rational integers $b_1, \ldots, b_{n-1}$ and $b_n (\neq 0)$ with absolute values at most B and B' respectively. Clearly, on taking $\delta = 1/B$ and assuming $B' \leqq B$, the number on the right becomes at least $C^{-\log A \log B}$ for some effectively computable C, and this bound is best possible with respect to A when B is fixed and with respect to B when A is fixed. Corollaries relating to the theory of Diophantine equations will be discussed in §3.

The proofs of the above theorems depend upon several new developments in the earlier works. In particular, the underlying auxiliary functions are now considerably more involved; the argument leading to Theorem 2, for instance, utilizes a function of the form

$$\sum_{\lambda_{-1}=0}^{L_{-1}} \cdots \sum_{\lambda_n=0}^{L_n} p(\lambda_{-1}, \ldots, \lambda_n)\ \Lambda(z_0)\ \alpha_1^{\gamma_1 z_1 \cdots} \alpha_{n-1}^{\gamma_{n-1} z_{n-1}},$$

where the $p(\lambda_{-1}, \ldots, \lambda_n)$ are, as usual, rational integers, $\gamma_r = \lambda_r + b_r \lambda_n$ and

$$\Lambda(z) = \Delta(z + \lambda_{-1}; h, \lambda_0 + 1, m_0) \prod_{r=1}^{n-1} \Delta(\gamma_r; m_r),$$

$$\Delta(z; k) = \frac{1}{k!}(z+1)\cdots(z+k), \quad \Delta(z; k, l, m) = \frac{1}{m!}\frac{d^m}{dz^m}(\Delta(z; k))^l.$$

Further, the inductive nature of the expositions is substantially modified and the ultimate contradiction is obtained now by an appeal to certain algebraic lemmas

relating to Kummer theory, quite different from the techniques employed previously. The reader is referred to the original memoirs for details.

3. Diophantine Equations

In my earlier survey, I discussed the fundamental theorem of Thue on $f(x, y) = m$, where f denotes an irreducible binary form with integer coefficients and degree $n \geqq 3$. More especially, I described how the theorem could be made effective, and indeed how one could establish an upper bound

$$\max(|x|, |y|) < C \exp\{(\log m)^{\kappa}\},$$

applicable for all integer solutions x, y, where $\kappa > n$ and C is computable in terms of κ and the coefficients of f. In view of Theorem 2, one can now strengthen the number on the right to Cm^c, where c can be computed like C, and this gives at once

Theorem 3. *For any algebraic number α with degree $n \geqq 3$ there exist positive effectively computable numbers c, κ depending only on α, with $\kappa < n$, such that*

$$|\alpha - p/q| > cq^{-\kappa}$$

for all rationals p/q $(q > 0)$.

Feldman [13] first obtained this result from a special case of Theorem 2, involving certain restrictions on α_n, and his arguments rested on rather different adaptations in the basic theory of linear forms in logarithms; yet another approach, employing p-adic analysis, was described by Sprindžuk [19], [20] at about the same time.

Several other new theorems on rational approximations to algebraic numbers follow from the general result cited after the enunciation of Theorem 2. Thus, for instance, it shows that

$$|\alpha - pp'/qq'| > Q^{-\kappa \log \ \log Q'},$$

where p', q' are comprised solely of powers of fixed sets of primes and Q, Q' are the maxima of the absolute values of p, q and p', q' respectively; this furnishes a further improvement on Ridout's generalization of Roth's theorem (cf. the recent survey of Schmidt [18]). Furthermore one sees that

$$|\alpha^{1/m} - p/q| > cq^{-\kappa \log m}$$

for any algebraic number α, where c, κ are positive numbers effectively computable in terms of α, and this is sharper than the Thue-Siegel inequality when the integer m is large. The latter theorem recalls to mind the very first effective results in this context, derived by means of special properties of Gauss's hypergeometric function (see [1]). When applicable, this method gives surprisingly strong estimates for the solutions of Diophantine equations, and it has not been dormant. In particular, Feldman [12] and Osgood [15], [16] have widely applied ideas of this nature to study effectively certain equations of norm form in several variables.

4. Class Numbers

I described at Stony Brook the transcendental method for determining all the imaginary quadratic fields with class number 1, and I remarked also that the same techniques could be used to treat the analogous problem for class number 2 when the discriminants of the fields are even. Since then, a complete resolution of the class number 2 problem has been obtained, and I should like to indicate the main new idea very briefly. A fuller account is provided by the text of the lecture I delivered a year or so ago in Washington [4].

If $Q((-d)^{1/2})$ has class number 2 and odd discriminant $-d < -15$, then $d = pq$, where p, q are primes congruent to 1 and 3 (mod 4) respectively. Denoting by $\chi'(n)$ one of the generic characters associated with forms of discriminant $-d$ and writing

$$\chi_{pq}(n) = \left(\frac{-pq}{n}\right), \quad \chi_p(n) = \left(\frac{p}{n}\right), \quad \chi_q(n) = \left(\frac{-q}{n}\right), \quad \chi(n) = \left(\frac{k}{n}\right),$$

where k is an integer $\equiv 1 \pmod 4$ and $(k, pq) = 1$, we obtain

$$L(1,\chi)L(1,\chi\chi_{pq}) + L(1,\chi\chi_p)L(1,\chi\chi_q) = \sum \chi(f)/f\,,$$

where $f = f(x, y)$ denotes the principal form with discriminant $-d$, and the sum is over all integers x, y not both 0. Now if k is not a prime power, for instance if $k = 21$, then the sum on the right approximates to a rational multiple of π^2, and on substituting for the L-functions on the left from Dirichlet's formulae we obtain an inequality of the type considered in Theorem 1; this leads at once to the desired effective bound for d.

By somewhat similar techniques, Schinzel and I [8] have recently shown that every genus of primitive binary quadratic forms with discriminant D represents a positive integer $\leqq c(\varepsilon)|D|^{3/8+\varepsilon}$ for any $\varepsilon > 0$, where $c(\varepsilon)$ depends only on ε. Our proof involves Siegel's theorem on L-functions and so does not enable $c(\varepsilon)$ to be effectively computed when $\varepsilon < \frac{1}{8}$; on the other hand, an effective estimate would, as we show, yield a complete determination of all the "numeri idonei" of Euler, and, of course, this would include the class number 1 and 2 results to which I have just referred.

5. Elliptic Functions

The main result on elliptic functions cited in [1] has been extended recently by Coates [11].

Theorem 4. *Any nonvanishing linear combination of* ω_1, ω_2, η_1, η_2 *and* $2\pi i$ *with algebraic coefficients is transcendental.*

Here ω_1, ω_2 denote a pair of fundamental periods of a Weierstrass $\mathcal{P}$-function with algebraic invariants g_2, g_3 and $\eta_1 = 2\zeta\left(\frac{1}{2}\omega_1\right)$, $\eta_2 = 2\zeta\left(\frac{1}{2}\omega_2\right)$, where $\zeta(z)$ denotes the associated Weierstrass ζ-function. The new feature in Theorem 4 is the

inclusion of $2\pi i$, this extension having been gained, however, at the cost of some restriction in the hypotheses. The result is of particular interest in view of the Legendre relation $\omega_1\eta_2 - \omega_2\eta_1 = 2\pi i$, showing that the five numbers in question are algebraically dependent. Furthermore, one sees that the theorem includes the transcendence of such numbers as $\pi + \omega$ and $\pi + \eta$ for any period ω of $\mathcal{P}(z)$ and quasi-period η of $\zeta(z)$.

Some quantitative estimates in connexion with Theorem 4 have recently been derived by a student of mine, D. W. Masser; in particular, he has proved [14]:

Theorem 5. *For any positive integer n and any $\varepsilon > 0$, we have*

$$|\mathcal{P}(n)| < Cn^{(\log\ \log n)^{7+\varepsilon}}$$

where C depends only on g_2, g_3 and ε.

Moreover he has shown that a similar estimate obtains for $\mathcal{P}(\pi + n)$ and indeed for $\mathcal{P}(\alpha)$, where α is any nonzero algebraic number. Theorem 5 compares well with the lower bound $|\mathcal{P}(n)| > Cn$ valid for some $C > 0$ and infinitely many n, and it improves upon the result mentioned in [1], where an unspecified power of $\log n$ occurred in place of $\log\log n$. It seems likely that this general area of study will be considerably developed in the next few years (cf. [10]).

6. Further Results and Problems

In a lecture at the same conference in Stony Brook to which I referred at the beginning, Chowla raised the problem whether there exists a rational-valued function $f(n)$, periodic with prime period p, such that $\sum f(n)/n = 0$. He proved some twenty years ago that this could not hold for odd functions f if $\frac{1}{2}(p-1)$ is prime, a condition subsequently removed by Siegel, and recently he showed that the same is true for even functions f if $f(0) = 0$. In a forthcoming paper [7] by Birch, Wirsing and myself, it is shown that there is in fact no function f with these properties. The arguments involve an appeal to the basic result on the linear independence of the logarithms of algebraic numbers, but otherwise the proof runs on classical lines. Our work enables us to treat more generally functions f that take algebraic values and are periodic with any modulus q, and we prove thereby

Theorem 6. *If $(q, \phi(q)) = 1$ and χ runs through all nonprincipal characters* mod *q then the $L(1, \chi)$ are linearly independent over the rationals.*

Theorem 6 plainly generalizes Dirichlet's famous result on the nonvanishing of $L(1, \chi)$; it does not, however, give a new proof of this result, for the latter is, in fact, utilized in the demonstration. It would be of much interest to know whether the theorem is valid when $(q, \phi(q)) > 1$.

Finally, I should like to discuss some possible future avenues of investigation. First, one would like to have a theorem of the nature of Theorem 1 in which A

denotes the height of all the α's and not just α_n; some work in this direction has been carried out by Ramachandra [17] and his pupil T. N. Shorey, and they have applied their results to certain questions in prime number theory. But, at the moment, the theorems are rather special and one would hope for considerable improvements here. Secondly, it is almost certain that Theorems 1 and 2 have natural p-adic analogues, and these would enable many of the Diophantine results obtained earlier to be strengthened. In particular, they would give an inequality of the form $||(3/2)^n|| > 2^{-\delta n}$, valid for all $n > n_0$, where n_0 is effectively computable, δ is an absolute constant with $0 < \delta < 1$ and $||x||$ denotes the distance of x from the nearest integer. If, moreover, the value of δ were such that $2^{-\delta} > \frac{3}{4}$ then this would settle an outstanding question in connexion with Waring's problem. But, of course, it may be difficult to obtain such a precise value of δ from the present analysis. Thirdly, one would like to obtain a value of κ in Theorem 3 depending only on n and indeed of the same order of magnitude as the Siegel exponent; this would naturally lead to an effective determination of all the integer points on a curve of arbitrary genus, that is, to a complete solution to the first problem mentioned at the end of [1]. Since the magnitude of κ depends on the value of C in Theorem 2, this again reflects on the basic theory of linear forms in logarithms. And lastly, one would like an extension of Theorem 2 in which $b_1, \ldots, b_{n-1}$ denote arbitrary algebraic numbers and not merely rational integers; this too seems difficult to obtain with our present techniques.

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EFFECTIVE METHODS IN THE THEORY OF NUMBERS / DIOPHANTINE PROBLEMS

Although the theory described in the preceding papers has progressed greatly in the intervening 25 years or so since they were published and modern surveys of the field would look very different, it became apparent on reading through them that I could not possibly reflect again the novelty of the results and the excitement of their discovery that is evident in these works. They have therefore been left as they stand and I give now a short note to indicate some of the main developments.

The basic theory of logarithmic forms has been much refined. The best result to date, at least as far as the fundamental rational case is concerned, is proved in Baker and Wüstholz [3]. It is shown that if

$$\Lambda = b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n \neq 0\,,$$

in the now familiar notation, then

$$\log |\Lambda| > -(16nd\,)^{2(n+2)} \log A_1 \cdots \log A_n \log B\,,$$

where the A's signify the respective classical heights of the α's and B denotes a bound for the absolute values of the b's; further, as usual, d signifies the degree of the number field generated by the α's over the rationals. A still stronger result is given in [3] in terms of the logarithmic Weil height; this has several nice properties, for instance, it is semi-multiplicative, and it has become the standard height employed in the field. The proof of the inequality depends on many new ingredients, among them Kummer theory, Δ-functions, successive minima, Blaschke products and multiplicity estimates on group varieties.

The latter estimates have themselves been the outcome of a remarkable series of discoveries. They originate from techniques involving commutative algebra introduced by Nesterenko [17] in connection with studies on E-functions; they were developed by Brownawell and Masser [4], Masser and Wüstholz [11], Philippon [18] and especially by Wüstholz [29]. He obtained, in 1983, the critical result extending a zero estimate on group varieties appertaining to a single differential operator to arbitrarily many such operators. The work has found widespread application; in particular it has yielded an abelian analogue of the famous Lindemann theorem and it has solved a long-standing problem concerning periods of elliptic integrals (see [26], [27], [28]). The key result is now the analytic subgroup theorem established in Wüstholz [30]; this furnishes a generalisation in terms of algebraic groups of the

basic qualitative theorem on logarithmic independence that customarily bears my name. Among the many sequels, Masser and Wüstholz have used this sphere of ideas to yield an isogeny result which makes effective one of the principal components in Faltings' well known proof of the Mordell conjecture (see [12] to [16]).

Another extensive area of application of logarithmic forms has been in connection with the solution of Diophantine equations. Indeed it now provides the standard method for effectively solving algebraic equations in two integer variables (see [1]). Wider aspects, appertaining for instance to norm form, discriminant form and index form equations, have been brought to light, most notably by Győry (see the abundant literature cited in [2]). My original work in this field proceeded by way of the S-unit equation and used linear forms in complex logarithms as above; this would still seem to be the most universal method. Recently, however, a number of writers have developed an alternative approach that, when applicable, is more direct (see [8], [23]); it is based on linear forms in elliptic logarithms. Here the best result to date is due to Hirata–Kohno [9] and the explicit version needed for determining the complete list of points on the relevant Diophantine curve has been worked out by S. David [5]. In either case, one meets the old problem of computing all solutions of a Diophantine inequality below a rather large bound; Davenport and I described a method of dealing with the problem based on a simple lemma in Diophantine approximation and two further computational techniques have been described subsequently, one due to Grinstead and Pinch [19], based on recurrence sequences, and the other due to Tzanakis and de Weger [24] based on an algorithm of A. K. Lenstra, H. W. Lenstra and L. Lovász. Now that the constants in our estimates for $\log |\Lambda|$ are much reduced and computers generally have become more powerful, these methods are seen to be very efficient — specific and fully worked instances are given e.g. in Gaál *et al.* [6], [7] and Tzanakis and de Weger [25]. It should be remarked that the p-adic theory of linear forms in logarithms also features in the practical solution of Diophantine equations; here the most precise results to date are due to Yu Kunrui [31]. Moreover, the fact that our lower bound for $\log |\Lambda|$ depends now on each of the $\log A$'s only to the first power, which is best possible, has led to the effective analysis of the remarkable class of exponential Diophantine equations; see the tract by Shorey and Tijdeman [21] and, for the Catalan equation in particular, the book by Ribenboim [20].

The theory has been applied in other diverse areas. They include p-adic L-functions (Ax and Brumer), Knot Theory (Riley), Modular Forms (Odoni), Ramanujan Functions (Murty) and Primitive Divisors (Schinzel and Stewart). Further, there is a close connection with the so-called *abc*-conjecture of Oesterlé and Masser (see [22]); in fact, before the conjecture itself was formulated, studies on logarithmic forms had already led Mason [10] to a statement and proof of its analogue for function fields. Indeed it has become evident that there is now considerable interplay between transcendence theory and many aspects of arithmetical algebraic geometry and this would seem to be a major trend for the future.

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Additional Comment (added in proof) The first of the three problems was solved by Goldfeld, Gross and Zagiet via the theory of elliptic curves; see *Bull. Amer. Math. Soc.* **13** (1985), 23–37, *Invertiones Math.* **84** (1986), 225–320. The other two problems remain open.

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THE WORK OF SERGE NOVIKOV

by
M. F. ATIYAH

It gives me great pleasure to report on the work of Serge Novikov. For many years he has been generally acknowleged as one of the most outstanding workers in the fields of Geometric and Algebraic Topology. In this rapidly developing area, which has attracted many brilliant young mathematicians, Novikov is perhaps unique in demonstrating great originality and very powerful technique both in its geometric and algebraic aspects.

Novikov made his first impact, as a very young man, by his calculation of the unitary cobordism ring of Thom (independently of similar work by Milnor). Essentially Thom had reduced a geometrical problem of classification of manifolds to a difficult problem of homotopy theory. Despite the great interest aroused by the work of Thom this problem had to wait several years before its successful solution by Milnor and Novikov. Many years later Novikov returned to this area and, combining cobordism with homotopy theory, he developed some very powerful algebraic machinery which gives one of the most refined tools at present available in Algebraic Topology. In his early work it was a question of applying homotopy to solve the geometric problem of cobordism; in this later work it was the reverse, cobordism was used to attack general homotopy theory.

On the purely geometric side I would like to single out a very beautiful and striking theorem of Novikov about foliations on the 3-dimensional sphere. Perhaps I should remind you that a foliation of a manifold is (roughly speaking) a decomposition into manifolds (of some smaller dimension) called the leaves of the foliation: one leaf passing through each point of the big manifold. If the leaves have dimension one then we are dealing with the trajectories (or integral curves) of a vector field, and closed trajectories are of course particularly interesting. In the general case a basic question therefore concerns the existence of *closed leaves*. Very little was known about this problem. Thus even in the simplest case of a foliation of the 3-sphere into 2-dimensional leaves the answer was not known until Novikov, in 1964, proved that every foliation in this case does indeed have a closed leaf (which is then necessarily a torus). Novikov's proof is very direct and involves many delicate geometric arguments. Nothing better has been proved since in this direction.

Undoubtedly the most important single result of Novikov, and one which combines in a remarkable degree both algebraic and geometric methods, is his

famous proof of the topological invariance of the Pontrjagin classes of a differentiable manifold. In order to explain this result and its significance I must try in a few minutes to summarize the history of manifold theory over the past 20 years. Fortunately, during this Congress you will be able to hear many more detailed and comprehensive surveys.

There are 3 different kinds or categories of manifold: differentiable, piece-wise linear (or combinatorial) and topological. For each category the main problem is to understand the structure or to give some kind of classification. There was no clear idea about the distinction between these 3 categories until Milnor produced his famous example of 2 different differentiable structures on the 7-sphere. After that the subject developed rapidly with important contributions from many people, including Novikov, so that in a few years the distinction between differentiable and piece-wise linear manifolds, and their classification, was very understood. However, there were still no real indications about the status of topological manifolds. Were they essentially similar to piece-wise linear manifolds or were they quite different? Nobody knew. In fact, there were no known invariants of topological manifolds except homotopy invariants. On the other hand, there were many invariants known for differentiable or piece-wise linear manifolds which were finer than homotopy invariants. Notable among these were the Pontrjagin classes. For a differentiable manifold these are cohomology classes which measure, in some sense, the amount of global twisting in the tangent spaces. For a manifold with a global parallelism like a torus they are zero. In the context of Riemannian geometry there is a generalized Gauss–Bonnet theorem which expresses them in terms of the curvature. In any case their definition relies heavily on differentiability. Around 1957 it was shown by Thom, Rohlin and Svarc, using important earlier work of Hirzebruch, that the Pontrjagin classes are actually piece-wise linear invariants (provided we use rational or real coefficients). When Novikov, in 1965, proved their topological invariance this was the first real indication that topological manifolds might be essentially similar to piece-wise linear ones. It was a big break-through and was quickly followed by very rapid progress which, in the past few years, through the work of many mathematicians — notably Kirby and Siebenmann — has resulted in fairly complete information about the topological piece-wise linear situation. Thus we now know that nearly all topological manifolds can be triangulated and essentially in a unique way. You will undoubtedly hear about this in the Congress lectures.

Perhaps you will understand Novikov's result more easily if I mention a purely geometrical theorem (not involving Pontrjagin classes) which lies at the heart of Novikov's proof. This is as follows:

Theorem (*). — If a differentiable manifold X is homeomorphic to a product $M \times R^n$ (where M is compact, simply-connected and has dimension ≥ 5) then X is diffeomorphic to a product $M' \times R^n$.

*This formulation is due to L. Siebenmann.

Here both M, M' are differentiable manifolds. The theorem thus asserts that a topological factorization implies a differentiable factorization: it is clearly a deep result. Combined with the earlier Thom–Hirzebruch work it leads easily to the invariance of the Pontrjagin classes.

I hope I have now indicated the importance of this result of Novikov's and its place in the general development of manifold theory. I would like also to stress the remarkable nature of the proof which combines very ingenious geometric ideas with considerable algebraic virtuosity. One aspect of the geometry is particularly worth mentioning. As is well-known many topological problems are very much easier if one is dealing with simply-connected spaces. Topologists are very happy when they can get rid of the fundamental group and its algebraic complications. Not so Novikov! Although the theorem above involves only simply-connected spaces, a key step in his proof consists in perversely introducing a fundamental group, rather in the way that (on a much more elementary level) puncturing the plane makes it non-simply-connected. This bold move has the effect of simplifying the geometry at the expense of complicating the algebra, but the complication is just manageable and the trick works beautifully. It is a real master stroke and completely unprecedented. Since then a somewhat analogous device has proved crucial in the important work of Kirby mentioned earlier.

I hope this brief report has given some idea of the real individuality of Novikov's work, its variety and its importance, all of which fully justifies the award of the Fields Medal. It is all the more remarkable when we remember that he worked in relative isolation from the main body of mathematicians in his particular field. We offer him our heartiest congratulations in the full confidence that he will continue, for many years to come, to produce mathematics of the highest order.

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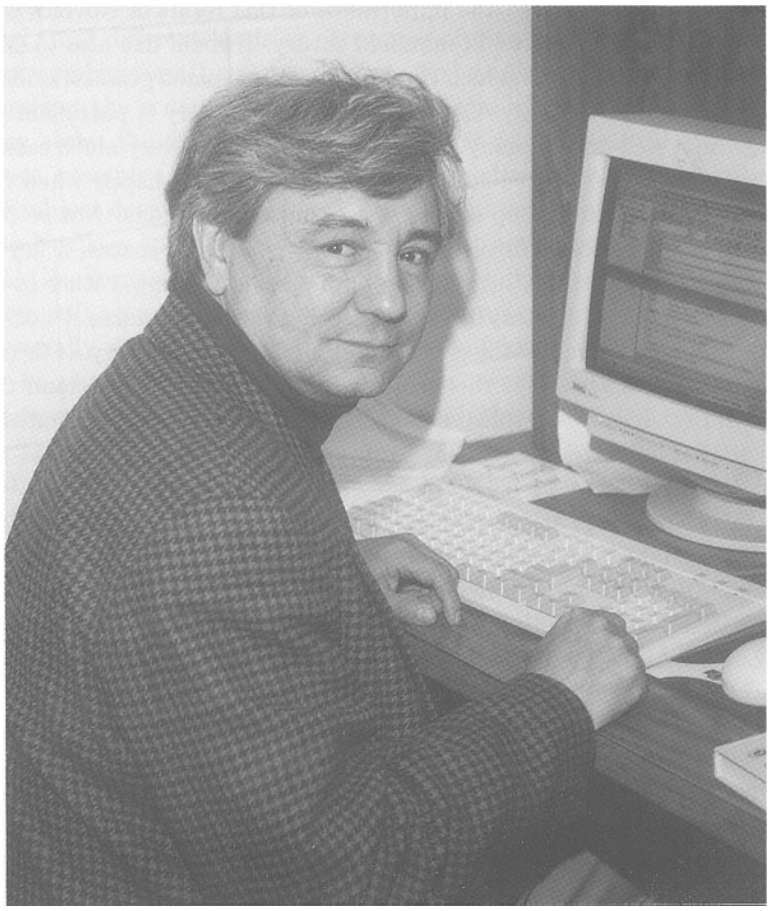

Sergei Novikov

SERGEI NOVIKOV

Novikov was born on March 20, 1938 in Gorki (Nizni Novgorod), into a family of outstanding mathematicians. His father, Petr Sergeevich Novikov (1901–1975), was an academician, an outstanding expert in mathematical logic, algebra, set theory, and function theory; his mother, Lyudmila Vsevolodovna Keldysh (1904–1976), was a professor, a well-known expert in geometric topology and set theory. Novikov received his mathematical education in the Faculty of Mathematics and Mechanics of Moscow University (1955–1960), and he has worked there since 1964 in the Department of Differential Geometry; since 1983 he has been head of the Department of Higher Geometry and Topology of Moscow University.

Novikov married Eleanor Tsoi (who also received her mathematical education at Moscow State University, 1955–1960). They have three children: two daughters, Irina (1964) and Maria (1965), and a son, Peter (1973).

In 1960 Novikov enrolled as a research student at the Steklov Institute of Mathematics, where his supervisor was M. M. Postnikov; since 1963 he has been on the staff there. He was awarded the degree of Ph.D. there in 1964, and that of Doctor of Science in 1965. In 1966 he was elected Corresponding Member of the Academy of Sciences of the USSR, and in 1981 a full member. Since 1984 he has been head of the Department of Geometry and Topology of the Mathematical Institute of the Academy of Sciences and in charge of the problem committee of *Geometriya I topologiya* (Geometry and Topology) at the Mathematics Division of the Academy of Sciences of the USSR. He has been head of the Mathematics Division at the L. D. Landau Institute of Theoretical Physics of the Academy of Sciences since 1971, where he works closely with the physicists. During the period 1985–1996 Novikov served as President of the Moscow Mathematical Society, and during 1986–1990 he was also a Vice-President of the International Association in Mathematical Physics.

In the Gorbachev–Elzyn era, Novikov started to visit western countries more actively. Before that it was very difficult for Soviet scientists to attend scientific conferences abroad. In 1970 Novikov was not permitted to attend a ceremony at the International Congress of Mathematicians in Nice where he was awarded the Fields Medal for his work in topology because he had signed some letters defending dissidents. In 1991 he was able to work for the first time in his life for a half year in Paris at the Laboratory of Theoretical Physics, Ec. Norm. Superior de Paris. Beginning in 1992 he regularly worked at the University of Maryland at College Park as a Visiting Professor. In September 1996 he became a full time professor at the University of Maryland at College Park in the Department of Mathematics

and the Institute for Physical Science and Technology (IPST). He also continues to work in Moscow for a period during the winter and summer.

Since 1971 his scientific work has played an important part in building a "bridge" between modern mathematics the theoretical physics. Some of Novikov's papers can be divided as follows:

Papers before 1971

Methods of calculating stable homotopy groups, Complex cobordism theory.

The classification of smooth simply-connected manifolds of dimension $n \geq 5$ with respect to diffeomorphisms. Topological Invariance of rational Pontryagyin classes, higher signatures.

The qualitative theory of foliations of codimension 1 on three-dimensional manifolds.

Papers after 1971

Methods of qualitative theory of dynamical systems in the theory of homogeneous cosmological models (of spatially homogeneous solutions to Einstein equations).

Periodic problems in the theory of solitons (non-linear waves) and in the spectral theory of linear operators, Riemann surfaces and Θ-functions in mathematical physics.

The Hamiltonian formalism of completely integrable systems, Hamiltonian hydrodynamic type systems, and applications of Riemannian geometry.

Ground states of a two-dimensional non-relativistic particle with spin 1/2 in a doubly-periodic topologically non-trivial magnetic field. Topological invariants for generic operators; Laplace transformations and exactly solvable two-dimensional Schrödinger operators in magnetic fields, and a discrete analogue of this theory.

Multi-valued functional in mechanics and quantum field theory. Analogue of the Morse theory for the closed 1-forms; foliations given by the closed 1-forms; the special case of 3-torus; and applications in the quantum theory of normal metal, observable topological numbers.

Analogues of the Fourier–Laurent series on Riemann surfaces, Virasoro algebras, operator construction of string theory.

Novikov's main area of current scientific interests: Geometry, Topology and Mathematical Physics.

Awards and Honors

1966–1981	Corresponding member of the Academy of Sciences of the USSR
1967	Lenin Prize
1970	Fields Medal of the International Mathematical Union

1981	Lobachevskii International Prize of the Academy of Sciences of the USSR
1981	Full Member of the Academy of Sciences of the USSR
1987	An Honorary Member of the London Math. Society
1988	Honorary Member of the Serbian Academy of Art and Sciences
1988	Honorary Doctor of the University of Athens
1991	Foreign Member of the "Academia de Lincei", Italy
1992	Member of Academia Europea
1994	Foreign Member of the National Academy of Sciences of the USA
1996	Member of Pontifical Academy of Sciences (Vatican)

Students of Sergei Novikov

More than 30 of Novikov's students have been awarded the Candidate Degree (equivalent to Ph.D.), and of these V. M. Buchstaber, A. S. Mishchenko, O. I. Bogoyavalenskii, I. M. Krichever, B. A. Dubrovin, G. G. Kasparov, F. A. Bogomolov, S. P. Tsarev, I. A. Taimanov, A. P. Veselov M. A. Brodskii, V. V. Vedenyapin, R. Nadiradze, V. L. Golo, S. M. Gusein–Zade have been awarded the degree of Doctor of Science (Scientific Degree, equivalent to the level of full professor in the former USSR and in Russia).

In addition to those mentioned above, other pupils of Novikov with the Candidate Degree (corresponding to Ph.D. level in the West) include I. A. Volodin, N. V. Panov, A. L. Brakhman, P. G. Grinevich, O. I. Mokhov, A. V. Zorich, F. A. Voronov, G. S. D. Grigoryan, A. S. Lyskova, E. Potemin, M. Pavlov, L. Alania, D. Millionshikov, V. Peresetski, I. Dynnikov, A. Maltsev, V. Sadov, Le Tu Thang, S. Piunikhin and A. Lazarev.

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RÔLE OF INTEGRABLE MODELS IN THE DEVELOPMENT OF MATHEMATICS

by
SERGEI NOVIKOV

The history of mathematics and theoretical physics shows that the starting ideas of the best mathematical methods were discovered in the process of solving integrable models.

Mathematical discoveries of the last twenty years will be especially discussed as by-products of the famous integrable systems of the soliton and quantum theories.

Starting as a Topologist

Before discussing certain models of mathematical physics and explaining their rôle in the development of modern mathematics, I want to say a few words about my own experiences as a mathematician. Let me start by reconstructing my career, not from the point when I was awarded the Fields Medal, but much earlier.

I started my mathematical life working in algebraic topology, and continued in this area for more than ten years; in fact, I still consider myself first as an algebraic topologist. When I started doing mathematics, in the mid-fifties, Russia was a very dark country, living behind the iron curtain. However, we had a very large and powerful mathematical school, whose leading person at that time was Andrei Kolmogorov in Moscow. He was the greatest mathematician, I think, after Poincaré, Hilbert and Hermann Weyl. A lot of famous mathematicians were his former students: Gel′fand, Arnol′d and many others (not me).

There was a common point of view in the Moscow mathematical school, concerning what was important and what was not important. The "important" areas of science were set theory, logic, functional analysis, and partial differential equations (not in the sense of solving models, but in the sense of proving rigorous theorems and establishing foundations). In Russian mathematics of that period — as in French mathematics — the main goal was to construct some kind of axiomatization, and the leading mathematicians were pursuing that. Topology was not existent in Russia in that period; there were only some remains of Pontrjagin's scientific school.

At the end of 1956, I was a second-level undergraduate student and had to choose one area — at least for some time — in order to be able to participate in seminars. I was attracted by an announcement posted in the Faculty of Mathematics and Mechanics of Moscow University. It was signed by Postnikov, Boltyanskiĭ, and

Albert Shvarts. (The latter was a graduate student, but he was not considered as a "young mathematician"; in Russia, people aged 25 were not "young" in that period.) It was written in the announcement that there was a very new and exciting science, namely modern algebraic topology, opposite the nonsense of point-set topology (maybe my translation is not very exact). These people were punished after that announcement; especially Shvarts (Postnikov and Boltyanskiĭ were professors, so life was somehow easier for them). Paul Alexandrov, the famous topologist — who just continued a science which was thirty years old at that time — was terrified. So there was no place for Shvarts to continue his job in Moscow University, and he had to leave. Then he started learning about Fredholm operators; later, he moved to quantum physics and participated in the discovery of instantons, in cooperation with Polyakov. He was the first to discover nontrivial topological quantum field theory, ten years ago. In some sense, he followed the same way in science as me, but he was the most active during that period.

My friends were Arnol′d and Sinaĭ, who were children of Kolmogorov's seminar, and Anosov, a child of Pontrjagin's seminar in control theory. Some people asked me why I was trying to learn such a strange science, which was "completely useless", instead of studying important sciences like probability or partial differential equations. Thus topology was completely outside of the interest of our community in Russia. Postnikov told me that there were no prospects in topology, yet I could perhaps find something in cohomological algebra. Only Shvarts was enthusiastic about topology; however, he left Moscow very soon after he finished his thesis.

I published my first paper when I was 21. I was not "young" at that period because people like, for example, Arnol′d, wrote their first papers at age 18 or 19. This was completely normal. I come from a mathematical family, and my mother complained that "Everybody has published scientific papers, except my son".

I first worked in homotopy theory. Postnikov and Dyn′kin had made a very good translation of a collection of famous papers, mainly by French mathematicians: Serre, Cartan, Thom, ... We learnt them in our seminar. I was impressed by the excellent papers of the leading person in homotopy theory at that time: Frank Adams (who died recently). He started as an extremely brilliant scientist, solving famous problems.

This was a very interesting period. For example, nontrivial Hopf algebras — which are now very popular in the framework of quantum groups — were discovered during that time, shortly after the axiomatization of the work of Hopf by Armand Borel. The first persons who wrote papers about Hopf algebras were Adams and Milnor; before them, Hopf algebras were just cohomology rings of H-spaces and Lie groups. My first papers were dedicated to applications of Hopf algebras to the computation of homotopy groups of spheres and Thom complexes, which are important in cobordism theory. After that, I moved to the theory of differentiable manifolds, under the influence of several people who started visiting Russia at that period. John Milnor, Fritz Hirzebruch and Steve Smale helped me to start differential topology. Dynamical systems also started, in connection with topology

and new kinds of algebra, after Milnor and Smale. When I write my memoirs, I will write something about all this, because I have found inaccuracies in many historical articles and books. For example, Smale's influence on the crucial point of the theory of dynamical systems in structural stability started in Russia. This relevant fact is missing in the historical literature, yet I was a direct witness of it.

Anosov and myself, together with Arnol′d, Sinaĭ, Shafarevich and Manin, organized a group of people who learnt different branches of mathematics from each other. Later Gel′fand's group joined us. People from partial differential equations started to interact with us after the discovery of the index of operators in the early sixties by Wolpert a strange person from Bielorussia who appeared in Moscow at that time. This was done before the Atiyah–Singer paper; in fact, Atiyah and Singer wrote their paper after the publicity that Gel′fand made of Wolpert's achievements.

Topology started to be recognized as something serious more or less after 1961. The main question that I wanted to answer was "For what are we working?". As I said, I had good connections. I consulted friends like Arnol′d and Anosov; as a result, I got involved with foliations, in connection with problems of dynamical systems which were originated by Smale. Other friends helped me in connection with index problems from Gel′fand's school, by teaching me about partial differential equations — I also wrote something in that area — and learning topology in their turn. People working in algebraic geometry were also extremely useful; they helped me in the use of certain algebraic concepts in topology. However, I found out very soon that, no matter how far I was moving into mathematics, I was not able to answer my basic question, concerning the goal of what we were doing. I found that the theory of partial differential equations was as abstract as topology, and probability even more (I never worked in probability, but my fried Sinaĭ explained this to me; he moved from that area to dynamical systems himself). Dynamical systems was a much more beautiful and newer area; however, it played no rôle in the real world either, because nobody knew enough about it; it was too hard for people working in natural sciences in that period.

The Split between Mathematics and Physics

Arnol′d taught me, in his seminar, analytical mechanics and elements of hydrodynamics, in the framework of classical mechanics (not of quantum mechanics). Indeed, in the mid-twenties, after the creation of quantum mechanics, there was a very serious split between mathematics and physics in Russia (and not only in Russia). The best mathematicians of Kolmogorov's period — with a very few exceptions — never knew even the mathematical language of theoretical physics. The new language of theoretical physics started to be constructed more or less in 1925. According to physicists of that period, the crucial point in the divergence between mathematics and physics was not the creation of relativity — the rôle of relativity was realized later — but the creation of quantum theory. In Moscow, Gel′fand was probably the only one who learnt the new physics. Sometimes

physicists participated in Gel′fand's seminar; however, in the late fifties Gel′fand stopped this job completely and physicists disappeared from that seminar for as long as twenty years.

I know a lot of childish tales from mathematicians about physicists, and from physicists about mathematicians, normally based on a lack of information. I have continually heard them for the last thirty years, even from great mathematicians or from the best physicists. I heard the latest yesterday: René Thom — one of my great teachers — spoke in his talk about the weakness of the "Landau theory of turbulence," which was pointed out to him by Arnol′d (who is also my friend, and with whom I have a lot of family connections). This was typically caused by a lack of information; one should not worry too much about these things happening. The story is that Landau never had any theory of turbulence. Landau had been interested in hydrodynamics since 1940, for twenty years at least, starting from his famous papers in superfluidity. For twenty-five years he was the only person who claimed that turbulence was purely a dynamical effect, a result of some global dynamics. It is a stupid idea, he said, to think that Navier–Stokes in nondeterministic: If one looks at a turbulent flow locally along time, one immediately observes that it is a very well defined flow; thus it is not reasonable to say that it is "something nondeterministic." Landau produced the idea that it is the result of something global, as a dynamical system. The weak point of his ideas was that nothing serious was known in the physics community about new complicated examples of dynamical systems (even in pure mathematics they appeared relatively late). He said that these were perhaps some complicated infinite-dimensional tori embedded in a functional space. This may be the origin of the tale. (I should add that I have no interest in criticizing bad mathematicians. The criticism is only interesting if it is addressed to a good scientist. I would be happy if there is a revenge with the same weapons.)

Arnol′d, who is in some sense my teacher in questions of mechanics, started coming to my lectures in the early sixties, when I was 23 years old (I remember that he was one of the three people who attended my lectures). He was shocked by the idea of *transversality* and *generic position*. Transversality was a completely new concept, even for people who were famous in the theory of functions of real variables in 1961. Thus Arnol′d also learned something from me, while I learnt mechanics from him.

He told me that Kolmogorov proposed him to improve the result which is now called the *Kolmogorov–Arnol′d–Moser theorem*. Kolmogorov found the basic ideas and invited Arnol′d to continue them and furnish a rigorous proof. Kolmogorov also asked him to learn mechanics. Thus he read a lot of books, starting from Appell and some Russian books written by people in classical mechanics; however, as he said, he could not understand what mechanics really was. Then he found the book of Landau and Lifshits (which was not yet famous at that time among the mathematical community). He told me that, after reading this book, he finally understood what mechanics was, and, after that, he understood how bad the book

was. Arnol′d himself wrote a brilliant book on mathematical understanding of classical mechanics. I would honestly say that I do not like that book, because he completely reconstructed the ideology. The book of Landau and his school was just a starting point to develop a great science; it contained many initial points allowing further progress. In Arnol′d's reconstruction, the mathematics is, of course, much better — it is a very good book for pure mathematicians —, but starting points for future research areas are missing. People who read Arnol′d's book arrive at an endpoint.

Learning Physics

My friend Manin had the same views as me about those books. We both independently decided, at the same time, to start learning quantum physics (my brother* used to tell me that mathematicians should know everything about quantum theory). I first tried to learn quantum field theory as mathematicians normally do, and found out that this task was completely impossible. It might even be stupid to do so. Instead, I very much like the style of Einstein's lectures or the best lectures of Landau. I understood that *naturality* was the base of that science, exactly as I had earlier realized in topological books. The topologists of that period, like Jean–Pierre Serre, René Thom or John Milnor, sometimes omitted definitions in their lectures; they just said "This definition is natural." I recognized this style of "naturality" in the best physicists; in the lectures of Einstein, in the best books of Landau. (Not all books of Landau are equally good, but the collection of all of them is very valuable. Our students who want to do theoretical physics must know all these books at the age of 22. It is their common starting point.)

We discussed with Manin some paradoxes and unclear features of quantum theory, about which we shared a common point of view. I remember Manin telling me that every mathematician would find unclear points, but it would be a mistake to stop at those points and stand on criticism against them. Many mathematicians, including my students, have important difficulties in learning theoretical physics. They want to learn it as if it were mathematics: If they find something that they do not understand, they stop. I may definitely say that physicists also find a lot of nonunderstandable things; however, one must go ahead and think about such things only after having done a lot of exercises and reached a certain level.

It is very difficult to carry out Hilbert's program and to write theoretical physics in an axiomatic style. Hilbert did important work after Einstein's discovery of general relativity. He realized that the Einstein equation was an Euler–Lagrange equation for some functional. Thus he confirmed, in the case of the Einstein equation, that the axioms for any fundamental physics theory have to be started from a Lagrangian principle. Hilbert's program was useful for Hilbert himself, because he

*Leonid Keldysh, a leading quantum solid state physicist in Russia.

used it in that way. However, I do not like the experiences of some of my friends — extremely good mathematical physicists — who work in Hilbert's program, trying to make physics rigorous. This is, I think, impossible. One may prove a good theorem here, a good theorem there, about some physical situations. However, I think that Richard Feynman is completely right when he claims that it cannot be done globally (perhaps it is sometimes possible locally). The development of physics is more rapid than the flux of theorems which try to axiomatize it. The percentage of things which may be done rigorously is going to zero; the number of good theorems is increasing, but the ratio is going down very rapidly.

I spent at least five years, between 1965 and 1970, just learning physics. Sometimes, half of my working time was dedicated to learning it as a student, from the earliest books of Landau, Lifshits, Einstein and others. In 1970, I wanted to make contact with physicists and people from Landau's school (they worked in the newborn Landau Institute, which was created in 1965). There were increasing rumours among physicists — even among engineers — that something very interesting was being developed, namely algebraic topology (of course, things like dynamical systems were also "topology" for them). Isaac Khalatnikov, who was director of the Institute, informed me that people at Landau Institute had some very good problems in general relativity. They needed a topologist, so I joined them. We started working together at the end of 1970. At that time, I knew general relativity, which I had learnt earlier as a part of differential geometry. (But not in mathematical books; definitely, the best books on general relativity are not mathematical. It is better to read books by physicists, even in order to understand what is the best mathematics therein. Einstein's lectures are suitable to start with. Also the books of Landau–Lifshits, Misner, Thorne, are extremely good.)

First Contributions to the Domain of Physics

The first paper which I was able to write was a joint paper with my collaborator Bogoyavlenskiĭ. He was my student at that time. We worked together in general relativity and just applied our knowledge about dynamical systems. My acquaintance with people in dynamical systems, which started ten years before, became fruitful in that period.

In this situation, I worked in the theory of homogeneous cosmological models, studying the space of homogeneous solutions of the Einstein equation, which leads to complicated dynamical systems, especially near cosmological singularities. There is an extremely unusual feature in this area, even from the point of view of the theory of dynamical systems. It is completely concrete; the ideas of genericity cannot be applied.

Physicists normally make the following criticism about such remarkable papers as the one by Ruelle and Takens. It is very good to construct abstractly nice examples of complicated dynamical systems. (A lot of such results were given before Ruelle and Takens, starting from Marston Morse in the thirties. Smale

and our people — like Anosov, Arnol′d and Sinaĭ— also found a lot of remarkable examples of this kind.) But when dealing with real Navier–Stokes equations or with real Einstein equations, the question is "Is this situation realizable or not?" I may definitely say that it has never been realized up to now in hydrodynamics.

Even remarkable specialists in ergodic theory, like Sinaĭ, were not able to understand, for ten years, how such Hamiltonian systems — like Einstein equations of homogeneous cosmological models — could lead to nontrivial ergodic properties, i.e., to nontrivial strange attractors. As far as I know, this is a unique example of a strange attractor which can be investigated analytically.

There is a particularly interesting geometry in the phase space describing the dynamical system of general relativity. In fact, some people from the Landau school had discovered these anomalous regimes before us; but nobody believed them, not even in the community of theoretical physicists. (I am not speaking about mathematics now; that job was totally done within the physics community.)

Bogoyavlenskiĭ and myself developed a certain technique of dynamical systems. As a special case of our technique, we were able to compare these strange anomalous regimes with strange attractors. They were computed analytically — not numerically — because of this very special strange geometry. But they were not *generic* attractors. In concrete systems, generic ideology does not work, because fundamental systems always have some important hidden symmetry.

We worked rigorously. This does not mean that we proved any theorem rigorously describing attractors. Rather, our skeleton was constructed in such a way that there were points with zero Lyapunov eigenvalues. We worked rigorously in the sense that our computations were done without arithmetical mistakes, in the process of solving those models.

At that time, my former student Bogoyavlenskiĭ presented his second dissertation in order to become full professor. It was the continuation of those methods on dynamical systems (later, he published a book in English about qualitative methods in gas dynamics and general relativity). The famous physicist, Zel′dovich, was the person who evaluated Bogoyavlenskiĭ's papers. He said that remarkable results had been obtained, not only in relativity, but also in gas dynamics. Moreover, after the official speech, Zel′dovich expressed to me unofficially how much he liked that technique, and how complicated it is for astrophysicists: nobody will use it for a long period! I have to say that, at that time, even the elementary Poincaré two-dimensional plane qualitative theory was considered as a high-level theory in the physics community. Only the best people were able to use it. The level of nonlinear science in the community of the best theoretical physicists was very low.

Modern Developments in Topology

My own work at the Landau Institute has been divided into two parts. One part was dedicated mainly to topology (I was paid to work as a topologist, and people just consulted me about "modern mathematics"). A lot of topology appeared in this

period, also related to instantons. I remember my fried Polyakov — who is six years younger than me — visiting me and asking "Did you hear anything about characteristic classes?" I told him that this was a very trivial concept from differential geometry. After I displayed the elementary formulae for dimension 4 (using tensor language), he said "It is quadratic!" Next day, he told me that he had found what had become the famous self-duality equation. Characteristic classes were quadratic, so it was possible to combine them with the Yang–Mills functional in that way!

I also interacted as a consultant with other people, leading to a famous discovery about the rôle of homotopy theory in liquid crystals and in anomalous kinds of superconductors in the early seventies. Fifty years before that, people had made interesting observations in optical experiments, like singularities in liquid crystals such as cholesterine (this is something that you can buy in a pharmacy and freeze up to -150 degrees in order to observe phases which display singularities). Nobody was able to realize before the early seventies that this phenomenon can be, in fact, explained by elementary homotopy theory. There were two groups working independently in that direction around 1974: one was in France; the other was at the Landau Institute.

I was not able to find anything substantially new and interesting in topology for myself until 1980. Before that, topology worked as it had been built. The new era began when topology started to apply discoveries of physicists inside topology itself. There was a huge noise in the physics community about topology during the seventies; also among applied physicsts, working in low-temperatures physics, liquid crystals, and related fields. It was a common opinion of physicists that the main new thing which physics borrowed from mathematics in the last ten years was topology. (You may find this assertion, for example, in an article written in the early 80's by Anderson, the famous applied physicist in superconductivity, and Nobel Prize winner). A lot of new things indeed appeared during the eighties, as I will next explain.

Thus I started producing new work in topology in the past decade only. Before that I sold my knowledge about topology to physicists. In fact, dynamical systems was more my own way of solving models. The new direction in my work started with the discovery of soliton theory, which was done in the late sixties. It was an extremely interesting finding, which led, among other things, to modern integrable systems, conformal field theory, and quantum group theory (as a late by-product).

Integrable Models in Classical Mathematics and Mechanics

Everyone knows the rôle of the famous two-body problem, solved by Newton, in the development of the mathematical methods of physics. For a long period after that, people used the method of the exact analytic solution for some differential equations as a principal tool in mathematical physics. They simplified their problem if it was (or looked) too difficult, and after that tried to find the exact solution.

A lot of work has been done in the process of searching for special "integrable cases" of famous problems, like the motion of the top, for example. All mathematical methods — like power and trigonometric series, Fourier–Laplace (and other) integral transformations, complex analysis and symmetry arguments — were discovered and developed for that in the nineteenth century. These methods led sometimes to remarkable negative results, i.e., to proofs that certain models are not solvable in principle.

Some strange integrable cases which do not admit any obvious symmetry were discovered in the nineteenth century: the integrability of geodesics on 2-dimensional ellipsoids in Euclidean 3-space (Jacobi), the motion of the top with special parameters for constant gravity (Kovalevskaya), and some others. Riemann surfaces and θ-functions of genus 2 played the leading rôle in their integrability. What kind of hidden symmetry can be found behind this? It was not completely clear until the discoveries of soliton theory.

We have to add to these discoveries a new understanding of the deep hidden algebraic symmetry in the two-body problem, based on the so-called Laplace–Runge–Lentz vector of integrals. The "hidden" group generated by it acts on the energy levels in the phase space; this group is isomorphic to SO(4) for the negative energy levels corresponding to the closed elliptic orbits, and isomorphic to SO(3, 1) for the positive levels corresponding to the hyperbolic noncompact orbits.

Generic spherically symmetric forces lead to the existence of nonperiodic orbits in an arbitrarily small neighborhood of any periodic orbit in the phase space (which is 6-dimensional for a particle in 3-space). The Kepler two-body problem is exceptional; all orbits are closed for any negative energy. As a consequence, any small spherically symmetric perturbation of the gravity forces leads to the famous displacement of the perihelium. This displacement was extremely important for the astronomical testing of celestial mechanics and general relativity. The group SO(3) is not enough for the periodicity of orbits! A very large hidden symmetry is needed here.

We shall discuss later the fundamental rôle of the same symmetry for applications of quantum mechanics to the structure of atoms, discovered in the 1920's by Pauli.

Integrable Models in Quantum and Statistical Mechanics (1925–1965)

The integrable models of classical mechanics were forgotten by almost everyone except a few very narrow experts in these very classical problems. A new era of qualitative methods started with Poincaré. But the very new important branches of modern physics, like special and general relativity and quantum mechanics, were discovered in the first quarter of the twentieth century. Some initial problems for the Einstein and Schrödinger equations were also explicitly solved in the classical style.

For two particles with opposite charges, the nonrelativistic electric force is mathematically the same as the gravity force. The exact solution of this quantum nonrelativistic "two-body" problem preserves the same type of hidden symmetry as in classical mechanics; there is the quantized Laplace–Runge–Lentz vector commuting with the Hamiltonian (energy operator) and generating the Lie algebra of the group SO(3, 1) for positive energy and SO(4) for negative energy. As a consequence, the spectrum of this quantum system (the Balmer spectrum) is more degenerate than the spherical symmetry requires (the group SO(3)). The stationary localized states have negative energy and can be considered as the quantized periodic orbits for the classical Kepler-type problem for electric forces. Together with the famous Pauli principle (two electrons cannot occupy the same state in any family of all possible states which present the orthogonal basis), this spectrum leads to an approximate explanation of the Mendeleev classification of elements.

One may observe that hidden symmetry of this sort is an exceptional property of the τ^{-2}-type forces (and also of the linear forces) but these exceptional cases appear in the most fundamental problems of classical and quantum theory as a first good approximation. There is a common belief in the community of theoretical physicists that *the most fundamental mathematical principles of physics should be described* (*at least in some good first approximation*) *by objects containing enormously large hidden symmetry.*

It is very difficult to classify the wide collection of concrete problems solved by physicists in the process of developing quantum theory between 1925 and 1965. We want to point out the well-known results of Bethe and Onsager, who solved some multiparticle one-dimensional quantum models and the models of statistical mechanics. The famous *Bethe Ansatz* for the construction of eigenfunctions was discovered and the two-dimensional Ising model was solved. But the influence of these discoveries on the mathematical methods of physics was fully recognized only much later, after the discovery of soliton theory in the process of quantization of its methods.

Soliton Theory

In the early sixties, people working in the theory of plasma observed that the Korteweg–de Vries (KdV) equation appears as the universal first approximation for the propagation of waves in many nonlinear media, combining nonlinearity and dispersion (if viscosity can be neglected). Before that, KdV was known for many years only as a very special system for waves in shallow water. People started to investigate the KdV equation in the sixties, and the most important discovery was made in the papers of Kruskal–Zabusky (1965), Gardner–Greene–Kruskal–Miura (1967), Lax (1968). This highly nontrivial system is in a sense exactly integrable by a very strange and new procedure (*inverse scattering transform*).

Let us be more precise now. The KdV system has a form

$$u_t = 6uu_x - u_{xxx}\,.$$

Its simplest solutions, known since the nineteenth century, are the *solitons*

$$u(x-vt) = -\frac{3a}{\mathrm{ch}^2(3a)^{\frac{1}{2}}(x+12\,at)}$$

and the *knoidal waves*

$$u(x-vt) = 2\wp(x-vt) + \text{constant}\,.$$

The soliton is localized and the knoidal wave is periodic in x. Here $\wp$ is the doubly-periodic Weierstrass elliptic function, whose degeneration is exactly the soliton.

Consider the Sturm–Liouville operator and the third-order operator

$$L = -\partial_x^2 + u(x,t)\,, \qquad A = -3\partial_x^3 + 4u\partial_x + 2u_x\,.$$

Their commutator

$$[L,A] = 6uu_x - u_{xxx} = Q_1$$

is multiplication by the function Q_1. It means that the *Lax equation*

$$\frac{\partial L}{\partial t} = [L,A]$$

is formally equivalent to the nonlinear KdV equation.

Inverse Scattering Transform

The famous GGKM procedure may be immediately deduced from the Lax equation. Suppose all functions $u(x,t)$ are *rapidly decreasing* for $x \to \pm\infty$. Consider the special solutions of the linear Sturm–Liouville equation with exponential asymptotics

$$L\psi = \lambda\psi\,, \quad L\varphi = \lambda\varphi\,, \quad k^2 = \lambda\,,$$

$$\psi_\pm \to e^{\pm ikx}\,, \quad x \to -\infty\,,$$

$$\varphi_\pm \to e^{\pm ikx}\,, \quad x \to +\infty\,.$$

There is a unimodular transformation from the basis ψ to φ:

$$\varphi_+ = a\psi_+ + b\psi_-$$

$$\varphi_- = \bar{b}\psi_+ + \bar{a}\psi_-$$

$$|a|^2 - |b|^2 = 1\,.$$

The coordinates $a(k)$, $b(k)$ determine the so-called *scattering data* for the Schrödinger operator L with localized potential. The potential may be completely reconstructed by the *inverse scattering transform* from the function $[a(k),\ b(k)]$ with the proper analytical properties, plus a finite number of "discrete data," as was known long ago in the fifties.

The GGKM theorem states that:

(a) $\dfrac{da(k)}{dt} = 0\,, \quad \dfrac{db(k)}{dt} = (ik)^3 b(k).$

(b) The discrete eigenvalues are the integrals of motion.

(c) Local densities of the integrals of motion for KdV may be constructed in the following way. Consider the formal solution for the Riccati equation

$$\chi_x + \chi^2 = u - \lambda\,, \quad \chi = k + \sum_{n\geq 1} \frac{P_n(u, u_x, \dots)}{(2k)^n}\,.$$

Then the integrals

$$I_m = \int P_{2m+3}\, dx$$

are the local conservative quantities for KdV.

(d) Exact (multisoliton) solutions will be obtained from the *reflexionless potentials* $b(k) \equiv 0$.

KdV Hierarchy. Hamiltonian Properties. Generalization

There is an infinite number of operators

$$A_0 = \partial_x\,, \quad A_1 = A\,, \quad \dots \quad A_n = \partial_x^{2n+1} + \cdots\,, \quad \dots$$

such that the commutator $[L, A_n]$ is multiplication by a certain polynomial

$$Q_n(u, u_x, u_{xx}, \ \dots, \ u_{2n+1})\,.$$

The *KdV hierarchy* is the collection of nonlinear systems

$$\frac{\partial u}{\partial t_n} = Q_n\,,$$

equivalent to the collection of Lax-type equations

$$\frac{\partial L}{\partial t_n} = [L, A_n]\,.$$

All these flows commute with each other; we may find a common solution

$$u(x, t_1, t_2, t_3, \ \dots\ )\,, \quad t_0 = x\,.$$

The detailed study of the polynomials Q_n was done in the late sixties by Gardner. In particular they have the following form (*Gardner form*)

$$Q_n = \tfrac{\partial}{\partial x}\left(\tfrac{\delta I_{n+1}}{\delta u(x)}\right)$$

$$I_{-1} = \textstyle\int u\, dx\,, \quad I_0 = \int u^2\, dx\,, \quad I_1 = \displaystyle\int \left(\tfrac{u_x^2}{2} + u^3\right) dx\,, \quad \dots$$

As observed by Gardner, Zakharov and Faddeev (1971), this is the form of the Hamiltonian equation corresponding to the GZF–Poisson bracket

$$\{u(x), u(y)\} = \delta'(x-y)$$

and to the Hamiltonian

$$H_n = I_{n+1}\,.$$

The inverse scattering transform may be treated as a functional analog of the transformation from $u(x)$ to the action-angle variables as in analytical mechanics, which were useful for the semiclassical quantization (Zakharov and Faddeev).

The quantization program was started after 1975 by Faddeev, Takhtadzhyan, Sklyanin and others in Leningrad. Different groups starting from 1971 discovered many new interesting nonlinear systems integrable by the Lax equation and inverse scattering transform. Some famous systems — well-known before — were solved by that method (for example, nonlinear Schrödinger, sine–Gordon, discrete Volterra system or discrete KdV, Toda lattice and many others, including some special two-dimensional systems which proved to be very important later).

Starting from 1976, different groups found other interesting properties and generalizations of the Hamiltonian formalism of KdV theory. Interesting new Poisson brackets with large hidden algebraic symmetry were discovered; for example, the Lenart–Magri second bracket for KdV, and the Gel′fand–Dikiĭ brackets for the generalizations of KdV hierarchy to the scalar operators of higher order.

The Periodic and Quasiperiodic Problems. Riemann Surface and θ-Functions

The appropriate approach to solving the periodic problem for KdV was found by the present author (1974). After that, it was completely solved by Novikov and Dubrovin (1974), Lax (1975), It.s and Matveev (1975), Mckean and Van Moerbeke (1975) and by Krichever (1976) for the (2 + 1)-system of KP-type (Kadomtsev–Petviashvili). The basis of this approach is the KdV-invariant class of "finite-gap" Schrödinger operators with periodic and quasiperiodic potentials, whose spectrum on the line $\mathbf{R}$ has a finite number of gaps only. These potentials satisfy the stationary KdV and higher KdV equations

$$\left[L, \sum_{j=0}^{N} c_j\, A_j\right] = 0\,,$$

where N is the number of finite gaps.

This system is a completely integrable finite-dimensional Hamiltonian system. Its solution was found explicitly in terms of the θ-functions of a Riemann surface whose genus is equal to N:

$$u(x,t) = \text{constant} - 2\partial_x^2 \log\theta\,(U_x + V_y + U_0)\,,$$

where U, V are certain N-vectors.

Finite-gap potentials form a dense family in the space of continous periodic functions (Marchenko–Ostrovskiĭ, 1977) and even in the space of quasiperiodic functions. For the periodic potential $u(x+T) = u(x)$ the Schrödinger operator L commutes with the shift $\hat{T} : x \to x + T$. Therefore, there is a common eigenfunction for the operators L and $\hat{T}$ (which is familiar in solid state physics)

$$L\Psi = \lambda\Psi$$
$$\hat{T}\Psi = e^{ipT}\Psi\,.$$

Here $p = p(\lambda)$ is some multivalued function of λ and the spectrum is exactly the set of all λ such that $p(\lambda) \in \mathbf{R}$.

There are exactly two Bloch eigenfunctions $\Psi_\pm$ for each complex value of λ (except for a countable number of branching points). The function Ψ is meromorphic on some hyperelliptic Riemann surface Γ (two-covering of the λ-plane), generically of infinite genus. For the real potential $u(x)$ on the line $\mathbf{R}$ the branching points are real; they exactly coincide with the endpoints of the spectrum. This is the *spectral curve* Γ.

The generic real nonsingular solution of the commutativity equation

$$\left[L, \sum_{j=0}^{N} c_j\, A_j\right] = 0$$

is quasiperiodic (it contains a dense family of periodic solutions with different periods). All of them have the spectral curve Γ of finite genus N and vice-versa. The potential $u(x)$ and the eigenfunction Ψ may be expressed in terms of θ-functions corresponding to the surface Γ. There was a very interesting formal algebraic investigation of the commuting ordinary differential operators in the twenties (Burchnall and Chaundy); a Riemann surface was discovered based on the polynomial relation between L and A, which they found for any commuting pair, with no formal connection between periodicity and our surface. There is a theorem stating that, for operators of relatively prime orders with periodic coefficients, these two surfaces in fact coincide. It may be not so in more complicated cases.

The general elementary idea between the appearance of Riemann surfaces in the theory of finite-dimensional integrable systems can now be explained. It is very useful to relate the original Lax representation for KdV-type systems to the compatibility condition of the two linear 2×2 systems whose coefficients depend on λ (*zero curvature equation*):

(a) LAX EQUATION (1968)
$$\frac{\partial L}{\partial t} = [L, A]\,.$$

(b) ZERO CURVATURE EQUATION (1974, for KdV hierarchy and sine–Gordon)

$$\frac{\partial \Psi}{\partial t} = \Lambda(\lambda)\Psi$$
$$\frac{\partial \Psi}{\partial x} = Q(\lambda)\Psi$$
$$\frac{\partial \Lambda}{\partial x} - \frac{\partial Q}{\partial t} = [\Lambda, Q]\,.$$

For the ordinary KdV we have:

$$Q = \begin{pmatrix} 0 & 1 \\ u-\lambda & 0 \end{pmatrix}$$

$$\Lambda_1 = \Lambda = \begin{pmatrix} -u_x & 2u+4\lambda \\ -4\lambda^2 + 2\lambda u - u_{xx} + 2u^2 & u_x \end{pmatrix}.$$

For higher KdV we have the same Q; the corresponding matrices Λ_n are polynomial in λ. For the stationary equation ($\partial_t = 0$) we have " Lax-type" representations for the finite-dimensional system

$$\frac{\partial \Lambda}{\partial x} = [Q, \Lambda]\,.$$

The Riemann surface Γ appears here. It has genus one for the ordinary stationary KdV, and the Weierstrass elliptic function appears:

$$\det(\Lambda - \mu I) = P(\lambda, \mu) = 0\,.$$

The function $\psi(x, P)$ is meromorphic on the surface Γ. Its coefficients are the integrals of the system (they do not depend on x),

For KdV and higher KdV, all matrices Λ_n have zero trace and the Riemann surfaces are hyperelliptic:

$$\mu^2 = R_{2n+1}(\lambda)\,,$$

where n is the genus of Γ.

The function ψ was explicitly found from the "algebro-geometrical data": The Riemann surface Γ and the poles of Ψ (Ψ has exactly n poles, which do not depend on x after the proper normalization). This approach works for all known nontrivial integrable systems. As is now know, this mechanism is valid also for the classical systems mentioned above (Jacobi, Clebsch, Kovalevskaya, ...). An important discovery was made by Krichever (1976), who improved this approach and found the generalizations to the (2 + 1)-systems like KP. *All Riemann surfaces appear in the theory of KP*. This last property is very important for applications of these ideas to different problems (like Schottky-type problems in the theory of θ-functions, analogs of Fourier–Laurent bases on Riemann surfaces — which are useful in string theory —, the formalism of τ-functions, etc). *The classical functional constructions were greatly extended after the periodic theory of solitons.*

Conclusion

To conclude this long discussion, we now present the collection of the different branches of mathematics and theoretical physics which were involved in the integrable soliton models. Important new connections between them were discovered.

(1) Nonlinear waves in continuum media (including plasma and nonlinear optics).
(2) Quantum theory; scattering theory and periodic crystals.
(3) Hamiltonian dynamics.
(4) Algebraic geometry of Riemann surfaces and Abelian varieties (θ-functions).

Quantization of the soliton methods led to the discovery of the *quantum inverse transform*, started by Faddeev, Sklyanin, Takhtadzhyan, Zamolodchikov, Belavin and others. An important new object, the so-called *Yang–Baxter equation*, started to play the leading rôle. Later (Sklyanin, Drinfel′d, Jimbo) Hopf algebras appeared here with the special *universal R-matrix* of Yang–Baxter, which led to the now very popular *quantum groups*.

Recall that from the self-dual Yang–Mills equation (Polyakov and others) the theory of instantons appeared, with remarkable applications in four-dimensional topology (Donaldson and other people in Atiyah's school). The theory of Yang–Baxter equations led to the Jones polynomials in the theory of knots. The famous 2-dimensional conformal field theories also gave us the remarkable collection of integrable models which are also the most beautiful new algebraic objects. They have deep connections with soliton theory and quantum groups.

As a result, I may formulate the following thesis: *A large part of the most important discoveries in mathematics and in the mathematical methods of physics was done in the process of developing the theory of integrable models.*

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Prepared from the author's text and the videotape of the talk, by Regina Martinez.

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THE WORK OF DAVID MUMFORD

by
J. TATE

It is a great pleasure for me to report on Mumford's work. However I feel there are many people more qualified than I to do this. I have consulted with some of them and would like to thank them all for their help, especially Oscar Zariski.

Mumford's major work has been a tremendously successful multi-pronged attack on problems of the existence and structure of varieties of moduli, that is, varieties whose points parametrize isomorphism classes of some type of geometric object. Besides this he has made several important contributions to the theory of algebraic surfaces. I shall begin by mentioning briefly some of the latter and then will devote most of this talk to a discussion of his work on moduli.

Mumford has carried forward, after Zariski, the project of making algebraic and rigorous the work of the Italian school on algebraic surfaces. He has done much to extend Enriques' theory of classification to characteristic $p > 0$, where many new difficulties appear. This work is impossible to describe in a few words and I shall say no more about it except to remark that our other Field's Medallist, Bombieri, has also made important contributions in this area, and that he and Mumford have recently been continuing their work in collaboration.

We have a good understanding of divisors on an algebraic variety, but our knowledge about algebraic cycles of codimension > 1 is still very meager. The first case is that of 0-cycles on an algebraic surface. In particular, what is the structure of the group of 0-cycles of degree 0 modulo the subgroup of cycles rationally equivalent to zero, i.e., which can be deformed to 0 by a deformation which is parametrized by a line. This group maps onto the Albanese variety of the surface, but what about the kernel of this map? Is it "finite-dimensional"? Severi thought so; but Mumford proved it is not, if the geometric genus of the surface is $\geqq 1$. Mumford's proof uses methods of Severi, and he remarks that in this case the techniques of the classical Italian algebraic geometers seem superior to their vaunted intuition. However, in other cases Mumford has used modern techniques to justify Italian intuition, as in the construction by him and M. Artin of examples of unirational varieties X which are not rational, based on 2-torsion in $H^3(X, \boldsymbol{Z})$.

Probably Mumford's most famous result on surfaces is his topological characterization of nonsingularity. Let P be a normal point on an algebraic surface V in a complex projective space. Mumford showed that if V is topologically a manifold

at P, then it is algebraically nonsingular there. Indeed, consider the intersection K of V with a small sphere about P. This intersection K is 3-dimensional and if V is a manifold at P, then K is a sphere and its fundamental group is trivial. Mumford showed how to compute this fundamental group $\pi_1(K)$ in terms of the diagram of the resolution of the singularity of V at P, and then he showed that $\pi_1(K)$ is not trivial unless the diagram is, i.e., unless V is nonsingular at P. A by-product of this proof is the fact that the Poincaré conjecture holds for the 3-manifolds which occur as K's. Mumford's paper was a critical step between the early work on singularities of branches of plane curves (where K is a torus knot) and fascinating later developments. Brieskorn showed that the analogs of Mumford's results are false in general for V of higher dimension. Consideration of the corresponding problem there led to the discovery of some beautiful relations between algebraic geometry and differential topology, including simple explicit equations for exotic spheres.

Let me now turn to Mumford's main interest, the theory of varieties of moduli. This is a central topic in algebraic geometry having its origins in the theory of elliptic integrals. The development of the algebraic and global aspects of this subject in recent years is due mainly to Mumford, who attacked it with a brilliant combination of classical, almost computational, methods and Grothendieck's new scheme-theoretic techniques.

Mumford's first approach was by the 19th century theory of invariants. In fact, he revived this moribund theory by considering its geometric significance. In pursuing an idea of Hilbert, Mumford was led to the crucial notion of "stable" objects in a moduli problem. The abstract setting behind this notion is the following: Suppose G is a reductive algebraic group acting on a variety V in projective space $\boldsymbol{P}_N$ by projective transformations. Then the action of G is induced by a linear and unimodular representation of some finite covering G^* of G on the affine cone $\boldsymbol{A}^{N+1}$ over the ambient $\boldsymbol{P}_N$. Mumford defines a point $x \in V$ to be *stable* for the action of G on V, relative to the embedding $V \subset \boldsymbol{P}_N$, if for one (and hence every) point $x^* \in \boldsymbol{A}^{N+1}$ over x, the orbit of x^* under G^* is closed in $\boldsymbol{A}^{N+1}$, and the stabilizer of x^* is a finite subgroup of G^*. His fundamental theorem is then that the set of stable points is an open set V_s in V, and V_s/G is a quasi-projective variety.

For example, suppose $V = (\boldsymbol{P}_n)^m$ is the variety of ordered m-tuples of points in projective n-space and G is PGL_n acting diagonally on V via the Segre embedding. Then a point $x = (x_1, x_2, \ldots, x_m) \in V$ is stable if and only if for each proper linear subspace $L \subset \boldsymbol{P}_n$, the number of points $x_i \in L$ is strictly less than $m(\dim L + 1)/(n+1)$. In case $n = 1$, for example, this means that an m-tuple of point on the projective line is unstable if more than half the points coalesce. The reason such m-tuples must be excluded is the following: Let $P_t = (tx_1, tx_2, \ldots, tx_r, x_{r+1}, \ldots, x_m)$ and $Q_t = (x_1, \ldots, x_r, t^{-1}x_{r+1}, \ldots, t^{-1}x_m)$, where the x_i are pairwise distinct. Then P_t is in the same orbit as Q_t, for $t \neq 0, \infty$, but $P_0 = (0, \ldots, 0, x_{r+1}, \ldots, x_m)$ is not

in the same orbit as $Q_0 = (x_1, \ldots, x_r, \infty, \ldots, \infty)$ unless $m = 2r$, and even then is not in general. Thus if we want a separated orbit space in which $\lim_{t\to 0}$ (Orbit P_t) is unique, we must exclude P_0 or Q_0; and it is natural to exclude the one with more than half its components equal.

Using the existence of the orbit spaces V_s/G, Mumford was able to construct a moduli scheme over the ring of integers for polarized abelian varieties, relative Picard schemes (following a suggestion of Grothendieck), and also moduli varieties for "stable" vector bundles on a curve in characteristic 0. The meaning of stability for a vector bundle is that all proper sub-bundles are less ample than the bundle itself, if we measure the ampleness of a bundle by the ratio of its degree to its rank. In the special example $V = (\boldsymbol{P}_n)^m$ mentioned above, the results can be proved by explicit computations which work in any characteristic and even over the ring of integers. But in its general abstract form Mumford's theory was limited to characteristic 0 because his proofs used the semisimplicity of linear representations. He conjectures that in characteristic p, linear representations of the classical semisimple groups have the property that complementary to a stable line in such a representation there is always a stable hypersurface (though not necessarily a stable hyperplane which would exist if the representation were semisimple). If this conjecture is true[1] then Mumford's treatment of geometric invariant theory would work in characteristic p. Seshadri has proved the conjecture in case of SL_2. He has also shown recently that the conjecture can be circumvented, by giving different more complicated proofs for the main results of the theory which work in any characteristic.

For moduli of abelian varieties and curves, Mumford has given more refined constructions than those furnished by geometric invariant theory. In three long papers in *Inventiones Mathematicae* he has developed an algebraic theory of theta functions. Classically, over the complex numbers, a theta function for an abelian variety A can be thought of as a complex function on the universal covering space $H_1(A, \boldsymbol{R})$ which transforms in a certain way under the action of $H_1(A, \boldsymbol{Z})$. For Mumford, over any algebraically closed field k, a theta function is a k-valued function on $\Pi_{l\in S} H_1(A, \boldsymbol{Q}_l)$ (étale homology) which transforms in a certain way under $\Pi_{l\in S} H_1(A, \boldsymbol{Z}_l)$. Here S is any finite set of primes $l \neq \mathrm{char}\,(k)$, though in treating some of the deeper aspects of the theory Mumford assumed $2 \in S$. In order to get an idea of what these theta functions accomplish let us consider a classical special case. Let A be an elliptic curve over $\boldsymbol{C}$ with its points of order 4 marked. Then we get a canonical embedding $A \hookrightarrow \boldsymbol{P}^3$ via the theta functions $\theta \begin{bmatrix} a \\ b \end{bmatrix}$; $a, b = 0, 1$. Let 0_A be the origin on A, whose coordinates in $\boldsymbol{P}_3$ are the "theta Nullwerte". Then A is the intersection of all quadric surfaces in $\boldsymbol{P}_3$ which pass through the orbit of 0_A under a certain action of $(\boldsymbol{Z}/4\boldsymbol{Z}) \times (\boldsymbol{Z}/4\boldsymbol{Z})$ on $\boldsymbol{P}_3$. Thus 0_A

[1](ADDED DURING CORRECTION OF PROOFS). The conjecture is true; shortly after the Congress, it was proved by W. Haboush in general and by E. Formanek and C. Procesi for $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$.

determines A and can be viewed as a "modulus". Moreover, 0_A lies on the quartic curve $\theta\begin{bmatrix}0\\0\end{bmatrix}^4 = \theta\begin{bmatrix}0\\1\end{bmatrix}^4 + \theta\begin{bmatrix}1\\0\end{bmatrix}^4$ in the plane $\theta\begin{bmatrix}1\\1\end{bmatrix} = 0$, and that curve minus a finite set of points is a variety of moduli for elliptic curves with their points of order 4 marked. Mumford's theory generalizes this construction to abelian varieties of any dimension, with points of any order $\geqq 3$ marked, in any characteristic $\neq 2$. The moduli varieties so obtained have a natural projective embedding, and their closure in that embedding is, essentially, an algebraic version of Satake's topological compactification of Siegel's moduli spaces. Besides these applications to moduli, the theory gives new tools for the study of a single abelian variety by furnishing canonical bases for all linear systems on it.

Next I want to mention briefly p-adic uniformization. Motivated by the study of the boundary of moduli varieties for curves, i.e., of how nonsingular curves can degenerate, Mumford was led to introduce p-adic Schottky groups, and to show how one can obtain certain p-adic curves of genus $\geqq 2$ transcendentally as the quotient by such groups of the p-adic projective line minus a Cantor set. The corresponding theory for genus 1 was discovered by the author, but the generalization to higher genus was far from obvious. Besides its significance for moduli, Mumford's construction is of interest in itself as a highly nontrivial example of "rigid" p-adic analysis.

The theta functions and p-adic uniformization give some insight into what happens on the boundary of the varieties of moduli of curves and abelian varieties, but a much more detailed picture can now be obtained by Mumford's theory of toroidal embeddings. This theory, which unifies ideas that had appeared earlier in the works of several investigators, reduces the study of certain types of varieties and singularities to combinatorial problems in a space of "exponents". The local model for a toroidal embedding is called a torus embedding. This is a compactification $\overline{V}$ of a torus V such that the action of V on itself by translation extends to an action of V on $\overline{V}$. The coordinate ring of V is linearly spanned by the monomials $x^a = x_1^{a_1}\, x_2^{a_2}\, \cdots x_n^{a_n}$, $n = \dim V$, with positive or negative integer exponents a_i. Viewing the exponent vectors a as integral points in Euclidean n-space, define a *rational cone* in that space to be a set consisting of r's such that $(r, a) \geqq 0$ for $a \in S$, where S is some finite set of exponent vectors. For each rational cone σ, the monomials x^a such that $(r, a) \geqq 0$ for all $r \in \sigma$ span the coordinate ring of an affine variety $V(\sigma)$ which contains V as an open dense subvariety, if σ contains no nonzero linear subspace of $\boldsymbol{R}^n$. Now if we decompose $\boldsymbol{R}^n$ into the union of a finite number of rational cones σ_α in such a way that each intersection $\sigma_\alpha \cap \sigma_\beta$ is a face of σ_α and σ_β, then the union of the $V(\sigma_\alpha)$ is a compactification of V of the type desired. All such compactifications $\overline{V}$ of V can be obtained in this way and the invariant sheaves of ideals on them can be described in terms of the decomposition into cones. One can also read off whether $\overline{V}$ is nonsingular, and if it is not one can desingularize it by suitably subdividing the decomposition. In short, there is a whole dictionary for

translating questions about the algebraic geometry of V and $\overline{V}$ into combinatorial questions about decompositions of $\boldsymbol{R}^n$ into rational cones.

Mumford with the help of his coworkers has used these techniques to prove the following semistable reduction theorem. If a family of varieties X_t over $\boldsymbol{C}$, in general nonsingular, is parametrized by a parameter t on a curve C, and if X_{t_0} is singular, then one can pull back the family to a ramified covering of C in a neighborhood of t_0 and blow it up over t_0 in such a way that the new singular fiber is of the stablest possible kind, i.e., is a divisor whose components have multiplicities 1 and cross transversally. The corresponding problem in characteristic p is open. For curves in characteristic p the result was proved by Mumford and Deligne and was a crucial step in their proof of the irreducibility of the moduli variety for curves of given genus.

Toroidal embeddings can also be used to construct explicit resolutions of the singularities of the projective varieties $\overline{D/\Gamma}$, where D is a bounded symmetric domain, Γ is an arithmetic group, and the bar denotes the "minimal" compactification of Baily and Borel. The construction of these resolutions is a big step forward. With them one has a powerful tool to analyse the behavior of functions at the "boundary", compute numerical invariants, and, generally, study the finer structure of these varieties.

I hope this report, incomplete as it is, gives some idea of Mumford's achievements and their importance. I heartily congratulate him on them and wish him well for the future!

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David Mumford

AUTOBIOGRAPHY OF DAVID MUMFORD

I was born in 1937 in an old English farm house. My father was British, a visionary with an international perspective, who started an experimental school in Tanzania based on the idea of appropriate technology and worked during my childhood for the newly created United Nations. He imbued me with old testament ideas of your obligation to fully use your skills and I learned science with greed. My mother came from a well-to-do New York family. I grew up on Long Island Sound, and went to Exeter and Harvard.

At Harvard, a classmate said "Come with me to hear Professor Zariski's first lecture, even though we won't understand a word" and Oscar Zariski bewitched me. When he spoke the words 'algebraic variety', there was a certain resonance in his voice that said distinctly that he was looking into a secret garden. I immediately wanted to be able to do this too. It led me to 25 years of struggling to make this world tangible and visible. Especially, I became obsessed with a kind of passion flower in this garden, the moduli spaces of Riemann. I was always trying to find new angles from which I could see them better. There were other amazing people in this garden, the acerbic and daunting Andre Weil, Alexander Grothendieck who truly seemed to be another species, and my fellow travelers, Artin, Griffiths and Hironaka, who made it seem progress was possible. I stayed at Harvard from my freshman year through most of my career as this was the world center for algebraic geometry.

At Radcliffe I met Erika Jentsch. She rescued me from what might have been an all too private world and we had a beautiful family of four children, Stephen, Peter, Jeremy and Suchitra. We spent summers in Maine, in the woods and on the sea sailing along the coast. She became a poet, kept winning prizes and I learned to go places as 'the spouse'. My life changed in the 80's. Erika passed away. I turned from algebraic geometry to an old love — is there a mathematical approach to understanding thought and the brain? This is applied mathematics and I have to say I don't think theorems are very important here. I met remarkable people who showed me the crucial role played by statistics, Grenander, Geman and Diaconis. My mother always said, as you get older, your horizons expand. With my second wife, Jenifer Gordon, we share seven children, now grown up with dreams and families of their own, and I took a new job at Brown, which is the world center for this approach to intelligence. The article reproduced here is a survey of this theory, meant to convince my mathematical colleagues I hadn't gone mad.

Oct. 17, 1996

PATTERN THEORY: A UNIFYING PERSPECTIVE

by
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1.1. Introduction

The term "Pattern Theory" was introduced by Ulf Grenander in the 70'sas a name for a field of applied mathematics which gave a theoreticalsetting for a large number of related ideas, techniques and resultsfrom fields such as computer vision, speech recognition, image andacoustic signal processing, pattern recognition and its statisticalside, neural nets and parts of artificial intelligence (see Grenander1976–81). When I first began to study computer vision about ten yearsago, I read parts of this book but did not really understand hisinsight. However, as I worked in the field, every time I felt I sawwhat was going on in a broader perspective or saw some theme whichseemed to pull together the field as a whole, it turned out that thistheme was part of what Grenander called pattern theory. It seems to menow that this is the right framework for these areas, and, as thesefields have been growing explosively, the time is ripe for making anattempt to reexamine recent progress and try to make the ideas behindthis unification better known. This article presents pattern theoryfrom my point of view, which may be somewhat narrower than Grenander's,updated with recent examples involving interesting new mathematics.

The problem that Pattern Theory aims to solve, which I would like tocall the 'Pattern Understanding Problem', may be described as follows:

the analysis of the patterns generated by the world in anymodality, with all their naturally occurring complexity and ambiguity,with the goal of reconstructing the processes, objects and events thatproduced them and of predicting these patterns when they reoccur.

These patterns may occur in the signals generated by one of the basic animal senses. For example *vision* usually refers to the analysis of patterns detected in the electromagnetic signals of wavelengths 400–700 nm incident at a point in space from different directions. *Hearing* refers to the analysis of the patterns present in the oscillations of 60–20,000 hertz in air pressure at a point in space as a function of time, both with and without human language. On the other hand, these patterns may occur in the data presented to a higher processing stage, in so-called cognitive thought. As an example, *medical expert systems* are concerned with the analysis of the patterns in the symptoms, history and tests presented by a patient: this is

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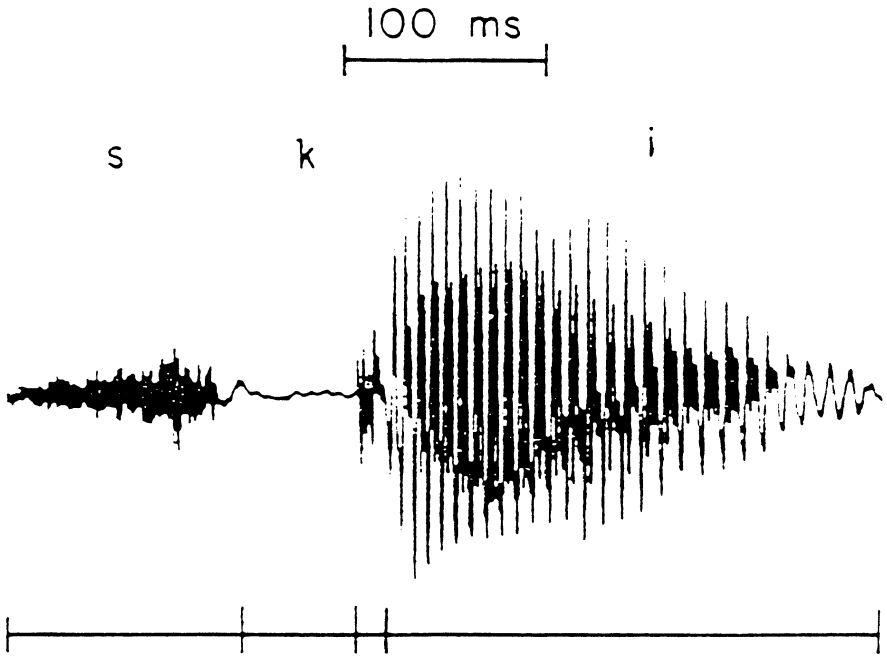

Figure 1.1 Acoustic waveform for an utterance of the word *SKI*.

a higher level modality, but still one in which the world generates confusing but structured data from which a doctor seeks to infer hidden processes and events. Finally, these patterns may be temporal patterns occurring in a sequence of motor actions and the resulting sensations, e.g. a sequence of complex signals 'move-feel-move-feel- etc.' involving motor commands and the resulting tactile sensation in either an animal or a robot.

Pattern Theory is one approach to solving this pattern understanding problem. It contains three essential components which we shall detail in Sections 1.2, 1.3 and 1.4 below. These concern firstly the abstract mathematical setup in which we frame the problem; secondly a hypothesis about the specific mathematical objects that arise in natural pattern understanding problems; and thirdly a general architecture for computing the solution.

Before launching into this description of pattern theory, we want to give two examples of simple sensory signals and the patterns that they exhibit. This will serve to motivate and make more concrete the theory which follows. The first example involves speech: Figure 1.1 shows the graph of the pressure $p(t)$ while the word "SKI" is being pronounced. Note how the signal shows four distinct wave forms: something close to white noise during the pronunciation of the sibilant 'S', then silence followed by a burst which conveys the plosive 'K', then an extended nearly musical note for the vowel 'I'. The latter has a fundamental frequency corresponding to the vibration of the vocal cords, with many harmonics whose power

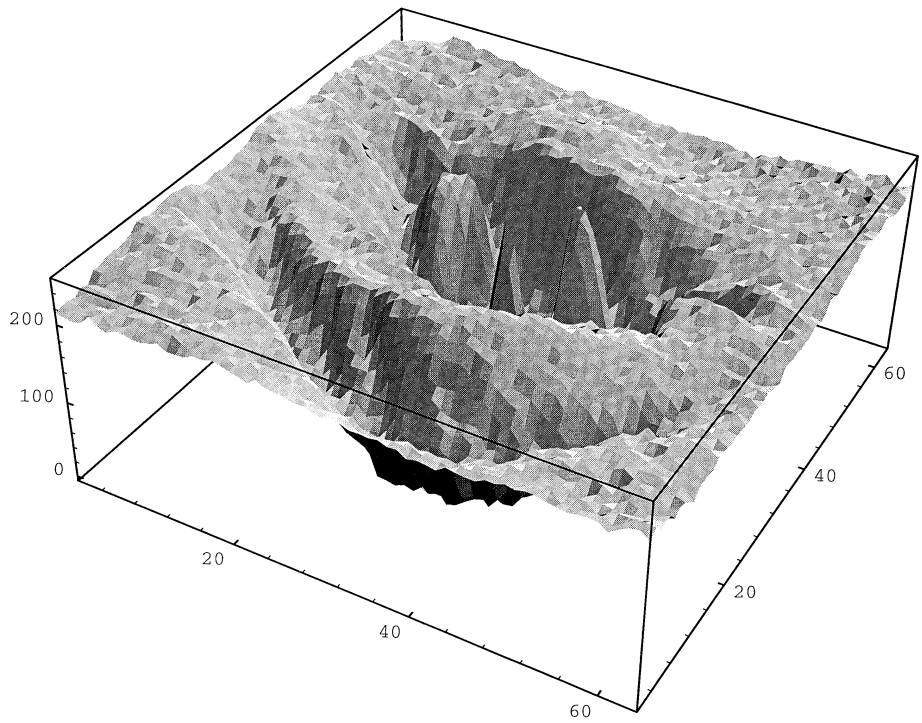

Figure 1.2a Visual waveform for an image of an eye.

peeks around three higher frequencies, the formants. Finally, the amplitude of the whole is modulated during the pronunciation of the word. In this example, the goal of auditory signal processing is to identify these four wave forms, characterize each in terms of its frequency power spectrum, its frequency and amplitude modulation, and then, drawing on a memory of speech sounds, identify each wave form as being produced by the corresponding configurations of the speaker's vocal tract, and finally label each with its identity as an english phoneme. In addition, one would like to describe explicitly the stress, pitch and the quality of the speaker's voice, using this later to help disambiguate the identity of the speaker and the intent of the utterance.

Figure 1.2a shows the graph of the light intensity $I(x, y)$ of a picture of a human eye: it would be hard to recognize this as an eye, but the black and white image defined by the same function is shown in Figure 1.2b. Note again how the domain of the signal is naturally decomposed into regions where I has different values or different spatial frequency behavior: the pupil, the iris, the whites of the eyes, the lashes, eyebrows and skin. These six regions are the six types of visible surface in this image, characterized in real world terms by distinct albedo, texture and geometry. The goal of visual signal processing is again to describe this function of two variables as being built up from simpler signals on subdomains, on which it either varies slowly

Figure 1.2b Identical waveform presented using variable intensity.

or is statistically regular, i.e. approximately stationary and hence to describe the distinct 3D visible surface patches. In other words, each surface patch is assumed to have an identifiable 'texture', which may be the result of its particular spatial power spectrum or may result from it being composed of some elementary units, called *textons*, which are repeated with various modifications. These modifications in particular include spatial distortion, contrast modulation and interaction with larger scale structures. This description of the signal may be computed either prior to or concurrent with a comparison of the signal with remembered eye shapes, which include a description of the expected range of variation of eyes, specific descriptions of the eyes of well-known people, etc.

Note that in both of these examples, something rather remarkable happened. The simplest description of the signal as an abstract function, e.g. having subdomains on which it is relatively homogeneous, leads naturally to a description of the

Figure 1.3 A challenge image for computers to recognize.

true processes and objects that produced the signal. We will discuss this further below, in connection with the idea of 'minimum description length'.

In order to understand what the field of pattern theory is all about, itis necessary to begin by addressing a major misconception — namelythat the whole problem is essentially trivial. The history of computerspeech and image recognition projects, like the history of AI, is along one of ambitious projects which attained their goals withcarefully tailored artificial input but which failed as soon as more of the complexities of real world data were present. The source of this misconception, I believe, is our subjective impression of perceiving instantaneously and effortlessly the significance of the patterns in a signal, e.g. the word being spoken or which face is being seen. Many psychological experiments however have shown that what we perceive is not the true sensory signal, but a rational reconstruction of what the signal should be. This means that the messy ambiguous raw signal never makes

it to our consciousness but gets overlaid with a clearly and precisely patterned version whose computation demands extensive use of memories, expectations and logic. An example of how misleading our impressions are is shown in Figure 1.3: the reader will instantly recognize that it is an image of an old man sitting on a park bench. But ask yourself — how did you know that? His face is almost totally obscure, with his hand merging with his nose; the most distinct shape is that of his hat, which by itself could be almost anything; even his jacket merges in many places with the background because of its creases and the way light strikes them, so no simple algorithm is going to trace its contour without getting lost. However, when you glance at the picture, in your mind's eye, you 'see' the face and its parts distinctly, the man's jacket is a perfectly clear coherent shape, whose creases in fact contribute to your perception of its 3-dimensional structure instead of confusing you. The ambiguities which must, in fact, have been present in the early stages of your processing of this image never become conscious because you have found an explanation of every peculiarity of the image, a match with remembered shapes and lighting effects. In fact, the problems of pattern theory are hard, and although major progress has been made in both speech and vision in the last 5 years, a robust solution has not been achieved!

1.2. Mathematical Formulation of the Problem

To make the field of pattern theory precise, we need to formulate it mathematically. There are three parts to this which all appear in Grenander's work: the first is the description of the players in the field, the fundamental mathematical objects which appear in the pattern understanding problem. The second is to restrict the possible generality of these objects by using something about the nature of the world. This gives us a more circumscribed, more focussed set of problems to study. Finally, the goal of the field being the reconstruction of hidden facts about the world, we aim primarily for algorithms, not theorems, and the last part is the general framework for these algorithms. In this section, we look at the first part, the basic mathematical objects of pattern theory. This formulation of the pattern understanding problem is not unique to pattern theory. It has been introduced many times, e.g. in the hidden Markov models of speech understanding, in work on expert systems, in connectionist analyses of neural nets, etc.

There are two essentially equivalent formulations, one using Bayesian statistics and one using information theory. The statistical approach (see for instance D. Geman, 1991) is this: consider all possible signals $f(\mathbf{x})$ which may be perceived. These may be considered as elements of a space Ω_{obs} of functions $f : \mathcal{D} \to \mathcal{V}$. For instance, speech is defined by pressure $P : [t_1, t_2] \to \mathbf{R}_+$ as a function of time, color vision is defined by intensity I on a domain $\mathcal{D} \subset S^2$ of visible rays with values in the convex cone of colors $\mathcal{V}_{\text{RGB}} \subset \mathbf{R}^3$, or these functions may be sampled on a discrete subset of the above domains, or their values may be approximated to finite precision, etc. The first basic assumption of the statistical approach is that nature

determines a probability p_{obs} on a suitable σ-algebra of subsets of Ω_{obs}, and that, in life, one observes random samples from this distribution. These signals, however, are highly structured as a result of their being produced by a world containing many processes, objects and events which don't appear explicitly in the signal. This means many more random variables are needed to describe the state of the world. The second assumption is that the possible states w of the world form a second probability space Ω_{wld} and that there is a big probability distribution $p_{o,w}$ on $\Omega_{\text{obs}} \times \Omega_{\text{wld}}$ describing the probability of both observing f and the world being in state w. Then p_{obs} should be the marginal distribution of $p_{o,w}$ on Ω_{obs}. The mathematical problem, in this setup, is to infer the state of the world w, given the measurement f, and for this we may use Bayes's rule:

$$p(w|f) = \frac{p(f|w) \cdot p(w)}{p(f)} \tag{1.1}$$

leading to the *maximum a posteriori reconstruction of the state of the world*†:

$$\text{MAP estimate of } w = \arg\max_w [p(f|w) \cdot p(w)] \tag{1.2}$$

To use the statistical approach, therefore, we must construct the probability space $(\Omega_{\text{obs}} \times \Omega_{\text{wld}}, p_{o,w})$ and finding algorithms to compute the MAP-estimate.

The information theoretic approach has its roots in work of Barlow (1961) (see also Rissanen, 1989). Assume D and V, hence Ω_{obs} are finite. The idea is that instead of writing out any particular perceptual signal f in raw form as a table of values, we seek a method of encoding f which minimizes its expected length in bits: i.e. we take advantage of the patterns possessed by most f to encode them in a compressed form. We consider coding schemes which involve choosing various auxiliary variables w and then encoding the particular f using these w (e.g. w might determine a specific typical signal f_w and we then need only to encode the deviation $(f - f_w)$). We write this:

$$\text{length}(\text{code}(f, w)) = \text{length}(\text{code}(w)) + \text{length}(\text{code}(f \text{ using } w))\,. \tag{1.3}$$

The mathematical problem, in the information theoretic setup, is, for a given f, to find the w leading to the shortest encoding of f, and moreover, to find the encoding *scheme* leading to the shortest expected coding of all f's. This optimal choice of w is called the *minimum description length* or MDL estimate of w:

$$\text{MDL est. of } w = \arg\min_w [\text{len}(\text{code}(w)) + \text{len}(\text{code}(f \text{ using } w))]\,. \tag{1.4}$$

Finding the optimal encoding scheme for all the signals of the world is obviously impractical and perhaps even contradictory (recall the problem of the finding the smallest integer not definable in less than twenty words!). What is really meant by

†Other statistical procedures can be used for estimating the state of the world: e.g. taking the mean of various world variables in the posterior distribution, or minimum risk estimates, etc.

MDL is that we seek some approximation to the optimal encoding, using coding schemes constrained in various ways†. Pattern theory proposes that there are quite specific kinds of encoding schemes which very often give good results and which may, therefore, be built into our thinking: we shall discuss these in the next section.

There is a close link between the Bayesian and the information-theoretic approaches which comes from Shannon's optimal coding theorem. This theorem states that given a class of signals f, the coding scheme for such signals for which a random signal has the smallest expected length satisfies:

$$\text{len}(\text{code}(f)) = -\log_2 p(f) \tag{1.5}$$

(where fractional bit lengths are achieved by actually coding several f's at once, and doing this the LHS gets asymptotically close to the RHS when longer and longer sequences of signals are encoded at once). We may apply Shannon's theorem both to encoding w and to encoding f, given w. For these encodings $\text{len}(\text{code}(w) = -\log_2 p(w)$ and $\text{len}(\text{code}(f \text{ using } w) = -\log_2 p(f|w)$. Therefore, taking $\log_2$ of equation (1.2), we get equation (1.4) and find that the MAP estimate of w is the same as the MDL estimate.

There is an additional wrinkle in the link between the two approaches. Why shouldn't the search for optimal encodings of world signals lead you to odd combinatorial coding schemes which have nothing to do with what is actually happening in the world? In the previous paragraph, we have assumed that the encoding scheme for f was based on encoding the true world variables w and using them to encode f. But one can imagine discovering some strange numerology (like Kepler's hypothesis for spacing the orbits of the planets via the inscribed and circumscribed spheres of the platonic polyhedra) which gave a concise description of some class of signals or measurements but had nothing to do with the true objects or processes of the world. This does not seem to happen in real life! We encountered an example of this in Section 1.1 where we noticed that finding time intervals in which a speech signal had a nearly constant spectrogram, hence was concisely codable, also gave us the time intervals in which the mouth of the speaker was in a particular position, i.e. a specific phoneme was being articulated. Similarly, finding parts of images where the texture was nearly constant usually gives coherent 3D surface patches. In fact, it seems that the search for short encodings leads you *automatically, without prior knowledge of the world* to the same hidden variables on which the Bayesian theory is based. Insofar as this can be relied on, the information-theoretic approach has the great advantage over the Bayesian approach that it does not require that you have *a prior* knowledge of the physics, chemistry, biology, sociology, etc. of the world, but gives you a way of discovering these facts.

†One could argue that the hypothesis of vengeful sky gods with human emotions was an early MDL hypothesis for the deeper processes of the world, and that modern science only bettered its description length when people sought to describe the quantitative signals of the world with more bits and with much larger samples including those outliers which we call 'experiments'.

A very simple example may be useful here. Suppose five different bird songs are heard regularly in your back yard. You can assign a short distinctive code to each such song, so that instead of having to remember the whole song from scratch each time, you just say to yourself something like "Aha, song #3 again". Note that in doing so, you have automatically learned a world variable at the same time: the number or code you use for each song is, in effect, a name for a species, and you have rediscovered part of Linnaean biology. Moreover, if one bird is the most frequent singer, you will probably use the shortest code, e.g. "song #1", for that bird. In this way, you are also learning the probability of different values for the variable "song #x", as in Shannon's fundamental theorem. In Section 1.5.4, we will give a more extended example of how this works.

1.3. Four Universal Transformations of Perceptual Signals

The above formulation of the pattern understanding problem provides a framework in which to analyze signals, but it says nothing about the nature of the patterns which are to be expected, what distortions, complexities and ambiguities are to be expected, hence what sorts of probability spaces Ω_{obs} are we likely to encounter, how shall we encode them, etc. What gives the theory its characteristic flavor is the hypothesis that the world does not have an infinite repertoire of different tricks which it uses to disguise what is going on. Consider the coding schemes used by engineers for the transmission of electrical signals: they use a small number of well-defined transformations such as AM and FM encoding, pulse coding, etc. to convert information into a signal which can be efficiently communicated. Analogous to this, the world produces sound to be heard, light to be seen, surfaces to be felt, etc. which are all, in various ways, reflections of its structure. We may think of these signals as the productions of a particularly perverse engineer, who sets us the problem of decoding this message, e.g. of recognizing a friend's face or estimating the trajectory of oncoming traffic, etc. The second contention of pattern theory is that such signals are derived from the world by *four types of transformations or deformations*, which occur again and again in different guises. In the terminology of Grenander (1976) simple unambiguous signals from the world are referred to as *pure images* and the transformations on them are called *deformations*, which produce the actually observed perceptual signals which he called *deformed images*. The bad news is that these four transformations produce much more complex effects than the coding schemes of engineers, hence the difficulty of decoding them by the standard tricks of electrical engineering. The good news is that these transformations are not arbitrary recursive operations which produce unlearnable complexity.

A very similar situation occurs in the study of the syntax of languages. In the formal study of the learnability of the syntax of language, Gold's theorem gives very strong restrictions on what languages can be learned if their syntax is at all general (see Osherson & Weinstein (1984) for an excellent exposition). In

contrast, Chomsky (1981) has suggested that all languages have essentially the same syntax, with individual languages differing only by the setting of a small number of parameters. In fact, transformational grammar has a very similar structure to pattern theory: each sentence has an underlying deep structure, analogous to Grenander's pure images. It is subjected to a restricted set of transformations, analogous to Grenander's deformations. And finally one observes the spoken surface form of the sentence, analogous to Grenander's deformed image.

The study of perceptual signals suggests a small number of special transformations, or deformations, that the languages in which perceptual signals speak are of very special types. The exact set of these is not completely clear at this point and my choices are not exactly the same as Grenander's. But to make progress, we must make some hypothesis and so I give here a set of four transformations which seem to me to suffice. These are:

(i) *Noise and blur.* These effects are the bread and butter of standard signal processing, caused for instance by sampling error, background noise and imperfections in your measuring instrument such as imperfect lenses, veins in front of the retina, dust and rust. A typical form of this transformation is given by the formula:

$$I \to (I * \sigma)(x_i) + n_i \tag{1.6}$$

where σ is a blurring kernel, x_i are the points where thesignal is sampled and n_i is some kind of additive noise, e.g. Gaussian, but of course much more complex formulae are possible. Especially significant is that Gaussian noise is usually a poor model of the noise effects, for example when the noise is caused by finer detail which is not being resolved. Rosenfeld calls such an *n clutter*, which conveys what it often represents. The key feature of noise, in whatever guise, is that it has no significant remaining patterns, hence cannot be compressed significantly by recoding. These transformations are part of what Grenander calls *changes in contrast.* When they are present, the unblurred noiseless I should be one of the variables w as getting rid of noise and blur reveals the hidden processes of the world more clearly.

(ii) *Superposition.* Signals typically can be decomposed into simpler components. Fourier analysis is the best studied example of this, in which a signal is written as a linear combination of sinusoidal functions. But the whole development of wavelets has shown that there are many other such expansions, appropriate for particular classes of signals. Most such superpositions have the property that the various components have different scales, but some may combine several on the same scale (e.g. one can superimpose 2D Gabor functions with different orientations; for another example, faces with arbitrary illumination can be approximated by the superposition of about half a dozen 'eigenfaces'). Most such superpositions are linear, but some may be more complex (e.g. in amplitude modulation, a low frequency signal plus a large enough positive constant is multiplied by a high frequency 'carrier'). In images, local properties

such as sharp edges and texture details may be constructed by adding Gabor functions or model step edges with small support, while global properties like slowly varying shading or large shapes may be obtained by adding slowly varying functions with large support. In speech, information is conveyed by the highest frequency formants, by the lower frequency vibration of the vocal cords and the even slower modulation of stress. A typical form of this transformation is given by the formula:

$$I_1, \ldots, I_n \longrightarrow (I_1 + \cdots + I_n) \quad \text{or} \quad \sigma(I_1, \ldots, I_n) \tag{1.7}$$

where $I_1, \ldots, I_n$ represent component signals often in disjoint frequency bands, which can be combined either additively or by some more complex rule σ. The individual components I_k of I should be included in the variables w.

(iii) *Domain warping.* Two signals generated by the same object or event in different contexts typically differ because of expansions or contractions of their domains, possibly at varying rates: phonemes may be pronounced faster or slower, the image of a face is distorted by varying expression and viewing angle. In speech, this is called 'time warping' and in vision, this is modeled by 'flexible templates'. In both cases, a diffeomorphism of the domain of the signal brings the variants much closer to each other, so that this transformation is given by:

$$I \longrightarrow (I \circ \psi) \tag{1.8}$$

where ψ represents a diffeomorphism of the domain of I. These transformations are what Grenander calls *background deformations*. The diffeomorphism ψ should be one of the variables w.

(iv) *Interruptions.* Natural signals are usually analyzed best after being broken up into pieces consisting of their restrictions to subdomains. This is because the world itself is made up of many objects and events and different parts of the signal are caused by different objects or events. For instance, an image may show different objects partially occluding each other at their edges, as in Figure 1.3 where the old man is an object which occludes part of the park bench or as in a tiger seen behind a fragmented foreground of occluding leaves. In speech, the phonemes naturally break up the signal and, on a larger scale, one speaker or unexpected sound may interrupt another. Such a transformation is given by a formula like:

$$I_1, I_2 \longrightarrow (I_1|_{D'}, I_2|_{D-D'}) \tag{1.9}$$

where I_2 represents the background signal which is interrupted by the signal I_1 on a part D' of its domain D, (or I_2 may only be defined on $D - D'$). This type of deformation is called *incomplete observations* by Grenander. The components I_k and the domain D' should be included in the variables w.

What makes pattern theory hard is not that any of the above transformations are that hard to detect and decode in isolation, but rather that all four of them tend to coexist, and then the decoding becomes hard.

1.4. Pattern Analysis Requires Pattern Synthesis

Taking the Bayesian definition of the objects of pattern theory, we note that the probability distribution $(\Omega_{\text{obs}} \times \Omega_{\text{wld}}, p_{o,w})$ allows you to do two things. On the one hand, we can use it to define the MAP-estimate of the state of the world; but we can also sample from it, possibly fixing some of the world variables w, using this distribution to construct sample signals f generated by various classes of objects or events. A good test of whether your prior has captured all the patterns in some class of signals is to see if these samples are good imitations of life. For this reason, Grenander's idea was that the analysis of the patterns in a signal and the synthesis of these signals are inseparable problems, using a common probabilistic model: computer vision should not be separated from computer graphics, nor speech recognition from speech generation.

Many of the early algorithms in pattern recognition were purely *bottom-up*. For example, one class of algorithms started with a signal, computed a vector of 'features', numerical quantities thought to be the essential attributes of the signal, and then compared these feature vectors with those expected for signals in various categories. This was used to classify images of alpha-numeric characters or phonemes for instance. Such algorithms give no way of reversing the process, of generating typical signals. The problem these algorithms encountered was that they had no way of dealing with anything unexpected, such as a smudge on the paper partially obscuring a character, or a cough in the middle of speech. These algorithms did not say what signals were expected, only what distinguished typical signals in each category.

In contrast, a second class of algorithms works by actively reconstructing the signal being analyzed. In addition to the bottom-up stage, there is a *top-down* stage in which a signal with the detected properties is synthesized and compared to the present input signal. What needs to be checked is whether the input signal agrees with the synthesized signal to within normal tolerances, or whether the residual is so great that the input has not been correctly or fully analyzed. This architecture is especially important for dealing with the fourth type of transformation: interruptions. When these are present, the features of the two parts of the signal get confused. Only when the obscuring signal is explicitly labelled and removed, can the features of the background signal be computed. We may describe this top-down stage as 'pattern reconstruction' in distinction to the bottom-up purely pattern recognition stage. A flow chart for such algorithms is shown in Figure 1.4.

The question of whether the correct interpretation of real world signals can be solved by a purely bottom-up algorithm, or whether it requires an iterative

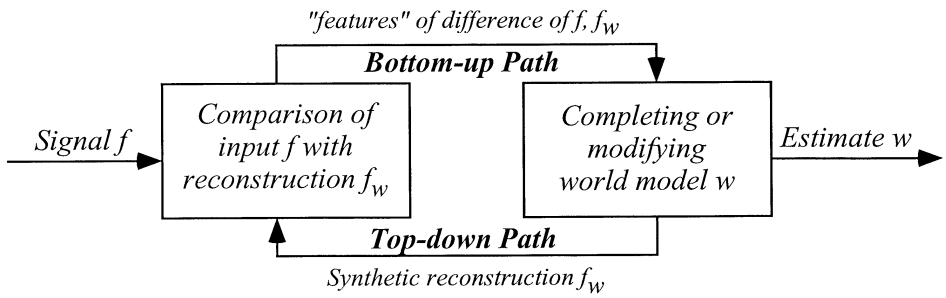

Figure 1.4 The fundamental architecture of pattern theory.

bottom-up/top-down architecture has been widely debated for a long time. The first person, to my knowledge, to introduce the above type of iterative architecture, was Donald MacKay (1956). On the other hand, Marr was a strong believer in the purely feedforward architecture, claiming that one needed merely to develop better algorithms to deal with things like interruptions in a purely feedforward way. A strong argument for the necessity of a top-down stage for the recognition of heavily degraded signals, such as faces in deep shadow, is given in Cavanagh (1991). Neural net theory has gone in both directions: while 'back propagation' nets categorize in an exclusively bottom-up manner (only using feedback in their learning phase), the 'attractive neural nets' with symmetric connections (Hopfield, 1982; D. Amit, 1989) seek not merely to categorize but also to construct the prototype ideal version of the category by a kind of pattern completion which they call 'associative memory'. What these nets do not do is to go back and attempt to compare this reconstruction with the actual input to see if the full input has been 'explained'. One demonstration system that does this is Grossberg and Carpenter's 'adaptive resonance theory' (Carpenter & Grossberg, 1987). A proposal for the neuroanatomical substrate for such bottom-up/top-down loops in mammalian cortex is put forth in Mumford (1991-92). One reason Marr rejected the complex top-down feedback architecture is that it seemed to take too long to converge to be biologically plausible. This argument, however, ignores the fact that a large proportion of the time, we can anticipate the next stimulus, either by extrapolating from the preceding stream of stimuli or by drawing on memories of shapes and sounds. In this situation, the top-down pathway may actively synthesize a guess for the next stimulus even before it arrives, and convergence is fast unless something totally unexpected happens.

The third part of our definition of pattern theory is the hypothesis that no practical feedforward algorithm exists for computing the most likely values of the world variables w from signals f. But that if your algorithm explicitly models the generation process, starting with a guess for w (or a set of guesses), then generating an f_w, then deforming w, combining and extending these guesses, you can solve the problem of computing the most likely w in a reasonable time.

1.5. Examples

In this section, we want to present several examples of interesting mathematics which have come out of pattern theory, in attempting to come to grips with one or another of the above universal transformations. These examples are from vision because this is the field I know best, but many of these ideas are used in speech recognition too.

1.5.1. *Pyramids and wavelets*

The problem of analyzing functions that convey information on more than one scale, has arisen in many disciplines. The classical method of separating additively combined scales is, of course, Fourier analysis. But what is usually required is to analyze a function locally simultaneously in its original domain *and* in the domain of its Fourier transform, and Fourier analysis does not do this. In vision, moving closer or farther from an object by a factor σ changes the image $I(x, y)$ of the object into the new image $I(\sigma x, \sigma y)$, thus any feature which occurs on one scale in one image is equally likely to occur at any other scale in a second image. In computer vision, at least as far back as the early 70's, this problem led to the idea of analyzing an image by means of a 'pyramid', e.g. Uhr, 1972; Rosenfeld & Thurston, 1971. In its original incarnation, the main idea was to compute a series of progressively coarser resolution images by blurring and resampling, e.g. a set of $(2^n \times 2^n)$-pixel images, for $n = 10, 9, \ldots, 1$. Putting these together, the resulting data structure looks like an exponentially tapering pyramid. Instead of running algorithms that took time proportional to the width of the image, one ran the algorithms up and down the pyramid, possibly in parallel at different pixels, in time proportional to *log*(width). Typical algorithms that were studied at this time are morphological ones, involving for instance linking and marking extended contours, which have nothing to do with filtering or linear expansions. The bottom layer of this so-called *Gaussian pyramid* held the original image, with both high and low frequency components, although it was used only to add local or high-frequency information.

In the early 80's, the idea of using the pyramid to separate band pass components of a signal and thus to expand that signal arose both in computer vision (Burt & Adelson, 1983) (where they *subtracted* successive layers of the Gaussian pyramid, producing what they called the *Laplacian pyramid*) and in petroleum geology (Grossman & Morlet, 1984). Figure 1.5 shows this Laplacian pyramid for a face: note that the high-frequency differences show textures and sharp edges, while the low frequency differences show large shapes. This work led directly to the idea of wavelets and wavelet expansions which now seem to be the most natural way to analyze a signal locally in both space and frequency. Mathematically, the idea is simply to expand an arbitrary function $f(\mathbf{x})$ of n variables as a sum:

$$f(\mathbf{x}) = \sum_{\text{scale } k \in Z} \left[\sum_{\vec{n} \in \text{lattice} L} \sum_{\text{fin.\# of } \alpha} a_{k,\vec{n},\alpha} \psi_\alpha(\lambda^k \mathbf{x} + \vec{n}) \right] \tag{1.10}$$

Figure 1.5 The Gaussian and Laplacian pyramids for a face.

where the ψ_α are suitable functions, called wavelets, with mean 0. Usually $\lambda = 2$, and, at least in dimension 1, there is a single α and wavelet ψ_α. The original expansions of Burt and Adelson, which are not quite of this form, have been reinvestigated from a more mathematical point of view recently in Mallat (1989). The basic link between the expansion in (1.10) and pyramids is this: define a space V_m to be the set of f's whose expansions involve only terms with $k \leq m$. This defines a 'multi-resolution ladder' of subspaces of functions with more and more detail:

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2(\mathbf{R}^n) \tag{1.11}$$

such that $f(x) \longmapsto f(2x)$ maps V_m isomorphically onto V_{m+1}. Then one may think of V_m as functions which have been blurred and sampled at a spacing 2^{-m}: i.e. the level of the pyramid of $(2^m \times 2^m)$-pixel images. The mathematical development of the theory of these expansions is due especially to Meyer and Daubechies (see Meyer, 1986; Daubechies, 1988; Daubechies, 1990), who showed that (i) with *very* careful choice of ψ, this expansion is even an orthogonal one, (ii) for many more ψ, the functions on the right form an unconditional but not orthogonal basis of $L^2(\mathbf{R}^n)$ and (iii) for even more ψ, the functions on the right form a 'frame', a set of functions that spans $L^2(\mathbf{R}^n)$ and gives a canonical though non-unique expansion of every f.

From the perspective of pattern theory, we want to make two comments on the theory of wavelets. The first is that they fit in very naturally with the idea of minimum description length. Looked at from the point of view of optimal linear encoding of visual and speech signals (i.e. encoding by linear combinations of the function values), the idea of wavelet expansions is very appealing. This was pointed

out early on by Burt and Adelson and data compression has been one of the main applications of wavelet theory ever since. Moreover, its further development leads beyond the classical idea of expanding a function in terms of a fixed basis to the idea of using a much larger spanning set which *oversamples* a function space and using suitably chosen subsets of this set in terms of which to expand or approximate the given function (see Coifman, Meyer & Wickerhouser, 1990, where *wavelet libraries* are introduced). Even though the data needed to describe this expansion or approximation is now both the particular subset chosen and the coefficients, this may be a more efficient code. If so, it should lead us to the correct variables w for describing the world (cf. Section 1.2): for example, expanding a speech signal using wavelet libraries, different bases would naturally be used in the time domains during which different phonemes were being pronounced — thus the break up of the signal into phonemes is discovered as a consequence of the search for efficient coding! It also appears that nature uses wavelet type encoding: there are severe size restrictions on the optic nerve connecting the retina with the higher parts of the brain and the visual signal is indeed transmitted using something like a Burt–Adelson wavelet expansion (Dowling, 1987).

The second point is that wavelets, even in their oversampled form, are still just the linear side of pyramid multi-scale analysis. In our description of multi-scale transformations of signals in Section 1.3, we pointed out that the two scales can be combined by multiplication or a more general function σ as well as by addition. To decode such a transformation, we need to perform some local non-linear step, such as rectification or auto-correlation, at each level of the pyramid before blurring and resampling. An even more challenging and non-linear extension is to a *multi-scale description of shapes*: e.g. subsets $S \subset \mathbf{R}^2$ with smooth boundary. The analog of blurring a signal is to let the boundary of S evolve by diffusion proportional to its curvature (see Gage & Hamilton, 1986; Grayson, 1987; Kimia, Tannenbaum & Zucker, 1993). Although there is no theory of this at present, one should certainly have a multi-scale description of S starting from its coarse diffused form — which is nearly round — and adding detailed features at each scale. In yet another direction, face recognition algorithms have been based on matching a crude blurry face template at a low resolution, and then refining this match, especially at key parts of the face like the eyes. This is the kind of general pyramid algorithm that Rosenfeld proposed many years ago (Rosenfeld & Thurston, 1971), many of which have been successfully implemented by Peter Burt and his group at the Sarnoff Laboratory (Burt & Adelson, 1983).

1.5.2. *Segmentation as a free-boundary value problem*

A quite different mathematical theory has arisen out of the search for algorithms to detect transformations of the 4th kind, interruptions. Evidence for an interruption or a discontinuity in a perceptual signal comes from two sources: the relative homogeneity of the signal on either side of the boundary and the presence of a

large change in the signal across the boundary. One approach is to model this as a variational problem: assuming that a blurred and noisy signal f is defined on a domain $D \subset \mathbf{R}^n$, one seeks a set $\Gamma \subset D$ and a smoothed version g of f which is allowed to be discontinuous on Γ such that:

- g is as close as possible to f,
- g has the smallest possible gradient on $D - \Gamma$,
- Γ has the smallest possible $(n-1)$-volume.

These conditions define a variational problem, namely to minimize the functional

$$E(g, \Gamma) = \mu^2 \int_D \cdots \int (f-g)^2 + \int_{D-\Gamma} \cdots \int \|\nabla g\|^2 + \nu|\Gamma| \tag{1.12}$$

where μ and ν are suitable constants weighting the three terms and $|\Gamma|$ is the $(n-1)$-volume of Γ. The g minimizing E may be understood as the optimal piecewise smooth approximation to the quite general function f. In Grenander's terms, the function g is the pure image and f is the deformed image; I like to call g a *cartoon* for the signal f. The Γ minimizing E is a candidate for the boundaries of parts of the domain D of f where different objects or events are detected. Segmenting the domain of perceptual signals by such variational problems was proposed independently by S. and D. Geman and by A. Blake and A. Zisserman (see Geman & Geman, 1984, and Blake & Zisserman, 1987) for functions on discrete lattices, and was extended by Mumford & Shah (1989) to the continuous case. In the case of visual signals, the domain D is 2-dimensional and we want to decompose D into the parts on which different objects in the world are projected. When you reach the edge of an object as seen from the image plane, the signal f typically will be more or less discontinuous (depending on noise and blur and the lighting effects caused by the grazing rays emitted by the surface as it curves away from the viewer). An example of the solution of this variational problem is shown in Figure 1.6: Figure 1.6a is the original image of the eye, 1.6b shows cartoon g and 1.6c shows the boundaries Γ. This is a case where the algorithm succeeds in finding the 'correct' segmentation, but it doesn't always work so well. Figure 1.7 gives the same treatment as Figure 1.6, to the 'oldman' image. Note that the algorithm fails to find the perceptually correct segmentation in several ways: the man's face is connected to his black coat and the black bar of the bench and the highlights on the back of his coat are treated as separate objects. One reason is that the surfaces of objects are often textured, hence the signal they emit is only statistically homogeneous. More sophisticated variational problems are needed to segment textured objects (see below).

From a mathematical standpoint, it is important to know if this variational problem is well-posed. It has been proven that E has a minimum if Γ is allowed to be a closed rectifiable set of finite Hausdorff $(n-1)$-dimensional measure and g is taken in a certain space SBV ('special bounded variation', which means that the

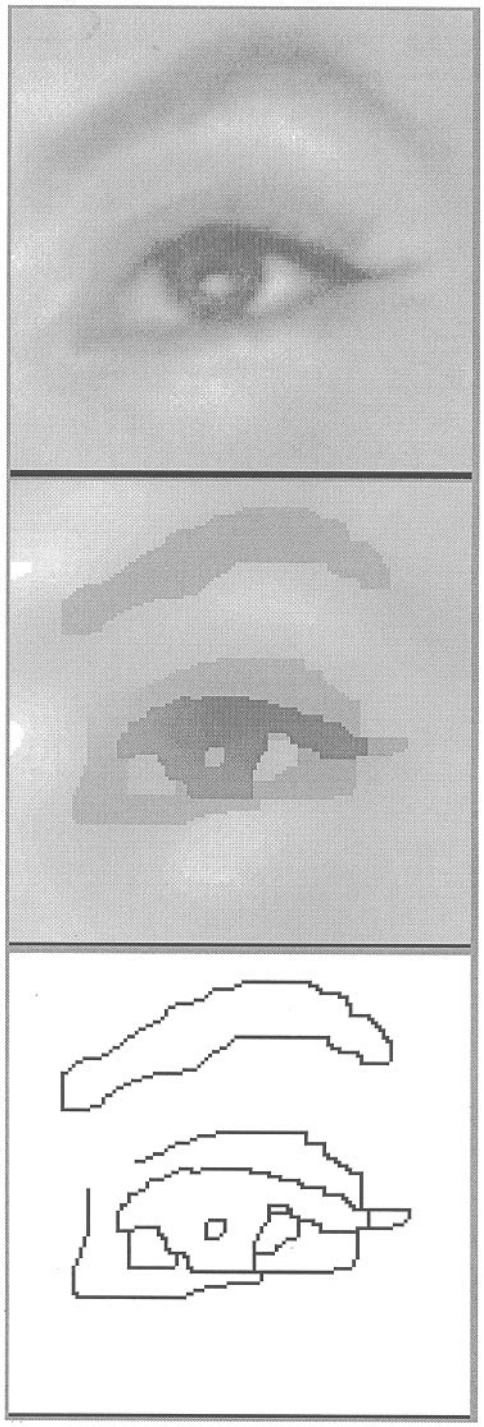

Figure 1.6 Segmentation of the eye-image via optimal piecewise smooth approximation.

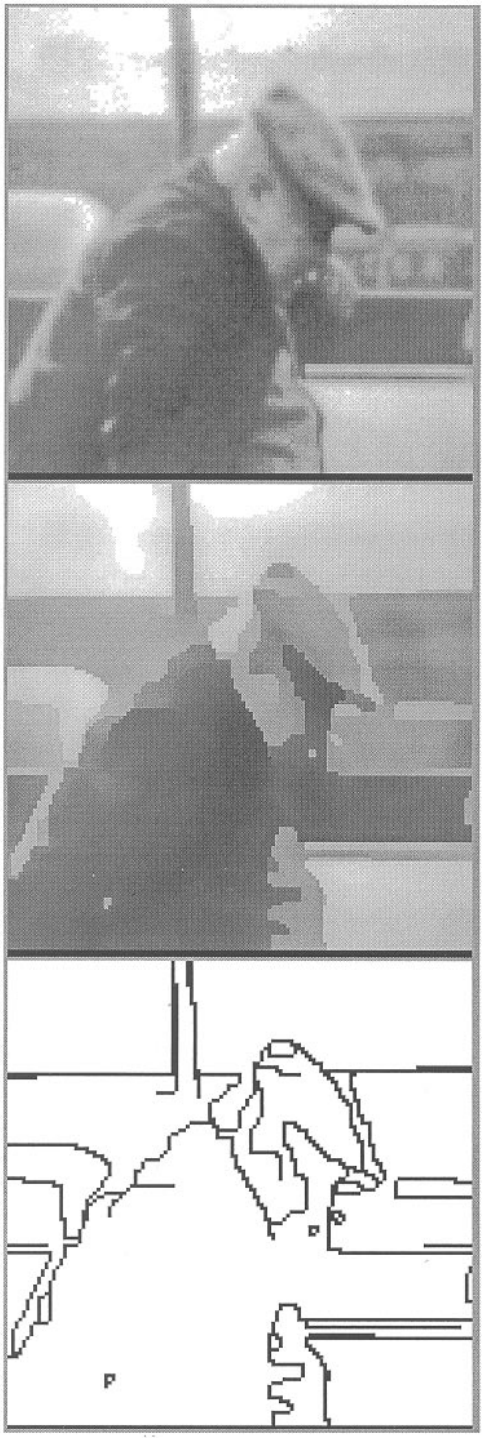

Figure 1.7 Segmentation of the oldman-image via optimal piecewise smooth approximation.

distributional derivative of g is the sum of an L^2-vector field plus a totally singular distribution supported on Γ) (see DeGiorgi, Carriero & Leaci, 1988; Ambrosio & Tortorelli, 1989; Dal Maso, Morel & Solimini, 1989). Unfortunately, it seems hard to check whether these minima are 'nice' when f is, e.g. whether, when $n = 2$, Γ is made up of a finite number of differentiable arcs, though Shah and I have conjectured that this is true. Of course, if the signal is replaced by a sampled version, the problem is finite dimensional and certainly well-posed.

This variational problem fits very nicely into both the Bayesian framework and the information theoretic one. Geman and Geman introduced it, for discrete domains, in the Bayesian setting. The basic idea is to define probability spaces by Gibbs fields. Let $D = \{x_\alpha\}$ be the domain, $\{f_\alpha\}$ and $\{g_\alpha\}$ the values of f and g at x_α. To describe Γ, for each pair of 'adjacent pixels' α and β, let $\ell_{\alpha,\beta} = 1$ or 0 depending on whether or not Γ separates the pixels α and β: these random variables are called the *line process*. Then we define an initial probability distribution on the random variables $\{\ell_{\alpha,\beta}\}$ by the formula:

$$p(\{\ell_{\alpha,\beta}\}) = \frac{e^{-\nu(\sum \ell_{\alpha,\beta})}}{Z_1} \tag{1.13}$$

where Z_1 is the usual normalizing constant. This just means that boundaries Γ get less and less probable, the bigger they are. Next, we put a joint probability distribution on $\{g_\alpha\}$ and on the line process by the formula:

$$p(\{g_\alpha\}, \{\ell_{\alpha,\beta}\}) = \frac{e^{-\sum_{\alpha,\beta \text{ adj}} (1-\ell_{\alpha,\beta})\cdot(g_\alpha - g_\beta)^2}}{Z_2}\,. \tag{1.14}$$

This is a discrete form of the previous E: if adjacent pixels α and β are *not* separated by Γ, then $\ell_{\alpha,\beta} = 0$ and the probability of $\{g_\alpha\}$ goes down as $|g_\alpha - g_\beta|$ gets larger, but if they *are* separated, then $\ell_{\alpha,\beta} = 1$ and g_α and g_β are independent. Together, the last two equations define an intuitive prior on $\{g_\alpha, \ell_{\alpha,\beta}\}$ enforcing the idea that g is smooth except across the boundary Γ. The data term in the Bayesian approach makes the observations $\{f_\alpha\}$ equal to the model $\{g_\alpha\}$ plus Gaussian noise, i.e. it defines the conditional probability by the formula:

$$p(\{f_\alpha\}|\{g_\alpha, \ell_{\alpha,\beta}\}) = \frac{e^{-\mu^2\cdot\sum_\alpha (f_\alpha - g_\alpha)^2}}{Z_3}\,. \tag{1.15}$$

Multiplying (1.12), (1.13) and (1.14) defines a probability space $(\Omega_{\text{obs}} \times \Omega_{\text{wld}}, p_{o,w})$ as in Section 1.2 and taking $-log$ of this probability, we get back a discrete version of E up to a constant. Thus the MAP-estimate of the world variables $\{g_\alpha, \ell_{\alpha,\beta}\}$ is essentially the minimum of the functional E.

This probability space is closely analogous to that introduced in physics in the Ising model. In terms of this analogy, the discontinuities Γ of the signal are exactly the interfaces between different phases of some material in statistical mechanics

(specifically in the Ising model of where the magnetic field of the iron atoms are oriented up or down).

From the information-theoretic perspective, we want to interpret E as the bit length of a suitable encoding of the image $\{f_\alpha\}$. These ideas have not been fully developed, but for the simplified model in which Γ is assumed to divide up the domain into pieces $\{D_k\}$ on which the image is approximately a constant $\{g_k\}$, this interpretation was pointed out by Leclerc (1989). We imagine encoding the image by starting with a 'chain code' for Γ: the length of this code will be proportional to its length $|\Gamma|$. Then we encode the constants $\{g_k\}$ up to some accuracy by a constant times the number of these pieces k. Finally, we encode the deviation of the image from these constants by Shannon's optimal encoding based on the assumption that $f_\alpha = g_k +$ Gaussian noise n_α. The length of this encoding will be a constant times the first term in E. (If g is not locally constant, we may go on to interpret the second term in E as follows: consider the Neumann boundary value problem for the laplacian Δ acting on the domain $D - \Gamma$. We may expand g in terms of its eigenfunctions, and encode g by Shannon's optimal encoding assuming these coefficients are independently normally distributed with variances going down with the corresponding eigenvalues. The length will be this second term.)

Many variants of this Gibbs field or 'energy functional' approach to perceptual signal processing have been investigated. Some of these seek to incorporate texture segmentation, e.g. Geman *et al.* (1990) and Lee *et al.* (1992) (which proposes an algorithm that should also segment most phonemes in speech) and others to deal with the asymmetry of boundaries caused by the 3D-world: at a boundary, one side is in front, the other in back (Nitzberg & Mumford, 1990). The 'Hidden Markov Models' used in speech recognition are Gibbs fields are of this type. To clarify the relationship, recall that HMM's are based on modelling speech by a Markov chain whose underlying graph is made up of subgraphs, one for each phoneme and whose states predict the power spectrum of the speech signal in local time intervals. Assuming a specific speech signal f is being modelled, HMM-theory computes the MAP sample of this chain conditional on the observed power spectra. Note that any sample of the chain defines a segmentation of time by the set $\Gamma = \{t_k\}$ of times at which the sample moves from the subchain for one phoneme to another, and each interval $t_k \leq t \leq t_{k+1}$ is associated to a specific phoneme a_k. Let A be the string $\{a_1 a_2, \ldots, a_N\}$. Taking $-log$ of the probability, the MAP estimate of the chain is the pair $\{\Gamma, A\}$ minimizing an energy E of the form:

$$E(A, \Gamma) = \sum_k \text{dist.}(f\big|_{t_k}^{t_{k+1}}, \text{phoneme } a_k) + \nu|\Gamma| \tag{1.16}$$

which is clearly analogous to the E's defined above.

Finally, some physiological theories have been proposed in which various areas of cortex (e.g. V1 and V2) compute the segmentation of images by an algorithm analogous to minimizing (1.12) (Grossberg & Mignolla, 1985). It has also been used in computing depth from stereo (see Belhumeur, this volume; Geiger *et al.*,

1992), computing the so-called optical flow field, the vector field of moving objects across the focal plane (Yuille & Grzywacz, 1989; Hildreth, 1984) and many other applications.

We have not mentioned the problem of actually computing or approximating the minimum of energy functionals like E. Four methods have been proposed: in case $n = 1$, we can use *dynamic programming* to find the global minimum fast and efficiently. This applies to the speech applications and is one reason why speech recognition is considerably ahead of image analysis. For any n, Geman & Geman (1984) applied a Monte Carlo algorithm due to Kirkpatrick *et al.* (1983) known as *simulated annealing*. Making this work is something of a black art, as the theoretical bounds on its correctness are astronomical; still it is always an easy thing to try as a first step. A third method, which seems the most reliable at this point, is the *graduated non-convexity* method introduced in Blake & Zisserman (1987). It is based on putting the functional E in a family E_t such that $E = E_0$ and E_1 is a convex functional, hence has a unique local minimum. One then starts with the minimum of E_1 and follows it as $t \to 0$. The final idea is related to the third and that is to use *mean field theory* as in statistical physics: this often leads to approximations to the Gibbs field which allow us to put E in a family becoming convex in the limit (see Geiger & Yuille, 1989).

1.5.3. *Random diffeomorphisms and template matching*

The third example concerns the identification of objects in an image, putting them in categories such as 'the letter A', 'a hammer' or 'my Grandmother's face'. One of the biggest obstacles in these problems is the variability of the shapes which belong to such categories. This variability is caused, for example, by changes in the orientation of the object and the viewpoint of the camera, changes in individual objects such as varying expressions on a face and differences between objects of the same category such as different fonts for characters, different brands of hammer, etc. If the shapes were not too variable, one could hope to introduce average examples of each letter, of each tool, of the faces of everyone you know — 'templates' for each of these objects — and recognize each such object as it is perceived by comparing it to the various templates stored in memory. Unfortunately, the variations are usually too large for this to work, and, worse than that, some variations occur commonly, while others do not (e.g. faces get wrinkled but never become wavy like water). What we need to do is to explicitly model the common variations and use our knowledge to see if a suitably varied template fits! A large part of this variation can be modelled by domain warping, the third of the transformations introduced in Section 1.3 and this leads one to study *deformable templates*, templates whose parts can be changed in size and orientation and shifted relative to each other. These were first introduced in computer vision by Fischler & Elschlager (1973) who called them 'templates with springs' but the idea is well-known in biology, e.g. in the famous

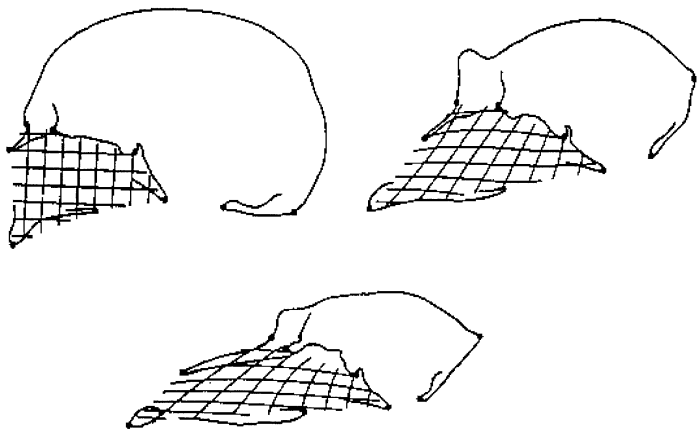

Figure 1.8a Diffeomorphism between primate skulls.

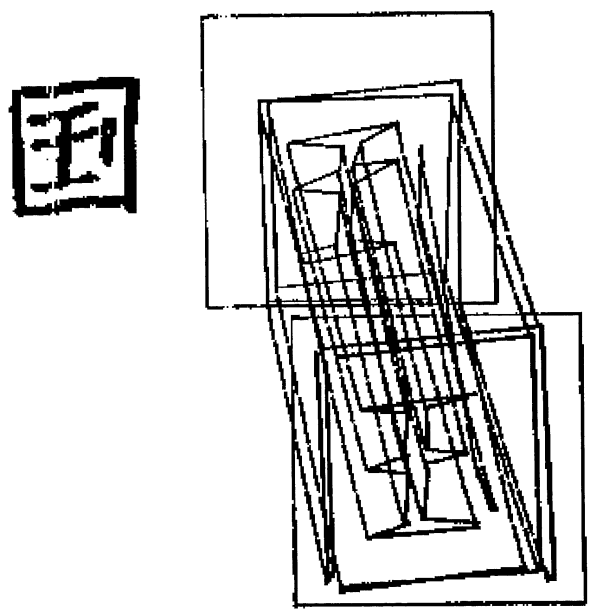

Figure 1.8b Diffeomorphism between kanji.

and beautiful book by Thompson (1917) (see Figure 1.8a, showing the deformations between three primate skulls).

Mathematically, we can describe flexible templates as follows. We must construct four things: (i) a standard image I_T on a domain D_T which can be a set of pixels or can also be reduced to a graph of 'parts' of the object, (ii) a space of allowable maps $\psi : D_T \to D$ or $(D \cup \{\text{missing}\})$, (iii) a measure $E(\psi)$ of the degree of deformation in the map ψ, the 'stretching of the springs' and (iv) a measure of

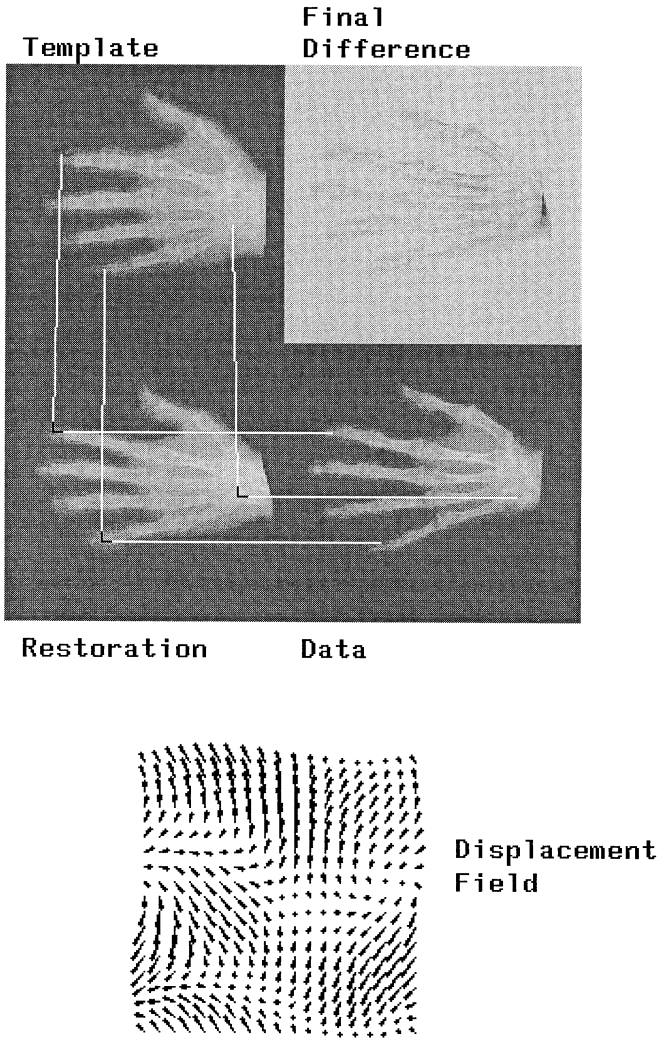

Figure 1.8c Diffeomorphism between X-rays of hands.

the difference d between the standard image I_T and the deformation $\psi^*(I)$ of the observed image I. Here ψ is typically a diffeomorphism, 'missing' is an extra element in the range of ψ to allow certain parts of the standard image to be missing in the observed image and $\psi^*(I)$ is a 'pull-back' of I which may be just the composition of I and ψ if D_T is a set of pixels, or may be some set of local 'features' of I when D_T is a graph of parts. The basic problem is then to compute:

$$\arg\min_{\psi}[d(\psi^*(I), I_T) + E(\psi)]\,, \tag{1.17}$$

in order to find the optimal match of the template with the observed image.

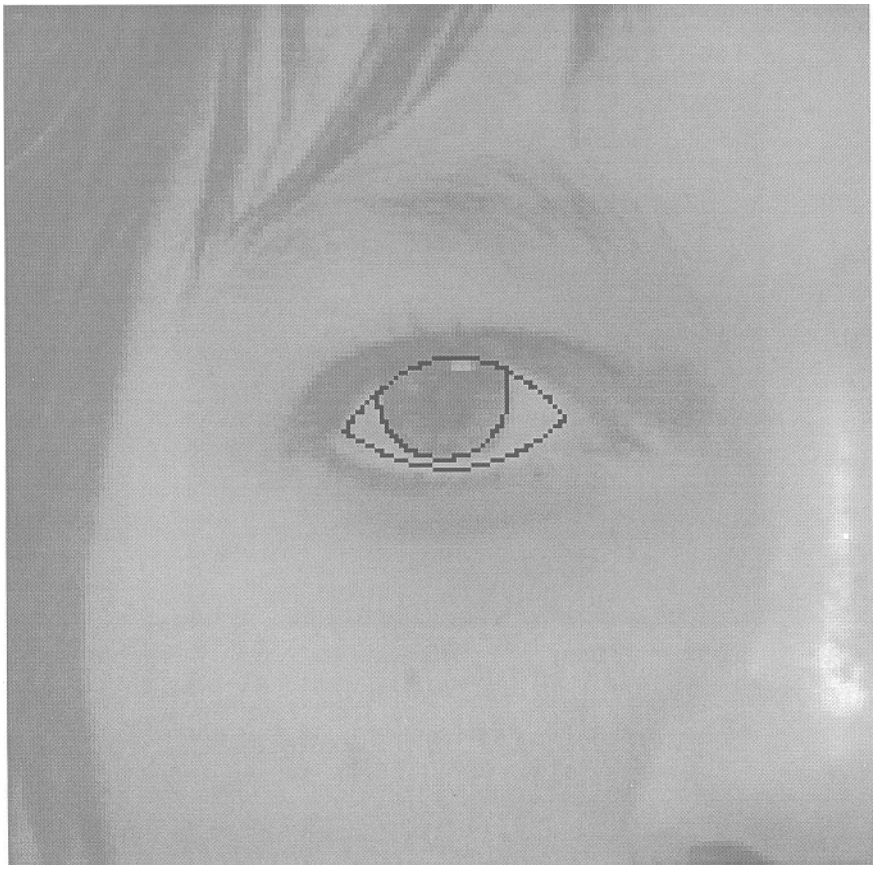

Figure 1.8d Diffeomorphism from a cartoon eye to a real eye.

Figures 1.8b, 1.8c and 1.8d show three examples of such matches. Figure 1.8b from Yamamoto & Rosenfeld (1982) applies these ideas to the recognition of chinese characters or kanji. In this application D_T is a 1-dimensional polygonal skeleton of the outline of the character, and ψ is a piecewise linear embedding of D_T in the domain D of the character image. Figure 1.8c from Y. Amit (1991) applies these ideas to tracing a hand in an X-ray by comparing it with a standard hand. Here ψ is a small deformation of the identity defined by a wavelet expansion of its (x, y)-coordinates and the prior $E(\psi)$ is a weighted L^2-norm of the expansion coefficients. Finally Figure 1.8d from Yuille, Hallinan & Cohen (1992) applies these ideas to the recognition of eyes. Here D_T has two parts, a pair of parabolas representing the outline of the eye and a black circle on a white ground representing the iris/pupil on the eyeball. ψ is linear on each parabola and on the circle, but the range of the first may *occlude* the range on the second to allow the iris/pupil to be partially hidden. This is incorporated in a careful definition of d.

An interesting mathematical side of this theory is the need for a careful definition and comparative study of various priors on the spaces of diffeomorphisms ψ. One can, for instance, define various measures $E(\psi)$ based (i) on the square norm of the Jacobian, as in harmonic map theory, (ii) on the area of the graph, as in geometric measure theory, (iii) on the stress of the map as in elasticity theory or (iv) on second derivatives of ψ, which give more control over the minima. Mumford (1991) discusses some of these measures, but the best approach is unclear and the restriction of maps to be diffeomorphisms is not always natural. An interesting neurophysiological aside is that the anatomy of the cortex of mammals seems well equipped to perform domain warping. The circuitry of the cortex is based on two types of connections: local connections within disjoint subsets of the cortex known as *cortical areas*, and global connections, called *pathways*, between the two distinct areas. The pathways occur in pairs, setting up maps which are crudely homeomorphisms between the cortical surfaces of the two areas which are inverse to each other. These pathways are not exactly point to point maps, however, because of the multiple synapses of their axons, hence the pair of inverse pathways may be able to shift a pattern of excitation by small amounts in any direction.

1.5.4. *The Stereo correspondence problem via minimum description length*

As described in Section 1.2, there are two approaches to the problems of pattern theory: the first is to use all the geometry, physics, chemistry, biology and sociology that we know about the world and try to define from this high-level knowledge an appropriate probabilistic model $(\Omega_{\text{obs}} \times \Omega_{\text{wld}}, p_{o,w})$ of the world and our observations. The second involves *learning this model* using only the patterns and the internal structure of the signals that are presented to us. Almost all research to date in computer vision falls in the first category, while the standard approach to speech recognition starts with the first but significantly improves on it using the 'EM-algorithm', a learning algorithm in the second category.

However, newborn animals seem to rely as strongly on learning their environment as on a genetically transmitted knowledge of it: it is not hard to imagine that a baby growing up in a virtual reality governed by quite unusual physics would learn these just as rapidly as the physics of its ancestral world. Humans can read scanning electron microscope images, which are produced by totally different reflectance rules from normal images. All of this suggests the possibility of discovering universal pattern analysis algorithms which learn patterns from scratch. One of the great appeals of the idea of pattern theory is the hope that the structure of the world can be discovered in this way. It is in this spirit that we present the final example. It is not an extensive theory like the previous three, but illustrates how the minimum description length principle can lead one to uncover the hidden structure of the world in a remarkably direct way. Closely related ideas can be found in Becker & Hinton (1992).

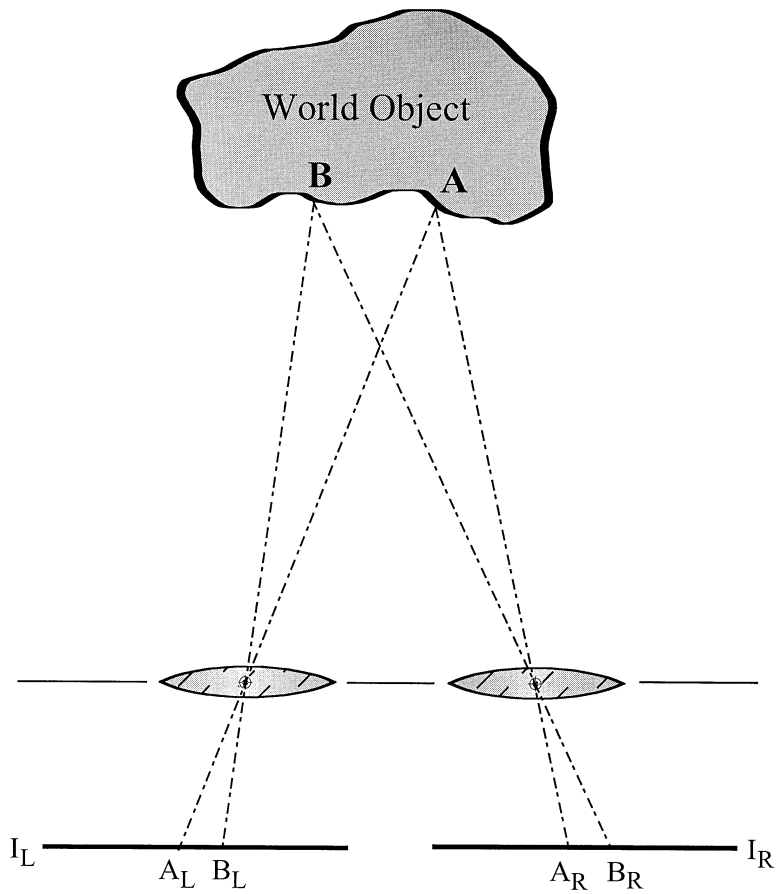

Figure 1.9 The geometry of stereo vision, in a plane through the centers of the two lenses.

We are concerned with the problem of stereo vision. If we view the world with two eyes or with two cameras separated by a known distance, and either identically oriented or with a known difference of orientations, then we can use trigonometry to infer the 3-dimensional structure of the world: see Figure 1.9. More precisely, the two imaging systems produce two images, I_L and I_R (the left and right images). Suppose a point A in the world visible in both images appears as $A_L \in D_L$ and $A_R \in D_R$ in the domains of the two images. The coordinates of A_L and A_R plus the known geometry of the imaging system give the 3-dimensional coordinates of A. However, to use this, we need to first find the pair of corresponding points A_L and A_R: finding these is called the *correspondence problem.* Notice from Figure 1.9 that the geometry of the imaging system gives us one simplification: all points A in a fixed 3-dimensional plane π, through the centers of the two lenses, are seen as points $A_L \in \ell_L$ and $A_R \in \ell_R$, where ℓ_L and ℓ_R are the intersections of π with the two focal planes, and are called *epipolar lines.* Moreover, when we are looking at a single

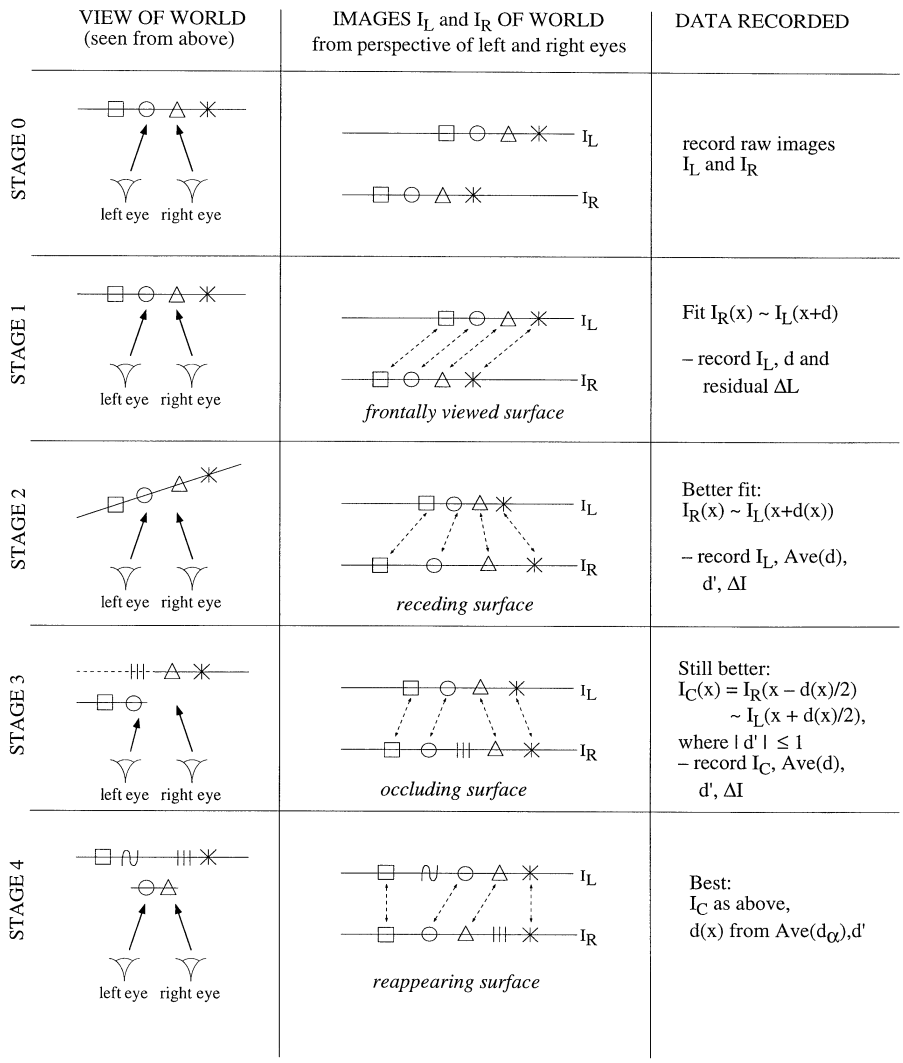

Figure 1.10 Discovering the world via MDL.

relatively smooth surface S in the 3-dimensional world, S is visible from the left and right eye as subdomains $S_L \subset D_L$ and $S_R \subset D_R$ and the corresponding points on these subdomains define a diffeomorphism $\psi : S_L \to S_R$ carrying each epipolar line in the left domain to the corresponding epipolar line in the right. This leads us to a problem like that in the last section. But there is a further twist: at the edges of objects, each of the two eyes can typically see a little further around one edge, producing pixels in one domain D_L or D_R with no corresponding pixel in the other domain. In this way, the domain is often segmented into subdomains corresponding to distinct objects. (See Belhumeur (1993)).

My claim is that the minimum description length principle alone leads you naturally to discover all this structure, without any prior knowledge of 3-dimensions. The argument is summarized in Figure 1.10. In this figure, I have represented a series of increasingly complex stereo images in diagrammatic form. Firstly, in order to represent the essentials concisely, I have used only a single pair of epipolar lines ℓ_L and ℓ_R instead of the full domains D_L and D_R. Secondly, instead of graphing the complex intensity function, we have used small symbols (squares, circles, triangles, stars, etc.) to denote local intensity functions with various characteristics. Thus a square on both lines represents local intensities which are similar functions. On the left, at each stage in Figure 1.10, we see the plane π in the world, with the visible surface points, and the left and right eyes. In the middle, we see the left and right images I_L and I_R which this scene produces, as well as dotted lines connecting corresponding points A_L and A_R. On the right we give a method of encoding the image data.

Stage 0 represents a simple flat object seen from the front: it produces images I_L and I_R, but we assume that our pattern analysis begins with naively encoding the images independently. At stage 1, the same scene is seen, but now the analysis uses the much more concise method of encoding only I_L, the fixed translation d by which corresponding points differ and a possible small residual $\Delta I(x) = I_R(x) - I_L(x+d)$. Clearly this is more concise. At stage 2, the scene is more complex: a surface of varying distance is seen, hence the displacement between corresponding points (called the *disparity*) is not constant. To adapt the previous encoding to this situation, one could take a mean value of d and have a bigger residual ΔI. But this residual could be quite big and a better scheme is replace the fixed d by a function $d(x)$ and encode I_L, the mean and derivative $(\overline{d}, d')$ of d and the residual ΔI. Now in stage 3, we encounter a new wrinkle: the scene consists in two surfaces, one occluding the other. Notice that a little bit of the back surface is visible to one eye only. To include this complexity, we go over to a more symmetrical treatment of the two eyes and encode a combined *cyclopean* image $I_C(x)$, where

$$I_C(x) = I_R\left(x - \frac{d(x)}{2}\right), I_L\left(x + \frac{d(x)}{2}\right) \text{ or their average} \tag{1.18}$$

depending on whether the point is visible only to the right eye, only to the left eye or to both eyes. To make this representation unique, it is easy to see that we must require that $|d'(x)| \leq 1$. Then we encode the scene via $(I_C, \overline{d}, d', \Delta I)$. In the final stage 4, we introduce the possibility of a surface disappearing behind another *and then reappearing*. The point is that each surface has its own average disparity, and it now becomes more efficient to record d by several means $\overline{d}_\alpha$, one for each surface, and the derivative d'. Thus we see how the search for minimum length encoding leads us naturally, first to the third coordinate of world points, then to smooth descriptions of surfaces in terms of their tangent planes and finally to explicit labelling of distinct surfaces in the visible field.

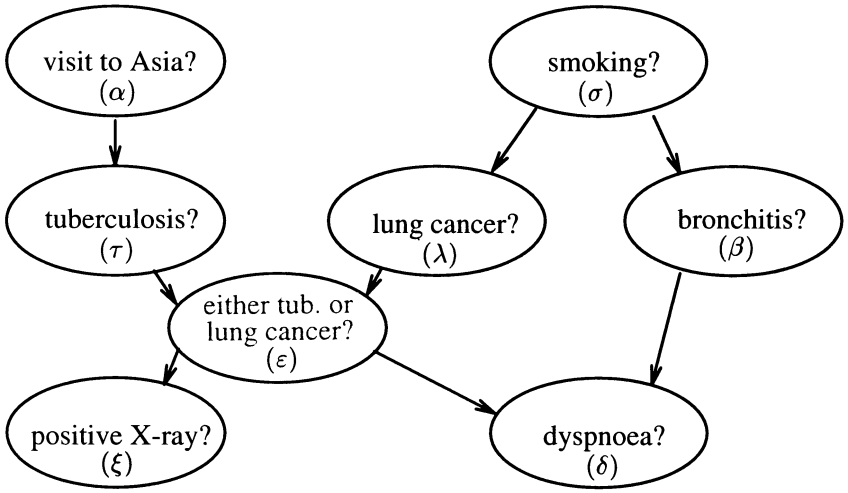

Figure 1.11 Causal graph in a toy medical expert system.

Although this approach might seem very abstract and impossible to implement biologically, G. Hinton (unpublished) has developed neural net theories incorporating both MDL and feed-back. These might be able to learn stereo exactly as outlined in this section.

1.6. Pattern Theory and Cognitive Information Processing

The examples of the last section all concern pattern theory as a theory for analyzing sensory input. The examples come from vision, but most of the ideas could apply to hearing or touch too. The purpose of this section is to ask the question: to what extent is pattern theory relevant to all cognitive information processing, both 'higher level' thinking as studied in cognitive psychology and AI, and the output stages of an intelligent agent, motor control and action planning. I believe that in many ways the same ideas are applicable on a theoretical level and that there is physiological evidence that the same algorithms are applied throughout the cortex.

In the introduction, we gave medical expert systems as another example of pattern theory. In this extension, we considered the data available to a physician — symptoms, test results and the patient's history — as an encoded version of the full state of the world, a 'deformed image' in Grenander's terminology. The full state of the world, the 'pure image', in this case means the diseases and processes present in the patient. Inferring these hidden random variables can and has been studied as a problem in Bayesian statistics, exactly as in Section 1.2: see, for instance, Pearl (1988) and Lauritzen & Spiegelhalter (1988). In particular, describing the probability distribution on all the random variables as a Gibbs field, as in Section 1.5, has been shown to be a powerful technique for introducing realistic models of the probability distribution in the real world. Figure 1.11, from the article Lauritzen

& Spiegelhalter (1988), shows a simplified set of such random variables and the graph on which a Gibbs distribution can be based. Whether or not pattern theory extends in an essential way to these types of problems hinges on whether the transformations described in Section 1.3 generate the kind of probability distributions encountered with higher level variables. To answer this, it is essential to look at test cases which are not too artificially simplified (as is done all too often in AI), but which incorporates the typical sorts of complexities and complications of the real world. While I do not think this question can be definitively answered at present, I want to make a case that the four types of transformations of Section 1.3 are indeed encoding mechanisms encountered at all levels of cognitive information processing.

The first class of transformations, noise and blur, certainly occur at all levels of thought. In the medical example, errors in tests, the inadequacies of language in conveying the nature of a pain or symptom, etc. all belong to this category. Uncertainty over facts, misinterpretations and confusing factors are within this class. The simplest model leads to multi-dimensional normal distributions on a vector P of 'features' being analyzed.

The fourth category of transformation, 'interruptions', are also obviously universal. In any cognitive sphere, the problem of separating the relevant factors for a specific event or situation being analyzed from the extraneous factors involved with everything else in the world, is clearly central. The world is a complex place with many, many things happening simultaneously, and highlighting the 'figure' against the 'ground' is not just a sensory problem, but one encountered at every level. Another way this problem crops up is that a complex of symptoms may result from one underlying cause or from several, and, if several causes are present, their effects have to be teased apart in the process of pattern analysis. As proposed in Section 1.4, pattern synthesis — actively comparing the results of one cause with the presenting symptoms P followed by analysis of the residual, the unexplained symptoms, is a universal algorithmic approach to these problems.

The second of the transformations, 'multi-scale superposition', can be applied to higher level variables as follows: philosophers, psychologists and AI researchers have all proposed systematizing the study of concepts and categories by organizing them in hierarchies. Thus psychologists (see Rosch, 1978) propose distinguishing *superordinate categories, basic level categories and subordinate categories*: for instance, a particular pet might belong to the superordinate category 'animal', the basic-level category 'dog' and the subordinate category 'terrier'. In AI, this leads to graphical structures called *semantic nets* for codifying the relationships between categories (see Findler, 1979). These nets always include ordered links between categories, called *isa* links, meaning that one category is a special case of another. I want to propose that cognitive multi-scale superposition is precisely the fact that to analyze a specific situation or thing, some aspects result from the situation belonging to very general categories, others from very specific facts about the situation that put it in very precise categories. Thus sensory thinking requires we deal with large

shapes with various overall properties, supplemented with details about their various parts, precise data on location, proportions, etc.; cognitive thinking requires we deal with large ideas with various general properties, supplemented with details about specific aspects, precise facts about occurrence, relationships, etc.

Finally, how about 'domain warping'? Consider a specific example first. Associated to a cold is a variety of several dozen related symptoms. A person may, however, be described as having a sore throat, a chest cold, flu, etc.: in each case the profile of their symptoms shifts. This may be modelled by a map from symptom to symptom, carrying for instance the modal symptom of soreness of throat to that of coughing. The more general cognitive process captured by domain warping is that of making an *analogy*. In an analogy, one situation with a set of participants in a specific relationship to each other is mapped to another situation with new participants in the same relationship. This map is the ψ in Section 1.5, and the constraints on ψ, such as being a diffeomorphism, are now that it preserve the appropriate relationships. The idea of domain warping applying to cognitive concepts seems to suggest that higher level concepts should form some kind of geometric space. At first this sounds crazy, but it should be remembered that the entire cortex, high and low level areas alike, has the structure of a 2-dimensional sheet. This 2-dimensional structure is used in a multitude of ways to organize sensory and motor processes efficiently: in some cases, sensory maps (like the retinal response and patterns of tactile responses) are laid out geometrically. In other cases, interleaved stripes carry intra-hemispheric and inter-hemispheric connections. In still other cases, there are 'blobs' in which related responses cluster. But, in all cases, adjacency in this 2D sheet allows a larger degree of cross-talk and interaction than with non-adjacent areas and this seems to be used to develop responses to related patterns. My suggestion is: is this spatial adjacency used to structure abstract thought too[†]?

To conclude, we want to discuss briefly how pattern theory helps the analysis of motor control and action planning, the output stage of a robot. Control theory has long been recognized as the major mathematical tool for analyzing these problems but it is not, in fact, all that different from pattern theory. In Figure 1.12a, we give the customary diagram of what control theory does. The controller is a black box which compares the current observation of the state of the world with the desired state and issues an updated motor command, which in turn affects the black box called the world. This diagram is very similar to Figure 1.4, which

[†]I have argued elsewhere that the remarkable anatomical uniformity of the neo-cortex suggests that common mechanisms, such as the 4 universal transformations of pattern theory, are used throughout the cortex (Mumford, 1991–92; 1993). The referree has pointed out that 'the uniformity of structure may reflect common machinery at a lower level. For example, different computers may have similar basic mechanisms at the level of registers, buses, etc., which is a low level of data handling. Similarly in the brain, the apparent uniformity of structure may be at the level of common lower-level mechanisms rather than the level of dealing with universal transformations.' This is a certainly an alternative possibility, quite opposite to my conjectural link between the high-level analysis of pattern theory and the circuitry of the neo-cortex.

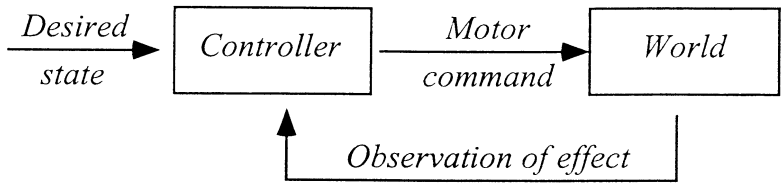

Figure 1.12a The flow chart of control theory.

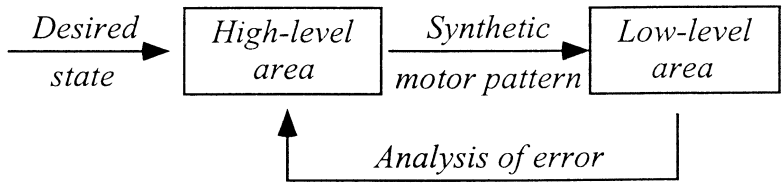

Figure 1.12b A motor task via pattern theory.

described how pattern analysis and pattern synthesis formed a loop used in the algorithm for reconstructing the hidden world variables from the observed sensory ones. Figure 1.12b presents the modification of Figure 1.4 to a motor task. Here a high-level area or 'black box' is in a loop with a low-level area: the high-level area compares the desired state with an analysis of the error and generates an updated motor command sequence by pattern synthesis. The low-level area, either by actually carrying out an action and observing its consequences, or by internal simulation, finds that it falls short in various ways, and send its pattern analysis of this error back up. Notice that the four transformations of Section 1.3 will occur or should be used in the top-down pattern synthesis step. Noise and blur are the inevitable consequences of the inability to control muscles perfectly, or eliminate external uncontrollable interference. Domain warping is the bread-and-butter of control theory — speeding up or slowing down an action by modifying the forces in order that it optimizes performance. Multi-scale superposition is what hierarchical control is all about: building up an action first in large steps, then refining these steps in their parts, eventually leading to detailed motor commands. Finally, interruptions are the terminations of specific control programs, either by success or by unexpected events, where quite new programs take over. In general, we seek to anticipate these and set up successor programs beforehand, hence we need to actively synthesize these as much as possible.

In summary, my belief is that pattern theory contains the germs of a universal theory of thought, one which stands in opposition to the accepted analysis of thought in terms of logic. The extraordinary similarity of the structure of all parts of the human cortex to each other and of human cortex with the cortex of the most primitive mammals suggests that a relatively simple universal principal governs its

operation, even in complex processes like language (see Mumford, 1991–92, 1993) where these physiological links are developed: pattern theory is a proposal for what these principles may be.

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THE WORK OF GREGORI ALEKSANDROVITCH MARGULIS

by

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The work of Margulis belongs to combinatorics, differential geometry, ergodic theory, the theory of dynamical systems and the theory of discrete subgroups of real and p-adic Lie groups. In this report, I shall concentrate on the last aspect which covers his main results.

1. Discrete subgroups of Lie groups. The origin. Discrete subgroups of Lie groups were first considered by Poincaré, Fricke and Klein in their work on Riemann surfaces: if M is a Riemann surface of genus $\geqslant 2$, its universal covering is the Lobatchevski plane (or Poincaré half-plane), therefore the fundamental group of M can be identified with a discrete subgroup Γ of $\mathrm{PSL}_2(\boldsymbol{R})$; the problem of uniformization and the theory of differentials on M lead to the study of automorphic forms relative to Γ.

Other discrete subgroups of Lie groups, such as $\mathrm{SL}_n(\boldsymbol{Z})$ (in $\mathrm{SL}_n(\boldsymbol{R})$) and the group of "units" of a rational quadratic form (in the corresponding orthogonal group) play an essential role in the theory of quadratic forms (reduction theory) developed by Hermite, Minkowski, Siegel and others. In constructing a space of moduli for abelian varieties, Siegel was led to consider the "modular group" $\mathrm{Sp}_{2n}(\boldsymbol{Z})$, a discrete subgroup of $\mathrm{Sp}_{2n}(\boldsymbol{R})$.

The group $\mathrm{SL}_n(\boldsymbol{Z})$, the group of units of a rational quadratic form and the modular group are special instances of "arithmetic groups", as defind by A. Borel and Harish–Chandra. A well-known theorem of those authors, generalizing classical results of Fricke, Klein, Siegel and others, asserts that if Γ is an arithmetic subgroup of a semi-simple Lie group G, then the volume of G/Γ (for any G-invariant measure) is finite; we say that Γ has *finite covolume* in G. The same holds for $G{=}\mathrm{PSL}_2(\boldsymbol{R})$ if Γ is the fundamental group of a Riemann surface "with null boundary" (for instance, a compact surface minus a finite subset).

Already Poincaré wondered about the possibility of describing all discrete subgroups of finite covolume in a Lie group G. The profusion of such subgroups in $G{=}\mathrm{PSL}_2(\boldsymbol{R})$ makes one at first doubt of any such possibility. However, $\mathrm{PSL}_2(\boldsymbol{R})$ was for a long time the only simple Lie group which was known to contain

non-arithmetic discrete subgroups of finite covolume, and further examples discovered in 1965 by Makarov and Vinberg involved only few other Lie groups, thus adding credit to conjectures of Selberg and Pyatetski–Shapiro to the effect that "for most semisimple Lie groups" discrete subgroups of finite covolume are necessarily arithmetic. Margulis' most spectacular achievement has been the complete solution of that problem and, in particular, the proof of the conjectures in question.

2. The noncocompact case. Selberg's conjecture. Let G be a semisimple Lie group. To avoid inessential technicalities, we assume that G is the group of real points of a real simply connected algebraic group $\mathcal{G}$ which we suppose embedded in some $\mathrm{GL}_n(\boldsymbol{R})$, and that G has no compact factor. Let Γ be a discrete subgroup of G with finite covolume and irreducible in the sense that its projection in any nontrivial proper direct factor of G is nondiscrete. Suppose that the real rank of G is $\geqslant 2$ (this means that G is not a covering group of the group of motions of a real, complex or quaternionic hyperbolic space or of an "octonionic" hyperbolic plane) and that G/Γ *is not compact*. Then, Selberg's conjecture asserts that Γ is arithmetic which, in this case, means the following: there is a base in $\boldsymbol{R}^n$ with respect to which $\mathcal{G}$ is defined by polynomial equations with rational coefficients and such that Γ is commensurable with $\mathcal{G}(\boldsymbol{Z}) = G \cap GL_n(\boldsymbol{Z})$ (i.e. $\Gamma \cap \mathcal{G}(\boldsymbol{Z})$ has finite index in both Γ and $\mathcal{G}(\boldsymbol{Z})$). Selberg himself proved that result in the special case where G is a direct product of (at least two) copies of $\mathrm{SL}_2(\boldsymbol{R})$.

A first important step toward the understanding of noncompact discrete subgroups of finite covolume was the proof by Každan and Margulis [2] of a related, more special conjecture of Selberg: *under the above assumptions* (except that no hypothesis is made on $\mathrm{rk}_R G$), Γ *contains nontrivial unipotent elements of* G (i.e. elements all of whose eigenvalues are 1). This was a vast generalization of results already known for $\mathrm{SL}_2(\boldsymbol{R})$ and products of copies of $\mathrm{SL}_2(\boldsymbol{R})$ (Selberg); in view of a fundamental theorem of Borel and Harish–Chandra ("Godement's conjecture"), it had to be true if Γ was to be arithmetic. Let us also note in passing another remarkable byproduct of Každan–Margulis' method: *given* G, *there exists a neighborhood* W *of the identity in* G *such that for every* Γ (*cocompact or not*), *some conjugate of* Γ *intersects* W *only at the identity*; in particular, *the volume of* G/Γ *cannot be arbitrarily small* (for a given Haar measure in G). For $G{=}\mathrm{SL}_2(\boldsymbol{R})$, the last assertion had been proved by Siegel, who had also given the exact lower bound of vol (G/Γ) in that case. A. Borel reported on those results of Každan and Margulis at Bourbaki Seminar [26].

The existence of unipotent elements in Γ was giving a hold on its structure. In [6], Margulis announced, among others, the following result which was soon recognized by the experts as a crucial step for the proof of Selberg's conjecture:

in the space of lattices in $\boldsymbol{R}^n$, *the orbits of a one-parameter unipotent subsemigroup of* $\mathrm{GL}_n(\boldsymbol{R})$ *"do not tend to infinity"* (*in other words, a closed orbit is periodic*).

For a couple of years, Margulis' proof remained unpublished and every attempt by other specialists to supply it failed. When it finally appeared in [9], the proof came as a great surprise, both for being rather short and using no sophisticated technique: it can be read without any special knowledge and gives a good idea of the extraordinary inventiveness shown by Margulis throughout his work.

Using unipotent element, it is relatively easy to show that, G and Γ being as above, there is a $\boldsymbol{Q}$-structure $\mathcal{G}$ on G such that $\Gamma \subset \mathcal{G}(\boldsymbol{Q})$. The main point of Selberg's conjecture is then to show that the matrix coefficients of the elements of Γ have bounded denominators. In [15], Margulis announced a complete proof of the conjecture and gave the details under the additional assumption that the $\boldsymbol{Q}$-rank of $\mathcal{G}$ is at least 2. Another proof under the same restriction was given independently by M. S. Raghunathan. The much more difficult case of a $\boldsymbol{Q}$-rank one group is treated by Margulis in [19], by means of a very subtle and delicate analysis of the set of unipotent elements contained in Γ. The main techniques used in [15] and [19] are those of algebraic group theory and p-adic approximation.

3. The cocompact case. Rigidity. Margulis was invited to give an address at the Vancouver Congress, no doubt with the idea that he would expose his solution of Selberg's conjecture. Instead, prevented (at this time) from attending the Congress, he sent a report on completely new and totally unexpected results on the cocompact case [18].

That case, about which nothing was known before, presented two great additional difficulties which nobody knew how to handle. On the one hand, if G/Γ is compact, Γ contains no unipotent element, so that the main technique used in the other case is not available. But there is another basic difficulty in the very notion of arithmetic group: let G, Γ be as in Sec. 2 except that G/Γ is no longer assumed to be noncompact; then Γ is said to be arithmetic if there exist an algebraic linear semi-simple simply connected group $\mathcal{H}$ defined over $\boldsymbol{Q}$ and a homomorphism $\alpha : \mathcal{H}(\boldsymbol{R}) \to G$ with compact kernel such that Γ is commensurable with $\alpha(\mathcal{H}(\boldsymbol{Z}))$. The point is that in the noncocompact case, α is necessarily an isomorphism. In the general case, there is *a priori* no way of knowing what $\mathcal{H}$ will be (in fact, for a given $G, \mathcal{H}$ tan have an arbitrarily large dimension). A conjecture, more or less formulated by Pyatetski–Shapiro at the 1966 Congress in Moscow, to the effect that also in the cocompact case, assuming again $\mathrm{rk}_R G \geqslant 2$, Γ had to be arithmetic, was certainly more daring at the time and seemed completely out of reach. It was the proof of that conjecture that Margulis sent, without warning, to the Vancouver Congress.

Arithmetic subgroups of Lie groups are in some sense "rigid"; intuitively, this follows from the impossibility to alter an algebraic number continuously without destroying the algebraicity. On the other hand, theorems of Selberg, Weil and Mostow showed that in semi-simple Lie groups different from $\mathrm{SL}_2(\boldsymbol{R})$ (up to local isomorphism) cocompact discrete subgroups are rigid, and Selberg had observed

that rigidity implies a "certain amount of arithmeticity": in fact, it is readily seen to imply that Γ is contained in $\mathcal{G}(\boldsymbol{K})$ for some algebraic group $\mathcal{G}$ and some number field $\boldsymbol{K}$. As before, the crux of the matter is the proof that the matrix coefficients of the elements of Γ have bounded denominators. This is achieved by Margulis through a "superrigidity" theorem which, for groups of real rank at least 2, is a vast generalization of Weil's and Mostow's rigidity theorems:

Assume $\mathrm{rk}_R G \geqslant 2$, *let F be a locally compact nondiscrete field and let $\varrho : \Gamma \to \mathrm{GL}_n(F)$ be a linear representation such that $\varrho(\Gamma)$ is not relatively compact and that its Zariski closure is connected; then $F = \boldsymbol{R}$ or C and ϱ extends to a rational representation of $\mathcal{G}$.*

The proof of this theorem is relatively short (considering the power of the result), but is a succession of extraordinarily ingenious arguments using a great variety of very strong techniques belonging to ergodic theory (the "multiplicative ergodic theorem"), the theory of unitary representations, the theory of functional spaces (spaces of measurable maps), algebraic geometry, the structure theory of semi-simple algebraic groups, etc. In 1975–1976, I devoted my course at the Collège de France to those results of Margulis; I believe that I learned more mathematics during that year than in any other year of my life. A summary of the main ideas of that beautiful piece of work is given in [27].

Another, quite different proof of the superrigidity theorem and its application to arithmeticity (both in the cocompact and the noncocompact case) — using the work of H. Furstenberg — can be found in [20].

4. Other results.

4.1. *S-arithmetic groups.* Let K be a number field, S a finite set of places of K including all places at infinity, $\mathfrak{O}$ the ring of elements of K which are integral at all finite places not belonging to S, $\mathcal{H} \subset \mathcal{GL}_n$ a simply connected semisimple linear algebraic group defined over K and $\mathcal{H}(\mathfrak{O}) = \mathcal{H} \cap \mathrm{GL}_n(\mathfrak{O})$. Then, $\mathcal{H}(\mathfrak{O})$ injects as a discrete subgroup of finite covolume in the product $H = \Pi_{v \in s} \mathcal{H}(K_v)$, where K_v denotes the completion of K at v.

(Example: if $\mathfrak{O}$ is the ring of rational numbers whose denominator is a power of 2, $\mathrm{SL}_n(\mathfrak{O})$ is a discrete subgroup of finite covolume of $\mathrm{SL}_n(\boldsymbol{R}) \times \mathrm{SL}_n(\boldsymbol{Q}_2)$). Now let G be a direct product of simply connected semi-simple real or p-adic Lie groups. A discrete subgroup Γ of G is called S-arithmetic if there exist $K, S, \mathcal{H}$ as above and a homomorphism $\alpha : H \to G$ with compact kernel such that Γ and $\alpha(\mathcal{H}(\mathfrak{O}))$ are commensurable. All results of [18], stated above for ordinary Lie groups and arithmetic groups, are in fact proved by Margulis in the more general framework described here. In particular, he show that *if G is as above, if the rank of G (i.e. the sum of the relative ranks of its factors) is at least 2 and if Γ is a discrete subgroup of finite covolume in G, which is irreducible (as defined in n° 2), then Γ is S-arithmetic.*

4.2. "*Abstract*" *isomorphisms.* The very general and powerful superrigidity theorem (in the framework of 4.1) has far-reaching consequences besides the arithmeticity. For instance, it enables Margulis to solve almost completely the problem of "abstract isomorphisms" between groups of points of algebraic simple groups over number fields or arithmetic subrings of such fields; his result embodies, in the arithmetic case, all those obtained before on that problem by Dieudonné, O'Meara and his school, A. Borel and me, and goes considerably further.

4.3. *Normal subgroups.* Let G be as in 4.1 and let Γ be an irreducible discrete subgroup of G of finite covolume. Margulis was able to show (cf. [27]) that *if* rk $G \geqslant 2$, *then every noncentral normal subgroup of* Γ *has finite index.* (In fact, the conditions of Margulis' theorem are more general: under suitable hypotheses, G is allowed to have factors defined over locally compact local fields of finite characteristic.) So far, the only results known in that direction — results of Mennicke, Bass, Milnor, Serre, Raghunathan — were connected with the congruence subgroup problem and valid only in the cases where that problem has a positive solution.

4.4. *Action on trees.* In a paper which appeared in the Springer Lecture Notes, no 372, Serre showed that the group of integral points of a simple Chevalley group-scheme of rank $\geqslant 2$ cannot act without fixed point on a tree; this also means that such a group is not an amalgam in a nontrivial way. Serre points out that his method of proof does not extend to congruence subgroups and asks whether the result generalizes to such subgroups or to other arithmetic groups. With his own methods, Margulis was able to solve at once the problem in its widest generality: *if* G *is as in* 4.1, *of rank at least 2, and if* Γ *is an irreducible discrete subgroup of finite covolume in* G*, then* Γ *cannot act without fixed point on a tree.*

5. Conclusion. Margulis has completely or almost completely solved a number of important problems in the theory of discrete subgroups of Lie groups, problems whose roots lie deep in the past and whose relevance goes far beyond that theory itself. It is not exaggerated to say that, on several occasions, he has bewildered the experts by solving questions which appeared to be completely out of reach at the time. He managed that through his mastery of a great variety of techniques used with extraordinary resources of skill and ingenuity. The new and most powerful methods he has invented have already had other important applications besides those for which they were created and, considering their generality, I have no doubt that they will have many more in the future.

I wish to conclude this report by a nonmathematical comment. This is probably neither the time nor the place to start a polemic. However, I cannot but express my deep disappointment — no doubt shared by many people here — in the absence of Margulis from this ceremony. In view of the symbolic meaning of this city of Helsinki, [1] I had indeed grounds to hope that I would have a chance at last to meet a

[1] The address was delivered in Finlandia Hall, where the 1975 Helsinki Agreements were concluded.

mathematician whom I know only through his work and for whom I have the greatest respect and admiration.

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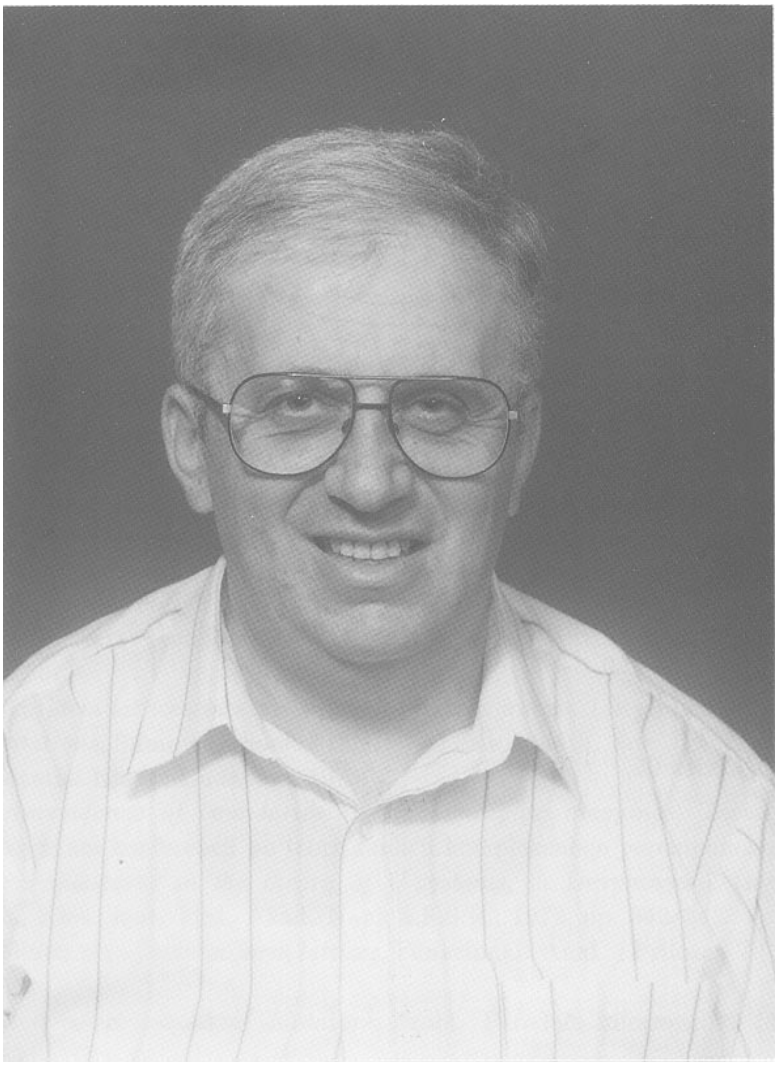

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OPPENHEIM CONJECTURE

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The main purpose of this paper is to give a survey of results and methods connected with a conjecture on values of indefinite irrational quadratic forms at integer points stated by Oppenheim in 1929 and proved by the author in 1986. The different approaches to this and related conjectures (and theorems) involve analytic number theory, the theory of Lie groups and algebraic groups, ergodic theory, representation theory, reduction theory, geometry of numbers and some other topics. A comprehensive survey of the methods related to the Oppenheim conjecture is thus a long story.

Let B be a real nondegenerate indefinite quadratic form in n variables. We will say that B is *rational* if it is a multiple of a form with rational coefficients and *irrational* otherwise. According to Meyer's theorem (see [Ca3]) if B is rational and $n \geq 5$ then B represents zero over $\mathbb{Z}$ nontrivially, i.e. there exist integers $x_1, \ldots, x_n$ not all equal to 0 such that $B(x_1, \ldots, x_n) = 0$. Let us set

$$m(B) = \inf\{|B(x)| \mid x \in \mathbb{Z}^n, x \neq 0\}\,.$$

Then Meyer's theorem is equivalent to the statement that if B is rational and $n \geq 5$ then $m(B) = 0$. In 1929 A. Oppenheim [Op1,2] conjectured that if $n \geq 5$ then $m(B) = 0$ also for irrational B. Later it was realized that $m(B)$ should be equal to 0 under a weaker condition $n \geq 3$ (for diagonal forms it was stated as a conjecture in [DavH]; let us remind the reader that a quadratic form Q is called *diagonal* if $Q(x_1, \ldots, x_n) = \lambda_1 x_1^2 + \cdots + \lambda_n x_n^2$). Of course if n is 3 or 4 one has to assume that B is irrational because there exist rational nondegenerate indefinite quadratic forms in 3 and 4 variables which do not represent zero over $\mathbb{Z}$ nontrivially. Thus the Oppenheim conjecture with the generalization for $n = 3$ and $n = 4$ states that if $n \geq 3$ and B is not proportional to a form with rational coefficients, then for any $\varepsilon > 0$ there exist integers $x_1, \ldots, x_n$ not all equal to 0 such that $|B(x_1, \ldots, x_n)| < \varepsilon$. (Let us note that the condition "$n \geq 3$" cannot be replaced by the condition "$n \geq 2$". To see this, consider the form $x_1^2 - \lambda x_2^2$ where λ is an irrational positive number such that $\sqrt{\lambda}$ has a continued fraction development with bounded partial quotients; for example $\lambda = (1 + \sqrt{3})^2 = 4 + 2\sqrt{3}$.) This conjecture was proved by studying

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orbits of the orthogonal group $SO(2,1)$ on the space $\Omega \cong SL(3,\mathbb{R})/SL(3,\mathbb{Z})$ of unimodular lattices in $\mathbb{R}^3$ (see [Marg3–6]).

Before the Oppenheim conjecture was proved it was extensively studied mostly using analytic number theory methods. We will describe this earlier development in Sec. 1. In Sec. 2, we begin with a 1955 paper of Cassels and Swinnerton-Dyer and proceed to describe results related to the Oppenheim conjecture up to the present. In Sec. 3 we present a proof of a strengthened Oppenheim conjecture based largely on a consideration of unipotent flows on homogeneous spaces. We point out other phenomena related to unipotent flows.

Let us now make the following remark. If B is a real irrational nondegenerate indefinite quadratic form in n variables and $m < n$ then $\mathbb{R}^n$ contains a rational subspace L of dimension m such that the restriction of B to L is irrational nondegenerate and indefinite (this is a standard fact; the proof can be found in Sec. 5 of [DanM1]). Hence if the Oppenheim conjecture is proved for some n_0, then it is proved for all $n \geq n_0$. In particular, it is enough to prove the conjecture for $n = 3$.

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1. Analytic Number Theory Methods

1.1. Chowla's result and the distribution of values of positive definite quadratic forms at integer points. The first result on the Oppenheim conjecture was obtained in 1934 by Chowla [Ch] who proved the conjecture for indefinite diagonal forms

$$B(x_1,\dots,x_n) = \lambda_1 x_1^2 + \dots + \lambda_r x_r^2 + \lambda_{r+1} x_{r+1}^2 + \dots + \lambda_n x_n^2$$

such that $n \geq 9$ and all the ratios $\lambda_i/\lambda_j (i \neq j)$ are irrational. His proof used a theorem of Jarnik and Walfisz on the number of integer points in a large ellipsoid. Indeed Chowla's proof can be applied also in the following cases: (a) $r \geq 5$, $\lambda_i > 0,\ 1 \leq i \leq r$, and at least one ratio $\lambda_i/\lambda_j, 1 \leq i < j \leq r$ is irrational; (b) $n-r \geq 5, \lambda_i < 0,\ r+1 \leq i \leq n$, and at least one ratio $\lambda_i/\lambda_j,\ r+1 \leq i < j \leq n$, is irrational.

Let us describe now the result of Jarnik and Walfisz. Let Q be a positive definite real quadratic form in n variables and let $A_Q(X)$ denote the number of integer solutions of the inequality $Q(x) \leq X$. One can interpret $A_Q(X)$ as the number of points from the lattice $\mathbb{Z}^n$ in the n-dimensional ellipsoid $\{x \in \mathbb{R}^n \mid Q(x) \leq X\}$. Let $\mathcal{T}_Q(X)$ denote the volume of this ellipsoid. We have

$$\mathcal{T}_Q(X) = \mathcal{C}_Q X^{n/2}, \mathcal{C}_Q = \frac{\pi^{n/2}}{\sqrt{D}\Gamma\left(\frac{n}{2}+1\right)} \tag{1}$$

where D is the determinant of Q. In 1930 Jarnik and Walfisz [JW] proved that if $n \geq 5$ and Q is diagonal and irrational then

$$A_Q(X) = \mathcal{T}_Q(X) + o(X^{n/2-1})\,. \tag{2}$$

They showed also that "o " cannot be replaced by any specific function. It means that for any positive function $\varphi(X), X > 0$, with $\lim_{X\to\infty} \varphi(X) = 0$, there exists a diagonal irrational form Q such that

$$\limsup_{X\to\infty} \frac{|A_Q(X) - \mathcal{T}_Q(X)|}{X^{n/2-1}\varphi(X)} = \infty\,.$$

In fact the set of all such Q is of second category in the Baire sense (as a subset of the space of positive definite diagonal forms). This can be easily proved by a standard topological argument using the fact that for any rational Q

$$\limsup_{X\to\infty} \frac{|A_Q(X) - \mathcal{T}_Q(X)|}{X^{n/2-1}} > 0\,.$$

Indeed the only case $n = 5$ was considered in [JW], because for $n \geq 6$ the estimate (2) had been proved in an earlier paper by Jarnik. The proof of (2) in this case $n = 5$ given in [JW] uses a modification of Jarnik's method.

It follows immediately from (1) and (2) that for any fixed $\varepsilon > 0$

$$A_Q(X+\varepsilon) - A_Q(X) \sim \varepsilon \mathcal{C}_Q \frac{n}{2} X^{n/2-1} \text{ as } X \to \infty\,.$$

In particular, the gaps between successive values of the quadratic form Q must tend to 0 or, equivalently,

$$A_Q(X+\varepsilon) - A_Q(X) > 0 \tag{3}$$

for every positive ε and all $X \geq X_0(\varepsilon)$.

Now let $B, \lambda_1, \ldots, \lambda_n$ be the same as at the beginning of this subsection. We assume that $n \geq 9$ and all the ratios $\lambda_i/\lambda_j (i \neq j)$ are irrational. The Chowla's argument is the following one. Of nine positive or negative numbers, at least five must have the same sign. Hence we may assume that $\lambda_1, \ldots, \lambda_5$ are positive and that at least one of the numbers $\lambda_6, \ldots, \lambda_n$ is negative. Now applying (3) to the quadratic form

$$Q(x_1, \ldots, x_5) = \lambda_1 x_1^2 + \cdots + \lambda_5 x_5^2$$

we get that there exist $y \geq X_0(\varepsilon), y > 0$, and integers $m_1, \ldots, m_n$ such that

$$\sum_{s=6}^{n} \lambda_s m_s^2 = -y$$

and

$$y < \lambda_1 m_1^2 + \cdots + \lambda_5 m_5^2 \leq y + \varepsilon\,.$$

Then $|B(m_1, \ldots, m_n)| < \varepsilon$ where the m's are not all zero; this is the desired statement.

It was conjectured in [DavL] that (2) and (3) are true for a general positive definite irrational quadratic form in $n \geq 5$ variables. These conjectures have not

yet been proved even under the assumption that n is very large. However, in 1972 Davenport and Lewis, in the same paper [DavL], partially resolved the question regarding gaps (that is, the generalization of (3)). They proved the following:

1.1.1. Theorem. *There exists an integer n_0 (absolute) with the following property:*

Let $Q(x) = Q(x_1, \dots, x_n)$ be a positive definite quadratic form with real coefficients and suppose that $n \geq n_0$. Then, if $x_1^*, \dots, x_n^*$ are integers with $\max_i |x_i^*|$ sufficiently large, there exist integers $x_1, \dots, x_n$, not all zero, such that

$$|Q(x + x^*) - Q(x^*)| < 1\,. \tag{4}$$

Clearly, in this theorem one can replace 1 by any positive ε. As it is noticed in [DavL] and [Lew] the result of Davenport and Lewis is imperfect in two ways. (a) One could have $Q(x+x^*) = Q(x^*)$ even if Q is irrational, and indeed this may well happen if Q represents a rational form in 4 variables (i.e. if there exists a rational 4-dimensional subspace L such that the restriction of Q to L is proportional to a form with integral coefficients). (b) Even if (a) does not occur, no deduction can be made about the gaps since the result does not prevent that the elements of $Q(\mathbb{Z}^n)$ occur in clumps with large gaps between the clumps.

1.1.2. The proof of Theorem 1.1.1 uses the Hardy–Littlewood circle method and representation properties of positive definite diagonal forms with integral coefficients. In 1986 Cook and Raghavan [CoR2] gave an estimate for the size of n_0 and also gave a lower bound for the number of solutions of the inequality (4). Their proof is similar to the proof given in [DavL]. It is also noticed in [CoR2] that Davenport and Lewis overlooked the (trivial) solution $x = -2x^*$ for (4). Let us state the theorem of Cook and Raghavan.

1.1.2. Theorem [CoR2, Theorem 2]. *There exists an integer $n_0 \leq 995$ and a constant $\tau > 0$ with the following property:*

Let $Q(x) = Q(x_1, \dots, x_n)$ be a positive definite quadratic form with real coefficients and suppose that $n \geq n_0$. Then, if $x_1^*, \dots, x_n^*$ are integers with $\max |x_i^*|$ sufficiently large, then there exist at least $[\|x^*\|^\tau]$ integer points $x \in \mathbb{Z}^n$ such that

$$|Q(x + x^*) - Q(x)| < 1\,.$$

1.2. Diagonal forms in $n \geq 5$ variables. In 1946, Davenport and Heilbronn proved the Oppenheim conjecture for diagonal forms in $n \geq 5$ variables. Strictly speaking, they considered only forms in five variables. But the case $n \geq 5$ immediately reduces to the case $n = 5$ by a trivial argument. In fact, Davenport and Heilbronn proved the following stronger theorem which gives lower bounds for the number of integral solutions of some quadratic inequalities in some domains.

1.2.1. Theorem [DavH]. *Let $\lambda_1, \dots, \lambda_5$ be real numbers, not all of the same sign, and none of them zero, such that at least one of the ratios λ_r/λ_s is irrational. Let*

$$B(x_1, \dots, x_5) = \lambda_1 x_1^2 + \cdots + \lambda_5 x_5^2 .$$

Then there exist arbitrarily large integers P such that the inequalities

$$1 \le x_1 \le P, \dots, 1 \le x_5 \le P \ , \ |B(x_1, \dots, x_5)| < 1$$

have more than γP^3 integral solutions, where $\gamma = \gamma(\lambda_1, \dots, \lambda_5) > 0$.

1.2.2. Remarks. (a) There is a footnote in [DavH] on page 186 that the same method, together with the use of an inequality due to Hua (Quarterly J. of Math., 9 (1938), 199–202), proves the corresponding theorem for $\lambda_1 x_1^k + \cdots + \lambda_s x_s^k$, where $s = 2^k + 1$.

(b) Clearly the inequality $|B(x)| < \varepsilon$ is equivalent to the inequality $|B_\varepsilon(x)| < 1$ where $B_\varepsilon = B/\varepsilon$. Therefore Theorem 1.4 implies that for any $\varepsilon > 0$ there exist arbitrarily large integers P such that the inequalities

$$1 \le x_1 \le P, \dots, 1 \le x_5 \le P \ , \ |B(x_1, \dots, x_5)| < \varepsilon$$

have more than $\gamma_\varepsilon P^3$ integral solutions where $\gamma_\varepsilon = \gamma(\varepsilon^{-1}\lambda_1, \dots, \varepsilon^{-1}\lambda_5) > 0$.

1.2.3. The Davenport–Heilbronn proof uses a modification of the Hardy–Littlewood method. We will describe now this proof following the original paper [DavH] and Lewis's survey [Lew].

We assume, without loss of generality, that λ_1/λ_2 is irrational. The constants implied by the symbol O depend only on $\varepsilon, \lambda_1, \dots, \lambda_5$.

Let $e(x) = e^{2\pi i x}$. For any positive integer P we define

$$S(\alpha) = S_P(\alpha) = \sum_{x=1}^{P} e(\alpha x^2), I(\alpha) = I_P(\alpha) = \int_0^P e(\alpha x^2) dx .$$

Direct computations or standard facts about Fourier transform show that for any $t \in \mathbb{R}$

$$\int_{-\infty}^{\infty} e(\alpha t) \left(\frac{\sin \pi\alpha}{\pi\alpha} \right)^2 d\alpha = \max(0, 1 - |t|) . \tag{5}$$

It is clear that

$$e(B(x_1, \dots, x_5)) = e(\lambda_1 x_1^2 + \cdots + \lambda_5 x_5^2) = e(\lambda_1 x_1^2) \cdot \ldots \cdot e(\lambda_5 x_5^2) .$$

Therefore the equality (5) implies the following two equalities:

$$\sum_{x_1=1}^{P} \cdots \sum_{x_5=1}^{P} \max(0, 1 - |B(x_1, \ldots, x_5)|)$$

$$= \int_{-\infty}^{\infty} S(\lambda_1\alpha) \ldots S(\lambda_5\alpha) \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 d\alpha, \tag{6}$$

$$I = \int_0^P \cdots \int_0^P \max(0, 1 - |B(x_1, \ldots, x_5)|) dx_1 \ldots dx_5$$

$$= \int_{-\infty}^{\infty} I(\lambda_1\alpha) \ldots I(\lambda_5\alpha) \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 d\alpha \tag{7}$$

A straightforward integration shows that

$$I > \gamma_1 P^3 \tag{8}$$

where $\gamma_1 = \gamma_1(\lambda_1, \ldots, \lambda_5) > 0$ is a positive constant independent of P [DavH, Lemma 10]. In fact, it is not difficult to prove the following asymptotic formula,

$$I \sim \beta P^3 \tag{9}$$

where

$$\beta = \int_{L\cap\Omega} \frac{dA}{\|\nabla B\|},$$

L is the lightcone $B = 0$, $\Omega = \{(x_1, \ldots, x_5) \mid 0 \le x_1 \le 1, \ldots, 0 \le x_5 \le 1\}$ and dA is the area element on L.

In view of (6), (7) and (8), the theorem will be proved if we show that for some $\delta > 0$ there exist infinitely large numbers P such that the difference between the right-hand sides of (6) and (7) is $O(P^{3-\delta})$. To do this we divide the remaining part of the proof into three steps.

Step 1. If $\alpha = O(P^{-\frac{3}{2}})$ then $S(\alpha) - I(\alpha) = O(1)$ [DavH, Lemma 5]. But it is trivial that $I(\alpha) \le P$ for any α. Hence

$$\prod_{r=1}^{5} S(\lambda_r\alpha) - \prod_{r=1}^{5} I(\lambda_r\alpha) = O(P^4) \text{ on } |\alpha| < P^{-\frac{3}{2}}. \tag{10}$$

Further,

$$S(\alpha) = O(|\alpha|^{-\frac{1}{2}-\varepsilon}) \text{ if } \alpha = O(P^{-1}), \alpha \ne 0, \tag{11}$$

for any $\varepsilon > 0$ [DavH, Lemma 7] and

$$I(\alpha) = \frac{1}{2|\alpha|^{\frac{1}{2}}} \int_0^{P^2|\alpha|} e(\pm x) x^{-\frac{1}{2}} dx = O(|\alpha|^{-\frac{1}{2}}) \tag{12}$$

for any $\alpha \ne 0$. It follows easily from the estimates (10), (11) and (12) that

$$\int_{-P^{-1}}^{P^{-1}} S(\lambda_1\alpha)\dots S(\lambda_5\alpha)\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2 d\alpha$$

$$= \int_{-\infty}^{\infty} I(\lambda_1\alpha)\dots I(\lambda_5\alpha)\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2 + O(P^{\frac{5}{2}})\,. \tag{13}$$

Step 2. Let $r(t)$ denote the number of representations of an integer t as the sum of two integral squares. It is well known that $r(t) = O(t^\varepsilon)$ for any $\varepsilon > 0$ [DavH, Lemma 2]. Hence for any $m \in \mathbb{R}$ and any $\varepsilon > 0$

$$\begin{aligned}\int_m^{m+1} |S(\alpha)|^4 d\alpha &= \int_0^1 |S(\alpha)|^4 d\alpha \\ &= \int_0^1 |\sum_{x=1}^{P}\sum_{y=1}^{P} e(\alpha(x^2+y^2))|^2 d\alpha \\ &\le \sum_{t=1}^{2P^2} r^2(t) = O(P^{2+\varepsilon})\,. \end{aligned} \tag{14}$$

It follows easily from (14) [DavH, Lemma 8] that for $\mu \ge 0$ and $1 \le r \le 5$

$$\int_\mu^\infty |S(\lambda_r\alpha)|^4 \left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2 d\alpha = O\left(\frac{P^{2+\varepsilon}}{\mu+1}\right)\,. \tag{15}$$

Using the trivial estimate $|S(\alpha)| \le P$ and the standard inequality comparing arithmetic and geometric means we get from (15) the following two estimates:

$$\int_{|\alpha|>\mu} S(\lambda_1\alpha)\dots S(\lambda_5\alpha)\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2 d\alpha = O\left(\frac{P^{3+\varepsilon}}{\mu+1}\right)\,, \tag{16}$$

$$\begin{aligned}&\int_{\mu_1<|\alpha|<\mu_2} S(\lambda_1\alpha)\dots S(\lambda_5\alpha)\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2 d\alpha \\ &= \sup\{\min(|S(\lambda_1\alpha)|, |S(\lambda_2\alpha)|)\ |\mu_1 < |\alpha| < \mu_2\}(\mu_2-\mu_1)O(P^{2+\varepsilon})\end{aligned} \tag{17}$$

for any $\mu \ge 0$ and $0 \le \mu_1 < \mu_2$.

Step 3. In previous two steps, P was an arbitrary positive integer. We now restrict P to a sequence of squares of the denominators of the partial fraction convergents to the irrational number λ_1/λ_2. Then there exist integers a, q such that the greatest common divisor (a, q) is 1,

$$|\lambda_1/\lambda_2 - a/q| < \frac{1}{q^2} \text{ and } P = q^2\,. \tag{18}$$

Let ρ be an absolute constant with $0 < \rho < \frac{1}{10}$. Assume that we can show

$$\min(|S(\lambda_1\alpha)|, |S(\lambda_2\alpha)|) = O(P^{1-\rho+\varepsilon}) \text{ on } P^{-1} \le |\alpha| \le P^\rho\,. \tag{19}$$

Then using (6), (7), (13), (16) and (17), we get the desired estimate:

$$\begin{aligned}
&\sum_{x_1=1}^{P}\cdots\sum_{x_5=1}^{P}\max(0,1-|B(x_1,\dots,x_5)|)\\
&-\int_0^P\cdots\int_0^P\max(0,1-|B(x_1,\dots,x_5))dx_1\dots dx_5\\
&=\int_{-\infty}^{\infty}S(\lambda_1\alpha)\dots S(\lambda_5\alpha)\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2d\alpha\\
&-\int_{-\infty}^{\infty}I(\lambda_1\alpha)\dots I(\lambda_5\alpha)\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2d\alpha\\
&=O(P^{3-\rho+\varepsilon}) \qquad (20)
\end{aligned}$$

(we use (16) for $\mu=P^\rho$ and (17) for $\mu_1=P^{-1}$ and $\mu_2=P^\rho$).

To prove (19) we take α with $P^{-1}\le|\alpha|\le P^\delta$ and consider rational approximations $a_1/q_1, a_2/q_2$ to $\lambda_1\alpha$ and $\lambda_2\alpha$, respectively, such that

$$\begin{cases}(a_1,q_1)=1, q_1\le P, |\lambda_1\alpha-a_1/q_1|\le P^{-1}q_1^{-1},\\ (a_2,q_2)=1, q_2\le P, |\lambda_2\alpha-a_2/q_2|\le P^{-1}q_2^{-1}\end{cases}. \qquad (21)$$

If $a_1=0$ or $a_2=0$ then (19) follows from (11). If $q_1\ge P^{2\rho}$ or $q_2\ge P^{2\rho}$ then (19) follows from Weyl's inequality for exponential sums [DavH, Lemma 6]. We may therefore suppose that

$$a_1\ne 0, a_2\ne 0, q_1<P^{2\rho}, q_2<P^{2\rho}\,. \qquad (22)$$

We can rewrite the inequalities (21) as

$$\lambda_1\alpha=\frac{a_1}{q_1}(1+b_1), \lambda_2\alpha=\frac{a_2}{q_2}(1+b_2), |b_1|\le P^{-1}, |b_2|\le P^{-1}\,. \qquad (23)$$

It follows from (18) and (23) that

$$\frac{a_1q_2}{a_2q_1}-\frac{a}{q}=O(P^{-1})\,. \qquad (24)$$

Since $|\alpha|<P^\rho$, (21) implies that $a_2/q_2=O(P^\rho)$. But $0<\rho<1/10$ and $q=P^{1/2}$. Hence it follows from (22) that $a_2q_1q=O(P^{5\rho+\frac12})=o(P)=o(q^2)$. It shows that (24) is impossible for sufficiently large P because $(a,q)=1$. This completes the proof of Theorem 1.2.1.

1.3. A modification of the theorem of Davenport and Heilbronn. The kernel $\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2$ is the Fourier transform of the function $q(t)=\max(0,1-|t|)$. We can replace q by another function. Let $\varphi(t)$ be a continuous function on $\mathbb{R}$ with a compact support, let $\hat\varphi(\alpha)$ denote the Fourier transform of φ and let

$$A_\varphi=\sup\{(1+\alpha^2)|\hat\varphi(\alpha)|\mid -\infty<\alpha<\infty\}\,.$$

We assume that $A_\varphi < \infty$. Then the estimates (13), (15), (16) and (17) will be still true if we replace $\left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2$ by $\hat{\varphi}(\alpha)$ and multiply the constants implied by the symbol O by A_φ. Therefore the following analogue of (20) is true:

$$\begin{aligned}
&\sum_{x_1=1}^{P} \cdots \sum_{x_5=1}^{P} \varphi(B(x_1,\ldots,x_5) \\
&- \int_0^P \cdots \int_0^P \varphi(B(x_1,\ldots,x_5))dx_1\ldots dx_5 \\
&= \int_{-\infty}^{\infty} S(\lambda_1\alpha)\ldots S(\lambda_5\alpha)\hat{\varphi}(\alpha)d\alpha \\
&- \int_{-\infty}^{\infty} I(\lambda_1\alpha)\ldots I(\lambda_5\alpha)\hat{\varphi}(\alpha)d\alpha \\
&= A_\varphi \cdot O(P^{3-\rho+\varepsilon})\,. \qquad (25)
\end{aligned}$$

Let us fix real numbers $a < b$. Let us denote $P^{-\frac{\rho}{2}}$ by θ and consider the following two functions:

$$\varphi_1(t) = \begin{cases} 0 & \text{if } t \le a \text{ or } t \ge b \\ 1 & \text{if } a+\theta \le t \le b-\theta \\ \frac{t-a}{\theta} & \text{if } a \le t \le a+\theta \\ \frac{b-t}{\theta} & \text{if } b-\theta \le t \le b, \end{cases}$$

$$\varphi_2(t) = \begin{cases} 0 & \text{if } t \le a-\theta \text{ or } t \ge b+\theta \\ 1 & \text{if } a \le t \le b \\ \frac{a-t}{\theta} & \text{if } a-\theta \le t \le a \\ \frac{t-b}{\theta} & \text{if } b \le t \le b+\theta \end{cases}.$$

Thus φ_1, φ_2 are piecewise linear functions and $\varphi_1 \le \chi_{[a,b]} \le \varphi_2$ where $\chi_{[a,b]}$ denotes the characteristic function of the segment $[a,b]$. We also have that $A_{\varphi_1} = O(\theta^{-1}) = O(P^{\frac{\rho}{2}})$, $A_{\varphi_2} = O(\theta^{-1}) = O(P^{\frac{\rho}{2}})$ and

$$\int_0^P \cdots \int_0^P (\varphi_1 - \varphi_2)(B(x_1,\ldots,x_5))dx_1\ldots dx_5 = O(\theta P^3) = O(P^{3-\frac{\rho}{2}})$$

(the last estimate can be obtained by a straightforward integration). Let us introduce the following notation:

$$\Omega = \{(x_1,\ldots,x_5) \mid 0 < x_1 \le 1, \ldots, 0 < x_5 \le 1\}$$

and

$$V^B_{(a,b)} = \{v = (x_1,\ldots,x_5) \in \mathbb{R}^5 \mid a < B(v) < b\}\,.$$

Then using the above observations we get from (25) that

$$
\begin{aligned}
|V^B_{(a,b)} \cap P\Omega \cap \mathbb{Z}^n| - \ & \mathrm{Vol}(V^B_{(a,b)} \cap P\Omega) \\
&= \sum_{x_1=1}^{P} \cdots \sum_{x_5=1}^{P} \chi_{[a,b]}(B(x_1,\dots,x_5)) \\
&- \int_0^P \cdots \int_0^P \chi_{[a,b]}(B(x_1,\dots,x_5))dx_1\dots dx_5 = O(P^{3-\frac{\rho}{2}+\varepsilon}) . \qquad (26)
\end{aligned}
$$

Let us recall that ρ is an arbitrary constant with $0 < \rho < \frac{1}{10}$ and ε is an arbitrary positive number. Thus we obtained the following modification of the theorem of Davenport and Heilbronn.

1.3.1. Theorem. *Let $\lambda_1, \dots, \lambda_5$ be real numbers, not all of the same sign, and none of them zero, such that λ_1/λ_2 is irrational. Let*

$$B(x_1,\dots,x_5) = \lambda_1 x_1^2 + \cdots + \lambda_5 x_5^2$$

and let P belong to a sequence of squares of the denominators of the partial convergents to the irrational number λ_1/λ_2. Let us fix real numbers $a < b$. Then the number of the integral solutions of the inequalities

$$1 \le x_1 \le P, \dots, 1 \le x_5 \le P, \ a < B(x_1,\dots,x_5) < b$$

is

$$
\begin{aligned}
&\mathrm{Vol}(V^B_{(a,b)} \cap P\Omega) + O(P^{3-\frac{\rho}{2}}) \\
&= \mathrm{Vol}\{v = (x_1,\dots,x_5) \in \mathbb{R}^5 \mid 1 \le x_1 \le P, \dots, 1 \le x_5 \le P, a < B(v) < b\} \\
&\quad + O(P^{3-\frac{\rho}{2}})
\end{aligned}
$$

where ρ is an arbitrary constant with $0 < \rho < 1/10$.

1.3.2. Remarks. (a) It is not difficult to show that

$$\mathrm{Vol}(V^B_{(a,b)} \cap P\Omega) = \beta P^3 + O(P^2)$$

where $\beta > 0$ is the same as in (9).

(b) Theorem 1.3.1 (without the remainder term) was proved in [Br].

1.4. Results for general quadratic forms. Watson, in a 1953 paper [Wa2], extended the result of Davenport and Heilbronn to forms which include a single cross-product term (say $\lambda_6 x_4 x_5$). As in [DavH], the proof in [Wa2] is based on the estimates of trigonometric sums and it seems that one can obtain an analogue of Theorem 1.3.1 for forms considered in [Wa2] (for some positive ρ). In another 1953 paper [Wa1], Watson proved the Oppenheim conjecture for the following two types of diagonal quadratic forms in three and four variables:

(A) $x^2 - a\theta y^2 - (a\theta + 1)z^2$ where a is a positive integer and θ is the positive root of the equation $\theta^2 = a\theta + 1$;

(B) $x^2 + dy^2 - \theta^2(z^2 + dw^2)$ where d is a positive integer, θ is any number in the quadratic field $\mathbb{Q}(\sqrt{N})$ and N is a non-square integer of the form $u^2 + dv^2$ with integral u, v.

In fact a more precise result is proved in [Wa1] for the forms of type (A). It is shown that there exists a number $\mathcal{C} = \mathcal{C}(a)$ such that the inequalities

$$0 < x \le X, 0 < y \le X, 0 < z \le X,\ |x^2 - a\theta y^2 - (a\theta + 1)z^2| < \mathcal{C}X^{-2}$$

are soluble in integers x, y, z for any positive integer X. The approach in [Wa1] is based on some elementary properties of continued fractions.

In 1956–57 Davenport [Dav1,3] proved the conjecture for quadratic forms of signature $(r, n-r)$ provided that either

$$r \ge 16 \text{ and } n - r \ge 16, \text{ or}$$
$$13 \le r \le 15 \text{ and } n > (9r - 20)/(r - 12).$$

As in [DavH] the proof in [Dav1,3] uses a modification of the Hardy–Littlewood method with the replacement of the kernel $(\sin \pi\alpha/\pi\alpha)^2$ by the kernel

$$K(\alpha) = \frac{4}{3}\frac{\sin 4\pi\alpha/3}{4\pi\alpha/3}\left(\frac{\sin 2\pi\alpha/3m}{2\pi\alpha/3m}\right)^m .$$

More precisely, this modification leads to the conclusion that either the desired result holds, or the restriction of the form αQ (for a certain real α) to a certain 5-dimensional sublattice of $\mathbb{Z}^n$ is an indefinite form with almost integral coefficients. In the latter case, Davenport uses the estimate of Cassels for the magnitude of the least solutions of homogeneous quadratic equations (see Theorem 1.4.1 below). A similar approach was used in a 1959 paper [DavR] by Davenport and Ridout where the Oppenheim conjecture was proved under the weaker hypothesis that

$$n \ge 21 \text{ and } \min(r, n-r) \ge 6\,.$$

A different method was used by Birch and Davenport in [BiD3] where they proved the conjecture under the hypothesis that either

$$1 \le \min(r, n-r) \le 4 \text{ and } n \ge 21, \text{ or}$$
$$\min(r, n-r) > 4 \text{ and } n \ge 17 + \min(r, n-r)\,.$$

The idea of [BiD3] is to prove that the restriction of B to a certain 5-dimensional sublattice of $\mathbb{Z}^n$ is an indefinite form which is almost diagonal. To the almost diagonal form in 5 variables Birch and Davenport apply a quantitative improvement of Theorem 1.2.1 (see Theorem 1.4.2 below). Their method was very soon modified by Ridout [Ri] who settled the case

$$n \ge 21 \text{ and } \min(r, n-r) = 5\,.$$

The combination of the above-mentioned results from [DavR], [BiD1] and [Ri] proves the Oppenheim conjecture for $n \geq 21$. For a rather detailed description of the papers [Dav1,3], [DavR], [BiD1] and [Ri] see the above-mentioned survey [Lew].

In 1975 using the methods of linear and half-dimensional sieves, Iwaniec [Iw] proved the conjecture for forms in 4 variables of the type

$$x_1^2 + x_2^2 - \theta(x_3^2 + x_4^2)$$

where θ is a real positive irrational number [Iw]. In 1986 R. Baker and Schlickewei [BaS] settled the following cases:

(a) $n = 18, r = 9$;
(b) $n = 19, 8 \leq r \leq 11$;
(c) $n = 20, 7 \leq r \leq 13$.

As it is noticed in [BaS], their method is an elaboration of that of Davenport and Ridout [DavR] and the new weapon is Schlickewei's extension of the work [BiD1] of Birch and Davenport on quadratic equations in several variables.

It seems that the methods of analytic number theory are not sufficient to prove the Oppenheim conjecture for general quadratic forms in a small number of variables. In connection with this let us quote a remark from the just mentioned paper [BaS] that the Oppenheim conjecture "does not seem likely to be settled soon at the present rate of progress".

Let us state now the above-mentioned results of Cassels on the least solutions of quadratic equations and of Birch and Davenport on a quantitative improvement of Theorem 1.2.1 of Davenport and Heilbronn.

1.4.1 Theorem [Ca1], [Dav2]). *Let*

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij} x_i x_j$$

be a quadratic form in n variables ($n \geq 2$) with integral coefficients. If the equation $f = 0$ has a non-trivial solution in integers, then it has a solution $(x_1, \ldots, x_n) \in \mathbb{Z}^n$ satisfying

$$0 < \sum_i x_i^2 \leq \gamma_{n-1}^{n-1} \left(2 \sum_{i,j} f_{ij}^2 \right)^{\frac{1}{2}(n-1)},$$

where γ_{n-1} is Hermite's constant, defined as the upper bound of the minima of positive definite quadratic forms in $n-1$ variables of determinant 1.

This formulation is taken from [Dav2]. An example given by Kneser [Ca1] shows that the exponent $\frac{1}{2}(n-1)$ cannot be improved for any n. Watson [Wa3] proved that for quadratic forms of signature (r, s), the exponent $\frac{1}{2}(n-1)$ can be replaced by

$$\theta = \max\left(2, \frac{1}{2}r, \frac{1}{2}s\right),$$

but cannot be replaced by any number smaller than $\frac{1}{2}h$ where an integer h is defined by

$$h \min(r,s) \le \max(r,s) < (h+1)\min(r,s)\,.$$

1.4.2. Theorem [BiD2]. *For $\delta > 0$ there exists $\mathcal{C}_\delta$ with the following property. For any real $\lambda_1, \dots, \lambda_5$ not all of the same sign and all of absolute value 1 at least, there exist integers $x_1, \dots, x_n$ which satisfy both inequalities*

$$|\lambda_1 x_1^2 + \cdots + \lambda_5 x_5^2| < 1 \quad \textit{and} \quad 0 < |\lambda_1| x_1^2 + \cdots + |\lambda_5| x_5^2 < \mathcal{C}_\delta |\lambda_1 \dots \lambda_5|^{1+\delta}\,.$$

An important part in the proof of this theorem is a modification of Theorem 1.4.1 given in [BiD2].

1.4.3. As before, let B be a real nondegenerate indefinite quadratic form in n variables. As it is noticed in [Lew] Birch, Davenport and Ridout actually showed in the series of papers [Dav1,3], [DavR], [BiD3] and [Ri] that $B(\mathbb{Z}^n)$ contains 0 as a nonisolated accumulation point if B is irrational and $n \ge 21$. On the other hand, Oppenheim [Op4] proved in 1952 that if $n \ge 3$ and $\{x \in \mathbb{Z}^n \mid 0 < |B(x)| < \varepsilon\} \ne \emptyset$ for any $\varepsilon > 0$ then $B(\mathbb{Z}^n)$ is dense in $\mathbb{R}$. Combining these two results we get that $B(\mathbb{Z}^n)$ is dense in $\mathbb{R}$ if B is irrational and $n \ge 21$.

Let us mention another result of Oppenheim which he also obtained in 1952. He proved that if B is irrational, $n \ge 4$ and $B(x) = 0$ for some nonzero $x \in \mathbb{Z}^n$ then $B(\mathbb{Z}^n)$ is dense in $\mathbb{R}$ (this result follows from Theorem 1 in [Op4], Theorems 1 and 2 in [Op5], and Theorem 2 in [Op6] for $n \ge 4$).

2. Oppenheim Conjecture And Subgroup Actions On Homogeneous Spaces

2.1 Paper [CaS] of Cassels and Swinnerton-Dyer. Let G be a Lie group, and Γ a discrete subgroup of G. Consider the action of a subgroup H of G on G/Γ by left translations. If H is one-parameter or cyclic then the study of this action becomes the usual theory of flows on homogeneous spaces. However, for our purposes we have to consider multidimensional H. In particular, it turns out that the Oppenheim conjecture is equivalent to the statement that any relatively compact orbit of $SO(2,1)$ in $SL(3,\mathbb{R})/SL(3,\mathbb{Z})$ is compact (see Proposition 2.2.4 below). While preparing this survey the author realized that in implicit form this equivalence appears already in Sec. 10 of an old paper [CaS] of Cassels and Swinnerton-Dyer. Though the language of the theory of dynamical systems is not used there, the paper [CaS] seems to be one of the first papers (maybe even the first one) on dynamical properties actions of multidimensional subgroups on homogeneous spaces. Let us first reproduce the abstract of [CaS].

"Isolation theorems for the minima of factorizable homogeneous ternary cubic forms and of indefinite ternary quadratic forms of a new strong type are proved. The problems whether there exist such forms with positive minima other than multiples

of forms with integer coefficients are shown to be equivalent to problems in the geometry numbers of a superficially different type. A contribution is made to the study of the problem whether there exist real φ, ψ such that $x|\varphi x - y||\psi x - z|$ has a positive lower bound for all integers $x > 0, y, z$. The methods used have wide validity."

As we can see from this abstract two types of forms are considered in [CaS]. The first type consists of products of three linear forms in 3 variables, and the second one consists of indefinite ternary quadratic forms. Let us state first some of the results from [CaS] about products of linear forms. For any function f on $\mathbb{R}^n$ we will denote $\inf\{|f((x)||x \in \mathbb{Z}^n, x \neq 0\}$ by $m(f)$.

2.1.1. Theorem [CaS, Theorem 2]. *Let $f(x, y, z) = L_1L_2L_3$ be the product of three real linear forms which represent zero (over $\mathbb{Q}$) only trivially, and suppose that f has integer coefficients. Let (δ_1, δ_2) be any open interval however small. Then there is a neighbourhood of f (in the space of products of three linear forms) such that all forms f^* in the neighbourhood which are not multiples of f itself take some value (on $\mathbb{Z}^3$) in the interval (δ_1, δ_2). In particular, to any given $\delta > 0$ we can choose a neighbourhood in which $m(f^*) < \delta$.*

Let us sketch the proof of this theorem. Since $f = L_1L_2L_3$ has integer coefficients and $f(x) \neq 0$ for any $x \in \mathbb{Z}^3 - \{0\}$, there exist $S, T \in SL(3, \mathbb{Z})$ and two independent units η_1, ζ_1 of a totally real cubic field K, with conjugates $\eta_2, \zeta_2,\ \eta_3, \zeta_3$ such that

$$S^mT^nL_j = \eta_j^m\zeta_j^nL_j \qquad (j = 1, 2, 3)$$

for each pair of integers m, n. Let us write

$$\begin{aligned} f^* &= (1 + \varepsilon_0)L_1^*L_2^*L_3^*, \\ L_1^* &= L_1 + \varepsilon_{12}L_2 + \varepsilon_{13}L_3, \\ L_2^* &= \varepsilon_{21}L_1 + L_2 + \varepsilon_{23}L_3, \\ L_3^* &= \varepsilon_{31}L_1 + \varepsilon_{32}L_2 + L_3. \end{aligned}$$

We may suppose without loss of generality that $\varepsilon_0 = 0$ and that

$$\varepsilon_{31} = \max|\varepsilon_{ij}| > 0\,.$$

Replacing if necessary S by S^mT^n for suitable m, n, we may also suppose that

$$\eta_1 > \eta_2 > \eta_3\,.$$

Then, using the independence of units η_1 and ζ_1, it is not difficult to show that the upper topological limit of the sequence of sets

$$\{S^mT^nf_r^* \mid m > 0, n \in \mathbb{Z}\}$$

contains $f + \lambda L_1^2 L_2$ for any $\lambda > 0$ when f_r^* is a form of the same type as f^* and $f_r^* \to f$. But $(S^m T^n f_r^*)(\mathbb{Z}^3) = f_r^*(\mathbb{Z}^3)$. Hence if $f_r^*(\mathbb{Z}^3) \cap (\delta_1, \delta_2) = \emptyset$ for every r, we have that

$$(f + \lambda L_1^2 L_2)(\mathbb{Z}^3) \cap (\delta_1, \delta_2) = \emptyset$$

for every $\lambda > 0$. This contradicts the existence of $w \in \mathbb{Z}^3$ such that

$$f(w) < \delta_1, L_1(w) \neq 0, \text{ and } L_2(w) > 0\,.$$

2.1.2. Theorem [CaS, Theorem 5]. *Assume that $m(L_1L_2L_3) = 0$ for any real linear forms L_1, L_2, L_3 in x, y, z such that $L_1L_2L_3$ is not a multiple of a form with integer coefficients. Then for any D_0 however large, there are only a finite number of inequivalent sets of forms L_1, L_2, L_3 with determinant $\leq D_0$ such that $m(L_1L_2L_3) = 1$.*

Here two sets of forms are considered *equivalent* if the corresponding products $L_1L_2L_3$ can be transformed one into the other by an integral unimodular transformation on x, y, z. Let us reproduce the argument from [CaS] which deduces Theorem 2.1.2 from Theorem 2.1.1.

If there are infinitely many inequivalent sets of forms L_1, L_2, L_3 with determinant $\leq D_0$ such that $m(L_1L_2L_3) = 1$, then there are infinitely many lattices F in $\mathbb{R}^3$ of determinant at most D_0 such that $X_1X_2X_3 \geq 1$ for any $(X_1, X_2, X_3) \in F - \{0\}$; and none of these lattices is obtainable from another by a diagonal transformation $X_j \to \lambda_j X_j$. By Mahler's compactness criterion the set of these lattices must contain a convergent subsequence $\{\Delta_r\}$. (Mahler's compactness criterion states that the set A of lattices in $\mathbb{R}^n$ is relatively compact in the space of lattices if and only if there exist $D > 0$ and a neighborhood U of 0 in $\mathbb{R}^n$ such that, for any lattice $\Lambda \in A$, the determinant of Λ is not greater than D and $\Lambda \cap U = \{0\}$.) Let us choose a basis of the limit lattice Δ and write X_1, X_2, X_3 in this basis. Then we get a set of forms L_1, L_2, L_3 with determinant $\leq D_0$ such that $m(L_1L_2L_3) \geq 1$. By the assumption in the formulation of Theorem 2.1.2, $L_1L_2L_3$ is a multiple of a form with integer coefficients. But this is in contradiction with Theorem 2.1.1 and the choice of the subsequence $\{\Delta_r\}$.

2.1.3. Theorem [CaS, Theorem 6]. *As in Theorem 2.1.2, assume that $m(L_1L_2L_3) = 0$ for any real linear forms L_1, L_2, L_3 in x, y, z such that $L_1L_2L_3$ is not a multiple of a form with integer coefficients. Then*

$$(*) \qquad \liminf_{n\to\infty} n\|n\alpha\|\|n\beta\| = 0$$

for any real α and β where $\|x\|$ denotes the distance from x to the closest integer.

To deduce Theorem 2.1.3 from Theorem 2.1.1 we apply Mahler's compactness criterion to the sequence $\{A^n\Lambda \mid n \in \mathbb{N}^+\}$ where Λ is the lattice in (X_1, X_2, X_3)-space with points

$$\begin{bmatrix} X_1 = x \\ X_2 = \alpha x - y \\ X_3 = \beta x - z \end{bmatrix} (x, y, z \quad \text{integers})$$

and $A(X_1, X_2, X_3) = (\frac{1}{4}X_1, 2X_2, 2X_3)$.

Remark. The statement (*) is the famous Littlewood conjecture still not settled.

2.1.4. Theorem [CaS, Theorem 4]. *The following two statements are equivalent*:

(a) *There exist real linear forms L_1, L_2, L_3 in x, y, z such that $L_1L_2L_3$ is not a multiple of a form with integer coefficients and $m(L_1L_2L_3) = 1$.*
(b) *There exist real linear forms M_1, M_2, M_3 in x, y, z such that*

$$\min\{m(M_1M_2M_3), m(M_1M_2(M_2 + M_3))\} = 1\,.$$

The proof of this theorem is analogous to the proof of the corresponding Theorem 2.1.8 for ternary quadratic forms and uses the following:

2.1.5. Lemma [CaS, Lemma 6]. *Let L_1, L_2, L_3 be three real linear forms in x_1, x_2, x_3 of non-zero determinant, each of which represents zero (over $\mathbb{Q}$) only trivially. Suppose there exist a transformation $T \in SL_3(\mathbb{Z})$ (other than the identity) and constants c_1, c_2, c_3 such that*

$$c_j > 0, c_1c_2c_3 = 1$$

and

$$TL_j = c_jL_j\ ,\ 1 \le j \le 3\,.$$

Then there is a multiple of $L_1L_2L_3$ with integer coefficients.

2.1.6. Let us state now theorems from [CaS] about indefinite ternary quadratic forms.

Theorem [CaS, Theorem 8]. *Let $B(x, y, z)$ be a non-singular indefinite ternary quadratic form with integer coefficients, and let (δ_1, δ_2) be any open interval however small. Then there is a neighborhood of B such that all (quadratic) forms B^* in the neighborhood which are not multiples of B itself take some value in the interval (δ_1, δ_2).*

The proof of this theorem is similar to the proof of Theorem 2.1.1. The only difference is that instead of commuting matrices $S, T \in SL(3, \mathbb{Z})$ one has to consider $S, T \in SL(3, \mathbb{Z})$ such that $SB = TB = B$, S (resp. T) has an eigenvalue $\lambda > 1$ (resp. $\mu > 1$), and λ and μ are elements of distinct quadratic fields. (Recall that if $SB = B$ then S has three eigenvalues $\lambda, \pm 1, \pm\lambda^{-1}$.)

2.1.7. The following theorem is deduced from Theorem 2.1.6 in the same way as Theorem 2.1.2 is deduced from Theorem 2.1.1. As before we say that a quadratic

form is rational if it is a multiple of a form with rational coefficients and irrational otherwise. Two forms B_1 and B_2 are considered *equivalent* if B_1 can be transformed into $aB_2, a \neq 0$, by an integral unimodular transformation of x, y, z.

Theorem [CaS, Theorem 10]. *Assume that $m(B) = 0$ for any irrational indefinite ternary quadratic form B. Then for any D_0 however large there are only a finite number of inequivalent indefinite ternary quadratic forms B with determinant at most D_0 such that $m(B) = 1$.*

2.1.8. Another result is the following:

Theorem [CaS, Theorem 9]. *The following two statements are equivalent*:

(a) *There is an irrational indefinite ternary quadratic form B such that $m(B) > 0$.*
(b) *There are ternary linear forms M_1, M_2, M_3 such that*

$$\min\{m(M_2^2 - M_1M_3), m(M_2^2 - M_3(M_1 + M_3))\} = 1\,.$$

The implication (b) $\Rightarrow$ (a) is straightforward. We will explain how (a) implies (b) in the next subsection 2.2.

2.2. Paper [CaS] (continuation). For any real quadratic form B in n variables, let us denote the special orthogonal group

$$SO(B) = \{g \in SL(n, \mathbb{R}) \mid gB = B\} \text{ by } H_B\,,$$

and

$$\inf\{|B(x)| | x \in \Lambda, x \neq 0\} \text{ by } m(B, \Lambda)\,.$$

It is clear that $m(B, h\Lambda) = m(B, \Lambda)$ for any $h \in H_B$. This remark and Mahler's compactness criterion imply the following lemma (which was not explicitly stated in [CaS] but was implicitly used several times in Sec. 10 of [CaS]).

2.2.1. Lemma. *Let B be a real quadratic form in n variables, and Λ a lattice in $\mathbb{R}^n$ (resp. Ψ a set of unimodular lattices). Then the orbit $H_B\Lambda$ (resp. the set $H_B\Psi$) is relatively compact in the space Ω_n of lattices if and only if $m(B, \Lambda) > 0$ (resp. $\inf\{m(B, \Lambda) \mid \Lambda \in \Psi\} > 0$). In particular, $H_B\mathbb{Z}^n$ is relatively compact if and only if $m(B) > 0$.*

2.2.2. Lemma. *Let $B(x_1, x_2, x_3)$ be an indefinite quadratic form not representing 0 (over $\mathbb{Q}$).*

(a) [CaS, Lemma 14]. *Assume that $H_B \cap SL(3,\mathbb{Z})$ contains two non-commuting linear transformations S and T each of which has three distinct eigenvalues. Then the form B is rational.*
(b) *Assume that $H_B/H_B \cap SL(3,\mathbb{Z})$ is compact. Then the form B is rational.*

The connected component of identity of the group H_B is isomorphic to $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm E\}$, and under this isomorphism hyperbolic elements of $PSL(2,\mathbb{R})$ correspond to elements with three positive distinct eigenvalues. On the other hand, any cocompact discrete subgroup of $PSL(2,\mathbb{R})$ contains non-commuting hyperbolic elements. Therefore (b) follows from (a). Let us sketch the proof of (a) given in [CaS].

Let $\lambda_1 = \lambda, \lambda_2 = 1, \lambda_3 = \lambda^{-1}$ be the eigenvalues of S. Then there exist linear forms ξ_1, ξ_2, ξ_3 such that

$$S\xi_j = \lambda_j \xi_j \ , \ 1 \le j \le 3 \, ,$$

ξ_2 has rational coefficients and the coefficients of ξ_1, ξ_3 lie in a quadratic field and are conjugates. On the other hand, $SB = B$ and the linear subspace of S-invariant quadratic forms on $\mathbb{R}^3$ has dimension 2. Hence

$$B(x_1, x_2, x_3) = \rho\xi_2^2 + \sigma\xi_1\xi_3 \, ,$$

where ρ, σ are real numbers and ξ_2^2 and $\xi_1\xi_3$ have rational coefficients. Applying the same argument for T instead of S, we get that there exist $\lambda_1^* = \lambda^*, \lambda_2^* = 1, \lambda_3^* = \lambda^{*-1}$ and linear forms $\xi_1^*, \xi_2^*, \xi_3^*$ such that

$$T\xi_j^* = \lambda_j^* \xi_j^* \ , \ 1 \le j \le 3 \, ,$$
$$B(x_1, x_2, x_3) = \rho^* \xi_2^{*2} + \sigma^* \xi_1^* \xi_3^*$$

where ρ^*, σ^* are real numbers and ξ_2^* and $\xi_1^*\xi_3^*$ have rational coefficients. Since S and T are non-commuting elements of H_B with three distinct eigenvalues, we have that ξ_2^* is not a multiple of ξ_2. Hence after a suitable coordinate change we may assume that

$$B(x_1, x_2, x_3) = \rho_1 x_1^2 + \sigma\xi_1\xi_3 = \rho_1^* x_2^2 + \sigma^* \xi_1^* \xi_3^*$$

where $\rho_1, \rho_1^*, \sigma, \sigma^*$ are real non-zero numbers and $\xi_1\xi_3$, $\xi_1^*\xi_3^*$ are quadratic forms in x_1, x_2, x_3 with rational coefficients. Now, comparing the coefficients on both sides and using the non-singularity of B, we see that $\rho_1/\sigma^*, \rho_1^*/\sigma$ and σ/σ^* are rational numbers. This implies that B is rational.

2.2.3. If B_1 and B_2 are two indefinite ternary quadratic forms then there exist $g \in SL(3,\mathbb{R})$ and $\lambda \neq 0$ such that $gB_1 = \lambda B_2$. Thus, instead of studying $B(\mathbb{Z}^3)$ and $H_B\mathbb{Z}^3$ for a general ternary quadratic form B, we can fix one such form B_0 and consider $B_0(\Lambda)$ and $H_{B_0}\Lambda$ for a general unimodular lattice $\Lambda \in \Omega_3 = SL(3,\mathbb{R})/SL(3,\mathbb{Z})$. Now, using Lemmas 2.2.1 and 2.2.2 we get the following:

2.2.4. Proposition. *Let us fix an indefinite ternary quadratic form B_0, and let us denote H_{B_0} by H and the stabilizer $\{h \in H \mid Hz = z\}$, $z \in \Omega_3 = SL(3,\mathbb{R})/SL(3,\mathbb{Z})$, by H_z. Then the following two statements are equivalent:*

(a) *If B is an irrational indefinite ternary quadratic form then $m(B) = 0$ (i.e. the Oppenheim conjecture is true).*
(b) *If $z \in \Omega_3$ and the orbit Hz is relatively compact in Ω_3, then the quotient space H/H_z is compact.*

Remarks. (I) To prove the implication (a) $\Rightarrow$ (b) we have to use the classical fact that $H_B/H_B \cap SL(3,\mathbb{Z})$ is compact if a form B is rational and does not represent 0.

(II) Let G be a second countable locally compact group, Γ a discrete subgroup of G, F a closed subgroup of G, and $z \in G/\Gamma$. It is well known that the orbit Fz is closed if and only if the natural map $F/F_z \to Fz$ is proper. Therefore (b) is equivalent to the statement:

(b′) Any relatively compact orbit of H on Ω_3 is compact.

2.2.5. Let us write

$$w = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The set of values of the form $x_2^2 - x_3(x_1 + x_3)$ on a lattice $L \subset \mathbb{R}^3$ coincides with the set of values of the form $x_2^2 - x_1x_3$ on wL. Therefore the statement (b) in the formulation of Theorem 2.1.8 is equivalent to the following statement:

(c) There is a lattice L in $\mathbb{R}^3$ such that

$$\min(m(B_0, L), m(B_0, wL)) = 1$$

where $B_0(x_1, x_2, x_3) = x_2^2 - x_1x_3$.

Combining this observation with Lemma 2.2.1 and Proposition 2.2.4 (together with Remark (II)) we get that the implication (a) $\Rightarrow$ (b) in Theorem 2.1.8 is a consequence of the following assertion:

2.2.6. Proposition. *Let $B_0(x_1, x_2, x_3) = x_2^2 - x_1x_3$, $H = H_{B_0}$ and $z \in \Omega_3 = SL(3,\mathbb{R})/SL(3,\mathbb{Z})$. Assume that the orbit Hz is relatively compact in Ω_3 but not closed. Then there is $y \in \overline{Hz}$ such that $wy \in \overline{Hz}$ (as usual $\bar{A}$ denotes the closure of A).*

This proposition is easily derived from the following two lemmas.

2.2.7. Lemma. *Let G be a second countable locally compact group, Γ a discrete subgroup of G, and F a closed subgroup of G.*

(a) *If $Y \subset G/\Gamma$, $FY{=}Y$ and Y is not closed, then the closure of the set $\{g \in G - F \mid gY \cap Y \neq \emptyset\}$ contains e. In particular if $z \in G/\Gamma$ and*

the natural map $F/F_z \to Fz$ *is not proper, then the closure of the set* $\{g \in G - F \mid gFz \cap Fz \neq \emptyset\}$ *contains* e.

(b) *Let* Y *and* Y' *be closed* F*-invariant subsets of* G/Γ, *and let* $M \subset G$. *Suppose that* Y *is compact and* $mY \cap Y' \neq \emptyset$ *for any* $m \in M$. *Then* $gY \cap Y' \neq \emptyset$ *for any* $g \in \overline{FMF}$.

To prove (a) it is enough to notice that if $\{g_i\} \subset G$, $\lim_{i\to\infty} g_i = e$ and $g_i g_j^{-1} \in F$ for all i and j, then $g_i \in F$ for any i. To prove (b), consider the set $S = \{g \in G \mid gY \cap Y' \neq \emptyset\}$. Since Y is compact and Y' is closed, the set S is closed. On the other hand $M \subset S$ and (since Y and Y' are F-invariant) $FSF = S$. Hence $S \supset \overline{FMF}$.

2.2.8. Lemma. *Let* $B_0(x_1, x_2, x_3) = x_2^2 - x_1x_3$, $H = H_{B_0}$, *and* $M \subset SL(3,\mathbb{R}) - H$. *Suppose that* $e \in \bar{M}$. *Then* $w \in \overline{HMH}$.

Let $\{g_n\}$ be a sequence of elements of M such that $g_n \to e$ as $n \to \infty$. It is easy to check that there exist $h_n, h'_n \in H$ such that $\|h_n\| < c, h'_n \to e$ as $n \to \infty$, and

$$\|h_n g h'_n h_n^{-1} - e\| \leq c|(h_n g h'_n h_n^{-1})_{13}|$$

where c is an absolute constant. Therefore for every $\varepsilon > 0$, we can find $g = (g_{ij}) \in HMH$ such that

$$\|g - e\| < \varepsilon \text{ and } \|g - e\| < c|g_{13}|\,.$$

Consider an element

$$g^* = \begin{pmatrix} |g_{13}|^{-\frac{1}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & |g_{13}|^{\frac{1}{2}} \end{pmatrix} g \begin{pmatrix} |g_{13}|^{\frac{1}{2}}\mathrm{sgn}g_{13} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & |g_{13}|^{-\frac{1}{2}}\mathrm{sgn}g_{13} \end{pmatrix}.$$

The first and last factors are elements of H. Hence $g^* \in HMH$. Clearly

$$\|g^*_{ii} - 1\| \leq c\varepsilon\ ,\ g^*_{13} = 1\,,$$
$$\|g^*_{ij}\| \leq c\varepsilon^{\frac{1}{2}} \qquad \text{otherwise}\,.$$

It implies that $g^* \to w$ when $\varepsilon \to 0$. This proves the lemma.

2.2.9. Remarks. (a) In Proposition 2.2.6 and Lemma 2.2.8, B_0 can be replaced by any indefinite ternary quadratic form of the type

$$ax_1x_3 + bx_2^2 + cx_2x_3 + dx_3^2\,.$$

Also, w can be replaced by

$$w(t) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) The above proof of Theorem 2.1.8 is essentially a translation of the proof from [CaS], though these two proofs superficially look quite different.

2.3. Closures of orbits of orthogonal groups and integer solutions of quadratic inequalities. Proposition 2.2.4 establishes the equivalence between the Oppenheim conjecture and the statement that any relatively compact orbit of $SO(2,1)$ in $SL(3,\mathbb{R})/SL(3,\mathbb{Z})$ is compact. As it is noticed in 2.1, in implicit form this equivalence appears already in Sec. 10 of [CaS] (and, in fact, is used to prove some of the results of the paper [CaS]). However the language of the theory of dynamical systems is not used in [CaS]. Because of that, for a long time the relation between the Oppenheim conjecture and problems in dynamical systems remained unknown to most (and probably all) people studying subgroup actions on homogeneous spaces. This changed when, in the mid-seventies, M.S. Raghunathan rediscovered the above-mentioned equivalence and noticed that the Oppenheim conjecture would follow from a conjecture about closures of orbits of unipotent subgroups (see [Dan3]). The Raghunathan conjecture states that if G is a connected Lie groups, Γ a lattice in G (that is, Γ is a discrete subgroup such that G/Γ carries a G-invariant probability measure), and U an Ad-unipotent subgroup of G (that is, $\mathrm{Ad}u$ is a unipotent linear transformation for any $u \in U$), then for any $x \in G/\Gamma$ there exists a closed connected subgroup $L = L(x)$ containing U such that the closure of the orbit Ux coincides with Lx. In the literature it was first stated in [Dan3] and in a more general form in [Marg5] (when the subgroup U is not necessarily unipotent but generated by unipotent elements). Raghunathan's conjecture and some other related conjectures will be discussed in Sec. 3.8.

Let us briefly describe the argument of Raghunathan reducing the Oppenheim conjecture to his conjecture.

Let B be a real indefinite quadratic form in $n \geq 3$ variables and let $H = SO(B)$. If U is a unipotent subgroup of H and the closure of $U\mathbb{Z}^n$ in the space of lattices coincides with the orbit $L_U\mathbb{Z}^n$ where L_U is a closed connected subgroup of $SL(n,\mathbb{R})$, then using Borel's density theorem [Bo1] it is not difficult to prove that L_U is an algebraic subgroup defined over $\mathbb{Q}$. Thus the number of all possible L_U is at most countable. This countability quite easily implies that there exists a unipotent subgroup $U_0 \subset SL(n,\mathbb{R})$ such that $L_{U_0} \supset L_U$ for any other unipotent subgroup U. But the connected component H^0 of identity in H is generated by unipotent elements (because $n \geq 3$). Hence $L_{U_0} \supset H^0$. On the other hand $H^0 = SO(B)^0$ is a maximal proper connected subgroup in $SL(n,\mathbb{R})$. Thus either $L_{U_0} = SL(n,\mathbb{R})$ or $L_{U_0} = H^0$. In the first case $m(B) = 0$ (according to Lemma 2.2.1) and in the second case B is rational (because L_{U_0} is defined over $\mathbb{Q}$).

Inspired by Raghunathan's observations the author started to work on the homogeneous space approach to the Oppenheim conjecture and eventually established the following theorem which proves the conjecture.

2.3.1. Theorem (see [Marg5, Theorem 1] or one of the papers [Marg3,4,6]). *Let B be a real irrational indefinite quadratic form in $n \geq 3$ variables. Then for any $\varepsilon > 0$ there exist integers $x_1, \ldots, x_n$ not all equal to 0 such that $|B(x_1, \ldots, x_n)| < \varepsilon$.*

2.3.2. The equivalence given by Proposition 2.2.4 is used in [Marg3-6], and indeed Theorem 2.3.1 is deduced there from the following reformulation of the statement (b) in Proposition 2.2.4.

Theorem [Marg5, Theorem 2]. *Let $G = SL(3, \mathbb{R})$ and $\Gamma = SL(3, \mathbb{Z})$. Let us denote by H the group of elements of G preserving the form $2x_1x_3 - x_2^2$ and by $\Omega_3 = G/\Gamma$ the space of lattices in $\mathbb{R}^3$ having determininant 1. Let G_y denote the stabilizer $\{g \in G \mid gy = y\}$ of $y \in \Omega_3$. If $z \in \Omega = G/\Gamma$ and the orbit Hz is relatively compact in Ω then the quotient space $H/H \cap G_z$ is compact.*

The proof of this theorem given in [Marg 5] depends to a large extent on studying the closures of orbits of unipotent subgroups but it does not use the argument of Raghunathan mentioned above.

2.3.3. In the 1952 paper [Op4] Oppenheim modified his conjecture replacing the inequality $|B(x)| < \varepsilon$ by a slightly stronger inequality $0 < |B(x)| < \varepsilon$. When informed by A. Borel of this fact, the author showed that the Oppenheim conjecture in this modified form can also be deduced from Theorem 2.3.2 (see [Marg5, Theorem 1′]). On the other hand, according to the result of Oppenheim proved in the same paper [Op4] and mentioned above in 1.5, if $n \geq 3$ and $\{x \in \mathbb{Z}^n \mid 0 < |B(x)| < \varepsilon\} \neq \emptyset$ for any $\varepsilon > 0$ then $B(\mathbb{Z}^n)$ is dense in $\mathbb{R}$. Thus we have the following:

Theorem. *If B is a real irrational indefinite quadratic form in $n \geq 3$ variables, then $B(\mathbb{Z}^n)$ is dense in $\mathbb{R}$ or, in other words, for any $a < b$ there exists integers $x_1, \ldots, x_n$ such that*

$$a < B(x_1, \ldots, x_n) < b\,.$$

2.3.4. An integer vector $x \in \mathbb{Z}^n$ is called *primitive* if $x \neq ky$ for any $y \in \mathbb{Z}^n$ and $k \in \mathbb{Z}$ with $|k| \geq 2$. The set of all primitive vectors in $\mathbb{Z}^n$ will be denoted by $\mathcal{P}(\mathbb{Z}^n)$. As in [Bo2] let us say that a subset $(x_1, \ldots, x_n)$ of $\mathbb{Z}^n$ $(m \leq n)$ is *primitive* if it is a part of a basis of $\mathbb{Z}^n$. If $m < n$, this condition is equivalent to the existence of $g \in SL_n(\mathbb{Z})$ such that $ge_i = x_i (i = 1, \ldots, m)$. We can state now a strengthening of Theorem 2.3.3.

2.3.4. Theorem. *Let B be a real irrational indefinite quadratic form in $n \geq 3$ variables, and let B_2 be the corresponding bilinear form defined by $B_2(v, w) = \frac{1}{4}\{B(v + w) - B(v - w)\}$ for all $v, w \in \mathbb{R}^n$.*

(a) [DanM1, Theorem 1]. *The set $\{B(x) \mid x \in \mathcal{P}(\mathbb{Z}^n)\}$ is dense in $\mathbb{R}$.*

(b) [BoP, Corollary 7.8] (see also [DanM1, Theorem 1] for $m \leq 2$). *Let $m < n$ and $y_1, \ldots, y_m$ be elements of $\mathbb{R}^n$. Then there exists a sequence $(x_{j,1}, \ldots, x_{j,m})$ $(j = 1, \ldots)$ of primitive subsets of $\mathbb{R}^n$ such that*

$$B_2(y_a, y_b) = \lim_{j\to\infty} B_2(x_{j,a}, x_{j,b}) \ (1 \leq a, b \leq m)\,.$$

(c) (see [BoP, Corollary 7.9] or [Bo2, Theorem 2]). *Let $c_i \in \mathbb{R}$ $(i = 1, \ldots, n-1)$. Then there exists a sequence $(x_{j,1}, \ldots, x_{j,n-1})$ $(j = 1, 2, \ldots)$ of primitive subsets of $\mathbb{Z}^n$ such that*

$$\lim_{j\to\infty} B(x_{j,i}) = c_i \quad (i = 1, \ldots, n-1)\,.$$

This is deduced from the following Theorem 2.3.5 by an extension of the argument reducing Theorems 2.3.1 and 2.3.3 to Theorem 2.3.2. Theorem 2.3.5 is in fact an easy consequence of a general result of M. Ratner on orbit closures (see Theorem 3.8.4 below). However, for $n = 3$ this theorem had been earlier proved in [DanM1]. The proof given in [DanM1] uses the technique which involves, as in [Marg5], finding orbits of larger subgroups inside closed sets invariant under unipotent subgroups.

2.3.5. Theorem. *Let B be a real indefinite quadratic form in $n \geq 3$ variables. Let us denote by H the special orthogonal group $SO(B)$ and by $\Omega_n = SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ the space of lattices in $\mathbb{R}^n$ with discriminant 1. Then any orbit of H in Ω_n either is closed and carries a H-invariant probability measure or is dense.*

2.3.6. Remarks. (a) Let $B(x_1, x_2, x_3) = 2x_1x_3 - x_2^2$, $H = SO(B)$, and let

$$w(t) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$W^+ = \{w(t) \mid t \geq 0\}$, $W^- = \{w(t) \mid t \leq 0\}$. It is easy to see that if L is a lattice in $\mathbb{R}^3$ then there exist $w_1 \in W^+, w_2 \in W^-$ and primitive vectors $x_1, x_2 \in \mathcal{P}(L)$ such that $B(w_1x_1) = B(w_2x_2) = 0$. Because of that, in order to prove Theorems 2.3.1, 2.3.3 and 2.3.4 (a), it is enough to prove that if $x \in \Omega_3$ then either Hx is closed and carries a H-invariant probability measure or there exist a $y \in \overline{Hx}$ such that W^+y or W^-y is contained in $\overline{Hx}$. Using this observation it is possible to simplify the proofs of Theorems 2.3.1, 2.3.3 and 2.3.4 (a). This was done in [DanM3] and [Dan8] for Theorem 2.3.4 (a) (see also Sec. 3.7 below) and in [Marg7] and [Si] for Theorems 2.3.1 and 2.3.3. It should also be mentioned that the proof given in [Dan8] does not depend on the axiom of choice (other proofs involve existence of minimal invariant subsets for various actions, which depends on Zorn's lemma and, in turn, on the axiom of choice).

(b) Let Q be a nondegenerate quadratic form in $n \geq 2$ variables. Borel and Prasad noted (see [Bo2, Proposition 4]) that Q is irrational if and only if, given $\varepsilon > 0$, there exist $x, y \in \mathbb{Z}^n$ such that $0 < |Q(x) - Q(y)| < \varepsilon$. This is related to the conjecture mentioned in 1.1 about the gaps between successive values of Q.

(c) Borel and Prasad generalized Theorems 2.3.1, 2.3.3 and 2.3.4 for a family $\{B_s\}$ where $s \in S$, S is a finite set of places of a number field k containing the set S_∞ of archimedean places, B_s is a quadratic form on k_s^n, and k_s is the completion of k at s (see [BoP] and [Bo2]). To prove these generalizations they use S-arithmetic analogs of Theorems 2.3.2 and 2.3.5.

2.4. Quantitative results. In connection with Theorem 2.3.3, it is natural to study the following problem. Let B be a real irrational indefinite quadratic form in $n \geq 3$ variables, let $a < b$, and let $N^B_{(a,b)}(T)$ denote the number of integral points v in a ball of radius T with $a < B(v) < b$. According to Theorem 2.3.3, $N^B_{(a,b)}(T) \to \infty$ when $T \to \infty$. It is well known that as $T \to \infty$

$$|\{v \in \mathbb{R}^n \mid \|v\| < T\}| \sim \ \mathrm{Vol}\{v \in \mathbb{Z}^n \mid \|v\| < T\}\,.$$

Therefore one might expect that

$$\lim_{T\to\infty} \frac{N^B_{(a,b)}(T)}{\mathrm{Vol}(\{v \in \mathbb{R}^n \mid a < Q(v) < b, \|v\| < T\})} = 1\,. \tag{1}$$

It was shown by Dani and the author that (1) is true but only if lim is replaced by lim inf. Rather surprisingly the asymptotic formula (1) holds if the signature of B is not (2,1) or (2,2) and does not hold for general irrational forms of these two signatures (it was proved recently by Eskin, Mozes and the author). Before we give the precise formulations, let us introduce some notations and make some preliminary remarks.

Let ν be a continuous positive function on the sphere $\{v \in \mathbb{R}^n \mid \|v\| = 1\}$ and let $\Omega = \{v \in \mathbb{R}^n \mid \|v\| < \nu(v/\|v\|)\}$. We denote by $T\Omega$ the dilate of Ω by T. Define the following sets:

$$V_{(a,b)}(\mathbb{R}) = V^B_{(a,b)}(\mathbb{R}) = \{x \in \mathbb{R}^n \mid a < B(x) < b\} \text{ and}$$
$$V_{(a,b)}(\mathbb{Z}) = V^B_{(a,b)}(\mathbb{Z}) = \{x \in \mathbb{Z}^n \mid a < B(x) < b\}\,.$$

One can verify that as $T \to \infty$

$$\mathrm{Vol}(V_{(a,b)}(\mathbb{R}) \cap T\Omega) \sim \lambda_{B,\Omega}(b-a)T^{n-2} \tag{2}$$

where

$$\lambda_{B,\Omega} = \int_{L\cap\Omega} \frac{dA}{\|\nabla B\|}\,, \tag{3}$$

L is the light cone $B = 0$ and dA is the area element on L.

Let $p \geq 2, q \geq 1, p \geq q$, let $\mathcal{O}(p,q)$ denote the space of quadratic forms of signature (p,q) and discriminant ± 1, let $n = p+q, (a,b)$ be an interval, and let $\mathcal{K}$ be a compact subset of $\mathcal{O}(p,q)$.

2.4.1. Theorem [DanM5, Corollary 5]. *(I) For any $\theta > 0$ there exists a finite subset $\mathcal{L} = \mathcal{L}(\theta, a, b)$ of $\mathcal{K}$ such that each form $B \in \mathcal{L}$ is rational and for any compact subset $\mathcal{F}$ of $\mathcal{K} - \mathcal{L}$ there exists $T_0 = T_0(\theta, a, b, \mathcal{F}) \geq 0$ such that for all B in $\mathcal{F}$ and $T \geq T_0$*

$$|V_{(a,b)}(\mathbb{Z}) \cap T\Omega| \geq (1-\theta)\mathrm{Vol}(V_{(a,b)}(\mathbb{R}) \cap T\Omega)\,. \tag{4}$$

(II) If $n \geq 5$, then for any $\varepsilon > 0$ there exist $c = c(\varepsilon, \mathcal{K}) > 0$ and $T_0 = T_0(\varepsilon, \mathcal{K}) > 0$ such that for all $B \in \mathcal{K}$ and $T \geq T_0$

$$|V_{(-\varepsilon,\varepsilon)}(\mathbb{Z}) \cap T\Omega| \geq c\ \mathrm{Vol}(V_{(-\varepsilon,\varepsilon)}(\mathbb{R}) \cap T\Omega)\,. \tag{5}$$

For a single B a similar estimate, but only with a positive constant rather than one arbitrarily close to 1, had been obtained earlier by Dani and Mozes, and also independently by Ratner (both unpublished).

2.4.2. Theorem (see [EMM1, Theorem 1] or [EMM2, Theorem 2.1]). *If $p \geq 3$ then as $T \to \infty$*

$$|V_{(a,b)}(\mathbb{Z}) \cap T\Omega| \sim \lambda_{B,\Omega}(b-a)T^{n-2} \tag{6}$$

for any irrational form $B \in \mathcal{O}(p,q)$ where $\lambda_{B,\Omega}$ is as in (3).

If the signature B is (2,1) or (2,2) then no universal formula like (6) holds. In fact, we have the following theorem:

2.4.3. Theorem (see [EMM1, Theorem 2] or [EMM2, Theorem 2.2]). *Let Ω_0 be the unit ball, and let $q = 1$ or 2. Then for every $\varepsilon > 0$ and every interval (a,b) there exists a quadratic irrational form B of signature $(2,q)$, and a constant $c > 0$ such that for an infinite sequence $T_j \to \infty$*

$$|V_{(a,b)}(\mathbb{Z}) \cap T\Omega_0| > cT_j^q(\log T_j)^{1-\varepsilon}\,. \tag{7}$$

The case $q = 1, b \leq 0$ of this theorem was noticed by Sarnak and worked out in detail in [Br]. The quadratic forms constructed are of the type $x_1^2 + x_2^2 - \alpha x_3^2$, or $x_1^2 + x_2^2 - \alpha(x_3^2 + x_4^2)$, where α is extremely well approximated by squares of rational numbers (in fact we can take α which belong to a set of second category in the Baire sense). Let us also note that in the statement of Theorem 2.4.2, $(\log T)^{1-\varepsilon}$ can be replaced by $\log T/\nu(T)$ where $\nu(T)$ is any unbounded increasing function.

However, in the (2,1) and (2,2) cases, there is an upper bound of the form $cT^q \log T$. This upper bound is effective, and is uniform over compact subsets of $\mathcal{O}(p,q)$. There is also an effective uniform upper bound for the case $p \geq 3$.

2.4.4. Theorem (see [EMM1, Theorem 3] or [EMM2, Theorem 2.3]). *If $p \geq 3$ there exists a constant $c = c(\mathcal{K}, a, b, \Omega)$ such that for any $B \in \mathcal{K}$ and all $T > 1$*

$$|V_{(a,b)}(\mathbb{Z}) \cap T\Omega| < cT^{n-2}\,. \tag{8}$$

If $p = 2$ and $q = 1$ or $q = 2$, then there exists a constant $c = c(\mathcal{K}, a, b, \Omega)$ such that for any $B \in \mathcal{K}$ and $T > 2$

$$|V_{(a,b)}(\mathbb{Z}) \cap T\Omega| < cT^{n-2} \log T\,. \tag{9}$$

Also, for the (2,1) and (2,2) case, the following "almost everywhere" result is true:

2.4.5. Theorem (see [EMM1, Theorem 4] or [EMM2, Theorem 2.5]). *The asymptotic formula* (6) *holds for almost all quadratic forms of signature* $(p,q) = (2,1)$ *or* $(2,2)$.

Sarnak [Sar] recently proved that (6) holds for almost all forms within the following two-parameter family of quadratic forms of signature (2,2).

$$(x_1^2 + 2bx_1x_2 + cx_2^2) - (x_3^2 + 2bx_3x_4 + cx_4^2)\,,$$

$b, c \in \mathbb{R}$, $c - b^2 > 0$. This family arises in problems related to quantum chaos.

Let us also state a "uniform" version of Theorem 2.4.2.

2.4.6. Theorem (see [EMM, Theorem 5] or [EMM2, Theorem 2.5]). *For any $\theta > 0$ there exists a finite subset $\mathcal{P} = \mathcal{P}(\theta, a, b)$ of $\mathcal{K}$ such that each $B \in \mathcal{P}$ is rational and for any compact subset $\mathcal{F}$ of $\mathcal{K} - \mathcal{P}$ there exists $T_0 = T_0(\theta, a, b, \mathcal{P}) \geq 0$ such that for all B in $\mathcal{F}$ and $T \geq T_0$*

$$(1-\theta)\lambda_{B,\Omega}(b-a)T^{n-2} \leq |V_{(a,b)}(\mathbb{Z}) \cap T\Omega| \leq (1+\theta)\lambda_{B,\Omega}(b-a)T^{n-2} \tag{10}$$

where $\lambda_{B,\Omega}$ is as in (3).

2.4.7. Remark. Let $\mathcal{P}(\mathbb{Z}^n)$ denote as before the set of primitive vectors in $\mathbb{Z}^n$. If we consider $|V_{(a,b)}(\mathbb{R}) \cap T\Omega \cap \mathcal{P}(\mathbb{Z}^n)|$ instead of $|V_{(a,b)}(\mathbb{Z}) \cap T\Omega|$ then Theorems 2.4.1, 2.4.2, 2.4.3, 2.4.5, and 2.4.6 hold provided one replaces $1-\theta$ by $(1-\theta)/\zeta(n)$ in Theorem 2.4.1 and $\lambda_{B,\Omega}$ by $\lambda'_{B,\Omega} = \lambda_{B,\Omega}/\zeta(n)$ in Theorems 2.4.2, 2.4.3, 2.4.5, and 2.4.6, where ζ is the Riemann zeta function.

2.4.8. Let $\Gamma = SL(n,\mathbb{Z}), G = SL(n,\mathbb{R})$, $\Omega_n = G/\Gamma = \{$ the space of unimodular matrices in $\mathbb{R}^n$ with discriminant $1\}$. One can associate to an integrable function ψ on $\mathbb{R}^n$ a function $\tilde{\psi}$ on $\Omega_n = G/\Gamma$ by setting

$$\tilde{\psi}(g\Gamma) = \sum_{v \in g\mathbb{Z}^n, v \neq 0} \tilde{\psi}(v) \ , \ g \in G \, . \tag{11}$$

According to a theorem of Siegel

$$\int_{\mathbb{R}^n} \psi dm^n = \int_{G/\Gamma} \tilde{\psi} d\mu \tag{12}$$

where m^n is the Lebesgue measure on $\mathbb{R}^n$ and μ is the G-invariant probability measure on G/Γ.

In [DanM5], the proof of Theorem 2.4.1 is based on the following identity which is immediate from the definitions:

$$\int_T^{\ell T} \int_F \sum_{v \in g\mathbb{Z}^n} \psi(u_t k v) d\sigma(k) dt = \int_T^{\ell T} \int_F \tilde{\psi}(u_t k g \Gamma) d\sigma(k) dt \tag{13}$$

where $\{u_t\}$ is a certain one-parameter unipotent subgroup of $SO(p,q)$, F is a Borel subset of the maximal compact subgroup K of $SO(p,q), \sigma$ is the normalized Haar measure on K, and ψ is a continuous function on $\mathbb{R}^n - \{0\}$ with compact support. The number $|V_{(a,b)}(\mathbb{Z}) \cap T\Omega|$ can be approximated by the sum over m of the integrals on the left-hand side of (13) for an appropriate choice of $g, \psi = \psi_i, F = F_i$, $1 \leq i \leq m$. The right-hand side of (13) can be estimated, uniformly when $g\Gamma$ belongs to certain compact subsets of $SL(n,\mathbb{R})/\Gamma$, using (12) and a refined version of Ratner's uniform distribution theorem (see Theorem 3.9.3 below). To prove the assertion (II) of Theorem 2.4.1 we have to use the following fact which is essentially equivalent to Meyer's theorem: if $n \geq 5$ then any closed orbit of $SO(p,q)$ in $SL(n,\mathbb{R})/\Gamma$ is unbounded.

Let us state the just-mentioned version of the uniform distribution theorem.

2.4.9. Theorem [DanM5, Theorem 3]**.** *Let G be a connected Lie group, Γ a lattice in G, and μ the G-invariant probability measure on G/Γ. For any closed subgroups H and W of G let $X(H,W) = \{g \in G \mid Wg \subset gH\}$. Let $U = \{u_t\}$ be an Ad-unipotent one-parameter subgroup of G and let φ be a bounded continuous function on G/Γ. Let $\mathcal{D}$ be a compact subset of G/Γ and let $\varepsilon > 0$ be given. Then there exist finitely many proper closed subgroups $H_1 = H_1(\varphi, \mathcal{D}, \varepsilon), \ldots, H_k = H_k(\varphi, \mathcal{D}, \varepsilon)$ such that $H_i \cap \Gamma$ is a lattice in H_i for all i, and compact subsets $C_1 = C_1(\varphi, \mathcal{D}, \varepsilon), \ldots, C_k = C_k(\varphi, \mathcal{D}, \varepsilon)$ of $X(H_1, U), \ldots, X(H_k, U)$ respectively, for which the following holds. For any compact subset F of $\mathcal{D} - \bigcup_{1 \leq i \leq k} C_i\Gamma/\Gamma$ there exist a $T_0 \geq 0$ such that for all $x \in F$ and $T > T_0$*

$$|\frac{1}{T} \int_0^T \varphi(u_t x) dt - \int_{G/\Gamma} \varphi d\mu| < \varepsilon \, .$$

2.4.10. The function $\tilde{\psi}$ defined by (12) is unbounded for any continuous nonzero function ψ on $\mathbb{R}^n$. Therefore we cannot use (13) and Theorem 2.4.9 to get the asymptotic of $|V_{(a,b)}(\mathbb{Z}) \cap T\Omega|$, because this theorem is proved (and in general true) only for bounded continuous functions. On the other hand, as it was done in [Dan M5] one can get lower bounds by considering bounded continuous function $f \leq \tilde{\psi}$ and applying Theorem 2.4.9 to f.

If $\delta > 0$ and h is a nonnegative function then for any $\varepsilon > 0$

$$h = h_\varepsilon + \varepsilon h^{1+\delta}$$

where h_ε is bounded. Therefore it would be possible to obtain the asymptotic of $|V_{(a,b)}(\mathbb{Z}) \cap T\Omega|$ from (13) and Theorem 2.4.9 if we knew that for some $\delta > 0$ and any continuous nonnegative function ψ on $\mathbb{R}^n$ with compact support,

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T \int_K \tilde{\psi}(u_t k g\Gamma)^{1+\delta} d\sigma(k) dt < \infty\,. \tag{14}$$

Using methods of the papers [Dan2], [Dan6] and [Marg 2] it is possible to prove Theorem 3.3.4 below which implies that for $\theta = 1/n^3$

$$\limsup_{T\to\infty} \left\{ \sup_{g\in L} \frac{1}{T} \int_0^T \tilde{\psi}(u_t g\Gamma)^\theta dt \right\} < \infty$$

for any compact subset $L \subset SL(n,\mathbb{R})$ and any unipotent subgroup $\{u_t\} \subset SL(n,\mathbb{R})$. But these methods are certainly not enough to prove (14). In [EMM1,2] we present a different argument and essentially prove (14) for all $0 \leq \delta < 1$ in the case where $p \geq 3$ and $q \geq 1$. If $(p,q) = (2,2)$ or $(p,q) = (2,1)$ then (14) is in general not true even for $\delta = 0$ but it becomes true for $\delta = 0$ if we replace in (14) the factor $1/T$ by $1/(T \log T)$. Let us note that for technical reasons we consider in [EMM1,2] not the integrals from (14) but the integrals of the form

$$\int \tilde{\psi}(a_t k)^{1+\delta} d\sigma(k) \tag{15}$$

where $\{a_t\}$ is a certain diagonalizable subgroup of $SO(p,q)$. Because of that we do not use Theorem 2.4.9 directly. Instead of that we first deduce the following result from Theorem 2.4.9. This relies on the well-known fact that in the hyperbolic plane large circles are well approximated by horocycles and that $SO(p,q)$ is a maximal proper subgroup in $SL(n,\mathbb{R})$.

2.4.11. Theorem [EMM2, Theorem 4.5]. *Let $\{a_t \mid t \in \mathbb{R}\}$ be a self-adjoint one-parameter subgroup of $SO(2,1)$. Let $p \geq 2$ and $q \geq 1$. Denote $p+q$ by n, $SO(p,q)$ by H, $SL(n,\mathbb{R})$ by G, and $SO(p) \times SO(q)$ by K. Let Γ be a lattice in G and let φ be any continuous function on G/Γ vanishing outside a compact set. Then for any $\varepsilon > 0$ and any bounded measurable function r on K and every compact subset $\mathcal{D}$ of G/Γ there exist finitely many points $x_1, \ldots, x_\ell \in G/\Gamma$ such that*

(i) *the orbits* $Hx_1, \ldots, Hx_\ell$ *are closed and have finite* H*-invariant measure*;

(ii) *for any compact subset* F *of* $\mathcal{D} - \bigcup_{1 \le i \le \ell} Hx_i$ *there exists* $T_0 > 0$ *such that for all* $x \in F$ *and* $t > T_0$

$$|\int_K \varphi(a_t kx) r(k) d\sigma(k) - \int_{G/\Gamma} \varphi d\mu \int_K r d\sigma| \le \varepsilon.$$

2.4.12. We will describe now the argument from [EMM2] how to obtain upper bounds for the integrals (15).

Let Δ be a lattice in $\mathbb{R}^n$. We say that a subspace L of $\mathbb{R}^n$ is Δ-*rational* if $L \cap \Delta$ is a lattice in L. For any Δ-*rational* subspace L, we denote by $d_\Delta(L)$ or simply by $d(L)$ the volume of $L/(L \cap \Delta)$. Let us note that $d(L)$ is equal to the norm of $e_1 \wedge \cdots \wedge e_\ell$ in the exterior power $\wedge^\ell(\mathbb{R}^n)$ where $\ell = \dim L$ and $(e_1, \ldots, e_\ell)$ is a basis over $\mathbb{Z}$ of $L \cap \Delta$. If $L = \{0\}$ we write $d(L) = 1$. It is clear that $d_\Delta(\mathbb{R}^n) = 1$ if and only if $\Delta \in \Omega_n = SL(n, \mathbb{R})/SL(n, \mathbb{Z})$.

Let us introduce the following notation:

$$\alpha_i(\Delta) = \sup\left\{\frac{1}{d(L)} \mid L \text{ is a } \Delta\text{-rational subspace of dimension } i\right\}, 0 \le i \le n\,,$$
$$\alpha(\Delta) = \max_{0 \le i \le n} \alpha_i(\Delta)\,.$$

It is a standard fact from the geometry of numbers that for any bounded function f on $\mathbb{R}^n$ vanishing outside a compact subset, there exists a positive constant $c = c(f)$ such that

$$\tilde{f}(\Delta) < c\alpha(\Delta) \tag{16}$$

for any lattice Δ in $\mathbb{R}^n$.

Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$. Let B_0 be the quadratic form defined by

$$B_0\left(\sum_{i=1}^n v_i e_i\right) = 2v_1 v_n + \sum_{i=2}^p v_i^2 - \sum_{i=p+1}^{n-1} v_i^2 \tag{17}$$

for all $v_1, \ldots, v_n \in \mathbb{R}$. The form B_0 has signature (p, q). Let $H = SO(B_0)$. For $t \in \mathbb{R}$, let a_t be the linear map so that $a_t e_1 = e^{-t} e_1, a_t e_n = e^t e_n$, and $a_t e_i = e_i$, $2 \le i \le n-1$. Then the one-parameter subgroup $\{a_t\}$ is contained in H. Let $\hat{K}$ be the subgroup of $SL(n, \mathbb{R})$ consisting of orthogonal matrices, and let $K = H \cap \hat{K}$. It is easy to check that K is a maximal compact subgroup of H, and consists of all $h \in H$ leaving invariant the subspace spanned by $\{e_1 + e_n, e_2, \ldots, e_p\}$.

Necessary upper bounds for the integrals (15) are deduced from (16) and the following main integrability estimates:

2.4.13. Theorem [EMM2, Theorems 3.2 and 3.3].

(a) *If* $p \geq 3, q \geq 1$ *and* $0 < s < 2$, *or if* $p = 2$, $q \geq 1$ *and* $0 < s < 1$, *then for any lattice* Δ *in* $\mathbb{R}^n$

$$\sup_{t>0} \int_K \alpha(a_t k\Delta)^s d\sigma(k) < \infty\,. \tag{18}$$

(b) *If* $p = 2$ *and* $q = 2$, *or if* $p = 2$ *and* $q = 1$, *then for any lattice* Δ *in* $\mathbb{R}^n$,

$$\sup_{t>1} \frac{1}{t} \int_K \alpha(a_t k\Delta) d\sigma(k) < \infty\,. \tag{19}$$

The upper bounds (18) *and* (19) *are uniform as* Δ *varies over compact sets the space of lattices.*

Sketch of the proof. For any $t \geq 0$ and any continuous action of the group H on a topological space X, we consider the averaging operator A_t:

$$(A_t f)(x) = \int_K f(a_t kx) d\sigma(k), x \in X\,.$$

If $X = H$, f is left K-invariant and $K\backslash H$ is $(n-1)$-dimensional hyperbolic space (or, equivalently, $q = 1$), then $(A_t f)(h)$ can be interpreted as the average of f over the sphere of radius $2t$ in $K\backslash H$, centered at Kh.

The main idea of the proof is to show that the α_i^s satisfy certain systems of integral inequalities which imply the desired bounds. If $0 < s < 2, p \geq 3$ and $0 < i < n$ or $p = 2, q = 2$ and $i = 1$ or 3, or if $0 < s < 1$, we prove that for any $c > 0$ there exist $t = t(s,c) > 0$ and $w = w(s,c) > 1$ such that

$$A_t \alpha_i^s \leq \frac{c}{2}\alpha_i^s + w^2 \max_{0<j\leq \min(n-i,i)} \sqrt{\alpha_{i+j}^s \alpha_{i-j}^s}\,. \tag{20}$$

The following inequalities also hold:

$$A_t \alpha_i^* \leq \alpha_i^* + w^2 \sqrt{\alpha_{3-i}^*}, 1 \leq i \leq 2\,, \tag{21}$$

for $(p,q) = (2,1)$ and

$$A_t \alpha_2^\# \leq \alpha_2^\# + w^2 \sqrt{\alpha_1 \alpha_3} \tag{22}$$

for $(p,q) = (2,2)$ and $i = 2$, where α_1^*, α_2^* and $\alpha_2^\#$ are suitably modified functions α_1 and α_2 (the ratios α_1/α_1^*, α_1^*/α_1, α_2/α_2^*, α_2^*/α_2, $\alpha_2/\alpha_2^\#$ and $\alpha_2^\#/\alpha_2$ are bounded).

Let $f_i(h) = \alpha_i(h\Delta), h \in H$. It follows from its definition that each α_i is K-invariant. Hence

$$f_i(Kh) = f_i(h), h \in H, 0 \leq i \leq n\,. \tag{23}$$

In view of (20) we have

$$A_t f_i^s \le \frac{c}{2} f_i^s + w^2 \max_{0<j\le \min(n-i,i)} \sqrt{f_{i+j}^s f_{i-j}^s}\,. \tag{24}$$

Let us denote $q(i) = i(n-i)$. Then by direct computations $2q(i) - q(i+j) - q(i-j) = -2j^2$. Therefore we get from (24) that for any $i, 0 < i < n$, and any positive $\varepsilon < 1$

$$\begin{aligned} A_t(\varepsilon^{q(i)} f_i^s) &\le \frac{c}{2}\varepsilon^{q(i)} f_i^s \\ &\quad + w^2 \max_{0<j\le \min(n-i,i)} \varepsilon^{q(i)-\frac{q(i+j)+q(i-j)}{2}} \sqrt{\varepsilon^{q(i+j)} f_{i+j}^s \varepsilon^{q(i-j)} f_{i-j}^s} \\ &\le \frac{c}{2}\varepsilon^{q(i)} f_i^s + \varepsilon w^2 \max_{0<j\le \min(n-i,i)} \sqrt{\varepsilon^{q(i+j)} f_{i+j}^s \varepsilon^{q(i-j)} f_{i-j}^s}\,. \end{aligned} \tag{25}$$

Consider the linear combination

$$f_{\varepsilon,s}(h) = \sum_{0\le i\le n} \varepsilon^{q(i)} f_i^s(h) = \sum_{0\le i\le n} \varepsilon^{q(i)} \alpha_i (h\Delta)^s\,. \tag{26}$$

Since $\varepsilon^{q(i)} f_i^s < f_{\varepsilon,s}, f_0 = 1$ and $f_n = 1/d(\Delta)$, the inequalities (25) imply the following inequality:

$$A_t f_{\varepsilon,s} < 1 + d(\Delta)^{-s} + \frac{c}{2} f_{\varepsilon,s} + n\varepsilon w^2 f_{\varepsilon,s}\,. \tag{27}$$

Taking $\varepsilon = \frac{c}{2nw^2}$ we see that

$$A_{t(s,c)} f_{\varepsilon,s} < c f_{\varepsilon,s} + b \tag{28}$$

where $b = 1 + d(\Delta)^{-s}$.

It is well known and easy to show that for every neighborhood V of e in H there exists a neighborhood U of e in K such that

$$a_t U a_r \subset K\, V a_t a_r\, K$$

for any $t \ge 0$ and $r \ge 0$. On the other hand the positive function $f_{\varepsilon,s}$ is left K-invariant (because of (23)) and the logarithm $\log f_{\varepsilon,s}$ is equicontinuous with respect to a left-invariant uniform structure on H. Using these facts and the inequality (28) for sufficiently small c, it is not difficult to show that

$$\sup_{t>0} (A_t f_{\varepsilon,s})(e) < \infty\,. \tag{29}$$

Combining (26) and (29) with the definition of A_t and α we get (18) (the assertion (a)). The proof of (b) is similar but we have to use (21) and (22), in addition to (20) (we first use (20) for $s = 1/2$ and after that use (21), (22) and (20) for $s = 1$).

To obtain (20) we have to apply the following two lemmas.

Lemma 1. *Let*

$$F(i) = \left\{x_1 \wedge \cdots \wedge x_i \mid x_1, \ldots, x_i \in \mathbb{R}^{p+q}\right\} \subset \wedge^i(\mathbb{R}^{p+q}) .$$

If $0 < s < 2, p \geq 3$ *and* $0 < i < n$ *or* $p = 2, q = 2$ *and* $i = 1$ *or 3, or if* $0 < s < 1$, *then*

$$\lim_{t\to\infty} \sup_{v\in F(i), \|v\|=1} \int_K \frac{d\sigma(k)}{\|a_t k v\|^s} = 0 . \tag{30}$$

Lemma 2. *For any two* Δ*-rational subspaces* L *and* M

$$d(L)d(M) \geq d(L \cap M)d(L + M) . \tag{31}$$

Lemma 1 is deduced from some standard facts about Lie groups and their finite dimensional representations. Lemma 2 easily follows from the inequality $\|x \wedge y\| \leq \|x\|\|y\|$, $x \in \wedge^i(\mathbb{R}^n), y \in \wedge^j(\mathbb{R}^n)$.

It follows from (30) that for any $c > 0$ there exist $t = t(s, c) > 0$ such that

$$\int_K \frac{d\sigma(k)}{d_{a_t k\Lambda}(a_t k L)^s} < \frac{c}{2} \frac{1}{d_\Lambda(L)^s} \tag{32}$$

for every lattice $\Lambda \subset \mathbb{R}^n$ and every Λ-rational subspace L of dimension i. Let $L_i = L_{i,\Lambda}$ be a Λ-rational subspace of dimension i such that

$$\frac{1}{d_\Lambda(L_i)} = \alpha_i(\Lambda) . \tag{33}$$

Let $w = e^t$. We have that

$$w^{-1} \leq \frac{\|a_t v\|}{\|v\|} \leq w, v \in \wedge^j(\mathbb{R}^n), 0 < j < n . \tag{34}$$

Let us denote the set of Λ-rational subspaces L of dimension i with $d_\Lambda(L) < w^2 d_\Lambda(L_i)$ by Ψ_i. We get from (34) that for a Λ-rational i-dimensional subspace $L \notin \Psi_i$

$$d_{a_t k\Lambda}(a_t k L) > d_{a_t k\Lambda}(a_t k L_i), k \in K . \tag{35}$$

It follows from (32), (33), (35) and the definition of α_i and A_t that

$$(A_t \alpha_i^s)(\Lambda) < \frac{c}{2} \alpha_i^s(\Lambda) \quad \text{if} \quad \Psi_i = \{L_i\} . \tag{36}$$

Assume now that $\Psi_i \neq \{L_i\}$. Let $M \in \Psi_i, M \neq L_i$. Then $\dim(M+L_i) = i+j, j > 0$. Now using (33), (34) and Lemma 2 we get that for any $k \in K$

$$\begin{aligned} \alpha_i(a_t k\Lambda) < w\alpha_i(\Lambda) = \frac{w}{d_\Lambda(L_i)} &< \frac{w^2}{\sqrt{d_\Lambda(L_i) d_\Lambda(M)}} \\ &\leq \frac{w^2}{\sqrt{d_\Lambda(L \cap M) d_\Lambda(L + M)}} \leq w^2 \sqrt{\alpha_{i+j}(\Lambda)\alpha_{i-j}(\Lambda)} . \end{aligned} \tag{37}$$

Hence

$$(A_t\alpha_i^s)(\Lambda) \le w^2 \max_{0<j\le\min(n-i,i)} \sqrt{\alpha_{i+j}^s\alpha_{i-j}^s} \quad \text{if} \quad \Psi_i \ne L_i\,. \tag{38}$$

Combining (36) and (38) we get (20).

2.4.14. Remark. Theorem 1.4.2 of Birch and Davenport implies that if B is a real indefinite form in five variables then for any $\delta > 0$ there exists $C = C(\delta, B)$ with the following property. For any $\varepsilon > 0$ there exist $x \in \mathbb{Z}^5, x \ne 0$, such that $|B(x)| < \varepsilon$ and $\|x\| < C\varepsilon^{-5-\delta}$. It would be interesting to prove a similar result for nondiagonal forms (maybe with the replacement of $\varepsilon^{-5-\delta}$ by $\varepsilon^{-n-\delta}$ for some n). It would also be interesting to obtain an analog of Theorem 1.3.1 for nondiagonal forms. At present it is not clear how the homogeneous space approach can be applied to solve these two problems.

2.5. Markov spectrum. Let $\mathcal{O}_n$ denote the set of all nondegenerate indefinite quadratic form in n variables, let $d(B)$ denote the discriminant of a form $B \in \mathcal{O}_n$, and let, as before, $m(B) = \inf\{B(x) \mid x \in \mathbb{Z}^n, x \ne 0\}$. Let us set $\mu(B) = m(B)^n/|d(B)|$. It is clear that $\mu(B) = \mu(B')$ if B and B' are equivalent (in the sense of 2.1.7). Let M_n denote $\mu(\mathcal{O}_n)$. The set M_n is called the *n-dimensional Markov spectrum*. It easily follows from Mahler's compactness criterion that M_n is bounded and closed.

In 1880, Markov [Mark1] described the intersection of M_2 with the segment $(4/9, \infty)$ and described corresponding quadratic forms. It follows from this description that $M_2 \cap (4/9, \infty)$ is a discrete countable subset of $(4/9, 4/5]$ and that, for any $a > 4/9$, there are only finitely many equivalence classes of forms $B \in \mathcal{O}_2$ with $\mu(B) > a$. On the other hand, the intersection $M_2 \cap [0, 4/9]$ is not countable and moreover has quite a complicated topological structure (see [Ca2], [CuF], [Po]).

It follows from Meyer's theorem and Theorem 2.3.1 that $M_n = \{0\}$ if $n \ge 5$. Theorems 2.1.7 and 2.3.1 imply that if n is 3 or 4 and $\varepsilon > 0$ then the set $M_n \cap (\varepsilon, \infty)$ is finite and moreover there are only finitely many equivalence classes of forms $B \in \Phi_n$ with $\mu(B) > \varepsilon$. For rational forms B, this result had been proved earlier by Vulakh [Vu2]. As noted in [Vu1], Vulakh obtained the complete description of spectra of nonzero minima of rational Hermitian forms.

For $n = 3$, Markov [Mark2] determined the first 4 values of $\mu(B)^{-1}$ and Venkov [Ven] determined the next 7 values. For $n = 4$ and forms of signature $(2, 2)$, Oppenheim [Op3] determined the first 7 values of $\mu(B)^{-1}$. For $n = 4$ and forms of signature $(3, 1)$ he [Op1] determined also the first value of $\mu(B)^{-1}$ which is $7/4$.

Recently, using a computer, Grunewald and Martini gave complete lists of representatives of 290 equivalence classes of $B \in \Phi_3$ for which $\mu(B) > 3/46$ and of representatives of 257 equivalence classes of $B \in \Phi_4$ for which $\mu(B) > 1/36$. On the

basis of these computations, they suggested the following conjectural asymptotic formulas:

$$\Phi_3(x) \approx 1,2x^2 \text{ and } \Phi_4(x) \approx 1,2x^{3/2} \text{ as } x \to \infty\,,$$

where

$$\Phi_n(x) = |\{B \in \mathcal{O}_n \mid \mu(B) \le x^{-1}\}|\,.$$

3. Unipotent Flows On Homogeneous Spaces

3.1. Growth properties of polynomials. It was observed a long time ago (already in [Marg 2] and probably even earlier) that certain growth properties of polynomials of bounded degrees play an important role in the study of unipotent flows on homogeneous spaces. We now state some of these properties in the form of several elementary lemmas which easily follow from Lagrange interpolation formula.

Let $\mathcal{P}_m$ (resp. $\mathcal{P}_m^+$) denote the space of all (resp. all non-negative) polynomials on $\mathbb{R}$ of degree at most m.

3.1.1. Lemma. *For any $k > 1$ and $m \in \mathbb{N}$ there exist $0 < \varepsilon_1(k,m) < \varepsilon_2(k,m)$ such that the following holds: Let $c > 0$ and $t_1 \le t_2$. If $P \in \mathcal{P}_m^+$ is such that $P(t) \le c$ for all $t \in [t_1, t_2]$ and $P(t_2) = c$ then the values of P at all points of one of the intervals*

$$[t_1 + k(t_2 - t_1), t_1 + k^2(t_2 - t_1)]\,,$$
$$[t_1 + k^3(t_2 - t_1), t_1 + k^4(t_2 - t_1)], \dots, [t_1 + k^{2m+1}(t_2 - t_1), t_1 + k^{2m+2}(t_2 - t_1)]$$

are greater than $c\varepsilon_1(k,m)$ and smaller than $c\varepsilon_2(k,m)$.

3.1.2. Lemma. *For any $k > 1$ and $m \in \mathbb{N}$ there exists $\bar{\varepsilon}(k,m) > 0$ such that the following holds: Let $c > 0$ and $t_1 \le t_2$. If $P \in \mathcal{P}_m^+$ is such that $P(t) = c$ for some $t \in [t_1, t_2], P(t_2) < c\bar{\varepsilon}(k,m)$ then there exists $t \in [t_2, t_1 + k(t_2 - t_1)]$ such that $P(t) = c\bar{\varepsilon}(k,m)$.*

3.1.3. Lemma. *For any $m \in \mathbb{N}$ and $\varepsilon > 0$ there exists $\alpha = \alpha(\varepsilon, m) > 0$ such that the following holds: Let $c > 0, t_1 \le t_2$, and $P \in \mathcal{P}_m$ be such that $|P(t)| < c$ for all $t \in [t_1, t_2]$. Then $|P(t) - P(t')| < \varepsilon c$ if $t_1 \le t < t' \le t_2$ and $t' - t < \alpha(t_2 - t_1)$.*

3.1.4. Lemma 3.1.3 can be viewed as a weak version of the following:

Lemma. *For any $m \in \mathbb{N}^+$ there exists $\beta = \beta(m) > 0$ such that the following holds: Let $c > 0$ and $t_1 \le t_2$. If $P \in \mathcal{P}_m$ is such that $|P(t)| < c$ for all $t \in [t_1, t_2]$ then*

$$|P^{(i)}(t)| < \frac{c\beta^i}{(t_2 - t_1)^i}$$

for any $t \in [t_1, t_2]$ and $i, 1 \le i \le n$, where $P^{(i)}$ denotes the i-th derivative of P.

3.1.5. Lemma. *For any $m \in \mathbb{N}^+$ and $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, m) > 0$ such that the following holds: Let $c > 0$ and $t_1 \leq t_2$. If $P \in \mathcal{P}_m^+$ is such that $P(t) = c$ for some $t \in [t_1, t_2]$ then*

$$\ell(\{t \in [t_1, t_2] \mid P(t) < \delta c\}) < \varepsilon(t_2 - t_1)$$

where ℓ denotes the Lebesgue measure on $\mathbb{R}$.

3.1.6. There is a slightly stronger version of Lemma 3.1.5 which can be easily deduced from Lemmas 3.1.3 and 3.1.5.

Lemma. *For any $m \in \mathbb{N}^+, \varepsilon > 0$ and $k > 1$ there exists $\delta' = \delta'(\varepsilon, m, k) > 0$ such that the following holds: Let $c > 0$ and $t_1 < t_2$. If $P \in \mathcal{P}_m^+$ is such that $P(t) = c$ for some $t \in [t_1, t_2]$ then*

$$\begin{aligned}&\ell(\{t \in [t_1, t_2] \mid P(t) < \delta' c\})\\ &\leq \varepsilon \ell(\{t \in [t_1, t_2] \mid ck^{-1} < P(t) < c\})\,.\end{aligned}$$

3.1.7. Remark. Lemmas 3.1.1–3.1.6. are standard and simple and, as it was noticed, they easily follow from Lagrange interpolation formula. There are slightly stronger versions of Lemmas 3.1.3 [Rat 5, Proposition 1.1] and 3.1.5 [DanM5, Lemma 4.1]. These versions can also be easily deduced from Lagrange interpolation formula. To prove Lemmas 3.1.1–3.1.6, one can also apply another standard argument which is based on the linear rescaling of variables and on the compactness of the unit ball in the space of polynomials on [0,1] of degree at most m (see for example [Marg 2] and [Dan 2] for Lemmas 3.1.1. and 3.1.2. and [Rat 6, Lemma 3.1] for Lemma 3.1.5).

3.2. Unipotent groups of linear transformations. Let $\{u(t) \mid t \in \mathbb{R}\}$ be a unipotent one-parameter subgroup of $SL(n, \mathbb{R})$. Matrix coefficients of $u(t)$ are polynomials in t of degree at most $n-1$, and $\|u(t)v\|^2$ is a polynomial in t of degree at most $2n-2$ for any $v \in \mathbb{R}^n$.

3.2.1. Applying Lemma 3.1.6 to the polynomials $P_v(t) = \|u(t)v\|^2$ we get the following:

Lemma. *For any $\varepsilon > 0$ and $k > 1$ there exists $\tilde{\delta} = \tilde{\delta}(\varepsilon, k, n) > 0$ such that the following holds: Let $c > 0, t_1 < t_2$ and $v \in \mathbb{R}^n$. If $\|u(t)v\| = c$ for some $t \in [t_1, t_2]$ then*

$$\begin{aligned}&\ell(\{t \in [t_1, t_2] \mid \|u(t)v\| < \tilde{\delta} c\})\\ &\leq \varepsilon \ell(\{t \in [t_1, t_2] \mid ck^{-1} < \|u(t)v\| < c\})\,.\end{aligned}$$

3.2.2. To put Lemma 3.2.1 in a more general context, let us consider a one-parameter group $\{T^t\}$ of homeomorphisms of a locally compact space X and a subset A of X (not necessarily closed). We say that A is *avoidable with respect to* $\{T^t\}$ if for any compact subset C of $X - A$ and any $\varepsilon > 0$ there exists a neighbourhood Ψ of A in X such that for any $x \in C$ and any $t_1 \leq 0 \leq t_2$, we have

$$\ell(\{t \in [t_1, t_2] \mid T^t x \in \Psi\}) \leq \varepsilon(t_2 - t_1)\,.$$

Lemma 3.2.1 essentially means that $\{0\}$ is avoidable with respect to $\{u(t)\}$. Let us note that $\{0\}$ is not avoidable with respect to any non-quasiunipotent one-parameter subgroup $\{a_t\}$ of $GL(n, \mathbb{R})$ (an element g of $GL(n, \mathbb{R})$ is called *quasiunipotent* if all eigenvalues of g have absolute value 1). Lemma 3.2.1 can be considered as a special case ($A = \{0\}$) of the following assertion which can be easily deduced from Proposition 4.2 in [DanM5].

Proposition. *Any algebraic subvariety A of $\mathbb{R}^n$ is avoidable with respect to $\{u(t)\}$.*

3.2.3. Lemma. *For any $\varepsilon > 0$ there exists $\tilde{\alpha} = \tilde{\alpha}(\varepsilon, n) > 0$ such that the following holds: Let $c > 0, t_1 \leq t_2$, and $v \in \mathbb{R}^n$ be such that $\|u(t)v\| < c$ for all $t \in [t_1, t_2]$. Then $\|u(t)v - u(t')v\| < \varepsilon c$ if $t_1 \leq t < t' \leq t_2$ and $t' - t < \tilde{\alpha}(t_2 - t_1)$.*

To prove this lemma, it is enough to notice that the coordinates of $u(t)v$ are polynomials in t of degree at most n and after that apply Lemma 3.1.3.

3.2.4. Let $V_0 = \{v \in \mathbb{R}^n \mid u(t)v = v \text{ for all } t \in \mathbb{R}\}$. Using an easy compactness argument one can deduce from Lemma 3.2.3 the following:

Lemma (cf. [Wi, Proposition 6.3]). *For any $\varepsilon > 0$ there exists $\theta = \theta(\varepsilon, n, \{u_t\}) > 0$ such that the following holds: Let $t_1 \leq 0 \leq t_2$, and $v \in \mathbb{R}^n$ be such that $\|v\| < \theta$ and $\|u(t)v\| < 1$ for all $t \in [t_1, t_2]$. Then for any $t \in [t_1, t_2]$ one can find $q_t \in V_0$ such that $\|u(t)v - q_t\| < \varepsilon$.*

3.2.5. We already defined V_0. Let $V_i = \{v \in \mathbb{R}^n \mid u(t)v - v \in V_{i-1} \text{ for all } t \in \mathbb{R}\}, 1 \leq i \leq n-1$. Then $\dim V_i/V_{i-1} \leq \dim V_{i-1}/V_{i-2}$ and $V_{n-1} = \mathbb{R}^n$. We can represent $\mathbb{R}^n$ as $V_0 \oplus V_1/V_0 \oplus \cdots \oplus V_{n-1}/V_{n-2}$. By Jordan decomposition, after the identification of V_i/V_{i-1} with a certain subspace $V_i' \subset V_{i-1}'$ of $V_0, 1 \leq i \leq n-1$, the following holds: for any $v \in \mathbb{R}^n$ there exists a polynomial function $\varphi_v : \mathbb{R} \to V_0$ of degree at most $n-1$ such that

$$\begin{aligned} u(t)v &= (\varphi_v(t), \varphi_v'(t), \ldots, \varphi_v^{(i)}(t), \ldots, \varphi_v^{(n-1)}(t)) \\ &\in \mathbb{R}^n = V_0 \oplus V_1/V_0 \oplus \cdots \oplus V_i/V_{i-1} \oplus \cdots \oplus V_{n-1}/V_{n-2}\,, \end{aligned}$$

Now applying Lemma 3.1.4 we get a stronger version of Lemma 3.2.4.

Lemma. *There exist $\tilde{\beta} = \tilde{\beta}(n) > 0$ and, for every $v \in \mathbb{R}^n$, a polynomial function $\varphi_v : \mathbb{R} \to V_0$ of degree at most $n-1$ with the following property: Let $c > 0$, $t_1 \le 0 \le t_2, t_1 \ne t_2$, and $v \in \mathbb{R}^n$ be such that $\|u(t)v\| < c$ for all $t \in [t_1, t_2]$. Then*

$$\|u(t)v - (\varphi_v(t), \dots, \varphi_v^{(i-1)}(t), 0, \dots, 0)\| < \frac{c\tilde{\beta}^i}{(t_2 - t_1)^i}$$

for any $t \in [t_1, t_2]$ and $1 \le i \le n-1$.

This lemma is similar to Corollary 3.1 in [Rat 2]. The proof given in [Rat 2] also uses Jordan decomposition.

3.2.6. Remark. Let $G = SL(2, \mathbb{R}), h(t) = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$, $H = \{h(t) \mid t \in \mathbb{R}\}$, and let d_G denote a left invariant metric on G. Lemma 3.2.4 (or Lemma 3.2.5) applied to the one-parameter unipotent group $\{\mathrm{Ad}h(t)\}$ implies a property of horocycle flows which Ratner terms the H-property (see Definition 1 in [Ra4]). This property in the form given in [Rat10] states that given $0 < \varepsilon < 1$ and $p, N > 0$ there exist $\delta(\varepsilon, p, N)$, $\alpha(\varepsilon) \in (0, 1)$ such that if $d_G(x, e) < \delta(\varepsilon, p, N)$ for some $x \in G - H$ then there are $L, T > 0$ with $N < L < T$, $L \ge \alpha(\varepsilon)T$ such that either $d_G(xh(t), h(t+p)) \le p\varepsilon$ for all $t \in [T - L, T]$ or $d_G(xh(t), h(t-p)) \le p\varepsilon$ for all $t \in [T - L, T]$. The H-property was generalized by Witte in [Wi].

3.2.7. It is possible to state and prove generalizations of Lemmas 3.2.1, 3.2.3–3.2.5 for multidimensional unipotent groups U. But for noncomutative U it involves a technical construction of subsets of U replacing intervals in the one-dimensional case. Let us state only a lemma which for one-parameter U easily follows from Lemma 3.2.5. In the general case this lemma can be proved by induction on n using the Lie–Kolchin theorem (see for example [Marg7, Sec. 5]).

Lemma. *Let U be a connected group of unipotent linear transformations of $\mathbb{R}^n$, and let $Y \subset \mathbb{R}^n$. Let $L = \{v \in \mathbb{R}^n \mid Uv = v\}$ and $p \in L \cap \bar{Y}$. Suppose that $L \cap Y = \emptyset$. Then $\overline{UY} \cap L$ contains the image of a nonconstant polynomial map $\varphi : \mathbb{R} \to L$ such that $\varphi(0) = p$.*

3.3. Recurrence to compact sets. Let $\Omega_n = SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ denote the space of unimodular lattices in $\mathbb{R}^n$. For any lattice $\Delta \in \Omega_n$ we define

$$\beta(\Delta) = \inf\{\|v\| \mid v \in \Lambda, v \ne 0\}.$$

According to Mahler's compactness criterion, a subset $K \subset \Omega_n$ is relatively compact in Ω_n if and only if $\inf\{\beta(\Delta) \mid \Delta \in K\} > 0$.

3.3.1. Theorem [Marg2]**.** *Let $\{u(t)\}$ be a one-parameter unipotent subgroup of $SL(n, \mathbb{R})$. For any lattice $\Delta \in \Omega_n$ there exists $\varepsilon(\Delta) > 0$ such that the set*

$\{t \geq 0 \mid \beta(u(t)\Delta) > \varepsilon(\Delta)\}$ *is unbounded. Equivalently, for any* $x \in \Omega_n$ *the "positive semi-orbit"* $\{u(t)x \mid t \geq 0\}$ *does not tend to infinity. That is, there exists a compact subset* $K = K(x) \subset \Omega_n$ *such that* $\{t \geq 0 \mid u(t)x \in K\}$ *is unbounded.*

For $n = 2$ the proof of this theorem is simple. Indeed, if $n = 2$, $\Delta \in \Omega_n$, v_1 and v_2 are nonzero primitive vectors in Δ, $v_1 \neq \pm v_2$, then the sets $\{t \in \mathbb{R} \mid \|u(t)v_1\| < 1\}$ and $\{t \in \mathbb{R} \mid \|u(t)v_2\| < 1\}$ are disjoint. After that it remains to apply Lemma 3.2.1. However for $n \geq 3$ the proof is much harder. It is based on a rather complicated induction argument and uses Lemmas 3.1.1 and 3.1.2 applied to the polynomials $P_L(t) = d^2_{u(t)\Delta}(u(t)L)$ where L is a Δ-rational subspace of $\mathbb{R}^n$ and $d_{u(t)\Delta}(u(t)L)$ is defined in 2.4.12.

3.3.2. Theorem [DanM5, Theorem 6.1]**.** *Let* G *be a connected Lie group,* Γ *a lattice in* G, F *a compact subset of* G/Γ *and* $\varepsilon > 0$. *Then there exists a compact subset* K *of* G/Γ *such that for any Ad-unipotent one-parameter subgroup* $\{u(t)\}$ *of* G, *any* $x \in F$, *and* $T \geq 0$,

$$\ell(\{t \in [0,T] \mid u(t)x \in K\} \geq (1-\varepsilon)T\,.$$

This theorem is essentially due to Dani. He proved it in [Dan5] for semisimple groups G of $\mathbb{R}$-rank 1 and in [Dan6] for arithmetic lattices. The general case can be easily reduced to these two cases using the arithmeticity theorem. In the case of arithmetic lattices the theorem can be considered as the quantitative version of Theorem 3.3.1 and the proof given in [Dan6] is similar in principle to the proof of Theorem 3.3.1 in [Marg2].

3.3.3. The following theorem is a strengthening of Theorem 3.3.1 and of Theorem 3.3.2 for arithmetic Γ.

Theorem (see [DanM1, Theorem 1.1] or [Dan6, Proposition 2.7])**.** *Given* $\varepsilon > 0$ *and* $\theta > 0$ *there exists* $\delta > 0$ *such that for any lattice* $\Delta \in \Omega_n$, *any unipotent one-parameter subgroup* $\{u(t)\}$ *of* $SL(n,\mathbb{R})$ *and* $T \geq 0$ *at least one of the following holds*:

(i) $\ell(\{t \in [0,T] \mid \beta(u(t)\Delta) \geq \delta\} \geq (1-\varepsilon)T$;
(ii) *there exists a* Δ*-rational subspace* L *of* $\mathbb{R}^n$ *such that* $P_L(t) < \theta$ *for all* $t \in [0,T]$, *where* $P_L(t) = d^2_{u(t)\Delta}(u(t)L)$ *denotes the same as in 3.3.1.*

Remark. About different generalizations of Theorems 3.3.1–3.3.3 see [Sha2] and [EMS2].

3.3.4. Using a modification of the approach developed in [Marg2] and [Dan6] it is possible to prove the following improvement of Theorem 3.3.3. It is stated for polynomial maps. Let us recall that any one-parameter unipotent subgroup $\{u(t)\} \subset \mathbb{R}^n$ can be considered as a polynomial map from $\mathbb{R}$ into $SL(n,\mathbb{R})$ of degree at most $n-1$. The proof will appear elsewhere.

Theorem. *Let $t_1 \le t_2$, $0 < \varepsilon < q \le 1$, and let $h : \mathbb{R} \to GL_n(\mathbb{R})$ be a polynomial map of degree not greater than r. Then at least one of the following holds*:

(i) $\ell(\{t \in [t_1, t_2] \mid \beta^2(h(t)\mathbb{Z}^n) < \varepsilon\} \le \varepsilon^{\frac{1}{2nr}} q^{-\frac{1}{2nr}} 4^{n+2} 2nr(t_2 - t_1)$;
(ii) *there exists a $\mathbb{Z}^n$-rational subspace L of $\mathbb{R}^n$ such that $d^2_{h(t)\mathbb{Z}^n}(h(t)L) < q$ for all $t \in [0, T]$.*

3.3.5. Using Theorem 3.3.3 it is not difficult to prove the following:

Corollary [DanM1, Corollary 1.3]**.** *Let $G = SL(n, \mathbb{R})$ and $\Gamma = SL(n, \mathbb{Z})$. There exists a compact subset C of $\Omega_n = G/\Gamma$ such that the following holds: Let U be a connected unipotent subgroup of G. Let $\mathcal{N}_G(U)$ be the normalizer of U in G and let $\{f(t)\}_{t \ge 0}$ be a curve in $\mathcal{N}_G(U)$ such that if L is a proper nonzero U-invariant subspace then L is invariant under $f(t)$ for all t and $|\det f(t) \mid_L| \to \infty$ as $t \to \infty$ ($\det f(t) \mid_L$ denotes the determinant of the restriction of $f(t)$ to L). Then for all $g \in G$, $C \cap Uf(t)g\Gamma/\Gamma$ is nonempty for all large t. If F is the subgroup generated by U and $\{f(t) \mid t \ge 0\}$, then every nonempty closed F-invariant subset contains a minimal closed F-invariant subset.*

3.3.6. Using an improvement of Theorem 3.3.2 obtained in [Dan6], Dani proved the following result.

Theorem (see [Dan6, Theorem 3.8] or [DanM1, Theorem 1.4]). *Let G be a connected Lie group and let Γ be a lattice in G. Then there exists a compact subset C of G/Γ such that for any closed connected Ad-unipotent subgroup U of G and any $g \in G$ either $C \cap Ug\Gamma/\Gamma$ is non-empty or $g^{-1}Ug$ is contained in a proper closed subgroup F such that $F\Gamma$ is closed and $F \cap \Gamma$ is a lattice in F.*

3.3.7. Corollary [DanM1, Corollary 1.5]**.** *Let G be a connected Lie group, Γ a lattice in G, and U a connected Ad-unipotent subgroup of G. Then any nonempty closed U-invariant subset of G/Γ contains a minimal closed U-invariant subset.*

3.3.8. Using Theorem 3.3.1 and its analog for non-arithmetic lattices, the author proved the following:

Theorem [Marg9, Theorem 1]. *Let G, Γ and U be the same as in Corollary 3.3.7. Then every minimal closed U-invariant subset of G/Γ is compact.*

The proof of this theorem also uses a general result about actions of nilpotent Lie groups on locally compact spaces.

3.4. Finiteness of ergodic measures. Suppose that G is a locally compact group acting continuously on a locally compact space X. We say that a subgroup H of G *has property* (D) *with respect to* X if for every H-invariant locally finite Borel measure μ on X, there exist Borel subsets $X_i \subset X$, $i \in \mathbb{N}^+$, such that $\mu(X_i) < \infty$,

$HX_i = X_i$ and $X = \cup X_i$. If H has property (D) with respect to X then every H-ergodic H-invariant locally finite Borel measure on X is finite. (When G and X are separable and metrizable, the converse is also true.)

3.4.1. Theorem (see [Dan2, Theorem 3.3] for arithmetic lattices and [Dan5, Theorem 4.3] in the general case)**.** *Let G be a connected Lie group and let Γ be a lattice in G. Then any unipotent subgroup U of G has property (D) with respect to G/Γ.*

This theorem is deduced from (a weak version of) Theorem 3.3.2 and the individual ergodic theorem.

3.4.2. We say that a subgroup H of a locally compact group G has *Mautner property* with respect to a subgroup F of G if, for any continuous unitary representation ρ of G on a Hilbert space W and any $w \in W$ such that $\rho(F)w = w$, we have that $\rho(H)w = w$. It is easy to see that if H has Mautner property with respect to F and F has property (D) with respect to X, then H has property (D) with respect to X. On the other hand, in view of a general result of Moore [Mo], if G is a connected Lie group and $H \subset G$ is a connected subgroup generated by its Ad-unipotent elements then H has Mautner property with respect to its maximal Ad-unipotent subgroup. Using these facts, one can deduce from Theorem 3.4.1 the following.

Theorem (cf. [Marg5, Theorem 7])**.** *Let G be a connected Lie group, Γ a lattice in G, and H a connected subgroup of G. Suppose that the quotient of H by its unipotent radical is semisimple. Then H has property (D) with respect to G/Γ.*

(By the unipotent radical of H we mean the maximal connected normal Ad-unipotent subgroup of H.)

3.4.3. The following is a special case of Theorem 3.4.2.

Theorem (cf. [Marg5, Theorem 8])**.** *Let G, Γ, and H be the same as in Theorem 3.4.2 and let $x \in G/\Gamma$. Suppose that the orbit Hx is closed in G/Γ. Then $H \cap G_x$ is a lattice in H, where $G_x = \{g \in G \mid gx = x\}$ is the stabilizer of x.*

3.4.4. Applying Theorem 3.4.3 to the case where $G = SL(n, \mathbb{R}), \Gamma = SL(n, \mathbb{Z})$, $x = e\Gamma$, and $H \subset G$ is the set of $\mathbb{R}$-rational points of a $\mathbb{Q}$-subgroup of G, we obtain the following special but important case of the theorem of Borel and Harish–Chandra on the finiteness of the volume of the quotient by an arithmetic subgroup.

Theorem. *Let H be a connected $\mathbb{Q}$-group. Suppose that the quotient of H by its unipotent radical is semisimple. Then $H(\mathbb{Z})$ is a lattice in $H(\mathbb{R})$.*

3.4.5. Combining Theorem 3.4.3 with Borel's density theorem [Bo1], it is easy to prove the following:

Proposition (cf. [BoP, Proposition 1.1]). *Let H be a connected subgroup of $SL(n,\mathbb{R})$ generated by its unipotent elements and such that $H\mathbb{Z}^n$ is closed in Ω_n. Then H is the connected component of the identity of the set of $\mathbb{R}$-rational points of a $\mathbb{Q}$-subgroup of $SL(n)$.*

3.4.6. Let us state now a result which can be considered as a generalization of Lemma 2.2.2 (b). It can be easily deduced from Proposition 3.4.5 (by a modification of the argument from Sec. 1.3 of [Marg5]).

Proposition. *Let B be a real indefinite quadratic form in $n \geq 3$ variables. Let us denote by H the special orthogonal group $SO(B)$. If the orbit $H\mathbb{Z}^n$ is closed in Ω_n then B is rational.*

3.5. General properties of minimal invariant sets. As before, in this and following subsections we denote by $\bar{A}$ the closure of a subset A of a topological space, by $\mathcal{N}_G(F)$ the normalizer of a subgroup F in a group G, and by e the identity element of G.

Let G be an arbitrary second countable locally compact group, and Ω a homogeneous space of G. The stabilizer $\{g \in G \mid gy = y\}$ of $y \in \Omega$ is denoted by G_y.

3.5.1. Lemma. *Let F be a closed subgroup of G, and Y a minimal closed F-invariant subset of Ω. Then for every nonempty open in Y subset U of Y and every compact subset K of Y, there exists a compact subset $L = L(U,K)$ of F such that $Ly \cap U \neq \emptyset$ for any $y \in K$.*

This lemma follows from the density of Fy in Y for every $y \in Y$.

3.5.2. Lemma. *Let F and H, $F \subset H$, be closed subgroups of G. Let Y be a minimal compact F-invariant subset of Ω, and let $y \in Y$. Suppose that there exists a sequence $\{f_n\} \subset F$ such that $\{f_n y\}$ is relatively compact in Ω_n but $\{f_n H_y\}$ is not relatively compact in H/H_y where $H_y = H \cap G_y$. Then the closure of $\{g \in G - H \mid gy \in Fy\}$ contains e.*

Proof. We can assume that the sequence $\{f_n\}$ tends to infinity in H/H_y. Let $W \subset G$ be a relatively compact neighborhood of e. By Lemma 3.5.1 there exist $\ell_n \in F$ and $g_n \in W$ such that $\ell_n f_n y = g_n y$ and the sequence $\{\ell_n\}$ is relatively compact in F. Then $\ell_n f_n H_y$ tends to infinity in H/H_y as $n \to \infty$. But $g_n^{-1}\ell_n f_n y = y$ and $\{g_n^{-1}\}$ is relatively compact in G. Therefore $g_n \notin H$ for almost all n. Thus $W \cap \{g \in G - H \mid gy \in Fy\} \neq \emptyset$. This proves the lemma.

3.5.3. We will need another simple result.

Lemma (see [Marg5, Lemmas 1 and 2] or [DanM1, Lemma 2.1]). *Let F, P and Q be closed subgroups of G such that $F \subset P \cap Q$. Let X and Y be closed subsets of Ω invariant under the actions of P and Q respectively. Suppose also that X is a compact minimal F-invariant subset. Let M be a subset of G such that $gX \cap Y$ is nonempty for all $g \in M$. Then $hX \subset Y$ for all $h \in \overline{QMP} \cap \mathcal{N}_G(F)$. If further $X = Y$ and $P = Q$ then X is invariant under the closed subgroup generated by $\overline{QMP} \cap \mathcal{N}_G(F)$.*

3.6. Topological limits of double cosets in algebraic groups. We define *a real algebraic group* to be a group of $\mathbb{R}$-rational points of an $\mathbb{R}$-group or, in other words, an algebraic subgroup of $GL(m, \mathbb{R})$.

3.6.1. Lemma (cf. [BoP, Proposition 2.3] and [Marg5, Lemma 5]). *Let G be a real algebraic group, U a connected unipotent subgroup of G and $M \subset G$. Suppose that $e \in \bar{M} - M$ and $M \subset G - \mathcal{N}_G(U)$. Then there is a rational map $\psi : U \to \mathcal{N}_G(U)$ such that $\psi(e) = e$, $\psi(U) \subset \overline{UMU}$ and $\psi(U) \not\subset KU$ for any compact subset K of $\mathcal{N}_G(U)$.*

Sketch of the proof. The connected unipotent subgroup U is algebraic and has no (nontrivial) rational characters. So, in view of Chevalley's theorem, there exists $n \in \mathbb{N}^+$, a faithful rational representation $\alpha : G \to GL(n, \mathbb{R})$ and $x_0 \in \mathbb{R}^n$ such that $U = \{g \in G \mid \alpha(g)x_0 = x_0\}$. Let $L = \{v \in \mathbb{R}^n \mid Uv = v\}$. It is easy to show that $L \cap \alpha(G)x_0 = \alpha(\mathcal{N}_G(U))x_0$. It remains to apply Lemma 3.2.7 to $\alpha(U)$, $Y = \alpha(M)x_0$ and $p = x_0$ and to notice that, since U is unipotent, there exists a rational section $s : G/U \to G$.

Remark. For $G = SL(2, \mathbb{R})$ the lemma had been known for a long time. In particular, this case of the lemma was used by A. Kirillov approximately in 1960 to prove that $SL(2, \mathbb{R})$ has Mautner property with respect to any nontrivial unipotent subgroup of $SL(2, \mathbb{R})$.

3.6.2. Combining Lemmas 3.5.3 and 3.6.1 we get the following:

Proposition. *Let G be a real algebraic group, U a connected unipotent subgroup of G, and Γ a discrete subgroup of G. Let X be a compact minimal U-invariant subset of G/Γ, and Y a closed U-invariant subset of G/Γ. Suppose that e belongs to the closure of the subset $\{g \in G - \mathcal{N}_G(U) \mid gX \cap Y \neq \emptyset\}$. Then there is a rational map $\psi : U \to \mathcal{N}_G(U)$ such that $\psi(e) = e, \psi(U) \not\subset KU$ for any compact subset K of $\mathcal{N}_G(U)$, and $hX \subset Y$ for any $h \in \psi(U)$. If further $X = Y$ then X is invariant under the closed subgroup generated by $\psi(U)$.*

3.6.3. Proposition. *Let G, U and Γ be the same as in Proposition 3.6. Let μ be a finite Borel U-ergodic U-invariant measure on G/Γ. Suppose that for any closed subset $X \subset G/\Gamma$ of μ-positive measure, the closure of the subset $\{g \in G - \mathcal{N}_G(U) \mid$*

$gX \cap X \neq \emptyset\}$ contains e. Then there is a rational map $\psi : U \to \mathcal{N}_G(U)$ such that $\psi(e) = e, \psi(U) \not\subset KU$ for any compact subset K of $\mathcal{N}_G(U)$, and the measure μ is invariant under $\psi(U)$.

This proposition can be considered as the measure theoretical analog of Proposition 3.6.2 (for $X = Y$). It is a weak form of Proposition 8.2 in [MargT2], and it also immediately follows from Ratner's measure classification theorem (see Theorem 3.9.1 below). The proof given in [MargT2] follows the same strategy as in the proof of Proposition 3.6.2 but it is technically much more complicated and involves a multidimensional generalization of the Birkhoff individual ergodic theorem. This generalization is in fact the replacement of (trivial) Lemma 3.5.1 which we implicitly used in the proof of Proposition 3.6.2.

Remark. There are certain similarities between the proof in [MargT2] of the above proposition and the proofs of [Rat2, Theorem 3.1, Lemma 3.3, Lemma 4.2] and [Rat3, Lemma 3.1] but, as far as the author can see, the proofs are prompted by different lines of thought.

3.6.4. Let B_0 be an indefinite quadratic form in n variables of the type

$$B_0(x, y, z) = 2xz - Q(y) = 2xz - \langle Ay, y\rangle, x, z \in \mathbb{R}, y \in \mathbb{R}^{n-2}\,,$$

where Q is a form in $n-2$ variables, A is the symmetric matrix corresponding to Q and $\langle\ ,\ \rangle$ is the standard inner product on $\mathbb{R}^{n-2}$. Let 1_i denote the unit $i \times i$ matrix and let, for $t \in \mathbb{R}$ and $c \in \mathbb{R}^{n-2}$.

$$d(t) = \begin{pmatrix} e^t & 0 & 0 \\ 0 & 1_{n-2} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}\,,$$

$$v_1(c) = \begin{pmatrix} 1 & Ac & \langle Ac, c\rangle/2 \\ 0 & 1_{n-2} & c \\ 0 & 0 & 1 \end{pmatrix}\,,$$

$$v_2(t) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1_{n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\,.$$

Let $H = SO(B_0)$,

$$\begin{aligned} D &= \{d(t) \mid t \in \mathbb{R}\}\ ,\ V_1 = \{v_1(c) \mid c \in \mathbb{R}^{n-2}\}\,, \\ V_2 &= \{v_2(t) \mid t \in \mathbb{R}\}\,, \\ V_2^+ &= \{v_2(t) \mid t \geq 0\}\ ,\ V_2^- = \{v_2(t) \mid t \leq 0\}\,, \end{aligned}$$

We note that D normalizes V_1 and V_2, and the subgroups D and V_1 are contained in H.

Lemma. *Let M be a subset of $G - HV_2$ such that $e \in \bar{M}$. Then $\overline{HMV_1}$ contains either V_2^+ or V_2^-.*

Proof. Let Φ denote the linear space of real quadratic forms in n variables. The group $G = SL(n, \mathbb{R})$ naturally acts on Φ. The stabilizer of B_0 under this action is H. Hence we can identify $H\backslash G$ and GB_0. Under this identification, Hg corresponds to $g^{-1}B_0$ and, in particular, $Hv_2(t)$ corresponds to $B_0 + tB_1$ where $B_1(x, y, z) = 2z^2$. Therefore it is enough to prove that if $Y \subset GB$, $B \in \bar{Y}$ and $Y \cap (B_0 + \mathbb{R}B_1) = \emptyset$ then $\overline{V_2Y}$ contains either $\{B_0 + tB_1 \mid t \geq 0\}$ or $\{B_0 + tB_1 \mid t \leq 0\}$. But $GB \cap (\mathbb{R}B_0 + \mathbb{R}B_1) \subset (\pm B_0 + \mathbb{R}B_1)$ (because forms from GB have the same discriminant as B). Thus in view of Lemma 3.2.7, it remains to show that

$$\mathbb{R}B_0 + \mathbb{R}B_1 = \{Q \in \Phi \mid V_1Q = Q\}. \tag{1}$$

It is not difficult to prove (1) by direct calculations. These caluclations can be simplified if we notice that the right-hand side in (1) is invariant under D, and hence it is spanned by eigenvectors of D. We omit the details.

3.7. Proof of Theorem 2.3.4 (a). We will use the notation introduced in 3.6.4. Let $G = SL(n, \mathbb{R})$, $\Gamma = SL(n, \mathbb{Z})$ and $\Omega_n = G/\Gamma$. According to Remark 2.3.6 (a), in order to prove Theorem 2.3.4 (a), it is enough to prove (for $n = 3$) the following:

3.7.1. Proposition. *If $x \in \Omega_n$ then either Hx is closed and carries a H-invariant probability measure or there exist a $y \in \overline{Hx}$ such that $\overline{Hx} \supset V_2y$.*

This proposition will be proved in 3.7.3. Before that we need to state Proposition 3.7.2.

3.7.2. Proposition. *Let $x \in \Omega_n$. Suppose that Hx is not closed. Then $\overline{Hx}$ contains a minimal compact V_1-invariant subset Z such that the closure of the set*

$$\{g \in G - HV_2 \mid gz \in \overline{Hx}\}$$

contains e for every $z \in Z$.

This proposition will be proved in 3.7.8 after we prove, in 3.7.4–3.7.7, some auxillary lemmas.

3.7.3. Proof of Proposition 3.7.1. Suppose that Hx is not closed. Let Z be a subset defined in Proposition 3.7.2. Then in view of Lemmas 3.5.3 and 3.6.4, $\overline{Hx}$ contains either V_2^+Z or V_2^-Z (we apply Lemma 3.5.3 to $P = F = V_1$, $Q = H, X = Z$ and $Y = \overline{Hx}$). On the other hand, it easily follows from Theorem 3.3.1 that the closure of any semi-orbit V_2^+x or $V_2^-x, x \in \Omega_n$, contains a V_2-orbit. Hence $\overline{Hx} \supset V_2y$ for some $y \in \Omega_n$. Now it remains to note that, according to Theorem 3.4.3, any closed H-orbit carries a H-invariant probability measure.

3.7.4. Lemma. (a) *There exists a compact subset C of Ω_n such that $DV_1C = \Omega_n$ and moreover for every $x \in \Omega_n, C \cap V_1 d(t)x$ is nonempty for all large t.*

(b) *Let S be a subgroup of G which contains DV_1. Then every closed S-invariant subset of Ω_n contains a minimal closed S-invariant subset.*

Proof. It is easy to show that any proper nonzero V_1-invariant subspace L of $\mathbb{R}^n$ is contained in $\{(0, y, z) \mid y \in \mathbb{R}^{n-2}, z \in \mathbb{R}\}$ and contains $\{(0, 0, z) \mid z \in \mathbb{R}\}$. Hence L is D-invariant and $\det d(t) \mid_L \to \infty$ as $t \to +\infty$. Now the assertion (a) follows from Corollary 3.3.5, and (b) follows from (a) and Zorn's lemma.

3.7.5. Let

$$L = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; C \in SO(Q) \right\} \subset H = SO(B_0)\,.$$

We note that L normalizes V_1 and V_2, and that DLV_1 is a closed normal subgroup of index 2 in $H \cap N_G(V_2)$. Let $F = DLV_1V_2$.

Lemma. *Let $g \in G$ and $\Lambda = F \cap g\Gamma g^{-1}$. Then at least one of the following holds*:

(i) *there exists a one-parameter subgroup $\{u(t)\}$ of V_1 such that $u(t)\Lambda$ tends to infinity in F/Λ as $t \to \infty$;*
(ii) *for every compact subset K of F/Λ, there exists $T = T(K)$ such that $K \cap V_1 d(t)\Lambda$ is empty for all $t > T$.*

Proof. Let us denote the Zariski closure of Λ by Φ. Then $g^{-1}\Phi g$ is the Zariski closure of the subgroup $g^{-1}\Lambda g$ of $\Gamma = SL(n, \mathbb{Z})$. Hence $g^{-1}\Phi g$ is is defined over $\mathbb{Q}$ and has no $\mathbb{Q}$-rational characters with infinite image. So, in view of Chevaley's theorem, there exist $m \in \mathbb{N}^+$, a faithful rational representation $\alpha : G \to GL(m, \mathbb{R})$ and $x_0 \in \mathbb{R}^m$ such that $\Phi = \{g \in G \mid \alpha(g)x_0 = x_0\}$. If $U = \{u(t)\}$ is a one-parameter unipotent subgroup of G then either $\alpha(U)x_0 = x_0$ or $\alpha(u(t))x_0$ tends to infinity in $\mathbb{R}^m$ as $t \to \infty$. This implies that if (i) does not hold then

$$\alpha(V_1)x_0 = x_0, \text{ and hence } V_1 \subset \Phi\,. \tag{2}$$

Thus we can assume that (2) is true. Then

$$\alpha(V_1 d(t)\Lambda)x_0 = \alpha(d(t)V_1\Lambda)x_0 = \alpha(d(t))x_0\,. \tag{3}$$

All eigenvalues of $\alpha(d(t))$ are real. Hence if $\alpha(D)x_0 \neq x_0$ then $\alpha(d(t))x_0$ tends either to infinity or to 0 as $t \to \infty$. This and (3) imply that if (ii) does not hold then $DV_1 \subset \Phi$. But any containing DV_1 closed subgroup of $F = DLV_1V_2$ is not unimodular (because D centralizes DL and $V_1V_2 \subset W = U_{d(-1)}$). Thus Φ and

$g^{-1}\Phi g$ are not unimodular. This contradicts the fact that $g^{-1}\Phi g$ is defined over $\mathbb{Q}$ and has no $\mathbb{Q}$-rational characters with infinite image.

3.7.6. Lemma. *Let Y be a minimal closed DLV_1-invariant subset of Ω_n, and let $y \in Y$. Then the closure of $\{g \in G - F \mid gy \in DLV_1 y\}$ contains e.*

Proof. We write $y = g_0\Gamma$. Then the stabilizer $F_y = F \cap G_y$ coincides with $F \cap g_0\Gamma g^{-1}$. In view of Theorem 3.3.1 and Lemma 3.7.4 (a), the following two assertions hold:

(i) for any one-parameter subgroup $\{u(t)\}$ of $V_1, u(t)y$ does not tend to infinity in Ω_n;

(ii) there exists a compact subset C of Ω_n such that $C \cap V_1 d(t)y$ is nonempty for all $t > 0$.

From these assertions and Lemma 3.7.5 we deduce the existence of a sequence $\{f_n\} \subset DV_1$ such that $\{f_n y\}$ is relatively compact in Ω_n but $\{f_n F_y\}$ is not relatively compact in F/F_y. Now the lemma follows from Lemma 3.5.2.

3.7.7. Lemma. *Let $h_1, h_2, h \in H$ and $t_1, t_2, t \in \mathbb{R}$. If $t_1 \neq 0$ and*

$$h_1 v_2(t_1) h_2 v_2(t_2) = h v_2(t)\,, \tag{4}$$

then $h_2 \in N_G(V_2)$.

Proof. As in the proof of Lemma 3.6.4, let $B_1(x, y, z) = 2z^2$. A direct computation shows that $v_2(s)B_1 = B_0 - sB_1, s \in \mathbb{R}$. But H preserves B_0 and (4) is equivalent to the equality $v_2(t_1)h_2 = h_1^{-1} h v_2(t - t_2)$. Hence

$$\begin{aligned} B_0 - t_1 B_1 = v_2(t_1) h_2 B_0 &= h_1^{-1} h v_2(t - t_2) B_0 \\ &= h_1^{-1} h (B_0 - (t - t_2) B_1) = B_0 - (t - t_2) h_1^{-1} h B_1\,. \end{aligned} \tag{5}$$

Thus $t_1 B_1 = (t - t_2) h_1^{-1} h B_1$. On the other hand, it is easy to check that if $h_0 \in H$ and $h_0 B_1$ is a multiple of B_1 then $h_0 \in N_G(V_2)$. Therefore $h_2^{-1} h \in N_G(V_2)$. It implies that

$$h_2 = v_2(-t_1) h_1^{-1} h v_2(t - t_2) \in V_2 N_G(V_2) V_2 = N_G(V_2)\,.$$

3.7.8. Proof of Proposition 3.7.2. Let us denote by X the set of $y \in \overline{Hx}$ such that the closure of the set $\{g \in G - H \mid gy \in \overline{Hx}\}$ contains e. Since Hx is not closed, $X \neq \emptyset$ (see Lemma 2.2.7 (a)). Then by Lemma 3.7.4 X contains a minimal closed DLV_1-invariant subset Y, and by Corollary 3.3.7 and Theorem 3.3.8 Y contains a minimal compact V_1-invariant subset Z. Take $z \in Z \subset X$ and assume that the closure of $\{g \in G - HV_2 \mid gz \in \overline{Hx}\}$ does not contain e. Then there exists a sequence $\{g_n\} \subset HV_2 - H$ such that $g_n z \in \overline{Hx}$ and $g_n \to e$ as $n \to \infty$. In view of Lemma 3.7.6, there exist $a_n \in DLV_1$ and $s_n \in G - F$ such that

$$s_n z = a_n z \text{ and } s_n \to e \text{ as } n \to \infty\,.$$

Replacing $\{g_n\}$ and $\{s_n\}$ by subsequences if necessary, we can assume that $a_n g_n a_n^{-1} \to e$ as $n \to \infty$ and that $s_n \in HV_2$. Since $g_n \in HV_2 - H$, $a_n \in DLV_1 \subset H$, and DLV_1 normalizes V_2, we have that $a_n g_n a_n^{-1} \in HV_2 - H$. But $s_n \in HV_2 - N_G(V_2)$ for all large n (because DLV_1 is open in $H \cap N_G(V_2)$, and therefore $F = DLV_1V_2$ is open in $HV_2 \cap N_G(V_2)$). Hence by Lemma 3.7.7 $a_n g_n a_n^{-1} s_n \notin HV_2$ for all large n. On the other hand $a_n g_n a_n^{-1} s_n z = a_n g_n z \in \overline{Hx}$ and $a_n g_n a_n^{-1} s_n \to e$ as $n \to \infty$. We have a contradiction with our previous assumption that the closure of $\{g \in G - HV_2 \mid gz \in \overline{Hz}\}$ does not contain e.

3.8. Conjectures of Dani and Raghunathan. Uniform distribution. In 1936 Hedlund [He] proved that if $G = SL(2, \mathbb{R})$, Γ is a lattice in G, and U is a unipotent one-parameter subgroup of G, then every U-orbit on G/Γ is either periodic or dense in G/Γ. In particular, if G/Γ is compact then all U-orbits are dense. In 1972 Furstenberg [Fu] proved that for the same G, Γ and U, the action of U on the compact space G/Γ is uniquely ergodic, i.e. the G-invariant probability measure is the only U-invariant probability measure on G/Γ.

Furstenberg's theorem was generalized by Bowen [Bow], Veech [Vee] and Ellis and Perrizo [EP]. In particular they showed that if G is a connected simple Lie group and Γ is a uniform lattice on G (that is, Γ is a lattice such that G/Γ is compact), then the action on G/Γ by left translations of any nontrivial horosperical subgroup U of G is uniquely ergodic. (By the *horospherical subgroup* corresponding to an element g of a Lie group F we mean the subgroup

$$U_g = \{u \in F \mid g^i u g^{-i} \to e \text{ as } j \to +\infty\}.$$

It is well known that U_g is a connected closed Ad-unipotent subgroup and g normalizes U_g. Any nontrivial connected unipotent subgroup of $SL(2, \mathbb{R})$ is horospherical.)

In a 1981 paper [Dan3] Dani formulated two conjectures. One is Raghunathan's conjecture which we mentioned in 2.3. The second conjecture is due to Dani himself and may be stated as follows (in a slightly stronger form than in [Dan3]). If G is a connected Lie group, Γ is a lattice in G, U is an Ad-unipotent subgroup of G, and μ is a finite Borel U-ergodic U-invariant measure on G/Γ, then there exists a closed subgroup F of G such that μ is F-invariant and $\operatorname{supp}\mu = Fx$ for some $x \in G/\Gamma$ (a measure for which this condition holds is called *algebraic*). In the same paper [Dan3] Dani proved his conjecture when G is reductive and U is a maximal horospherical subgroup of G. In another paper [Dan7], he proved Raghunathan's conjecture in the case when G is reductive and U is an arbitrary horospherical subgroup of G. Starkov [Sta1,2] proved Raghunathan's conjecture for solvable G. We remark that the proof given in [Dan7] is restricted to horospherical U and the proof given in [Sta1,2] cannot be applied for nonsolvable G.

The first result on Raghunathan's conjecture for nonhorospherical subgroups of semisimple groups was obtained by Dani and the author in [DanM2]. We proved the conjecture in the case when $G = SL(3, \mathbb{R})$ and $U = \{u(t)\}$ is a one-parameter unipotent subgroup of G such that $u(t) - 1$ has rank 2 for all $t \neq 0$. Though

this is only a very special case, the proof in [DanM2] together with the methods developed in [Marg5] and [DanM1] suggests an approach for proving the Raghunathan conjecture in general (cf. [Sha3]). This approach is based on the technique which involves (as in [Marg5] and [Dan1]) finding orbits of larger subgroups inside closed sets univariant under unipotent subgroups by studying the minimal invariant sets, and the limits of orbits of sequences of points tending to a minimal invariant set.

Let G be a Lie group, Γ a lattice in G, and $\{u_t \mid t \in \mathbb{R}\}$ a one-parameter subgroup of G. We say that the orbit $\{u_t x\}$ of $x \in G/\Gamma$ is *uniformly distributed with respect to a probability measure μ on G/Γ* if for any bounded continuous function f on G/Γ

$$\frac{1}{T}\int_0^T f(u_t x)dt \to \int_{G/\Gamma} f d\mu \text{ as } T \to \infty\,.$$

The above-mentioned result of Furstenberg implies that if $G = SL(2,\mathbb{R})$, Γ is a uniform lattice in G and $\{u_t\}$ is a one-parameter unipotent subgroup of G then every orbit of $\{u_t\}$ on G/Γ is uniformly distributed with respect to the G-invariant probability measure. For nonuniform lattices Γ in $G = SL(2,\mathbb{R})$ Dani and Smillie [DanS] proved that every nonperiodic orbit of $\{u_t\}$ on G/Γ is uniformly distributed with respect to the G-invariant probability measure. For $\Gamma = SL(2,\mathbb{Z})$ this result had been proved earlier in [Dan4]. In the paper [Dan4], Dani also stated another conjecture that, for any one-parameter unipotent subgroup $\{u_t\}$ of $SL(n,\mathbb{R})$ and any point $x \in G/\Gamma$, the orbit $\{u_t x\}$ is uniformly distributed with respect to a probability measure μ_x on $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$. This conjecture was reproduced in [Marg4] where it was stated for an arbitrary lattice in connected Lie group. For nilpotent Lie groups it was proved by Lesigne [Les1,2].

3.9. Ratner's results on unipotent flows. Conjectures of Dani and Raghunathan were eventually proved in full generality by Ratner. In fact, she proved the results for a larger class of actions. Let us first state Ratner's measure classification theorem for actions of connected Lie subgroups H. This theorem proved to be an important and useful tool for the study of (unipotent) flows on homogeneous spaces.

3.9.1. Theorem [Rat2–4]. *Let G be a connected Lie group and Γ a discrete subgroup of G (not necessarily a lattice). Let H be a Lie subgroup of G that is generated by the Ad-unipotent one-parameter subgroups contained in it. Then any finite H-ergodic H-invariant measure μ on G/Γ is algebraic.*

3.9.2. Remarks. (a) It immediately follows from the result of Moore on Mautner property (mentioned in 3.4.2) that μ is U-ergodic for any maximal unipotent subgroup U of H. Therefore it is enough to prove Theorem 3.9.1 for unipotent H.

(b) Some ingredients of Ratner's original proof of Theorem 3.9.1 are described for $SL(2,\mathbb{R})$ in [Rat6], and in the general case it is sketched in [Rat8]. We refer the reader also to a shorter proof given by Tomanov and the author (see [MargT1,2] for

the crucial case where G is algebraic and [MargT3] for a simple reduction to that case). The proof in [MargT2] is mostly based on some versions of Proposition 3.6.3. This proof bears a strong influence of Ratner's arguments but is substantially different in approach and methods.

3.9.3. Using Theorem 3.9.1 together with Theorem 3.3.2 and a simple result about the countability of a certain (depending on Γ) set of subgroups of G, Ratner proves in [Rat5] the following uniform distribution theorem which can be considered as the quantitative strengthening of Raghunathan's conjecture for one-parameter unipotent subgroups. (Ratner also proves in [Rat5] an analogous statement for cyclic unipotent subgroups.)

Theorem. *If G is a connected Lie group, Γ is a lattice in G, $\{u_t\}$ is a one-parameter Ad-unipotent subgroup of G and $x \in G/\Gamma$, then the orbit $\{u_t x\}$ is uniformly distributed with respect to an algebraic probability measure μ_x on G/Γ.*

This theorem proves the above-mentioned conjecture from [Dan4] and [Marg4]. An alternative approach to the proof of the uniform distribution theorem for the case of a "regular unipotent subgroup", also using Theorems 3.9.1 and 3.3.2, is given in [Sha1]. In another paper [Sha2] Shah proved an analogue of the uniform distribution theorem for multidimensional unipotent subgroups.

Let us recall that a refined version of Ratner's uniform distribution theorem is stated in 2.4.9. This version is due to Dani and the author. The proof which we give in [DanM5] uses, as in [Rat5] and [Sha2], Theorems 3.9.1 and 3.3.2. The reduction to these theorems is essentially based on a variation of the following assertion: Let H be a connected closed subgroup of G such that $H \cap \Gamma$ is a lattice in G, and let $X(H,U) = \{g \in G \mid Ug \subset gH\}$. Then the subset $X(H,U)\Gamma$ of G/Γ is avoidable with respect to $\{u_t\}$ (in the sense of 3.2.2). In the proof of (the variation of) this assertion we use Proposition 3.2.2.

3.9.4. The uniform distribution theorem, in combination with the countability result mentioned in 3.9.3, rather easily implies the following theorem. This theorem proves a generalized version of Raghunathan's conjecture mentioned in 2.3.

Theorem [Rat5]**.** *Let G and H be as in Theorem 3.9.1. Let Γ be a lattice in H. Then for any $x \in G/\Gamma$, there exists a closed connected subgroup $L = L(x)$ containing H such that $\overline{Hx} = Lx$ and there is an L-invariant probability measure supported on Lx.*

3.10. Remarks. (a) Let G be a connected semisimple Li group without compact factors, Γ an irreducible lattice in G, and g a semisimple element of G such that the horospherical subgroup $U = U_g$ corresponding to g is nontrivial. We are given $f \in L^2(U)$ with compact support and a uniformly continuous $\psi \in L^2(G/\Gamma)$. Then

according to [KM, Proposition 2.2.1], for any compact subset L of G/Γ and any $\varepsilon > 0$ there exists $T > 0$ such that

$$(*) \qquad |\int_U f(u)\psi(g^{-n}ux)dm(u) - \int_U f dm \int_{G/\Gamma} \psi d\mu| \leq \varepsilon$$

for all $x \in L$ and $t \geq T$, where m is Haar measure on U and μ is the G-invariant probability measure on G/Γ. Using this assertion in combination with Theorem 3.3.6, it is not very difficult to prove for horospherical subgroups of semisimple Lie groups the conjectures of Dani and Raghunathan and a suitable version of the uniform distribution theorem. This is one of the reasons why the horospherical subgroup case is much simpler than the general case. The proof of (*) uses mixing properties of the action of g on G/Γ and the fact that any neighborhood V of e in G contains another neighborhood W such that $g^{-n}WUg^n \subset VU$ for all positive n (so called "banana argument"). Let us note that similar methods were used in [Bow], [EMc], [EP], [Marc], [Marg1] and [Rat 2] (chronologically the first reference in this list is [Marg 1] where an analog of (*) for Anosov flows was proved using a version of the above argument).

(b) Recently Raghunathan informed the author that his interest in orbit closures was triggered by some early papers of J. S. Dani and S. G. Dani (in particular [DanJ] and [Dan1]). After S. G. Dani proved some of his results about actions of horospherical subgroups, Raghunathan began wondering about arbitrary unipotent subgroups. Finally, Dani's theorems on horospherical flows, the result from [Marg2] that unipotent one-parameter subgroups cannot have non-compact orbits in the space of lattices (Theorem 3.3.1 of the present paper) and some observations about the Oppenheim conjecture led Raghunathan to his conjecture. His hope was that one could use a downward induction on dimension starting with a horosphere (in some sense, this is the strategy in [DanM2]). He also hoped that unipotent flows are likely to have "manageable behavior" because of the slow divergence of orbits of unipotent one-parameter subgroups (in contrast to the exponential divergence of orbits of diagonalizable subgroups).

(c) Analogs of most of the above results from 2.3, 2.4.1 and 3.2–3.9 hold in p-adic and S-arithmetic cases (see [Bo2], [BoP], [MargT1–3], [Rat7–10]).

(d) We refer to surveys [Bo2], [Dan9], [Dan10], [Marg8] and [Rat10] for further information about the topics which we considered in Secs. 2 and 3.

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THE WORK OF ALAIN CONNES

by
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The theory of operator algebras, after being quietly nourished in somewhat isolation for 30 years or so, started a revolutionary development around the late 1960's. Alain Connes came into this field just when the smokes of the first stage of the revolution were settling down. He immediately led the field to breathtaking achievements beyond the expectation of experts.

His most remarkable contributions are: (1) general classification and a structure theorem for factors of type III, obtained in his thesis [12], (2) classification of automorphisms of the hyperfinite factor [29], which served as a preparation for the next contribution, (3) classification of injective factors [31], and (4) applications of the theory of C^*-algebras to foliations and differential geometry in general [44, 50] — a subject currently attracting a lot of attention.

In this report, I shall mostly concentrate on the first three aspects which form a well-established and most spectacular part of the theory of von Neumann algebras.

1. Classification of Type III Factors

In the latter half of 1930's, Murray and von Neumann initiated the study of what are now called von Neumann algebras (i.e. weakly closed *-sub-algebra of the *-algebra $L(H)$ of all bounded linear operators on a Hilbert space H) and classified the factors (i.e. von Neumann algebras with trivial centers) into the types I_n, $n = 1, 2, \ldots$, and I_∞ (isomorphic to $L(H)$ with $\dim H = n$ and ∞), II_1, II_∞ and III. (In the following we restrict our attention to von Neumann algebras M on separable Hilbert space H.) Only three type III factors (and only three type II_1 factors) had been known to be mutually non-isomorphic till 1967, when Powers showed the existence of a continuous family of mutually non-isomorphic type III factors.

Traces provided a tool for a systematic analysis of type II factors at an earlier stage, while the non-existence of traces made type III factors remain untractable till the late 1960's, when the Tomita–Takesaki theory was created and furnished a powerful tool for type III. To introduce notation, let M be a von Neumann algebra on a separable Hilbert space H and let $\Psi \in H$ be cyclic (i.e. $M\Psi$ be dense in H) and separating (i.e. such that $x\Psi = 0$ for $x \in M$ implies $x = 0$). The conjugate linear

operator $S_\Psi x\Psi \equiv x^*\Psi$, $x \in M$, has a closure $\bar{S}_\Psi$ and defines the positive selfadjoint operator $\Delta_\Psi \equiv S_\Psi^* \bar{S}_\Psi$, called the *modular operator*. The Tomita–Takesaki theory says that $x \in M$ implies $\sigma_t^\varphi(x) \equiv \Delta_\Psi^{it} x \Delta_\Psi^{-it} \in M$. The one-parameter group of (*-)automorphisms σ_t^ψ of M depends only on the positive linear functional $\psi(x) \equiv (x\Psi, \Psi)$ and is called the *group of modular automorphisms*.

Connes [12] has shown that the modular automorphisms for different ψ's are mutually related by inner automorphisms (in other words, the independence of σ_t^Ψ from ψ in the quotient Out M of the group Aut M of all automorphisms modulo the subgroup Int M of all inner automorphisms) and introduced the following two isomorphism invariants for M:

$$S(M) = \bigcap_\Psi \mathrm{Sp}\Delta_\Psi \quad (\mathrm{Sp}\ \text{denotes the spectrum}),$$

$$T(M) = \{t \in \boldsymbol{R} : \sigma_t^\psi \in \mathrm{Int}\boldsymbol{M}\}.$$

(In a more general case, $S(M) = \cap\, \mathrm{Sp}\Delta_\psi$, where the intersection is taken over faithful normal semifinite weights ψ.) It turns out that $S(M)\backslash\{0\}$ is a closed multiplicative subgroup of $\boldsymbol{R}_+^*$, and this leads to the classification of type III factors into the types III_λ, $0 \le \lambda \le 1$:

$$S(M) = \{\lambda^n : n \in \boldsymbol{Z}\} \cup \{0\} \quad \text{if } 0 < \lambda < 1,$$

$$S(M) = \boldsymbol{R}_+^* \quad \text{if } \lambda = 1, \qquad S(M) = \{0, 1\} \quad \text{if } \lambda = 0.$$

The Powers examples R_λ, $0 < \lambda < 1$, due to Powers, are of types III_λ and hence mutually non-isomorphic. The two invariants $r_\infty(M)$ and $\varrho(M)$ of Araki and Woods, introduced for a systematic classification of infinite tensor products of type I factors (including R_λ), are shown to be equivalent to $S(M)$ and $T(M)$ for them [7].

2. Structure Analysis of Type III Factors

Connes [12] went on and succeeded in analysis of the structures of type III_λ factors M, $0 \le \lambda < 1$, in terms of a type II von Neumann algebra N (with a non-trivial center) and an automorphism θ of N, such that M is the so-called *crossed product* $N \times_\theta \boldsymbol{Z}$ of N by θ.

For $0 < \lambda < 1$, θ should scale a trace τ of a type II_∞ factor N in the sense that $\tau(\theta x) = \lambda\tau(x)$ for all $x \in N_+$ and $N_i \times_{\theta_i} \boldsymbol{Z}$, $i = 1, 2$, are isomorphic if and only if there exists an isomorphism π of N_1 onto N_2 such that $\pi^{-1}\theta_2\pi\theta_1^{-1}$ is inner, or equivalently (in view of a later result of Connes and Takesaki), $\pi\theta_1\pi^{-1} = \theta_2$. This means that the pair (N, θ) is uniquely determined by M and the classification of M is reduced to that of the pair (N, θ).

For $\lambda = 0$, θ should scale down a trace τ of a type II_∞ von Neumann algebra N in the sense that $\tau(\theta x) \le \varrho\tau(x)$ for all $x \in N_+$ for some $\varrho < 1$ and, again, there is a somewhat more complicated uniqueness result for the pair (N, θ).

Motivated by the above results of Connes, a general structure theorem including the type III_1 has been obtained by Takesaki in terms of a one-parameter group θ_t of trace-scaling automorphisms of a type II_∞ von Neumann algebra N.

In the process of developing the above classification and structure theory, Connes introduced two important technical tools, namely the unitary Radon–Nikodym cocycle (equivalently, relative modular operators) useful in application to quantum statistical mechanics, non-commutative L_p theory, etc., and the Connes spectrum useful in the analysis of C^* dynamical systems.

3. Classification of Automorphisms of the Hyperfinite Factor

A von Neumann algebra, containing an ascending sequence of finite-dimensional subalgebras with a dense union, is called *approximately finite-dimensional* (AFD). AFD factors of type II_1, as shown by Murray and von Neumann, are all isomorphic to what is called the *hyperfinite factor*, denoted by R in the following. Connes [29] has given a complete classification of automorphisms of R modulo inner automorphisms (i.e. the conjugacy class of Out R). Namely, a complete set of isomorphism invariants in Out R for an $\alpha \in \mathrm{Aut}\, R$ is given by the pair of the outer period $p\ (= 2, 3, \ldots)$, which is the smallest $p > 0$ such that α^p is inner, defined to be 0 for outer aperiodic α, and the obstruction γ which is the p-th root of 1 (1 for $p = 0$) such that $\alpha^p = \mathrm{Ad}U$, $\alpha(U) = \gamma U$, where $(\mathrm{Ad}U)(x) = UxU^*$. Although the result is about a specific R, this factor R is in the bottom of all known AFD factors and the result that outer aperiodic automorphisms of R are all conjugate up to inner automorphisms is essential for the results described in the next section.

As a by-product, Connes [23] solved negatively one of the old problems on von Neumann algebras by exhibiting, for each $0 < \lambda < 1$, factors of type III_λ, not anti-isomorphic to themselves.

4. Classification of AFD Factors

A complete classification of AFD factors of type III_λ, $\lambda \neq 1$, is what I consider the most distinguished work of Connes. It turns out that an AFD factor of type III_λ is unique and is isomorphic to R_λ for each $0 < \lambda < 1$, while AFD factors of type III_0 are isomorphic to Krieger's factors associated with single non-singular ergodic transformations of the Lebesgue measure space, their isomorphism classes being in one-to-one correspondence with the metric equivalence classes of non-singular non-transitive ergodic flows on the Lebesgue measure space.

One of the most important technical ingredients of the proof is the equivalence of various concepts about a Neumann algebra which arose over years in the theory of von Neumann algebras. Murray and von Neumann found a factor N of type II_1 non-isomorphic to R (distinguished by Property Γ). In 1962, Schwartz distinguished N, R and $N \otimes R$ by the following property P: A von Neumann algebra M on a Hilbert space H has the property P iff for any $T \in L(H)$, the norm closed convex

hull of the uTu^* with u varying over all unitary operators in M intersects M'. Any AFD factors posseses Property P.

The Property P for M implies the existence of a projection of norm 1 from $L(H)$ to M'. This is the Hakeda–Tomiyama extension property for M', called Property E by Connes, and is stable under taking the intersection of a decreasing family, the weak closure of the union of an ascending family, the commutant, tensor products and crossed product by an amenable group. Thus the Property P implies Property E for M.

Any projection E of norm 1 from a C^*-algebra $\mathfrak{A}_1$ to its subalgebra $\mathfrak{A}_2$ is shown by Tomiyama to be a completely positive map satisfying the property of the conditional expectation: $E(axb) = aE(x)b$ for any $a, b \in \mathfrak{A}_2$ and $x \in \mathfrak{A}_1$. A C^*-algebra $\mathfrak{A}$ with unit is called *injective* if any completely positive unit preserving linear map θ from $\mathfrak{A}$ into another C^*-algebra $\mathfrak{B}$ with unit has an extension $\bar{\theta}$ to any C^*-algebra $\mathfrak{A}_1$ containing $\mathfrak{A}$ as a completely positive unit preserving linear map from $\mathfrak{A}_1$ into $\mathfrak{B}$. A von Neumann algebra M is injective if and only if it has Property E.

Effros and Lance called a von Neumann algebra M *semidiscrete* if the identity map from M into M is a weak pointwise limit of completely positive maps of finite rank and proved that M is injective if it is semidiscrete.

Connes [31] unified all these concepts by showing that they are all equivalent. The core result is the isomorphism of all injective factors of type II_1 to the unique hyperfinite factor R; it is established by a highly involved and technical proof, utilizing a theorem on tensor products of C^*-algebras, the property Γ of a factor which Murray and von Neumann introduced to distinguish some factors, properties of Aut N and Int N of a factor N, the ultra product R^ω for a free ultrafilter ω, an argument analogous to Day–Namioka proof of Fϕlner's characterization of amenable groups, etc.

The uniqueness of injective factors of type II_1 then implies the uniqueness of injective factors of type II_∞. Together with an earlier uniqueness result for trace-scaling automorphisms of $R \otimes B(H)$ (exhibiting the unique injective factor of type II_∞), it also implies the uniquencess of injective factors of type III_λ, $0 < \lambda < 1$. With the help of an earlier result of Krieger, injective factor of type III_0 are also completely classified by the isomorphism class of the so-called flow of weight. Thus Connes succeeded in a complete classification of AFD factors (which is as much as saying injective factors) except for the case of type III_1, which still remains open.

The work of Connes also shows that any continuous representation of a separable locally compact group G generates an injective von Neumann algebra if G/G_0 is amenable, where G_0 is the connected component of the identity (in particular, if G is connected or amenable).

5. Other Works

After his success in the almost complete classification of injective factors, Connes turned his attention to application of operator algebras to differential geometry.

Connes developed a non-commutative integration theory, which provides a method of integration over a family of ergodic orbits or over the set of leaves of a foliation. One significant outcome of this theory is an index theorem of foliation. I am sure that this subject will rapidly develop much further. For a survey of the present status, we refer to [44], [50].

The works on positive cones [13] provide a geometric characterization of von Neumann algebras through the associated natural positive cone in the Hilbert space and lead to some applications.

A work connected with Kazdan's property T [42] provides a simple example of continuously many non-isomorphic factors of type II, and answers a question of Murray and von Neumann about the fundamental group of a factor of type II_1.

I hope that I have conveyed to you some feeling about the incredible power of Alain Connes and the richness of his contributions.

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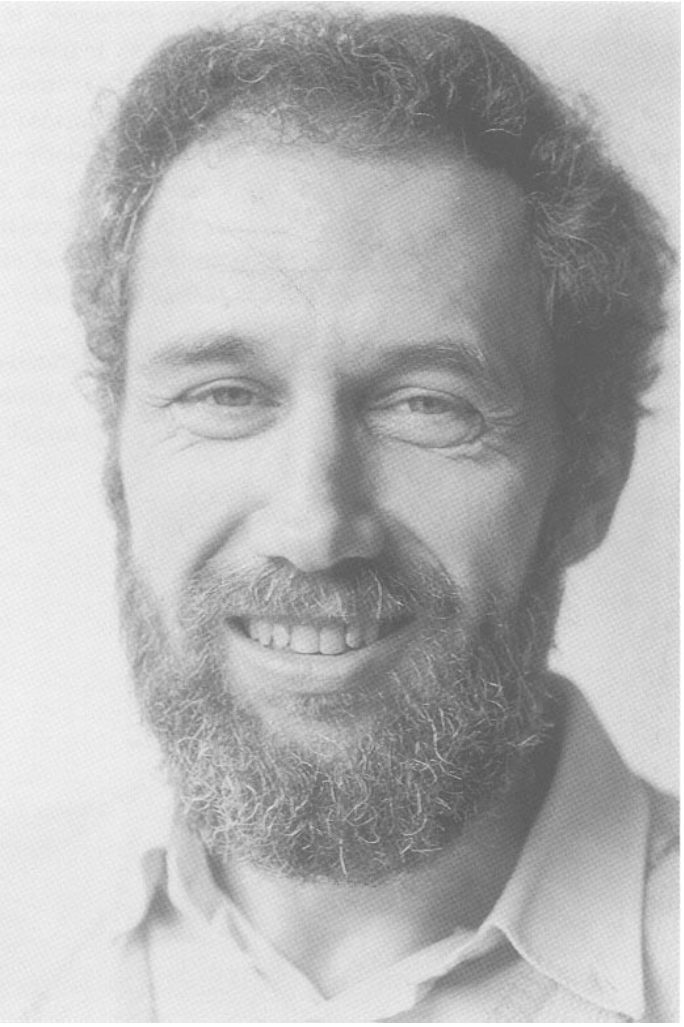

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July 8, 1996

BRISURE DE SYMÉTRIE SPONTANÉE ET GÉOMÉTRIE DU POINT DE VUE SPECTRAL

par
ALAIN CONNES

1. Généralités

La géométrie de Riemann admet pour données préalables une *variété* M dont les points $x \in M$ sont localement paramétrés par un nombre fini de coordonnées réelles x^μ, et la *métrique* donnée par l'élément de longueur infinitésimal

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu\,. \tag{1}$$

La distance entre deux points $x, y \in M$ est donnée par

$$d(x,y) = \text{Inf Longueur } \gamma\,, \tag{2}$$

où γ varie parmi les arcs joignant x à y, et

$$\text{Longueur } \gamma = \int_x^y ds\,. \tag{3}$$

La théorie de Riemann est à la fois assez souple pour fournir (au prix d'un changement de signe) un bon modèle de l'espace temps de la relativité générale et assez restrictive pour mériter le nom de géométrie. Le point essentiel est que le calcul différentiel et intégral permet de passer du local au global et que les notions simples de la géométrie euclidienne telle celle de droite continuent à garder un sens. L'équation des géodésiques:

$$\frac{d^2\, x^\mu}{dt^2} = -\Gamma^\mu_{\nu\rho}\, \frac{dx^\nu}{dt}\, \frac{dx^\rho}{dt} \tag{4}$$

(où $\Gamma^\mu_{\nu\rho} = \frac{1}{2}\, g^{\mu\alpha}(g_{\alpha\nu,\rho} + g_{\alpha\rho,\nu} - g_{\nu\rho,\alpha})$) pour la métrique $dx^2 + dy^2 + dz^2 - (1 - 2V(x,y,z))dt^2$ donne l'équation de Newton dans le potentiel V (cf. [W] pour un énoncé plus précis). Les données expérimentales récentes sur les pulsars binaires confirment [DT] la relativité générale et l'adéquation de la géométrie de Riemann comme modèle de l'espace temps à des échelles suffisamment grandes. La question ([R]) de l'adéquation de cette géométrie comme modèle de l'espace temps à très courte échelle est controversée mais la longueur de Planck

$$\ell_p = (G\hbar/c^3)^{1/2} \sim 10^{-33}\text{cm} \tag{5}$$

est considérée comme la limite naturelle sur la détermination précise des coordonnées d'espace temps d'un évènement. (Voir par exemple [F] ou [DFR] pour l'argument physique, utilisant la mécanique quantique, qui établit cette limite.)

Dans cet exposé nous présentons une nouvelle notion d'espace géométrique qui en abandonnant le rôle central joué par les *points* de l'espace permet une plus grande liberté dans la description de l'espace temps à courte échelle. Le cadre proposé est suffisamment général pour traiter les espaces discrets, les espaces riemanniens, les espaces de configurations de la théorie quantique des champs et les duaux des groupes discrets non nécessairement commutatifs. Le problème principal est d'adapter à ce cadre général les notions essentielles de la géométrie et en particulier le calcul infinitésimal. Le formalisme opératoriel de la mécanique quantique joint à l'analyse des divergences logarithmiques de la trace des opérateurs donnent la généralisation cherchée du calcul différentiel et intégral (Section II). Nous donnons quelques applications directes de ce calcul (Théorèmes 1, 2, 4).

La donnée d'un espace géométrique est celle d'un *triplet spectral*:

$$(\mathcal{A}, \mathcal{H}, D) \tag{6}$$

où $\mathcal{A}$ est une algèbre involutive d'opérateurs dans l'espace de Hilbert $\mathcal{H}$ et D un opérateur autoadjoint non borné dans $\mathcal{H}$. L'algèbre involutive $\mathcal{A}$ correspond à la donnée de l'espace M selon la dualité Espace $\leftrightarrow$ Algèbre classique en géométrie algébrique. L'opérateur $D^{-1} = ds$ correspond à l'élément de longueur infinitésimal de la géométrie de Riemann.

Il y a deux différences évidentes entre cette *géométrie spectrale* et la géométrie riemannienne. La première est que nous ne supposerons pas en général la commutativité de l'algèbre $\mathcal{A}$. La deuxième est que ds, étant un opérateur, ne commute pas avec les éléments de $\mathcal{A}$, même quand $\mathcal{A}$ est commutative.

Comme nous le verrons, des relations de commutation très simples entre ds et l'algèbre $\mathcal{A}$, jointes à la dualité de Poincaré caractérisent les triplets spectraux (6) qui proviennent de variétés riemanniennes (Théorème 6). Quand l'algèbre $\mathcal{A}$ est commutative sa fermeture normique dans $\mathcal{H}$ est l'algèbre des fonctions continues sur un espace compact M. Un point de M est un caractère de $\bar{\mathcal{A}}$, i.e. un homomorphisme de $\bar{\mathcal{A}}$ dans $\mathbb{C}$

$$\begin{gathered}\chi : \bar{\mathcal{A}} \to \mathbb{C}\ ,\ \chi(a+b) = \chi(a) + \chi(b)\ ,\ \chi(\lambda a) = \lambda\,\chi(a)\ ,\ \chi(ab) = \chi(a)\,\chi(b)\,, \\ \forall\, a, b \in \bar{\mathcal{A}}\ ,\ \forall\, \lambda \in \mathbb{C}\,.\end{gathered} \tag{7}$$

Soit par exemple $\mathcal{A}$ l'algèbre $\mathbb{C}\Gamma$ d'un groupe discret Γ agissant dans l'espace de Hilbert $\mathcal{H} = \ell^2(\Gamma)$ de la représentation régulière (gauche) de Γ. Quand le groupe Γ et donc l'algèbre $\mathcal{A}$ sont commutatifs les caractères de $\bar{\mathcal{A}}$ sont les éléments du dual de Pontrjagin de Γ

$$\hat{\Gamma} = \{\chi : \Gamma \to U(1)\ ;\ \chi(g_1\, g_2) = \chi(g_1)\,\chi(g_2) \qquad \forall\, g_1, g_2 \in \Gamma\}\,. \tag{8}$$

Les notions élémentaires de la géométrie différentielle pour l'espace $\hat{\Gamma}$ continuent à garder un sens dans le cas général où Γ n'est plus commutatif grâce au dictionnaire suivant dont la colonne de droite n'utilise pas la commutativité de l'algèbre $\mathcal{A}$:

Espace X	Algèbre $\mathcal{A}$
Fibré vectoriel	Module projectif de type fini
Forme différentielle de degré k	Cycle de Hochschild de dimension k
Courant de de Rham de dimension k	Cocycle de Hochschild de dimension k
Homologie de de Rham	Cohomologie cyclique de $\mathcal{A}$

L'intérêt de la généralisation ci-dessus au cas non commutatif est illustré par exemple par la preuve de la conjecture de Novikov pour les groupes Γ qui sont hyperboliques [CM1].

Dans le cas général la notion de point, donnée par (7) est de peu d'intérêt, par contre celle de mesure de probabilité garde tout son sens. Une telle mesure φ est une forme linéaire positive sur $\mathcal{A}$ telle que $\varphi(1) = 1$

$$\varphi : \bar{\mathcal{A}} \to \mathbb{C}\ ,\ \varphi(a^*a) \geq 0\,, \quad \forall\, a \in \bar{\mathcal{A}}\ ,\ \varphi(1) = 1\,. \tag{9}$$

Au lieu de mesurer les distances entre les points de l'espace par la formule (2) nous mesurons les distances entre états φ, ψ sur $\bar{\mathcal{A}}$ par une formule duale qui implique un *sup* au lieu d'un *inf* et n'utilise pas les arcs tracés dans l'espace

$$d(\varphi, \psi) = \operatorname{Sup}\{|\varphi(a) - \psi(a)|\,;\ a \in \mathcal{A}\ ,\ \|[D, a]\| \leq 1\}\,. \tag{10}$$

Vérifions que cette formule redonne la distance géodésique dans le cas riemannien. Soit M une variété riemannienne compacte munie d'une K-orientation, i.e. d'une structure spinorielle. Le triplet spectral $(\mathcal{A}, \mathcal{H}, D)$ associé est donné par la représentation

$$(f\,\xi)(x) = f(x)\,\xi(x) \qquad \forall\, x \in M\ ,\ f \in \mathcal{A}\ ,\ \xi \in \mathcal{H} \tag{11}$$

de l'algèbre des fonctions sur M dans l'espace de Hilbert

$$\mathcal{H} = L^2(M, S) \tag{12}$$

des sections de carré intégrable du fibré des spineurs.

L'opérateur D est l'opérateur de Dirac (cf. [L-M]). On vérifie immédiatement que le commutateur $[D, f]$, $f \in \mathcal{A}$ est l'opérateur de multiplication de Clifford par le gradient ∇f de f et que sa norme hilbertienne est:

$$\|[D, f]\| = \operatorname*{Sup}_{x \in M} \|\nabla f\| = \text{Norme lipschitzienne de } f\,. \tag{13}$$

Soient $x, y \in M$ et φ, ψ les caractères correspondants: $\varphi(f) = f(x)$, $\psi(f) = f(y)$. $\forall\, f \in \mathcal{A}$ la formule (10) donne le même résultat que la formule (2), i.e. donne la distance géodésique entre x et y.

Contrairement à (2) la formule duale (10) garde un sens en général et en particulier pour les espaces discrets ou totalement discontinus.

La notion usuelle de *dimension* d'un espace est remplacée par un *spectre de dimension* qui est un sous-ensemble de $\mathbb{C}$ dont la partie réelle est bornée supérieurement par $\alpha > 0$ si

$$\lambda_n^{-1} = O(n^{-\alpha}) \tag{14}$$

où λ_n est la n-ième valeur propre de $|D|$.

La relation entre le local et le global est donnée par la formule locale de l'indice (Théorème 4) ([CM2]).

La propriété caractéristique des *variétés différentiables* qui est transposée au cas non commutatif est la *dualité de Poincaré*. La dualité de Poincaré en homologie ordinaire est insuffisante pour caractériser le type d'homotopie des variétés ([Mi-S]) mais les résultats de D. Sullivan ([S2]) montrent (dans le cas PL, simplement connexe, de dimension ≥ 5 et en ignorant le nombre premier 2) qu'il suffit de remplacer l'homologie ordinaire par la KO-homologie.

De plus la K-homologie admet grâce aux résultats de Brown Douglas Fillmore, Atiyah et Kasparov une traduction algébrique très simple, donnée par

Espace X	Algèbre $\mathcal{A}$
$K_1(X)$	Classe d'homotopie stable de triplet spectral $(\mathcal{A}, \mathcal{H}, D)$
$K_0(X)$	Classe d'homotopie stable de triplet spectral $\mathbb{Z}/2$ gradué

(i.e. pour K_0 on suppose que $\mathcal{H}$ est $\mathbb{Z}/2$ gradué par γ, $\gamma = \gamma^*$, $\gamma^2 = 1$ et que $\gamma a = a\gamma \quad \forall\, a \in \mathcal{A}$, $\gamma D = -D\gamma$).

Cette description suffit pour la K-homologie complexe qui est périodique de période 2.

Dans le cas non commutatif la *classe fondamentale* d'un espace est une classe μ de KR-homologie pour l'algèbre $\mathcal{A} \otimes \mathcal{A}^0$ munie de l'involution

$$\tau(x \otimes y^0) = y^* \otimes (x^*)^0 \qquad \forall\, x, y \in \mathcal{A} \tag{15}$$

où $\mathcal{A}^0$ désigne l'algèbre opposée de $\mathcal{A}$. Le produit intersection de Kasparov [K] permet de formuler la dualité de Poincaré par l'invertibilité de μ. La KR-homologie est périodique de période 8 et la dimension modulo 8 est spécifiée par les règles de commutation suivantes, où J est une isométrie antilinéaire dans $\mathcal{H}$ qui implémente l'involution τ

$$JxJ^{-1} = \tau(x) \qquad \forall\, x \in \mathcal{A} \otimes \mathcal{A}^0\,. \tag{16}$$

On a $J^2 = \varepsilon$, $JD = \varepsilon' DJ$, $J\gamma = \varepsilon'' \gamma J$ où $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ et si n désigne la dimension modulo 8

$n =$	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

L'isométrie antilinéaire J est donnée dans le cas riemannien par l'opérateur de conjugaison de charge et dans le cas non commutatif par l'opérateur de Tomita [Ta] qui, dans le cas où une algèbre d'opérateurs $\mathcal{A}$ admet un vecteur cyclique qui est cyclique pour le commutant $\mathcal{A}'$, établit un antiisomorphisme

$$a \in \mathcal{A}'' \to Ja^* J^{-1} \in \mathcal{A}' \,. \tag{17}$$

La donnée de μ ne spécifie que la classe d'homotopie stable du triplet spectral $(\mathcal{A}, \mathcal{H}, D)$ muni de l'isométrie J (et de la $\mathbb{Z}/2$ graduation γ si n est pair). La non trivialité de cette classe d'homotopie est visible dans la forme d'intersection

$$K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z} \tag{18}$$

donnée par l'indice de Fredholm de D à coefficient dans $K(\mathcal{A} \otimes \mathcal{A}^0)$.

Pour comparer les triplets spectraux dans la classe μ, nous utiliserons la fonctionnelle spectrale suivante

$$\mathrm{Trace}\,(\overline{\omega}(D)) \tag{19}$$

où $\overline{\omega} : \mathbb{R} \to \mathbb{R}_+$ est une fonction positive convenable.

L'algèbre $\mathcal{A}$ une fois fixée, une géométrie spectrale est déterminée par la classe d'équivalence unitaire du triplet spectral $(\mathcal{A}, \mathcal{H}, D)$ avec l'isométrie J. Si l'on note π la représentation de $\mathcal{A}$ dans $\mathcal{H}$ l'équivalence unitaire entre $(\pi_1, \mathcal{H}_1, D_1, J_1)$ et $(\pi_2, \mathcal{H}_2, D_2, J_2)$ signifie qu'il existe un unitaire $U : \mathcal{H}_1 \to \mathcal{H}_2$ tel que

$$U\pi_1(a)U^* = \pi_2(a) \quad \forall\, a \in \mathcal{A}\,,\ UD_1U^* = D_2\,,\ UJ_1U^* = J_2 \tag{20}$$

(et $U\gamma_1 U^* = \gamma_2$ dans le cas où n est pair).

Le groupe $\mathrm{Aut}(\mathcal{A})$ des automorphismes de l'algèbre involutive $\mathcal{A}$ agit sur l'ensemble des géométries spectrales par composition

$$\pi'(a) = \pi(\alpha^{-1}(a)) \qquad \forall\, a \in \mathcal{A}\,,\ \alpha \in \mathrm{Aut}(\mathcal{A})\,. \tag{21}$$

Le sous-groupe $\mathrm{Aut}^+(\mathcal{A})$ des automorphismes qui préservent la classe μ agit sur la classe d'homotopie stable déterminée par μ et préserve par construction la fonctionnelle d'action (19). En général ce groupe est non compact, et il coïncide par exemple dans le cas riemannien avec le groupe $\mathrm{Diff}^+(M)$ des difféomorphismes qui

préservent la K-orientation, i.e. la structure spinorielle de M. A l'inverse le groupe d'isotropie d'une géométrie donnée, est automatiquement *compact* (pour $\mathcal{A}$ unifère). Ceci montre que la fonctionnelle d'action (19) donne automatiquement naissance au phénomène de brisure de symétrie spontanée (Figure 1).

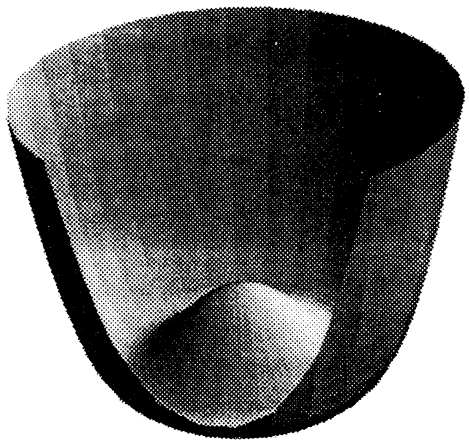

Figure 1

Nous montrerons que pour un choix convenable de l'algèbre $\mathcal{A}$ la fonctionnelle d'action (19) ajoutée au terme $\langle \xi, D\xi \rangle$, $\xi \in \mathcal{H}$ donne le modèle standard de Glashow–Weinberg–Salam couplé à la gravitation. L'algèbre $\mathcal{A}$ est le produit tensoriel de l'algèbre des fonctions sur un espace riemannien M par une algèbre non commutative de dimension finie dont les données phénoménologiques spécifient la géométrie spectrale.

2. Un Calcul Infinitésimal

Nous montrons comment le formalisme opératoriel de la mécanique quantique permet de donner un sens précis à la notion de variable infinitésimale. La notion d'infinitésimal est sensée avoir un sens intuitif évident. Elle résiste cependant fort bien aux essais de formalisation donnés par exemple par l'analyse non standard. Ainsi, pour prendre un exemple précis ([B-W]), soit $dp(x)$ la probabilité pour qu'une fléchette lancée au hasard sur la cible Ω termine sa course au point $x \in \Omega$ (Figure 2). Il est clair que $dp(x) < \varepsilon \quad \forall \varepsilon > 0$ et que néanmoins la réponse $dp(x) = 0$ n'est pas satisfaisante. Le formalisme usuel de la théorie de la mesure ou des formes différentielles contourne le problème en donnant un sens à l'expression

$$\int f(x)\, dp(x) \qquad f : \Omega \to \mathbb{C} \tag{1}$$

mais est insuffisant pour donner un sens par exemple à $e^{-\frac{1}{dp(x)}}$. La réponse, à savoir un réel non standard, fournie par l'analyse non standard, est également décevante:

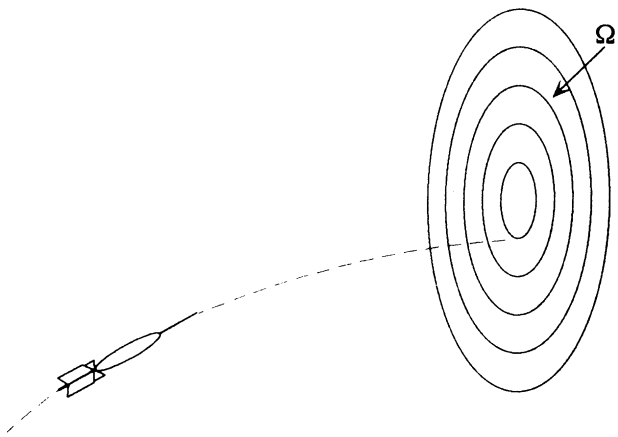

Figure 2

tout réel non standard détermine canoniquement un sous-ensemble non Lebesgue mesurable de l'intervalle $[0,1]$ de sorte qu'il est impossible ([Ste]) d'en exhiber un seul. Le formalisme que nous proposons donnera une réponse substantielle et calculable à cette question.

Le cadre est fixé par un espace de Hilbert séparable $\mathcal{H}$ décomposé comme somme directe de deux sous-espaces de dimension infinie. Donner cette décomposition revient à donner l'opérateur linéaire F dans $\mathcal{H}$ qui est l'identité, $F\xi = \xi$, sur le premier sous-espace et moins l'identité, $F\xi = -\xi$ sur le second; on a

$$F = F^* \,,\ F^2 = 1\,. \tag{2}$$

Le cadre ainsi déterminé est unique à isomorphisme près. Le début du dictionnaire qui traduit les notions classiques en language opératoriel est le suivant:

Classique	Quantique
Variable complexe	Opérateur dans $\mathcal{H}$
Variable réelle	Opérateur autoadjoint
Infinitésimal	Opérateur compact
Infinitésimal d'ordre α	Opérateur compact dont les valeurs caractéristiques μ_n vérifient $\mu_n = O(n^{-\alpha})\ ,\ n \to \infty$
Différentielle d'une variable réelle ou complexe	$đ\, f = [F, f] = Ff - fF$
Intégrale d'un infinitésimal d'ordre 1	$⨍ T =$ Coefficient de la divergence logarithmique dans la trace de T.

Les deux premières lignes du dictionnaire sont familières en mécanique quantique. L'ensemble des valeurs d'une variable complexe correspond au *spectre* d'un

opérateur. Le calcul fonctionnel holomorphe donne un sens à $f(T)$ pour toute fonction holomorphe f sur le spectre de T. Les fonctions holomorphes sont les seules à opérer dans cette généralité ce qui reflète la différence entre l'analyse complexe et l'analyse réelle où les fonctions boréliennes arbitraires opèrent. Quand $T = T^*$ est autoadjoint $f(T)$ a un sens pour toute fonction borélienne f. Notons que toute variable aléatoire usuelle X sur un espace de probabilité, (Ω, P) peut être trivialement considérée comme un opérateur autoadjoint. On prend $\mathcal{H} = L^2(\Omega, P)$ et

$$(T\xi)(p) = X(p)\,\xi(p) \qquad \forall\, p \in \Omega\ ,\ \xi \in \mathcal{H}\,. \tag{3}$$

La mesure spectrale de T redonne la probabilité P.

Passons à la troisième ligne du dictionnaire. Nous cherchons des "variables infinitésimales", i.e. des opérateurs T dans $\mathcal{H}$ tels que

$$\|T\| < \varepsilon \qquad \forall\, \varepsilon > 0\,, \tag{4}$$

où $\|T\| = \operatorname{Sup}\{\|T\xi\|\ ;\ \|\xi\| = 1\}$ est la norme d'opérateur. Bien entendu si l'on prend (4) au pied de la lettre on obtient $\|T\| = 0$ et $T = 0$ est la seule solution. Mais on peut affaiblir (4) de la manière suivante

$$\forall\, \varepsilon > 0\ ,\ \exists \text{ sous-espace de dimension finie } E \subset \mathcal{H} \text{ tel que } \|T/E^{\perp}\| < \varepsilon \tag{5}$$

où $E^{\perp}$ désigne l'orthogonal de E dans $\mathcal{H}$

$$E^{\perp} = \{\xi \in \mathcal{H}\ ;\ \langle \xi, \eta \rangle = 0 \quad \forall\, \eta \in E\} \tag{6}$$

qui est un sous-espace de codimension finie de $\mathcal{H}$. Le symbole $T/E^{\perp}$ désigne la restriction de T à ce sous-espace

$$T/E^{\perp} : E^{\perp} \to \mathcal{H}\,. \tag{7}$$

Les opérateurs qui satisfont la condition (5) sont les *opérateurs compacts*, i.e. sont caractérisés par la compacité normique de l'image de la boule unité de $\mathcal{H}$. L'opérateur T est compact ssi $|T| = \sqrt{T^*T}$ est compact, et ceci a lieu ssi le spectre de $|T|$ est une suite de valeurs propres $\mu_0 \geq \mu_1 \geq \mu_2 \ldots\ ,\ \mu_n \downarrow 0$.

Ces valeurs propres sont les valeurs caractéristiques de T et on a

$$\mu_n(T) = \operatorname{Inf}\{\|T - R\|\ ;\ R \text{ opérateur de rang } \leq n\} \tag{8}$$

$$\mu_n(T) = \operatorname{Inf}\{\|T/E^{\perp}\|\ ;\ \dim E \leq n\}\,. \tag{9}$$

Les opérateurs compacts forment un idéal bilatère $\mathcal{K}$ dans l'algèbre $\mathcal{L}(\mathcal{H})$ des opérateurs bornés dans $\mathcal{H}$ de sorte que les règles algébriques élémentaires du calcul infinitésimal sont vérifiées.

La taille d'un infinitésimal $T \in \mathcal{K}$ est gouvernée par l'ordre de décroissance de la suite $\mu_n = \mu_n(T)$, quand $n \to \infty$. En particulier pour tout réel positif α la condition

$$\mu_n(T) = O(n^{-\alpha}) \qquad \text{quand } n \to \infty \tag{10}$$

(i.e. il existe $C > 0$ tel que $\mu_n(T) \leq Cn^{-\alpha} \quad \forall\, n \geq 1$) définit les infinitésimaux d'ordre α. Ils forment de même un idéal bilatère, comme on le voit en utilisant (8), (cf. [Co]) et de plus

$$T_j \text{ d'ordre } \alpha_j \Rightarrow T_1 T_2 \text{ d'ordre } \alpha_1 + \alpha_2\,. \tag{11}$$

(Pour $\alpha < 1$ l'idéal correspondant est un idéal normé obtenu par interpolation réelle entre l'idéal $\mathcal{L}^1$ des opérateurs traçables et l'idéal $\mathcal{K}$ ([Co]).) Ainsi, hormis la commutativité, les propriétés intuitives du calcul infinitésimal sont vérifiées.

Comme la taille d'un infinitésimal est mesurée par une suite $\mu_n \to 0$ il pourrait sembler inutile d'utiliser le formalisme opératoriel. Il suffirait de remplacer l'idéal $\mathcal{K}$ de $\mathcal{L}(\mathcal{H})$ par l'idéal $c_0(\mathbb{N})$ des suites convergeant vers 0 dans l'algèbre $\ell^\infty(\mathbb{N})$ des suites bornées. Cette version commutative ne convient pas car tout élément de $\ell^\infty(\mathbb{N})$ a un spectre ponctuel et une mesure spectrale discrète. Ce n'est que la *non commutativité* de $\mathcal{L}(\mathcal{H})$ qui permet la coexistence de variables ayant un spectre de Lebesgue avec des variables infinitésimales.

En fait la ligne suivante du dictionnaire utilise de manière cruciale la non commutativité de $\mathcal{L}(\mathcal{H})$. La différentielle df d'une variable réelle ou complexe

$$df = \Sigma \frac{\partial f}{\partial x^\mu}\, dx^\mu \tag{12}$$

est remplacée par le commutateur

$$đ\, f = [F, f]\,. \tag{13}$$

Le passage de (11) à (12) est semblable à la transition du crochet de Poisson $\{f, g\}$ de deux observables f, g de la mécanique classique, au commutateur $[f, g] = fg - gf$ d'observables quantiques.

Etant donnée une algèbre $\mathcal{A}$ d'opérateurs dans $\mathcal{H}$ la *dimension* de l'espace correspondant (au sens du dictionnaire 1) est gouvernée par la taille des différentielles $đ\, f$, $f \in \mathcal{A}$. En dimension p on a

$$đ\, f \text{ d'ordre } \frac{1}{p}\,,\ \forall\, f \in \mathcal{A}\,. \tag{14}$$

Nous verrons très vite des exemples concrets où p est la dimension de Hausdorff d'un ensemble de Julia. Des manipulations algébriques très simples sur la fonctionnelle

$$\tau(f^0, \ldots, f^n) = \mathrm{Trace}\,(f^0\, đ\, f^1 \ldots đ\, f^n) \qquad n \text{ impair},\ n > p \tag{15}$$

montrent que τ est un cocycle cyclique et permettent de transposer les idées de la topologie différentielle en exploitant *l'intégralité* du cocycle τ, i.e. $\langle \tau, K_1(\mathcal{A})\rangle \subset \mathbb{Z}$.

Si le dictionnaire s'arrêtait là, il nous manquerait un outil vital du calcul infinitésimal, la *localité*, i.e. la possibilité de négliger les infinitésimaux d'ordre > 1 dans un calcul. Dans notre cadre les infinitésimaux d'ordre > 1 sont contenus dans l'idéal bilatère suivant

$$\left\{T \in \mathcal{K}\ ;\ \mu_n(T) = o\left(\frac{1}{n}\right)\right\} \tag{17}$$

où le petit o à la signification usuelle.

Ainsi, si nous utilisons la trace comme dans (16) pour intégrer nous rencontrons deux problèmes:

(a) les infinitésimaux d'ordre 1 ne sont pas dans le domaine de la trace,

(b) la trace des infinitésimaux d'ordre > 1 n'est pas nulle.

Le domaine naturel de la trace est l'idéal bilatère $\mathcal{L}^1(\mathcal{H})$ des opérateurs traçables

$$\mathcal{L}^1 = \left\{T \in \mathcal{K}\ ;\ \sum_o^\infty \mu_n(T) < \infty\right\}. \tag{18}$$

La trace d'un opérateur $T \in \mathcal{L}^1(\mathcal{H})$ est donnée par la somme

$$\mathrm{Trace}\,(T) = \sum \langle T\xi_i, \xi_i\rangle \tag{19}$$

indépendamment du choix de la base orthonormale (ξ_i) de $\mathcal{H}$. On a

$$\mathrm{Trace}\,(T) = \sum_o^\infty \mu_n(T) \qquad \forall\, T \geq 0\,. \tag{20}$$

Soit $T \geq 0$ un infinitésimal d'ordre 1, le seul contrôle sur $\mu_n(T)$ est

$$\mu_n(T) = O\left(\frac{1}{n}\right) \tag{21}$$

ce qui ne suffit pas pour assurer la finitude de (20). Ceci précise la nature du problème (a) et de même pour (b) puisque la trace ne s'annule pas pour le plus petit idéal de $\mathcal{L}(\mathcal{H})$, l'idéal $\mathcal{R}$ des opérateurs de rang fini.

Ces deux problèmes sont résolus par la trace de Dixmier [Dx], i.e. par l'analyse suivante de la divergence logarithmique des traces partielles

$$\mathrm{Trace}_N(T) = \sum_o^{N-1} \mu_n(T)\ ,\ T \geq 0\,. \tag{22}$$

Il est utile de définir $\mathrm{Trace}_\Lambda(T)$ pour tout $\Lambda > 0$ par la formule d'interpolation

$$\mathrm{Trace}_\Lambda(T) = \mathrm{Inf}\,\{\|A\|_1 + \Lambda\|B\|\ ;\ A + B = T\} \tag{23}$$

où $\|A\|_1$ est la norme $\mathcal{L}^1$ de A, $\|A\|_1 = \mathrm{Trace}\,|A|$. Cette formule coïncide avec (22) pour Λ entier et donne l'interpolation affine par morceaux. On a de plus ([Co])

$$\mathrm{Trace}_\Lambda(T_1 + T_2) \leq \mathrm{Trace}_\Lambda(T_1) + \mathrm{Trace}_\Lambda(T_2) \qquad \forall\, \Lambda \tag{24}$$

$$\mathrm{Trace}_{\Lambda_1+\Lambda_2}(T_1 + T_2) \geq \mathrm{Trace}_{\Lambda_1}(T_1) + \mathrm{Trace}_{\Lambda_2}(T_2) \qquad \forall\, \Lambda_1, \Lambda_2 \tag{25}$$

où T_1, T_2 sont *positifs* pour (25).
Soit $T > 0$ infinitésimal d'ordre 1 on a alors

$$\mathrm{Trace}_\Lambda(T) \leq C\, \log \Lambda \tag{26}$$

et la propriété remarquable *d'additivité asymptotique* du coefficient de la divergence logarithmique (26) est la suivante: ($T_j \geq 0$),

$$|\tau_\Lambda(T_1 + T_2) - \tau_\Lambda(T_1) - \tau_\Lambda(T_2)| \leq 3C\, \frac{\log(\log \Lambda)}{\log \Lambda} \tag{27}$$

où pour tout $T \geq 0$ on pose

$$\tau_\Lambda(T) = \frac{1}{\log \Lambda} \int_e^\Lambda \frac{\mathrm{Trace}_\mu(T)}{\log \mu}\, \frac{d\mu}{\mu} \tag{28}$$

qui est la moyenne de Cesaro sur le groupe $\mathbb{R}_+^*$ des échelles, de la fonction $\frac{\mathrm{Trace}_\mu(T)}{\log \mu}$.
Il résulte facilement de (27) que toute limite simple τ des fonctionnelles non linéaires τ_Λ définit une trace positive et linéaire sur l'idéal bilatère des infinitésimaux d'ordre 1

$$\begin{gathered} \tau(\lambda_1\, T_1 + \lambda_2\, T_2) = \lambda_1\, \tau(T_1) + \lambda_2\, \tau(T_2) \qquad \forall\, \lambda_j \in \mathbb{C} \\ \tau(T) \geq 0 \qquad \forall\, T \geq 0 \\ \tau(ST) = \tau(TS) \qquad \forall\, S \text{ borné} \\ \tau(T) = 0 \text{ si } \mu_n(T) = o\left(\frac{1}{n}\right). \end{gathered} \tag{29}$$

En pratique le choix du point limite τ est sans importance car dans tous les exemples importants (et en particulier comme corollaire des axiomes dans le cadre général, cf. Section IV) la condition suivante de *mesurabilité* est satisfaite:

$$\tau_\Lambda(T) \text{ est convergent quand } \Lambda \to \infty\,. \tag{30}$$

Pour les opérateurs mesurables la valeur de $\tau(T)$ est indépendante de τ et est notée

$$⨍ T\,. \tag{31}$$

Le premier exemple intéressant est celui des opérateurs pseudodifférentiels T sur une variété différentiable M. Quand T est d'ordre 1 (au sens de (21)) il est mesurable et $⨍ T$ est le résidu non commutatif de T ([Wo], [Ka]). Ce résidu a une expression locale très simple en terme du noyau distribution $k(x,y)$, $x,y \in M$. Quand T est d'ordre 1 (au sens de (21)) le noyau $k(x,y)$ admet une divergence logarithmique au voisinage de la diagonale

$$k(x,y) = a(x)\log|x-y| + 0(1) \tag{32}$$

où $|x-y|$ est la distance riemannienne dont le choix n'affecte pas la 1-densité $a(x)$. On a alors (à normalisation près)

$$⨍ T = \int_M a(x) \tag{33}$$

et le terme de droite de cette formule se prolonge de manière quasi évidente à tous les opérateurs pseudodifférentiels (cf. [Wo]) si l'on note que le noyau d'un tel opérateur admet un développement asymptotique de la forme

$$k(x,y) = \sum a_k(x, x-y) + a(x)\log|x-y| + 0(1) \tag{34}$$

où $a_k(x,\xi)$ est homogène de degré $-k$ en la variable ξ, et où la 1-densité $a(x)$ est définie de manière intrinsèque.

En fait le même principe de prolongement de $⨍$ à des infinitésimaux d'ordre < 1 s'applique aux opérateurs hypoelliptiques et plus généralement (cf. Théorème 4) aux triplets spectraux dont le spectre de dimension est simple.

Après cette description passons à des exemples. La variable infinitésimale $dp(x)$ qui donne la probabilité dans le jeu de fléchettes (Figure 2) est donnée par l'opérateur

$$dp = \Delta^{-1} \tag{35}$$

où Δ est le Laplacien de Dirichlet pour le domaine Ω. Il agit dans l'espace de Hilbert $L^2(\Omega)$ ainsi que l'algèbre des fonctions $f(x_1,x_2)$, $f : \Omega \to \mathbb{C}$, qui agissent par opérateurs de multiplication (cf. (3)). Le théorème de H. Weyl montre immédiatement que dp est d'ordre 1, que $f\,dp$ est mesurable et que

$$⨍ f\,dp = \int_\Omega f(x_1,x_2)\,dx_1 \wedge dx_2 \tag{36}$$

donne la probabilité usuelle.

Montrons maintenant comment utiliser notre calcul infinitésimal pour donner un sens à des expressions telles que l'aire d'une variété de dimension 4, qui sont dépourvues de sens dans le calcul usuel.

Il y a, à équivalence unitaire et multiplicité près, une seule quantification du calcul infinitésimal sur $\mathbb{R}$ qui soit invariante par translations et dilatations. Elle est

donnée par la représentation de l'algèbre des fonctions f sur $\mathbb{R}$ comme opérateurs de multiplication dans $L^2(\mathbb{R})$ (cf. (3)), alors que l'opérateur F dans $\mathcal{H} = L^2(\mathbb{R})$ est la transformation de Hilbert ([St])

$$(37) \qquad (f\xi)(s) = f(s)\,\xi(s) \qquad \forall\, s \in \mathbb{R}\ ,\ \xi \in L^2(\mathbb{R})\ ,\ (F\xi)(t) = \frac{1}{\pi i}\int \frac{\xi(s)}{s-t}\,ds\,.$$

On a une description unitairement équivalente pour $S^1 = P_1(\mathbb{R})$ avec $\mathcal{H} = L^2(S^1)$ et

$$(38) \qquad F\,e_n = \mathrm{Sign}\,(n)\,e_n\ ,\ e_n(\tau) = \exp\,(in\tau) \qquad \forall\,\tau \in S^1\ ,\ (\mathrm{Sign}\,(0) = 1)\,.$$

L'opérateur $đ\,f = [F, f]$, pour $f \in L^\infty(\mathbb{R})$, est représenté par le noyau $\frac{1}{\pi i}\,k(s,t)$, avec

$$(39) \qquad k(s,t) = \frac{f(s) - f(t)}{s-t}\,.$$

Comme f et F sont des opérateurs bornés il en est de même de $đ\,f = [F, f]$ pour toute f mesurable bornée sur S^1, ce qui donne un sens à $|đ\,f|^p$ pour tout $p > 0$. Soient par exemple $c \in \mathbb{C}$ et J l'ensemble de Julia associé à l'itération de la transformation

$$(40) \qquad \varphi(z) = z^2 + c\ ,\ J = \partial B\ ,\ B = \{z \in \mathbb{C}\ ;\ \operatorname*{Sup}_{n\in\mathbb{N}} |\varphi^n(z)| < \infty\}\,.$$

Pour c petit J est une courbe de Jordan et B la composante bornée de son complément. Soit $Z : S^1 \to J$ la restriction à $S^1 = \partial D$, $D = \{z \in \mathbb{C}\,,\ |z| < 1\}$ d'une équivalence conforme $D \sim B$. Comme (par un résultat de D. Sullivan) la dimension de Hausdorff p de J est > 1 (pour $c \neq 0$) la fonction Z n'est nulle part à variation bornée et la valeur absolue $|Z'|$ de la dérivée de Z au sens des distributions n'a pas de sens. Cependant $|đ\,Z|$ est bien défini et on a:

Théorème 1. — (a) *$|đ\,Z|$ est un infinitésimal d'ordre $\frac{1}{p}$.*

(b) *Pour toute fonction continue h sur J, l'opérateur $h(Z)\,|đ\,Z|^p$ est mesurable.*

(c) $\exists\,\lambda > 0$:

$$⨍ h(Z)\,|đ\,Z|^p = \lambda \int h\,d\Lambda_p \qquad \forall\, h \in C(J)\,,$$

où $d\Lambda_p$ désigne la mesure de Hausdorff sur J.

L'énoncé (a) utilise un résultat de V. V. Peller qui caractérise les fonctions f pour lesquelles $\mathrm{Trace}\,(|đ\,f|^\alpha) < \infty$. La constante λ gouverne le développement asymptotique de la distance dans $L^\infty(S^1)$ entre Z et les fonctions rationnelles ayant au plus n-pôles hors du disque unité. Cette constante est de l'ordre de $\sqrt{p-1}$ et s'annule pour $p = 1$. Cela tient à une propriété spécifique de la dimension 1, à savoir que pour

$f \in C^\infty(S^1)$ $\quad đ\, f$ n'est pas seulement d'ordre $(\dim S^1)^{-1} = 1$ mais est traçable, avec

$$\text{Trace}\,(f^0\, đ\, f^1) = \frac{1}{\pi i}\int_{S_1} f^0\, df^1 \qquad \forall\, f^0, f^1 \in C^\infty(S^1)\,. \tag{41}$$

En fait par un résultat classique de Kronecker $đ\, f$ est de rang fini ssi f est une fraction rationnelle (cf. [P]).

Le calcul différentiel quantique s'applique de la même manière à l'espace projectif $P_1(K)$ sur un corps local arbitraire K (i.e. un corps localement compact non discret) et est invariant par le groupe des transformations projectives. Les cas spéciaux $K = \mathbb{C}$ et $K = \mathbb{H}$ (corps des quaternions) sont des cas particuliers du calcul sur les variétés compactes conformes orientées de dimension paire, $M = M_{2n}$, qui se définit ainsi:

$$\mathcal{H} = L^2(M, \Lambda^n T^*)\ ,\ (f\xi)(p) = f(p)\,\xi(p) \qquad \forall\, f \in L^\infty(M)\ ,\ F = 2P - 1 \tag{42}$$

où le produit scalaire sur l'espace de Hilbert des formes différentielles de degré $n = \frac{1}{2}\dim M$ est donné par $\langle \omega_1, \omega_2\rangle = \int \omega_1 \wedge * \omega_2$ et ne dépend que de la structure conforme de M. L'opérateur P est le projecteur orthogonal sur le sous-espace des formes exactes.

Prenons d'abord $n = 1$. Un calcul immédiat donne

$$⨍ đ\, f\, đ\, g = -\frac{1}{\pi}\int df \wedge * dg \qquad \forall\, f, g \in C^\infty(M)\,. \tag{43}$$

Soit alors X une application (C^∞) de M dans l'espace $\mathbb{R}^N$ muni de la métrique riemannienne $g_{\mu\nu}\, dx^\mu\, dx^\nu$; on a

$$⨍ g_{\mu\nu}\, đ\, X^\mu\, đ\, X^\nu = -\frac{1}{\pi}\int_M g_{\mu\nu}\, dX^\mu \wedge * dX^\nu \tag{44}$$

où le terme de droite est l'action de Polyakov de la théorie des cordes. Pour $n = 4$ l'égalité (44) n'a pas lieu, l'action définie par le terme de droite n'est pas intéressante car elle n'est pas invariante conforme. Le terme de gauche est parfaitement défini par le calcul quantique et est invariant conforme: on a

Théorème 2. — *Soit X une application C^∞ de M_4 dans $(\mathbb{R}^N, g_{\mu\nu}\, dx^\mu\, dx^\nu)$,*

$$⨍ g_{\mu\nu}(X)\, đ\, X^\mu\, đ\, X^\nu = (16\pi^2)^{-1}\int_M g_{\mu\nu}(X)$$
$$\left\{\frac{1}{3}\, r\langle dX^\mu, dX^\nu\rangle - \Delta\langle dX^\mu, dX^\nu\rangle + \langle \nabla dX^\mu, \nabla dX^\nu\rangle - \frac{1}{2}\,(\Delta X^\mu)\,(\Delta X^\nu)\right\} dv$$

où pour écrire le terme de droite on utilise sur M une structure riemannienne η quelconque compatible avec la structure conforme. Ainsi la courbure scalaire r, le

Laplacien Δ et la connection de Levi Civita ∇ se réfèrent à η, mais le résultat n'en dépend pas.

Le Théorème 2 est à rapprocher de la formule suivante qui exprime l'action de Hilbert–Einstein comme l'aire d'une variété de dimension 4 (cf. [Kas] [K-W])

$$\fint ds^2 = \frac{-1}{96\pi^2} \int_{M_4} r\,\sqrt{g}\; d^4x \tag{45}$$

($dv = \sqrt{g}\; d^4x$ est la forme volume et $ds = D^{-1}$ est l'élément de longueur, i.e. l'inverse de l'opérateur de Dirac).

Quand la métrique $g_{\mu\nu}\, dx^\mu\, dx^\nu$ sur $\mathbb{R}^N$ est invariante par translations, la fonctionnelle d'action du Théorème 2 est donnée par l'opérateur de Paneitz sur M. C'est un opérateur d'ordre 4 qui joue le rôle du Laplacien en géométrie conforme ([B-O]). Son anomalie conforme a été calculée par T. Branson [B].

Reprenons le cas $n = 2$ et modifions la structure conforme de M par une différentielle de Beltrami $\mu(z,\bar z)\, d\bar z/dz$, $|\mu(z,\bar z)| < 1$ en utilisant pour définir les angles en $z \in M$

$$X \in T_z(M) \to \langle X, dz + \mu(z,\bar z)\, d\bar z\rangle \in \mathbb{C} \tag{46}$$

au lieu de $\langle X, dz\rangle$. Le calcul quantique sur M associé à la nouvelle structure conforme s'obtient simplement en remplaçant l'opérateur F par l'opérateur F',

$$F' = (\alpha F + \beta)(\beta F + \alpha)^{-1}\ ,\ \alpha = (1-m^2)^{-1/2}\ ,\ \beta = m(1-m^2)^{-1/2} \tag{47}$$

où m est l'opérateur dans $\mathcal{H} = L^2(M, \Lambda^1 T^*)$ donné par l'endomorphisme du fibré vectoriel $\Lambda^1 T^* = \Lambda^{(1,0)} \oplus \Lambda^{(0,1)}$ de matrice,

$$m(z,\bar z) = \begin{bmatrix} 0 & \bar\mu(z,\bar z)\, d\bar z/dz \\ \mu(z,\bar z)\, dz/d\bar z & 0 \end{bmatrix}. \tag{48}$$

Les propriétés cruciales de l'opérateur $m \in \mathcal{L}(\mathcal{H})$ sont

$$\|m\| < 1\ ,\ m = m^*\ ,\ m\, f = f\, m \qquad \forall\, f \in \mathcal{A} = C^\infty(M) \tag{49}$$

et la déformation (47) de F est un cas particulier de la:

Proposition 3. — *Soient $\mathcal{A}$ une algèbre involutive d'opérateurs dans $\mathcal{H}$ et $N = \mathcal{A}' = \{T \in \mathcal{L}(\mathcal{H})\,;\, Ta = aT \quad \forall\, a \in \mathcal{A}\}$ l'algèbre de von Neumann commutant de $\mathcal{A}$.*

(a) *L'égalité suivante définit une action du groupe $G = GL_1(N)$ des éléments inversibles de N sur les opérateurs F, $F = F^*$, $F^2 = 1$*

$$g(F) = (\alpha F + \beta)\,(\beta F + \alpha)^{-1} \qquad \forall\, g \in G$$

où $\alpha = \frac{1}{2}(g - (g^{-1})^*)$, $\beta = \frac{1}{2}(g + (g^{-1})^*)$.

(b) *On a* $[g(F), a] = Y[F, a]\, Y^* \quad \forall\, a \in \mathcal{A}$, *où* $Y = (\beta F + \alpha)^{*-1}$.

L'égalité (b) montre que pour tout idéal bilatère $J \subset \mathcal{L}(\mathcal{H})$ la condition

$$[F, a] \in J \tag{50}$$

est préservée par la déformation $F \to g(F)$. Comme seule la *mesurabilité* de la différentielle de Beltrami μ est requise pour que m vérifie (49), seule la mesurabilité de la structure conforme sur M est requise pour que le calcul quantique associé soit défini. De plus (b) montre que la condition de régularité sur $a \in L^\infty(M)$ définie par (50) ne dépend que de la structure quasiconforme de la variété M ([CST]). Un homéomorphisme local φ de $\mathbb{R}^n$ est *quasiconforme* ssi il existe $K < \infty$ tel que

$$H_\varphi(x) = \operatorname*{Lim\,sup}_{r \to 0} \ \frac{\max |\varphi(x) - \varphi(y)|\,;\ |x - y| = r}{\min |\varphi(x) - \varphi(y)|\,;\ |x - y| = r} \leq K\ ,\ \forall\, x \in \text{Domaine}\, \varphi\,. \tag{51}$$

Une structure quasiconforme sur une variété topologique M_n est donnée par un atlas quasiconforme. La discussion ci-dessus s'applique au cas général (n pair) ([CST]) et montre que le calcul quantique est bien défini pour toute variété quasiconforme. Le résultat de D. Sullivan [S] basé sur [Ki] montre que toute variété topologique M_n, $n \neq 4$ admet une structure quasiconforme. En utilisant le calcul quantique et la cohomologie cyclique à la place du calcul différentiel et de la théorie de Chern–Weil on obtient ([CST]) une formule locale pour les classes de Pontrjagin topologiques de M_n.

3. La Formule De L'indice Locale Et La Classe Fondamentale Transverse

Nous montrons dans cette section que le calcul infinitésimal ci-dessus permet le passage du local au global dans le cadre général des triplets spectraux $(\mathcal{A}, \mathcal{H}, D)$. Nous appliquons ensuite le résultat général au produit croisé d'une variété par le groupe des difféomorphismes.

Nous ferons l'hypothèse de régularité suivante sur $(\mathcal{A}, \mathcal{H}, D)$

$$a \text{ et } [D, a] \in \cap\, \text{Dom}\, \delta^k\ ,\ \forall\, a \in \mathcal{A} \tag{1}$$

où δ est la dérivation $\delta(T) = [|D|, T]$.

Nous désignerons par $\mathcal{B}$ l'algèbre engendrée par les $\delta^k(a)$, $\delta^k([D, a])$. La *dimension* d'un triplet spectral est bornée supérieurement par $p > 0$ ssi $a(D + i)^{-1}$ est un infinitésimal d'ordre $\frac{1}{p}$ pour tout $a \in \mathcal{A}$. Quand $\mathcal{A}$ est unifère cela ne dépend que du spectre de D.

La notion précise de dimension est définie comme le sous-ensemble $\Sigma \subset \mathbb{C}$ des singularités des fonctions analytiques

$$\zeta_b(z) = \text{Trace}\,(b|D|^{-z}) \qquad \text{Re}\, z > p\ ,\ b \in \mathcal{B}\,. \tag{2}$$

Nous supposerons que Σ est discret et simple, i.e. que les ζ_b se prolongent à $\mathbb{C}/\Sigma$ avec des pôles simples en Σ.

Nous renvoyons à [CM2] pour le cas de spectre multiple.

L'indice de Fredholm de l'opérateur D détermine une application additive, $K_1(\mathcal{A}) \xrightarrow{\varphi} \mathbb{Z}$ donnée par l'égalité

$$\varphi([u]) = \text{Indice}\,(PuP)\ ,\ u \in GL_1(\mathcal{A}) \tag{3}$$

où P est le projecteur $P = \frac{1+F}{2}$, $F = \text{Signe}\,(D)$.

Cette application est calculée par l'accouplement entre $K_1(\mathcal{A})$ et la classe de cohomologie du cocycle cyclique suivant

$$\tau(a^0, \ldots, a^n) = \text{Trace}\,(a^0[F, a^1] \ldots [F, a^n]) \qquad \forall\, a^j \in \mathcal{A} \tag{4}$$

où $F = \text{Signe}\ D$ et où n est un entier impair $n \geq p$.

Le problème est que τ est difficile à déterminer en général car la formule (4) implique la trace ordinaire au lieu de la trace locale $⨍$.

Ce problème est résolu par la formule suivante:

Théorème 4. ([CM2]) — *Soit* $(\mathcal{A}, \mathcal{H}, D)$ *un triplet spectral vérifiant les hypothèses* (1) *et* (2).

(a) *L'égalité* $⨍ P = \text{Res}_{z=0}\ \text{Trace}\,(P|D|^{-z})$ *définit une trace sur l'algèbre engendrée par* $\mathcal{A}$, $[D, \mathcal{A}]$ *et* $|D|^z$, $z \in \mathbb{C}$.

(b) *La formule suivante n'a qu'un nombre fini de termes non nuls et définit les composantes* $(\varphi_n)_{n=1,3,\ldots}$ *d'un cocycle dans le bicomplexe* (b, B) *de* $\mathcal{A}$

$$\varphi_n(a^0, \ldots, a^n) = \sum_k c_{n,k} ⨍ a^0[D, a^1]^{(k_1)} \ldots [D, a^n]^{(k_n)}\, |D|^{-n-2|k|} \quad \forall\, a^j \in \mathcal{A}$$

où l'on note $T^{(k)} = \nabla^k(T)$ *et* $\nabla(T) = D^2T - TD^2$, *et où* k *est un multiindice,* $c_{n,k} = (-1)^{|k|}\sqrt{2i}(k_1! \cdots k_n!)^{-1}\,((k_1+1)\ldots(k_1+k_2+\cdots+k_n+n))^{-1}\,\Gamma\left(|k| + \frac{n}{2}\right)$, $|k| = k_1 + \cdots + k_n$.

(c) *L'accouplement de la classe de cohomologie cyclique* $(\varphi_n) \in HC^*(\mathcal{A})$ *avec* $K_1(\mathcal{A})$ *donne l'indice de Fredholm de* D *à coefficient dans* $K_1(\mathcal{A})$.

Rappelons que le bicomplexe (b, B) est donné par les opérateurs suivants agissant sur les formes multilinéaires sur l'algèbre $\mathcal{A}$

$$(b\varphi)(a^0, \ldots, a^{n+1}) =$$

$$\sum_0^n (-1)^j\, \varphi(a^0, \ldots, a^j a^{j+1}, \ldots, a^{n+1}) + (-1)^{n+1}\, \varphi(a^{n+1}a^0, a^1, \ldots, a^n) \tag{5}$$

$$B = AB_0 \ , \ B_0\,\varphi(a^0,\dots,a^{n-1}) = \varphi(1,a^0,\dots,a^{n-1}) - (-1)^n\,\varphi(a^0,\dots,a^{n-1},1)$$

$$(A\psi)(a^0,\dots,a^{n-1}) = \sum_0^{n-1}(-1)^{(n-1)j}\,\psi(a^j,a^{j+1},\dots,a^{j-1})\,. \tag{6}$$

Nous renvoyons à [Co] pour la normalisation de l'accouplement entre HC^* et $K(\mathcal{A})$.

Remarques. — (a) L'énoncé du Théorème 4 reste valable si l'on remplace dans toutes les formules l'opérateur D par $D|D|^\alpha$, $\alpha \geq 0$.

(b) Dans le cas pair, c'est-à-dire si l'on suppose que $\mathcal{H}$ est $\mathbb{Z}/2$ gradué par γ, $\gamma = \gamma^*$, $\gamma^2 = 1$, $\gamma a = a\gamma \quad \forall\, a \in \mathcal{A}$, $\gamma D = -D\gamma$, on a une formule analogue pour un cocycle (φ_n), n pair qui donne l'indice de Fredholm de D à coefficient dans K_0. Cependant la composante φ_0 ne s'exprime pas en terme du résidu $⨍$ car elle est non locale pour $\mathcal{H}$ de dimension finie (cf. [CM2]).

(c) Quand le spectre de dimension Σ a de la multiplicité on a une formule analogue mais qui implique un nombre fini de termes correctifs, dont le nombre est borné indépendamment de la multiplicté (cf. [CM2]).

Le spectre de dimensions d'une variété V est $\{0,1,\dots,n\}$, $n = \dim V$, et est simple. La multiplicité apparait pour les variétés singulières et les ensembles de Cantor donnent des exemples de points complexes, $z \notin \mathbb{R}$ dans ce spectre. Nous discutons maintenant une construction géométrique générale pour laquelle les hypothèses (1) et (2) sont vérifiées. Il s'agit de construire la classe fondamentale en K-homologie d'une variété K-orientée M sans briser la symétrie du groupe $\mathrm{Diff}^+(M)$ des difféomorphismes de M qui préservent la K-orientation. De manière plus précise nous cherchons un triplet spectral, $(C^\infty(M),\mathcal{H},D)$ de la même classe de K-homologie que l'opérateur de Dirac associé à une métrique riemannienne (cf. I (11) et (12)) mais qui soit équivariant par rapport au groupe $\mathrm{Diff}^+(M)$ au sens de [K]. Cela signifie que l'on a une représentation unitaire $\varphi \to U(\varphi)$ de $\mathrm{Diff}^+(M)$ dans $\mathcal{H}$ telle que

$$U(\varphi)\,f\,U(\varphi)^{-1} = f\circ\varphi^{-1} \qquad \forall\, f \in C^\infty(M)\ ,\ \varphi \in \mathrm{Diff}^+(M) \tag{7}$$

et que

$$U(\varphi)\,D\,U(\varphi)^{-1} - D \text{ est borné pour tout } \varphi \in \mathrm{Diff}^+(M)\,. \tag{8}$$

Lorsque D est l'opérateur de Dirac associé à une structure riemannienne le symbole principal de D détermine cette métrique et les seuls difféomorphismes qui vérifient (8) sont les isométries.

La solution de ce problème est essentielle pour définir la géométrie transverse des feuilletages et elle est effectuée en 2 étapes. La première est l'utilisation ([Co1]) de la métrique de courbure négative de l'espace $GL(n)/O(n)$ et de l'opérateur "dual Dirac" de Miscenko et Kasparov pour se ramener à l'action de $\mathrm{Diff}^+(M)$ sur l'espace

total P du fibré des métriques de M. La deuxième, dont l'idée est due à Hilsum et Skandalis ([HS]) est l'utilisation des opérateurs hypoelliptiques pour construire l'opérateur D sur P.

On notera qu'alors que la géométrie équivariante obtenue pour P est de dimension finie et vérifie les hypothèses (1) (2) la géométrie obtenue sur M en utilisant le produit intersection avec le "dual Dirac" est de dimension infinie et θ-sommable

$$\text{Trace}\,(e^{-\beta D^2}) < \infty \qquad \forall\, \beta > 0\,. \tag{9}$$

Par construction, le fibré $P \xrightarrow{\pi} M$ est le quotient $F/O(n)$ du $GL(n)$ fibré principal F des repères sur M par l'action du groupe orthogonal $O(n) \subset GL(n)$. L'espace P admet la structure canonique suivante: le feuilletage vertical $V \subset TP$, $V = \text{Ker}\,\pi_*$ et les structures euclidiennes suivantes sur les fibrés V et $N = (TP)/V$. Le choix d'une métrique riemannienne $GL(n)$-invariante sur $GL(n)/O(n)$ détermine la métrique sur V et celle de N est la métrique tautologique: $p \in P$ détermine une métrique sur $T_{\pi(p)}(M)$ qui grâce à π_* est isomorphe à N_p.

Cette construction est fonctorielle pour les difféomorphismes de M.

Le calcul hypoelliptique adapté à cette structure est un cas particulier du calcul pseudodifférentiel sur les variétés de Heisenberg ([BG]). Il modifie simplement l'homogénéité des symboles $\sigma(p, \xi)$ en utilisant les homothéties:

$$\lambda \cdot \xi = (\lambda \xi_v, \lambda^2 \xi_n)\ ,\ \forall\, \lambda \in \mathbb{R}_+^* \tag{10}$$

où ξ_v, ξ_n sont les composantes verticales et normales du covecteur ξ. La formule (10) dépend de coordonnées locales (x_v, x_n) adaptées au feuilletage vertical mais le calcul pseudodifférentiel correspondant n'en dépend pas. Le symbole principal d'un opérateur hypoelliptique d'ordre k est une fonction, homogène de degré k pour (10), sur le fibré $V^* \oplus N^*$. Le noyau distribution $k(x, y)$ d'un opérateur pseudodifférentiel T dans le calcul hypoelliptique admet un développement au voisinage de la diagonale de la forme

$$k(x, y) \sim \sum a_j(x, x - y) + a(x) \log |x - y|' + 0(1) \tag{11}$$

où a_j est homogène de degré $-j$ en $x - y$ pour (10) et où la métrique $|x - y|'$ est localement de la forme

$$|x - y|' = ((x_v - y_v)^4 + (x_n - y_n)^2)^{1/4}\,. \tag{12}$$

Comme dans le calcul pseudodifférentiel ordinaire, le résidu se prolonge aux opérateurs de tout degré et est donné par l'égalité

$$\fint T = \frac{1}{v + 2m} \int a(x) \tag{13}$$

où la 1-densité $a(x)$ ne dépend pas du choix de la métrique $|\ |'$ et où $v = \dim V$, $m = \dim N$ de sorte que $v + 2m$ est la dimension de Hausdorff de l'espace métrique $(P, |\ |')$.

L'opérateur D est défini par l'équation $D|D| = Q$ où Q est l'opérateur différentiel hypoelliptique de degré 2 obtenu en combinant (quand v est pair) l'opérateur $d_V d_V^* - d_V^* d_V$ de signature où d_V est la différentiation verticale, avec l'opérateur de Dirac transverse. (On utilise le revêtement métaplectique $M\ell(n)$ de $GL(n)$ pour définir la structure spinorielle sur M.) La formule explicite de Q utilise une connection affine sur M mais le choix de cette connection n'affecte pas le *symbole principal hypoelliptique* de Q et donc de D ce qui assure l'invariance (8) de D par rapport aux difféomorphismes de M.

Donnons la formule explicite de Q dans le cas $n = 1$, i.e. pour $M = S^1$. On remplace P par la suspension $SP = \mathbb{R} \times P$ pour se ramener au cas où la dimension verticale v est paire. Un point de $SP = \mathbb{R} \times P$ est paramétré par 3 coordonnées $\alpha \in \mathbb{R}$ et $p = (s, \tau)$ où $\tau \in S^1$ et où $s \in \mathbb{R}$ définit la métrique $e^{2s}(d\tau)^2$ en $\tau \in S^1$.

On munit SP de la mesure $\nu = d\alpha\, ds\, d\tau$ et l'on représente l'algèbre $C_c^\infty(SP)$ par opérateurs de multiplication dans $\mathcal{H} = L^2(SP, \nu) \otimes \mathbb{C}^2$. La fonctorialité de la construction ci-dessus donne la repésentation unitaire suivante du groupe $\mathrm{Diff}^+(S^1)$

$$(U(\varphi)^{-1}\xi)(\alpha, s, \tau) = \varphi'(\tau)^{1/2}\, \xi(\alpha, s - \log \varphi'(\tau), \varphi(\tau))\,. \tag{14}$$

Enfin l'opérateur Q est donné par la formule

$$Q = -2\partial_\alpha \partial_s \sigma_1 + \frac{1}{i}\, e^{-s} \partial_\tau \sigma_2 + \left(\partial_s^2 - \partial_\alpha^2 - \frac{1}{4}\right) \sigma_3 \tag{15}$$

où $\sigma_1, \sigma_2, \sigma_3 \in M_2(\mathbb{C})$ sont les 3 matrices de Pauli.

L'opérateur ∂_τ est de *degré* 2 dans le calcul hypoelliptique et l'on vérifie que Q est hypoelliptique.

Un long calcul donne le résultat suivant ([CM3]):

Théorème 5. — *Soit $\mathcal{A}$ l'algèbre produit croisé de $C_c^\infty(SP)$ par* $\mathrm{Diff}^+(S^1)$.

(a) *Le triplet spectral $(\mathcal{A}, \mathcal{H}, D)$ (où $\mathcal{A}$ agit dans $\mathcal{H}$ par* (14) *et $D|D| = Q$) satisfait les hypothèses* (1) *et* (2) *et son spectre de dimension est $\Sigma = \{0, 1, 2, 3, 4\}$.*

(b) *La seule composante non nulle du cocycle associé (Théorème* 4) *est φ_3 et elle est cohomologue à 2ψ où ψ est le 3-cocycle cyclique classe fondamentale transverse du produit croisé.*

L'intégralité de 2ψ, i.e. de l'accouplement $\langle 2\psi, K_1(\mathcal{A})\rangle$ résulte alors du Théorème 4. Le 3-cocycle ψ est donné par (cf. [Co])

$$\begin{gathered} \psi(f^0 U(\varphi_0)\ ,\ f^1 U(\varphi_1)\ ,\ f^2 U(\varphi_2)\ ,\ f^3 U(\varphi_3)) = \\ \int h^0\, dh^1 \wedge dh^2 \wedge dh^3 \text{ si } \varphi_0\varphi_1\varphi_2\varphi_3 = 1 \\ \text{et } \ = 0 = \text{si } \varphi_0\varphi_1\varphi_2\varphi_3 \neq 1 \\ \text{avec } h^0 = f^0\ ,\ h^1 = (f^1)^{\varphi_0}\ ,\ h^2 = (f^2)^{\varphi_0\varphi_1}\ ,\ h^3 = (f^3)^{\varphi_0\varphi_1\varphi_2}\,. \end{gathered} \tag{16}$$

L'homologie entre φ_3 et 2ψ met en évidence l'action sur l'algèbre $\mathcal{A}$ de l'algèbre de Hopf engendrée par les transformations linéaires suivantes (pour la relation de δ_3 avec l'invariant de Godbillon Vey, voir [Co]) de $\mathcal{A}$

$$\begin{gathered}\delta_1(fU(\varphi)) = (\partial_\alpha f)\, U(\varphi)\ ,\ \delta_2(fU(\varphi)) = (\partial_s f)\, U(\varphi)\,,\\ \delta_3(fU(\varphi)) = f\, e^{-s}\, \partial_\tau \log(\varphi^{-1})'\, U(\varphi)\ ,\ X(fU(\varphi)) = e^{-s}(\partial_\tau\, f)\, U(\varphi)\end{gathered} \tag{17}$$

dont la compatibilité avec la multiplication de $\mathcal{A}$ est régie par le coproduit

$$\Delta\delta_j = \delta_j \otimes 1 + 1 \otimes \delta_j \qquad j = 1, 2, 3 \tag{18}$$

(i.e. les δ_j sont des dérivations de $\mathcal{A}$)

$$\Delta X = X \otimes 1 + 1 \otimes X - \delta_3 \otimes \delta_2 \tag{19}$$

où (19) montre que X est de degré 2.

4. La Notion De Variété Et Les Axiomes De La Géométrie

Commençons par spécifier la place de la géométrie riemannienne dans notre cadre en caractérisant (Théorème 6) les triplets spectraux correspondants. Soit $n \in \mathbb{N}$ la dimension, le triplet $(\mathcal{A}, \mathcal{H}, D)$ est supposé $\mathbb{Z}/2$ gradué par γ, $\gamma = \gamma^*$, $\gamma^2 = 1$ quand n est pair.

Les axiomes *commutatifs* sont les suivants:

(1) (Dimension) $ds = D^{-1}$ est infinitésimal d'ordre $\frac{1}{n}$.

(2) (Ordre un) $[[D, f], g] = 0 \quad \forall\, f, g \in \mathcal{A}$.

(3) (Régularité) Pour tout $f \in \mathcal{A}$, f et $[D, f]$ appartiennent à $\bigcap_k$ Domaine δ^k, où δ est la dérivation $\delta(T) = [|D|, T]$.

(4) (Orientabilité) Il existe un cycle de Hochschild $c \in Z_n(\mathcal{A}, \mathcal{A})$ tel que $\pi(c) = 1$ (n impair) ou $\pi(c) = \gamma$ (n pair), où π: $\mathcal{A}^{\otimes(n+1)} \to \mathcal{L}(\mathcal{H})$ est l'unique application linéaire telle que $\pi(a^0 \otimes a^1 \otimes \ldots \otimes a^n) = a^0[D, a^1] \ldots [D, a^n] \quad \forall\, a^j \in \mathcal{A}$.

(5) (Finitude) Le $\mathcal{A}$-module $\mathcal{E} = \bigcap_k$ Domaine D^k est projectif de type fini et l'égalité suivante définit une structure hermitienne sur $\mathcal{E}$

$$\langle a\xi, \eta\rangle = \fint a(\xi, \eta)\, ds^n \qquad \forall\, \xi, \eta \in \mathcal{E}\ ,\ a \in \mathcal{A}\,.$$

(6) (Dualité de Poincaré) La forme d'intersection $K_*(\mathcal{A}) \times K_*(\mathcal{A}) \to \mathbb{Z}$ donnée par la composition de l'indice de Fredholm de D avec la diagonale, $m_* : K_*(\mathcal{A}) \times K_*(\mathcal{A}) \to K_*(\mathcal{A} \otimes \mathcal{A}) \to K_*(\mathcal{A})$, est *inversible*.

(7) (Réalité) Il existe une isométrie antilinéaire J sur $\mathcal{H}$ telle que $Ja^*J^{-1} = a \quad \forall a \in \mathcal{A}$ et $J^2 = \varepsilon$, $JD = \varepsilon' DJ$, $J\gamma = \varepsilon'' \gamma J$ où la table des valeurs de $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ en fonction de n modulo 8 est donnée en (I.16).

Les axiomes (2) et (4) donnent la présentation de l'algèbre abstraite notée $(\mathcal{A}, ds)$ engendrée par $\mathcal{A}$ et $ds = D^{-1}$.

Théorème 6. — *Soit $\mathcal{A} = C^\infty(M)$ où M est une variété compacte de classe C^∞.*

(a) *Soit π une représentation unitaire de $(\mathcal{A}, ds)$ satisfaisant les conditions* (1) *à* (7). *Il existe alors une unique structure riemannienne g sur M telle que la distance géodésique soit donnée par :*

$$d(x,y) = \operatorname{Sup}\{|a(x) - a(y)| \ ; \ a \in \mathcal{A} \ , \ \|[D,a]\| \leq 1\} \, .$$

(b) *La métrique $g = g(\pi)$ ne dépend que de la classe d'équivalence unitaire de π et les fibres de l'application $\{$classe d'équivalence unitaire$\} \to g(\pi)$ forment un nombre fini d'espaces affines $\mathcal{A}_\sigma$ paramétrés par les structures spinorielles σ de M.*

(c) *La fonctionnelle $⨍ ds^{n-2}$ est quadratique et positive sur chaque $\mathcal{A}_\sigma$ où elle admet un unique minimum π_σ.*

(d) *π_σ est la représentation de $(\mathcal{A}, ds)$ dans $L^2(M, S_\sigma)$ donnée par les opérateurs de multiplication et l'opérateur de Dirac associé à la connection de Levi Civita de la métrique g.*

(e) *La valeur de $⨍ ds^{n-2}$ en π_σ est l'action de Hilbert–Einstein de la métrique g:*

$$⨍ ds^{n-2} = -c_n \int r \sqrt{g} \, d^n x \ , \ c_n = \tfrac{n-2}{12} \, (4\pi)^{-n/2} \, 2^{[n/2]} \, \Gamma\left(\tfrac{n}{2} + 1\right)^{-1} .$$

L'exemple le plus simple pour comprendre la signification du théorème et de vérifier que la géométrie du cercle S^1 de longueur 2π est entièrement spécifiée par la présentation:

$$U^{-1}[D,U] = 1 \ , \ \text{où } UU^* = U^*U = 1 \, . \tag{1}$$

L'algèbre $\mathcal{A}$ étant celle des fonctions C^∞ de l'opérateur unitaire U, on a $S^1 =$ Spectre $(\mathcal{A})$ et l'égalité (1) est le cas le plus simple de l'axiome 4.

Remarques. — (a) L'hypothèse $\mathcal{A} = C^\infty(M)$ devrait résulter des axiomes (1) – (7) (et de la commutativité de $\mathcal{A}$). Il résulte de (3) et (5) que si $\mathcal{A}''$ est l'algèbre de von Neumann engendrée par $\mathcal{A}$ on a:

$$\mathcal{A} = \left\{ T \in \mathcal{A}'' \ ; \ T \in \bigcap_{k>0} \operatorname{Dom} \delta^k \right\} , \tag{2}$$

ce qui montre que $\mathcal{A}$ est uniquement spécifiée dans $\mathcal{A}''$ par la donnée de D. Cela montre que $\mathcal{A}$ est stable par calcul fonctionnel C^∞ dans sa fermeture normique $A = \bar{\mathcal{A}}$ et en particulier que

$$\operatorname{Spectre} \mathcal{A} = \operatorname{Spectre} A \, . \tag{3}$$

Soit X cet espace compact; on devrait déduire des axiomes que l'application de X dans $\mathbb{R}^N$ donnée par les $a_i^j \in \mathcal{A}$ qui interviennent dans le cycle de Hochschild c de (4) est un plongement de X comme sous-variété C^∞ de $\mathbb{R}^N$ (cf. Proposition 15, p. 312 de [Co]).

(b) Rappelons qu'un cycle de Hochschild $c \in Z_n(\mathcal{A}, \mathcal{A})$ est un élément de $\mathcal{A}^{\otimes(n+1)}$, $c = \sum a_i^0 \otimes a_i^1 \ldots \otimes a_i^n$ tel que $bc = 0$, où b est l'application linéaire $b : \mathcal{A}^{\otimes n+1} \to \mathcal{A}^{\otimes n}$ telle que:

$$b(a^0 \otimes \ldots \otimes a^n) =$$

$$\sum_0^{n-1} (-1)^j \, a^0 \otimes \ldots \otimes a^j a^{j+1} \otimes \ldots \otimes a^n + (-1)^n \, a^n a^0 \otimes a^1 \otimes \ldots \otimes a^{n+1} \, .$$

La classe de Hochschild du cycle c détermine la *forme volume*.

(c) Nous utilisons la convention selon laquelle la courbure scalaire r est positive pour la sphère S^n, en particulier le signe de l'action $⨍ \, ds^{n-2}$ est le bon pour la formulation euclidienne de la gravitation. Par exemple, pour $n = 4$, l'action de Hilbert–Einstein $-\frac{1}{16\pi G} \int r \sqrt{g} \, d^4x$ coïncide avec l'aire $\frac{1}{\ell_p^2} ⨍ \, ds^2$ en unité de Planck.

(d) Quand M est une variété spinorielle l'application $\pi \to g(\pi)$ du théorème est surjective et si l'on fixe le cycle $c \in Z_n(\mathcal{A}, \mathcal{A})$ son image est l'ensemble des métriques dont la forme volume est fixée — (b).

(e) Si l'on supprime l'axiome 7 on a un résultat analogue au théorème en remplaçant les structures spinorielles par les structures spin^c ([LM]), mais l'on n'a plus unicité dans (c) à cause de la liberté dans le choix de la connection spinorielle.

(f) Il résulte de l'axiome 4 et de ([Co], Théorème 8, p. 309) que les opérateurs $a \, ds^n$, $a \in \mathcal{A}$ sont automatiquement mesurables de sorte que le symbole $⨍$ qui apparait dans 5 est bien défini.

Passons au cas général non commutatif. Étant donnée une algèbre involutive $\mathcal{A}$ d'opérateurs dans l'espace de Hilbert $\mathcal{H}$ la théorie de Tomita [Ta] associe à tout vecteur $\xi \in \mathcal{H}$ cyclique pour $\mathcal{A}$ et pour son commutant $\mathcal{A}'$,

$$\overline{\mathcal{A}\xi} = \mathcal{H} \ , \ \overline{\mathcal{A}'\xi} = \mathcal{H} \tag{4}$$

une involution antilinéaire isométrique $J : \mathcal{H} \to \mathcal{H}$ obtenue à partir de la décomposition polaire de l'opérateur

$$S \, a\xi = a^* \xi \qquad \forall \, a \in \mathcal{A} \tag{5}$$

et qui vérifie la propriété de commutativité suivante:

$$J\mathcal{A}'' J^{-1} = \mathcal{A}' \, . \tag{6}$$

On a donc en particulier $[a, b^0] = 0 \quad \forall \, a, b \in \mathcal{A}$ où

$$b^0 = J b^* J^{-1} \qquad \forall b \in \mathcal{A} \tag{7}$$

de sorte que $\mathcal{H}$ devient un $\mathcal{A}$-bimodule en utilisant la représentation de l'algèbre opposée $\mathcal{A}^0$ donnée par (7). Dans le cas commutatif on a $a^0 = a \quad \forall\, a \in \mathcal{A}$ de sorte que l'on ne perçoit pas la nuance entre module et bimodule.

Le théorème de Tomita est l'outil nécessaire pour assurer la substance des axiomes dans le cas général. Les axiomes (1) (3) et (5) sont inchangés, dans l'axiome de réalité (7) on remplace l'égalité $Ja^*J^{-1} = a \quad \forall\, a \in \mathcal{A}$ par

$$(7') \qquad [a, b^0] = 0 \qquad \forall\, a, b \in \mathcal{A} \text{ où } b^0 = Jb^*J^{-1}$$

et l'axiome (2) (ordre un) se formule ainsi

$$(2') \qquad [[D, a], b^0] = 0 \qquad \forall\, a, b \in \mathcal{A}\,.$$

(On notera que comme a et b^0 commutent $2'$ équivaut à $[[D, a^0], b] = 0 \quad \forall\, a, b \in \mathcal{A}$.)

L'axiome $(7')$ fait de $\mathcal{H}$ un $\mathcal{A}$-bimodule et donne une classe μ de KR^n-homologie pour l'algèbre $\mathcal{A} \otimes \mathcal{A}^0$ munie de l'automorphisme antilinéaire τ

$$\tau(x \otimes y^0) = y^* \otimes x^{*0}\,.$$

Le produit intersection de Kasparov [K] permet alors de formuler la dualité de Poincaré, comme l'invertibilité de μ

$$(6') \qquad \exists\, \beta \in KR_n(\mathcal{A}^0 \otimes \mathcal{A})\ ,\ \beta \otimes_{\mathcal{A}} \mu = \mathrm{id}_{\mathcal{A}^0}\ ,\ \mu \otimes_{\mathcal{A}^0} \beta = \mathrm{id}_{\mathcal{A}}\ .$$

Ceci implique l'isomorphisme $K_*(\mathcal{A}) \xrightarrow{\cap \mu} K^*(\mathcal{A})$. La forme d'intersection

$$K_*(\mathcal{A}) \times K_*(\mathcal{A}) \to \mathbb{Z}$$

est obtenue à partir de l'indice de Fredholm de D à coefficient dans $K_*(\mathcal{A} \otimes \mathcal{A}^0)$ et n'utilise plus l'application diagonale $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ qui n'est un homomorphisme que dans le cas commutatif. Cette forme d'intersection est quadratique ou symplectique selon la valeur de n modulo 8.

L'homologie de Hochschild à coefficient dans un bimodule garde tout sons sens dans le cas général et l'axiome (4) prend la forme suivante

$(4')$ Il existe un cycle de Hochschild $c \in Z_n(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^0)$ tel que $\pi(c) = 1$ (n impair) ou $\pi(c) = \gamma$ (n pair).

(Où $\mathcal{A} \otimes \mathcal{A}^0$ est le $\mathcal{A}$ bimodule obtenu par restriction à la sous-algèbre $\mathcal{A} \otimes 1 \subset \mathcal{A} \otimes \mathcal{A}^0$ de la structure de $\mathcal{A} \otimes \mathcal{A}^0$ bimodule de $\mathcal{A} \otimes \mathcal{A}^0$, i.e.

$$a(b \otimes c^0)d = abd \otimes c^0 \qquad \forall\, a, b, c, d \in \mathcal{A}\,.)$$

Les axiomes (1), (3) et (5) sont inchangés dans le cas non commutatif et la démonstration de la mesurabilité des opérateurs $a(ds)^n$, $a \in \mathcal{A}$ reste valable en général.

Nous adopterons les axiomes (1), (2′), (3), (4′), (5), (6′) et (7′) dans le cas général comme définition d'une *variété spectrale* de dimension n. L'algèbre $\mathcal{A}$ étant fixée nous parlerons de géométrie spectrale sur $\mathcal{A}$ comme dans I.20 et I.21. On démontre que l'algèbre de von Neumann $\mathcal{A}''$ engendrée par $\mathcal{A}$ dans $\mathcal{H}$ est automatiquement finie et hyperfinie et on a la liste complète de ces algèbres à isomorphisme près [Co]. L'algèbre $\mathcal{A}$ est stable par calcul fonctionnel C^∞ dans sa fermeture normique $A = \bar{\mathcal{A}}$ de sorte que $K_j(\mathcal{A}) \simeq K_j(A)$, i.e. $K_j(\mathcal{A})$ ne dépend que de la topologie sous-jacente (définie par la C^* algèbre A). L'entier $\chi = \langle \mu, \beta \rangle \in \mathbb{Z}$ donne la caractéristique d'Euler sous la forme

$$\chi = \operatorname{Rang} K_0(\mathcal{A}) - \operatorname{Rang} K_1(\mathcal{A})$$

et le Théorème 4 en donne une formule locale.

Le groupe $\operatorname{Aut}(\mathcal{A})$ des automorphismes de l'algèbre involutive $\mathcal{A}$ joue en général le rôle du groupe $\operatorname{Diff}(M)$ des difféomorphismes d'une variété M. (On a un isomorphisme canonique $\operatorname{Diff}(M) \overset{\alpha}{\to} \operatorname{Aut}(C^\infty(M))$ donné par

$$\alpha_\varphi(f) = f \circ \varphi^{-1} \qquad \forall\, f \in C^\infty(M)\ ,\ \varphi \in \operatorname{Diff}(M)\,.)$$

Dans le cas général non commutatif, parallèlement au sous-groupe normal $\operatorname{Int}\mathcal{A} \subset \operatorname{Aut}\mathcal{A}$ des automorphismes intérieurs de $\mathcal{A}$,

$$\alpha(f) = ufu^* \qquad \forall\, f \in \mathcal{A} \tag{8}$$

où u est un élément unitaire de $\mathcal{A}$ (i.e. $uu^* = u^*u = 1$), il existe un feuilletage naturel de l'espace des géométries spectrales sur $\mathcal{A}$ en classes d'équivalences formées des *déformations intérieures* d'une géométrie donnée. Une telle déformation est obtenue sans modifier ni la représentation de $\mathcal{A}$ dans $\mathcal{H}$ ni l'isométrie antilinéaire J par la formule

$$D \to D + A + JAJ^{-1} \tag{9}$$

où $A = A^*$ est un opérateur autoadjoint arbitraire de la forme

$$A = \Sigma\, a_i[D, b_i]\ ,\ a_i, b_i \in \mathcal{A}\,. \tag{10}$$

Le nouveau triplet spectral obtenu continue à vérifier les axiomes (1) — (7′).

L'action du groupe $\operatorname{Int}(\mathcal{A})$ sur les géométries spectrales (cf. I.21) se réduit à une transformation de jauge sur A, donnée par la formule

$$\gamma_u(A) = u[D, u^*] + uAu^*\,. \tag{11}$$

L'équivalence unitaire est implémentée par la représentation suivante du groupe unitaire de $\mathcal{A}$ dans $\mathcal{H}$,

$$u \to uJuJ^{-1} = u(u^*)^0\,. \tag{12}$$

La transformation (9) se réduit à l'identité dans le cas riemannien usuel. Pour obtenir un exemple non trivial, il suffit d'en faire le produit par l'unique géométrie spectrale sur l'algèbre de dimension finie $\mathcal{A}_F = M_N(\mathbb{C})$ des matrices $N \times N$ sur $\mathbb{C}$, $N \geq 2$. On a alors $\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F$, $\mathrm{Int}(\mathcal{A}) = C^\infty(M, PSU(N))$ et les déformations intérieures de la géométrie sont paramétrées par les potentiels de jauge pour une théorie de jauge de groupe $SU(N)$. L'espace $P(\mathcal{A})$ des états purs de l'algèbre $\mathcal{A}$ est le produit $P = M \times P_{N-1}(\mathbb{C})$ et la métrique sur $P(\mathcal{A})$ déterminée par la formule I.10 dépend du potentiel de jauge A. Elle coïncide avec la métrique de Carnot [G] sur P définie par la distribution horizontale de la connection associée à A (cf. [Co3]). Le groupe $\mathrm{Aut}(\mathcal{A})$ des automorphismes de $\mathcal{A}$ est le produit semi direct

$$\mathrm{Aut}(\mathcal{A}) = \mathcal{U} \rtimes \mathrm{Diff}(M) \tag{13}$$

du groupe $\mathrm{Int}(\mathcal{A})$ des transformations de jauges locales par le groupe des difféomorphismes. En dimension $n = 4$, les fonctionnelles d'action de Hilbert–Einstein pour la métrique riemannienne et de Yang–Mills pour le potentiel vecteur A apparaissent simplement, et avec les bons signes, dans le développement asymptotique en $\frac{1}{\Lambda}$ du nombre $N(\Lambda)$ de valeurs propres de D qui sont $\leq \Lambda$. On régularise cette expression en la remplaçant par

$$\mathrm{Trace}\, \overline{\varpi}\left(\frac{D}{\Lambda}\right) \tag{14}$$

où $\overline{\varpi} \in C_c^\infty(\mathbb{R})$ est une fonction paire qui vaut 1 sur l'intervalle $[-1, 1]$, (cf. [CC]). Les seuls autres termes non nuls du développement asymptotique sont un terme cosmologique, un terme de gravité de Weyl et un terme topologique.

Un exemple plus élaboré de variété spectrale est le tore non commutatif $\mathbb{T}^2_\theta$. Le paramètre $\theta \in \mathbb{R}/\mathbb{Z}$ définit la déformation suivante de l'algèbre des fonctions C^∞ sur le tore $\mathbb{T}^2$, de générateurs U, V. Les relations

$$VU = \exp 2\pi i\theta\, UV \quad \text{et} \quad UU^* = U^*U = 1\ ,\ VV^* = V^*V = 1 \tag{15}$$

définissent la structure d'algèbre involutive de $\mathcal{A}_\theta = \{\Sigma\, a_{n,m} U^n V^n\ ;\ a = (a_{n,m}) \in \mathcal{S}(\mathbb{Z}^2)\}$ où $S(\mathbb{Z}^2)$ est l'espace de Schwartz des suites à décroissance rapide. Comme pour les courbes elliptiques on utilise comme paramètre pour définir la géométrie de $\mathbb{T}^2_\theta$ un nombre complexe τ de partie imaginaire positive et, à isométrie près, cette géométrie ne dépend que de l'orbite de τ pour $PSL(2, \mathbb{Z})$ [Co]. Le phénomène nouveau qui apparaît est *l'équivalence de Morita* qui relie entre elles les algèbres $\mathcal{A}_{\theta_1}, \mathcal{A}_{\theta_2}$ lorsque θ_1 et θ_2 sont dans la même orbite de l'action de $PSL(2, \mathbb{Z})$ sur $\mathbb{R}$ [Ri].

Étant données une variété spectrale $(\mathcal{A}, \mathcal{H}, D)$ et une équivalence de Morita entre $\mathcal{A}$ et une algèbre $\mathcal{B}$ donnée par

$$\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E}) \tag{16}$$

où $\mathcal{E}$ est une $\mathcal{A}$-module à droite, projectif de type fini et hermitien, on obtient une géométrie spectrale sur $\mathcal{B}$ par le choix d'une *connection hermitienne* sur $\mathcal{E}$. Une telle connection ∇ est une application linéaire $\nabla : \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D$ vérifiant les règles ([Co])

$$\nabla(\xi a) = (\nabla\xi)a + \xi \otimes da \qquad \forall\, \xi \in \mathcal{E}\ ,\ a \in \mathcal{A} \tag{17}$$

$$(\xi, \nabla\eta) - (\nabla\xi, \eta) = d(\xi, \eta) \qquad \forall\, \xi, \eta \in \mathcal{E} \tag{18}$$

où $da = [D, a]$ et où $\Omega^1_D \subset \mathcal{L}(\mathcal{H})$ est le $\mathcal{A}$-bimodule formé par les opérateurs de la forme (10).

Toute algèbre $\mathcal{A}$ est Morita équivalente à elle-même (avec $\mathcal{E} = \mathcal{A}$) et quand on applique la construction ci-dessus on obtient les déformations intérieures de la géométrie spectrale.

5. La Géométrie Spectrale De L'espace Temps

L'information expérimentale et théorique dont on dispose sur la structure de l'espace temps est résumée par la fonctionnelle d'action suivante, $\mathcal{L} = \mathcal{L}_E + \mathcal{L}_G + \mathcal{L}_{G\varphi} + \mathcal{L}_\varphi + \mathcal{L}_{\varphi f} + \mathcal{L}_f$, où $\mathcal{L}_E = -\frac{1}{16\pi G}\int r\,\sqrt{g}\,d^4x$ est l'action de Hilbert–Einstein et les 5 autres termes constituent le modèle standard de la physique des particules, couplé de manière minimale à la gravitation. Outre la métrique $g_{\mu\nu}$ ce Lagrangien implique plusieurs champs de bosons et de fermions. Les bosons de spin 1 sont le photon γ, les bosons médiateurs $W^\pm$ et Z et les huit gluons. Les bosons de spin 0 sont les champs de Higgs φ qui sont introduits pour briser la parité et pour que le mécanisme de brisure de symétrie spontanée confère une masse aux diverses particules sans contredire la renormalisabilité des champs de jauge non abéliens. Tous les fermions sont de spin $\frac{1}{2}$ et forment 3 familles de quarks et leptons.

Les champs impliqués dans le modèle standard ont a priori un statut très différent de celui de la métrique $g_{\mu\nu}$. Le groupe de symétrie de ces champs, à savoir le groupe des transformations de jauge locales:

$$\mathcal{U} = C^\infty(M, U(1) \times SU(2) \times SU(3)) \tag{1}$$

est a priori très différent du groupe $\mathrm{Diff}(M)$ de symétries de $\mathcal{L}_E$. Le groupe de symétrie naturel de $\mathcal{L}$ est le produit semidirect $\mathcal{U} \rtimes \mathrm{Diff}(M) = G$. La première question à résoudre si l'on veut donner une signification purement géométrique à $\mathcal{L}$ est de trouver un espace géométrique X tel que $G = \mathrm{Diff}(X)$. Ceci détermine, en tenant compte du relèvement des difféomorphismes aux spineurs, l'algèbre $\mathcal{A}$:

$$\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F\ ,\ \mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})\,, \tag{2}$$

où l'algèbre involutive $\mathcal{A}_F$ est la somme directe des algèbres $\mathbb{C}, \mathbb{H}$ des quaternions et $M_3(\mathbb{C})$ des matrices 3×3 complexes.

L'algèbre $\mathcal{A}_F$ correspond à un espace *fini* dont les fermions du modèle standard et les paramètres de Yukawa (masses des fermions et matrice de mélange de Kobayashi

Maskawa) déterminent la géométrie spectrale de la manière suivante. L'espace de Hilbert $\mathcal{H}_F$ est de dimension finie et admet pour base la liste des fermions élémentaires. Par exemple pour la 1ère génération de leptons cette liste est

$$e_L, e_R, \nu_L, \bar{e}_L, \bar{e}_R, \bar{\nu}_L \,. \tag{3}$$

L'algèbre $\mathcal{A}_F$ admet une représentation naturelle dans $\mathcal{H}_F$ (cf. [Co3]) et en désignant par J_F l'unique involution antilinéaire qui échange f et $\bar{f}$ pour tout vecteur de la base, on a la commutation

$$[a, Jb^* J^{-1}] = 0 \qquad \forall\, a, b \in \mathcal{A}_F \,. \tag{4}$$

L'opérateur D_F est simplement donné par la matrice $\begin{bmatrix} Y & 0 \\ 0 & \bar{Y} \end{bmatrix}$ où Y est la matrice de couplage de Yukawa. De plus les propriétés particulières de Y assurent la commutation

$$[[D_F, a], b^0] = 0 \qquad \forall\, a, b \in \mathcal{A}_F \,. \tag{5}$$

La $\mathbb{Z}/2$ graduation naturelle de $\mathcal{H}_F$ vaut 1 pour les fermions gauches $(e_L, \nu_L \ldots)$ et -1 pour les fermions droits; on a

$$\gamma_F = \varepsilon\, \varepsilon^0 \text{ où } \varepsilon = (1, -1, 1) \in \mathcal{A}_F \,. \tag{6}$$

Nous renvoyons à [Co3] pour les vérifications des axiomes (1) — (7′). Le seul défaut est que le nombre de générations introduit une multiplicité dans la forme d'intersection, $K_0(\mathcal{A}) \times K_0(\mathcal{A}) \to \mathbb{Z}$, donnée par un multiple entier de la matrice 3×3

$$\begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} . \tag{7}$$

Nous reviendrons à la fin de cet exposé sur la signification de la géométrie spectrale $(\mathcal{A}_F, \mathcal{H}_F, D_F) = F$.

Le pas suivant consiste à calculer les déformations intérieures (formule III.9) de la géométrie produit $M \times F$ où M est une variété riemannienne spinorielle de dimension 4. Le calcul donne les bosons de jauge du modèle standard, $\gamma, W^{\pm}, Z$, les huits gluons et les champs de Higgs φ avec les bons nombres quantiques et montre que

$$\mathcal{L}_{\varphi f} + \mathcal{L}_f = \langle \psi, D\psi \rangle \tag{8}$$

où $D = D_0 + A + JAJ^{-1}$ est la déformation intérieure de la géométrie produit (donnée par l'opérateur $D_0 = \partial\!\!\!/ \otimes 1 + \gamma_5 \otimes D_F$).

La structure de produit de $M \times F$ donne une bigraduation de Ω^1_D et une décomposition $A = A^{(1,0)} + A^{(0,1)}$ de A qui correspond à la décomposition (8). Le terme

$A^{(1,0)}$ rassemble tous les bosons de spin 1 et le terme $A^{(0,1)}$ les bosons de Higgs qui apparaissent comme des termes de différence finie sur l'espace F. Cette bigraduation existe sur l'analogue Ω_D^2 des 2-formes ([Co]) et décompose la courbure $\theta = dA + A^2$ en trois termes $\theta = \theta^{(2,0)} + \theta^{(1,1)} + \theta^{(0,2)}$ 2 à 2 orthogonaux pour le produit scalaire

$$\langle \omega_1, \omega_2 \rangle = \fint \omega_1 \, \omega_2^* \, ds^4 \,. \tag{9}$$

Ainsi l'action de Yang–Mills, $\langle \theta, \theta \rangle = \fint \theta^2 \, ds^4$ se décompose comme somme de 3 termes et on démontre que ces termes sont respectivement $\mathcal{L}_G$, $\mathcal{L}_{G\varphi}$ et $\mathcal{L}_\varphi$ pour $(2,0)$, $(1,1)$ et $(0,2)$ respectivement [Co].

L'action de Yang-Mills $\fint \theta^2 \, ds^4$ utilise la décomposition $D = D_0 + A + JAJ^{-1}$ et n'est donc pas, a priori, une fonction ne dépendant que de la géométrie définie par D. Nous avons vu en III.14 que, dans un cas plus simple, la combinaison $\mathcal{L}_E + \mathcal{L}_G$ apparaît directement dans le développement asymptotique du nombre de valeurs propres inférieures à Λ. Le même principe (cf. [CC]) s'applique au modèle standard et conduit à la fonctionnelle suivante

$$\text{Trace}\left(\overline{\omega}\left(\frac{D}{\Lambda}\right)\right) + \langle \psi, D\psi \rangle \tag{10}$$

dont le développement asymptotique ([CC]) donne $\mathcal{L}+$ un terme de gravité de Weyl et un terme en $r\varphi^2$ qui est le seul terme que l'on peut rajouter à $\mathcal{L}$ sans altérer le modèle standard. Nous renvoyons à [CC] pour l'interprétation physique de ces résultats.

La géométrie finie F ci-dessus était dictée par les résultats expérimentaux et il reste à en comprendre la signification conceptuelle à partir de l'analogue des groupes de Lie en géométrie non commutative, i.e. la théorie des groupes quantiques. Le fait simple (cf. [M]) est que le revêtement spinoriel Spin(4) de $SO(4)$ n'est pas un revêtement maximal parmi les groupes quantiques. On a $\text{Spin}(4) = SU(2) \times SU(2)$ et même le groupe $SU(2)$ admet grâce aux résultats de Lusztig des revêtements finis de la forme (Frobenius à l'∞):

$$1 \to H \to SU(2)_q \to SU(2) \to 1 \,, \tag{11}$$

où q est une racine de l'unité, $q^m = 1$, m impair. Le cas le plus simple est $m = 3$, $q = \exp\left(\frac{2\pi i}{3}\right)$. Le groupe quantique fini H a une algèbre de Hopf de dimension finie très voisine de $\mathcal{A}_F$, et la représentation spinorielle de H définit un bimodule sur cette algèbre de Hopf de structure très voisine du bimodule $\mathcal{H}_F$ sur $\mathcal{A}_F$. Cela suggère d'étendre la géométrie spinorielle ([LM]) aux revêtements quantiques du groupe spinoriel, ce qui nécessite même pour parler de G-fibré principal, d'introduire un minimum de non commutativité (du style $C^\infty(M) \otimes \mathcal{A}_F$) dans l'algèbre des fonctions.

Mentionnons enfin que nous avons négligé dans cet exposé la nuance importante entre les signatures riemanniennes et lorentziennes.

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THE WORK OF W. THURSTON

by

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Thurston has fantastic geometric insight and vision; his ideas have completely revolutionized the study of topology in 2 and 3 dimensions, and brought about a new and fruitful interplay between analysis, topology and geometry.

The central new idea is that a very large class of closed 3-manifolds should carry a hyperbolic structure — be the quotient of hyperbolic space by a discrete group of isometries, or equivalently, carry a metric of constant negative curvature. Although this is a natural analogue of the situation for 2-manifolds, where such a result is given by Riemann's uniformization theorem, it is much less plausible — even counter-intuitive — in the 3-dimensional situation. The case of a manifold fibred over a circle with fibre a surface of genus exceeding 1 seems particularly implausible, and this was the case Thurston examined first. The fibration is determined by a homeomorphism h of the surface, and in seeking to put h (and hence its iterates) into normal form, he was led to consider the images of curves under high iterates of h: these may eventually become dense in some regions, leading to measured foliations. In general, he was led to consider a lamination, which is a disjoint union of injectively immersed curves, which may be dense in some regions and not in others. These ideas gave rise to a geometric model of Teichmüller space and its compactification, which revolutionized thinking in this already highly developed subject.

In this, Thurston was able to draw on his previous work on foliation theory. He swept through this subject producing startling new examples (the Godbillon-Vey invariant takes on uncountably many values), extending the Haefliger foliation theory to closed manifolds by an entirely novel geometric technique, calculating homology of classifying spaces of foliations and relating it to homology of diffeomorphism groups, etc. One dramatic example: any closed manifold of Euler characteristic zero admits a codimension 1 foliation.

The analysis of the diffeomorphisms of a surface concludes with a partition of the surface into well defined pieces, on each of which h has a structure of particularly simple form: the "generic" case is that in which h is Anosov. This partition gives a partition of the total space of the fibration mentioned above, and again we have a geometric structure on each piece. This leads to a reformulation of

the project, which occurred at a timely moment in the independent development of 3-dimensional topology.

In the late 1950's, Papakyriakopoulos obtained fundamental new results on embeddings of discs and spheres into 3-manifolds, one consequence of which was a unique decomposition of M by spheres into irreducible pieces. It seemed that these were largely determined by their fundamental groups, and much work went into studying properties of these, though even a basic question like "are the fundamental groups of knots residually finite?" remained unanswered. A method was developed by Haken and Waldhausen using successive decompositions of M by "essential" embedded surfaces to answer such questions: this gave excellent results in the cases to which it applied. These are the manifolds M formerly called "sufficiently large" but now, following Thurston, "Haken 3-manifolds". The condition is that there exists an embedded surface of positive genus whose fundamental group maps injectively to $\pi_1(M)$. If M has a boundary component of positive genus, or if the first Betti number is non-zero, such a surface can be constructed. Yet more particular are the examples given by Seifert fibre spaces. A close analysis of these by Jaco and Shalen and (independently) Johannsen led to a decomposition of M into a "Seifert" piece and an "atoroidal" piece: in the latter, every embedded torus is parallel to the boundary.

Thurston was now able to conjecture that every irreducible, atoroidal 3-manifold has a hyperbolic structure, and to prove it in the case of Haken manifolds. This includes, for example, the complement of any knot in S^3 (other than torus knots and companion knots) — which allows one to prove the residual finiteness mentioned above. It also led to the solution of the Smith Conjecture — that the fixed point set of a periodic homeomorphism of S^3 is always unknotted — a problem which had attracted a great deal of attention over a forty year period. In fact, Thurston formulates the general conjecture in more attractive terms: every compact 3-manifold has a canonical decomposition into pieces each of which has a geometric structure.

Here, a type of geometric structure is defined by a (simply-connected) model manifold X, with a Riemannian metric, and its group G_x of isometrics. To say that M has a structure of type X means that there are local coordinate charts from M to X, with transformations between different charts given on their overlaps by elements of G_X. From this, Thurston constructs a developing map φ: $\tilde{M} \to X$ from the universal covering of M: the structure on M is complete if φ is a homeomorphism. In order to be of interest, X must satisfy some conditions (e.g. G_X is transitive): such X he calls *geometries*. Thurston then shows that there are just 8 three-dimensional geometries: in addition to the sphere, Euclidean and hyperbolic space, and products of two-dimensional models with a line, there are 3 further cases, in each of which X is a Lie group. Although the hyperbolic structures are by far the most profound, this general theory of geometric structures has clarified and synthesized much previous work in the other cases also: the list of 8 three-dimensional geometries has not been previously obtained.

The main theorem proved to date is that every compact Haken 3-manifold admits a geometrical decomposition as above. However, hyperbolic structures have also been obtained for numerous non-Haken manifolds. There is also an extended conjecture: that if the manifold has a finite group of automorphisms, then a decomposition, and geometric structures on the pieces, can be found so that the group respects these. This is now known in most cases where a decomposition exists. Of particular difficulty are finite group actions on the 3-sphere. Thurston has shown that most of these are equivalent to orthogonal actions, but the fixed-point free case still eludes the method. Results on these problems have been obtained by other authors using minimal surface theory. Thurston's method involves using his main theorem to obtain a hyperbolic structure on the subset where the group acts freely.

As might be expected, the proof of the general result is long and involves many new ideas: it is not yet all available in detail. All I can do here is to mention a few of the ingredients.

Thurston showed that one can pass from any point in Teichmüller space to any other point by a unique "left earthquake". An example of an earthquake is an incomplete Dehn twist: cut a Riemann surface along a simple closed geodesic and then identify the banks after moving each point on one by the same distance. In a general earthquake the simple closed geodesic is replaced by a lamination. Thurston's earthquake theorem was used by his student Kerckhoff to solve the Nielsen realization problem (every finite subgroup of the Teichmüller modular group has a fixed point).

Discrete isometry groups of hyperbolic 3-space were first studied by Poincaré, who dubbed them "Kleinian groups". These have been much studied by analysts, and Ahlfors' finiteness theorem obtained in the 1960's was a fundamental result. Thurston has studied deformations of such groups (in order to patch together hyperbolic structures defined on two pieces of a manifold): this involves a deep study of limit sets. He has shown that a quasiconformal map on S^3 which conjugates one Kleinian group to another extends to H^3 as a quasiconformal volume preserving map with the same property. Typically, one of these groups is Fuchsian, with limit set S^1, but the image Jordan curve is fractal, with Hausdorff dimensional $d > 1$. It can be constructed as the boundary of a disc obtained by bending the standard D^2 along all the curves of a lamination. Thurston has also shown that for a large class of Kleinian groups (including "degenerate" ones), the limit set has measure zero: thus proving another conjecture which had resisted repeated earlier attempts.

Thurston's work has had an enormous influence on 3-dimensional topology. This area has a strong tradition of "bare hands" techniques and relatively little interaction with other subjects. Direct arguments remain essential, but 3-dimensional topology has now firmly rejoined the main stream of mathematics.

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THE WORK OF SIMON DONALDSON

by
MICHAEL F. ATIYAH

In 1982, when he was a second-year graduate student, Simon Donaldson proved a result [**1**] that stunned the mathematical world. Together with the important work of Michael Freedman (described by John Milnor), Donaldson's result implied that there are "exotic" 4-spaces, i.e., 4-dimensional differentiable manifolds which are topologically but not differentiably equivalent to the standard Euclidean 4-space R^4. What makes this result so surprising is that $n = 4$ is the only value for which such exotic n-spaces exist. These exotic 4-spaces have the remarkable property that (unlike R^4) they contain compact sets which cannot be contained inside any differentiably embedded 3-sphere!

To put this into historial perspective, let me remind you that in 1958 Milnor discovered exotic 7-spheres, and that in the 1960s the structure of differentiable manifolds of dimension ≥ 5 was actively developed by Milnor, Smale (both Fields Medallists), and others, to give a very satisfactory theory. Dimension 2 (Riemann surfaces) was classical, so this left dimensions 3 and 4 to be explored. At the last Congress, in Warsaw, Thurston received a Fields Medal for his remarkable results on 3-manifolds, and now at this Congress we reach 4-manifolds. I should emphasize that the stories in dimensions 3, 4 and $n \geq 5$ are totally different, with the low-dimensional cases being much more subtle and intricate.

Although I have highlighted the exotic 4-space as a spectacular corollary of the Freedman/Donaldson results, this is a by-product; their work is actually devoted to studying *closed* 4-manifolds. To such a 4-manifold, one associates standard topological invariants. In particular, for an *oriented* manifold, one gets a symmetric integer matrix of determinant ± 1 defined by the intersection properties of the 2-cycles (and depending on a choice of basis). Freedman showed that all such matrices can occur for topological 4-manifolds. Donaldson's result was that, among positive definite matrices, only those equivalent to the unit matrix can occur for differentiable 4-manifolds.[1] This is a severe restriction and shows that the differentiable and topological situations are totally different.

[1]Actually, in [**1**] Donaldson restricted himself to simply connected manifolds, but more recently he has succeeded in removing this restriction.

The surprise produced by Donaldson's result was accentuated by the fact that his methods were completely new and were borrowed from theoretical physics, in the form of the Yang–Mills equations. These equations are essentially a nonlinear generalization of Maxwell's equations for electro-magnetism, and they are the variational equations associated with a natural geometric functional. Differential geometers study connections and curvature in fibre bundles, and the Yang–Mills functional is just the L^2-norm of the curvature. If the group of the fibre bundle is the circle, we get back the linear Maxwell theory, but for nonabelian Lie groups, we get a nonlinear theory. Donaldson uses only the simplest nonabelian group, namely SU(2), although in principle other groups can and will perhaps be used.

Physicists are interested in these equations over Minkowski space-time, where they are hyperbolic, and also over Euclidean 4-space, where they are elliptic. In the Euclidean case, solutions giving the absolute minimum (for given boundary conditions at ∞) are of special interest, and they are called *instantons*.

Several mathematicians (including myself) worked on instantons and felt very pleased that they were able to assist physics in this way. Donaldson, on the other hand, conceived the daring idea of reversing this process and of using instantons on a general 4-manifold as a new geometrical tool. In this he has been brilliantly successful: he has unearthed totally new phenomena and simultaneously demonstrated that the Yang–Mills equations are beautifully adapted to studying and exploring this whole new field.

Of course, the use of differential equations in geometry is not new; the study of geodesics or minimal surfaces are classical examples. However, in these cases a solution of the differential equation (e.g., a minimal surface) is used as a geometrical object. Donaldson's use of instantons is quite different. I should explain that instantons as solutions of a minimization problem are not unique but typically depend on a finite number of continuous parameters, and it is the nonlinear space of these instanton parameters that Donaldson uses as a geometrical tool. The closest prior example of such an approach is the (linear) Hodge theory of harmonic forms. In fact, Hodge was directly motivated by Maxwell's equations, and instantons are a natural nonlinear generalization of harmonic forms. In the linear case the parameter space is of course linear and determined by its dimension, but in the nonlinear case there is much more information embodied in the parameter space, which is a topologically interesting manifold.

The success of Donaldson's program depends on having a thorough understanding of the analysis of the Yang–Mills equation. On needs existence, regularity, and convergence theorems, all of which are quite delicate, involving both local and global aspects. Fortunately, C. H. Taubes [**6, 7**] and K. Uhlenbeck [**8, 9**] have provided these analytical foundations, and so one can proceed to use instantons as an effective geometric tool. However, instantons cannot be bought off the shelf: to use them one has to understand and become involved with the full details of the analysis, and Donaldson has had to do this in order to put them to geometric use.

The Yang–Mills equations depend on fixing a background metric on the 4-manifold and, as in Hodge theory, Donaldson has to study the effect of varying the metric in order to derive results which depend only on the underlying manifold. Because of the nonlinearity, this is a more serious problem than in Hodge theory and great care is needed.

In fact, the Yang–Mills equations depend only on the conformal class of the metric and this conformal invariance is fundamental in physics where it implies the absence of a basic length scale. Analytically it is a source of difficulty making the equations a delicate border-line case where certain compactness arguments just fail, so that a sequence of instanton solutions can pick up Dirac delta functions in the limit. It is, however, just this delicate failure that Donaldson exploits geometrically: instead of the delta functions being regarded as undesirable singularities, they provide the key link between the 4-manifold and the instanton parameter space. One might say that the physicist's ambivalence to particles and fields is the essence of Donaldson's theory.

When Donaldson proved his first result it was by no means clear if this was some isolated case or whether instantons could be used more generally. Since then, however, Donaldson has, with great insight and skill, developed and exploited instantons with remarkable success. He has extended his results to the case of indefinite intersection matrices, providing further constraints on the topology of differentiable 4-manifolds. He has also, in the other direction, produced new invariants of 4-manifolds which can be used to distinguish smooth manifolds which are topologically equivalent. In particular, he has shown that complex algebraic surfaces (of complex dimension 2 and so of real dimension 4) appear to play a key role. In a very elegant paper [**2**] he proved an existence theorem which showed that, on an algebraic surface, instantons (or rather their parameter spaces) have a purely algebraic description, coinciding with what algebraic gometers call stable vector bundles. His new invariants can then be calculated algebraically and he used this [**3**] to exhibit two algebraic surfaces which are homeomorphic but not diffeomorphic. One of these surfaces is rational and his results strongly suggest that the rationality of an algebraic surface may be a differentiable property (it is *not* topological).

I indicated earlier that mathematicians has been working on the original physicists' problem of explicitly finding all instantons on Euclidean 4-space. In a short but decisive paper [**4**] Donaldson linked this problem with algebraic vector bundles on the complex projective plane (viewed as a compactification of $R^4 = C^2$). He also applied similar ideas [**5**] to solve a related but more difficult physical problem, that of magnetic monopoles. He proved the remarkably simple result that the parameter space of monopoles of magnetic charge k can be identified with the space of rational functions of a complex variable of degree k.

When Donaldson produced his first few results on 4-manifolds, the ideas were so new and foreign to geometers and topologists that they merely gazed in bewildered admiration. Slowly the message has gotten across and now Donaldson's ideas are beginning to be used by others in a variety of ways.

From what I have said you can see that Donaldson has opened up an entirely new area; unexpected and mysterious phenomena about the geometry of 4-dimensions have been discovered. Moreover, the methods are new and extremely subtle, using difficult nonlinear partial differential equations. On the other hand, this theory is firmly in the mainstream of mathematics, having intimate links with the past, incorporating ideas from theoretical physics, and tying in beautifully with algebraic geometry. It is remarkable and encouraging that such a young mathematician can understand and harness such a wide range of ideas and techniques in so short a time and put them to such brilliant use. It is an indication that mathematics has not lost its unity, or its vitality.

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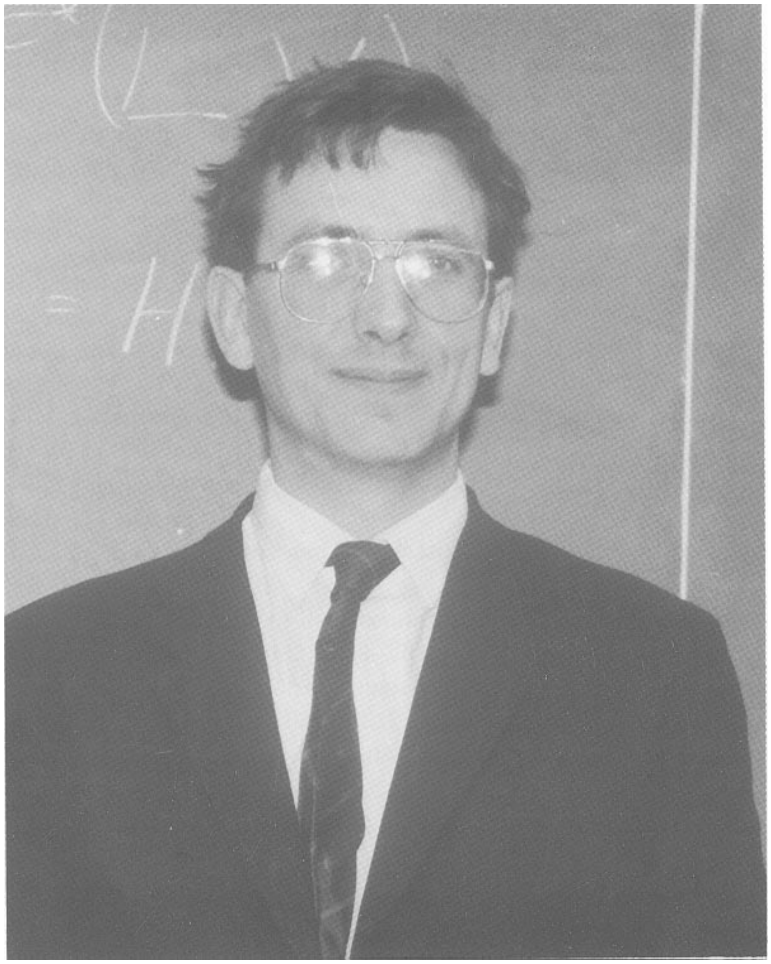

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REMARKS ON GAUGE THEORY, COMPLEX GEOMETRY AND 4–MANIFOLD TOPOLOGY

by

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This article is made up of a number of rather different parts. Nearly all my research work has fallen under two headings:

(1) Differential geometry of holomorphic vector bundles,
(2) Applications of gauge theory to 4-manifold topology.

These two areas intertwine, but have also developed into separate, fairly well-defined fields. In the last part of the article, which is the only part which contains any significant technical content, I will explain how — motivated by recent work of Tian — one can extend some of the ideas which are well known under heading (1) above to problems in Kahler–Einstein geometry. The second and third parts of the article deal with heading (2), and I will give a very rapid survey of the development of this field, mainly through reference to other survey articles, and describe a result of Fintushel and Stern in order to illustrate some of the problems of current research interest. In the first part of the article, as a piece of self-indulgence, I will reminisce briefly about my own early encounters with these two fields.

1. Reminiscences, 1980–83

When I arrived in Oxford in 1980 as a beginning doctoral student I was fortunate to be presented with a problem by my research supervisor, Nigel Hitchin, which turned out to be outstandingly fruitful and prescient. The problem concerned Yang–Mills connections on holomorphic vector bundles over Kahler manifolds. If X is a Kahler manifold and E is a holomorphic vector bundle over X then the curvature $F(A)$ of a compatible connection A on E is a bundle-valued form of type (1, 1). The Kahler metric defines a contraction $\Lambda : \Omega^{1,1} \to \Omega^0$ so one gets an endomorphism $\Lambda F(A)$ of E. The equation $\Lambda F(A) = 0$ is a special form of the Yang–Mills equations, which co-incides with the instanton equation in the case of complex dimension 2, and Hitchin suggested that the condition that a holomorphic bundle admit such a connection should be related to the algebro-geometric condition of *stability*. (Similar conjectures were made independently at about the same time by Kobayashi). One

of the main pieces of the evidence for the conjecture was the old (1963) result of Narasimhan and Seshadri, dealing with the case of Riemann surfaces, which had been cast in "Yang–Mills" form, and studied in great depth, by Atiyah and Bott shortly before [1]. I soon began to tinker with analytical approaches to Hitchin's conjecture, perhaps motivated by the renowned work of Yau, a few years before, on the Calabi conjecture. I also spent many hours with the paper of Eells and Sampson [15] on harmonic maps, since it seemed a good idea to try to find the desired connections as a limit of solutions to a nonlinear "heat" equation, in the manner pioneered by Eells and Sampson. In one way or another I put together various things in the year 1980–81 which seemed to me to be heading towards an analytical proof of Hitchin's conjecture, but in the autumn of 1981 this work was interrupted by the realisation that Yang–Mills theory, in the shape of the instanton equations, could be applied to problems in 4-manifold topology.

This topological turn came about as an accidental by-product of the project described above. In retrospect, one can certainly see that this development was bound to occur sooner or later. The decisive shift was really one of attitude: the main focus of the work by mathematicians in the late 1970's on the instanton equation involved the interaction with twistor geometry, culminating in the famous ADHM construction of all instantons over 4-sphere, and this discussion was tied to manifolds with special geometric structures, rather than general Riemannian 4-manifolds. For example large parts of the foundational paper [2] of Atiyah, Hitchin and Singer carry over to general 4-manifolds without any change, although the whole paper restricts attention to the class of "self-dual" manifolds. Perhaps the first work in which the general case was considered was that of Taubes [30], which proved an existence theorem for instantons using Taubes' celebrated "grafting" technique. In fact this change in emphasis was not completely clear-cut in Taubes' work: the first version of [30] restricted attention to a manifolds admitting a metric with a certain positive curvature property (which was exploited in a Weitzenboch formula). The revised, published, version relaxed this hypothesis to consider 4-manifolds with no self-dual harmonic forms. It is a very simple consequence of the Hodge theory, although this was not pointed out explicitly by Taubes in [30], that this condition is a purely topological one — it is the same as saying that the intersection form of the 4-manifold is negative definite.

I studied Taubes' paper in detail in 1980–81: it fitted in with my thinking about Hitchin's problem in the following way. In studying the nonlinear heat equation mentioned above the essential thing was to obtain analytical compactness theorems which would allow one to get some kind of limit. This is closely related to understanding the compactness of instanton moduli spaces: now the algebro-geometric literature contained various examples of moduli spaces of stable bundles, and one can observe in these examples that the moduli spaces have natural compactifications in which one adjoins points "at infinity" made up of configurations of point in the underlying complex surface. It was therefore natural to make the hypothesis,

assuming the conjectured relation between instantons and holomorphic bundles, that instanton moduli spaces over general 4-manifolds should be compactified by adjoining configurations of points — this is the now well-known "bubbling" phenomenon, the points correspond to sequences of connections whose curvature densities converge to delta-functions on the manifold. Taubes' grafting construction then appears as the inverse procedure, constructing connections "near to infinity" in the moduli space.

In fact the papers of Uhlenback [32, 33] which appeared about that time contained essentially all the analysis required to put this picture on a firm footing. The papers do not discuss "bubbling" explicitly — perhaps the arguments were supposed to be obvious to experts by analogy with the work of Sacks and Uhlenbeck [27] in the harmonic maps case — so it took some months before I really absorbed the material, and in that time it seemed natural to test the compactification hypothesis by contemplating various examples. How about the 4-manifold $\overline{\mathbf{CP}}^2$ (the complex projective plane with reversed orientation)? This has negative definite form, so Taubes' work applies. The deformation theory of [2] says that the moduli space M (in the simplest, charge 1, case) is 5-dimensional and it is natural to guess that Taubes' construction produces a collar $\overline{\mathbf{CP}}^2 \times (0,1) \subset M$ and the complement should be compact, according to the ideas above. But I had learnt about cobordism theory, so something funny must happen inside M, since the complex projective plane is not a boundary. The solution of course is that there is a reducible connection in the moduli space, where the bundle reduces to $S^1 \subset \mathrm{SU}(2)$, and this is a singular point. In fact, in view of the symmetry group $\mathrm{SU}(3)$ of the projective plane which must act on M, about the only thing that could happen is that M is a *cone* over $\overline{\mathbf{CP}}^2$, with the reducible connection as vertex. (This was later confirmed by explicit constructions of Buchdahl [6] and myself [8].) From this it was a short step to consider a general 4-manifold with definite intersection form, obtaining the picture described in the accompanying article, reproduced from the Proceedings of the 1983 International Congress, and leading to the conclusion that the intersection form must be standard. The point I wish to make is that the chain of reasoning was to a large extent a product of naivete; the initial impetus being the desire to test the compactification hypothesis for instanton moduli spaces. Moreover, it was not immediately clear to me what use the argument should be put to: I did not even know that there were any non-standard quadratic forms! At that time I shared an office in the Mathematical Institute with Mike Hopkins, and he put me straight on this point. Then one wondered if it was known for elementary reasons that 4-manifolds could not have non-standard intersection forms — I had no particular acquaintance with 4-manifold theory at that time. Another possibility seemed to lie in the orientation of the moduli space: maybe there were subtleties of index theory involved here; but this idea did not last long since Atiyah (who was supervising my work by this time) quickly furnished a proof that the moduli spaces were orientable. All in all, within a few months, it became clear that if this chain

of reasoning could be made solid then one would get important new results about smooth 4-manifolds, in sharp contrast with those of Freedman (which appeared, by coincidence, at much the same time) in the topological case.

I would like to mention one other ingredient in this chain of ideas which raises certain wider issues. This concerns the local structures of the instanton moduli spaces. Atiyah, Hitchin and Singer's deformation theory for instantons was adapted from the work of Kuranishi and others, stretching back to Riemann, on the deformations of complex structures on manifolds — the main ideas apply to a wide variety of structures. In these theories one has cohomology groups H^0, H^1, H^2, and the moduli space of structures is locally a manifold, modelled on H^1, if H^0 (which is the Lie algebra of the automorphism group) and H^2 (the obstruction space) vanish. Now the classical problems to which this theory is applied are typically very "rigid" — there is just *one* moduli space of complex structures associated to a particular smooth manifold; take it or leave it. The obstruction spaces H^2 (for example in the case of complex surfaces) are often very large, and it is hard to say much about the moduli spaces on general grounds. The instanton problem, at least once one moves away from 4-manifolds with special structures, is rather different since one has the infinite dimensional space of metrics on the underlying 4-manifold which appear as parameters: each metric gives a moduli space. One can go further and consider arbitrary deformations of the instanton equation of various kinds: provided the deformation is reasonably controlled these give moduli spaces of solutions which serve just as well for the topological arguments one wants to make. Thus the situation is very "flexible". With this shift in perspective one is studying the instanton moduli spaces essentially in the framework of "Fredholm differential topology" and there are general transversality arguments that mean one can reduce to the case when the obstruction spaces vanish. The arguments are not difficult, and go back at least to Smale [29]: the issue is more a matter of change of view-point than anything else. One can carry this "flexible" point of view over to other moduli problems: contemplating what the structure of a moduli space should be after some kind of generic deformation. One area in which this has been done is in the moduli spaces of complex curves, where the generic deformations arise naturally from the consideration of almost-complex structures. But in other problems it is not so clear what significance such generic deformations should have. Another issue is that in higher dimensions one encounters over-determined equations, reflected in further cohomology groups $H^i, i \geq 3$, and it is not so clear what the right differential topological framework should be. For example, in theory of the Casson invariant, which is closely allied to the instanton theory, one is familiar with the idea that the space of irreducible SU(2)-representations of a 3-manifold fundamental group $\pi_1(M^3)$ "should be" a finite set of points; in the sense that it becomes so after a generic perturbation, although the actual representation space could be a variety of higher dimension. It is not clear whether there is any useful extension of this notion to representations of the fundamental group of higher dimensional manifolds.

I will now close this reminisence with a few concluding remarks. First, for completeness, I will go back to the conjecture of Hitchin and Kobayashi. I put together a proof of the conjecture for the case of complex surfaces in mid-1983 [7]: this proof was really a hybrid of two different lines of attack, one of which was written up in [14] and the other, dealing with the higher dimensional case but using a substantial input from algebraic geometry, in [10]. Meanwhile Uhlenbeck and Yau, deploying more powerful analytical techniques, gave what is probably the most natural proof in [34]. Turning back to 4-manifolds: a striking development, which came out in late 1983 was in the work of Fintushel and Stern who found an approach which gave the main results on definite forms, also using instanton moduli spaces but for connections with structure group SO(3). Their approach avoided the analysis of the boundary of the moduli space, which was precisely the most difficult technical part of the original argument and as I have described above, the route by which the argument was discovered! Finally, I would like to mention the beginnings of the development of 4-manifold *invariants* defined by instanton moduli spaces. In their applications, these are complementary to the first results: rather than showing that some intersection forms do not occur for any smooth 4-manifold, one wants to distinguish manifolds with the same intersection form. The general scheme by which one could try to obtain invariants from instanton moduli spaces was fairly clear to me by the middle of 1983: that is, one considers the pairings

$$\langle\, \mu(\alpha_1) \cup \cdots \cup \mu(\alpha_r), [M] \,\rangle\,, \tag{1}$$

where $[M]$ is the fundamental homology class of a moduli space, interpreted via one's understanding of its compactification, and the $\mu(\alpha_i)$ are cohomology classes over the moduli space which one can construct by a general procedure (that I had learnt from the paper of Atiyah and Bott). The problem for some time was how to calculate any examples, and using these to obtain new information about 4-manifolds. But what seemed to be clear was the existence of the following dichotomy: *either* these pairings would give new information, which would be good, *or* they could be expressed in terms of some known data, in which case it would be a good problem to find formulae for them.

2. Gauge Theory and 4–Manifolds: Survey of Surveys

Following the reminiscence above, I will now pass rapidly over the ensuing decade. An enormous amount of work was done in this period, by many mathematicians, on the interaction between gauge theory (in the shape of the instanton equations) and 4-manifold topology; and there exist a number of surveys of these developments. My contribution [9] to the Proceedings of the 1986 International Congress was written when the foundations of the theory, and the first applications, were complete. This is also roughly the material covered in the book [14]. In the late 1980's there was a great volume of work, much of it devoted to calculations of particular invariants. Perhaps the first breakthrough in the way of calculations

was the work of Friedman and Morgan on elliptic surfaces [18]. One can also refer to the article [17], which contains many conjectures which set the direction for much of this research. My contribution [11] to the Proceedings of the 1992 European Congress surveys this period of very diverse developments. A particularly notable development in the late 1980's came through the work of Floer, leading to the Floer homology groups of 3-manifolds and the development of "cutting and pasting" formulae for instanton invariants. This is a topic which is not very well served, at present, in the literature, but there are various expositions of the ideas, for example [12] and the articles in the volume [19].

In a sense the survey article [11] was written at the "wrong" time, coming just before the dramatic discoveries by Kronheimer and Mrowka [23] of fundamental structural formulae for the instanton invariants (although this timing may perhaps add to its interest as a historial record). Let us recall that the simplest set of invariants of a 4-manifold X are polynomial functions on $H_2(X)$:

$$f_d(\alpha) = \langle\, \mu(\alpha)^d, M_X^{2d}\,\rangle\,, \tag{2}$$

where M_X^{2d} is a moduli space of instantons over X of dimension $2d$. Kronheimer and Mrowka's work applies to simply connected 4-manifolds X with $b^+(X)$, the dimension of a maximal positive subspace for the intersection form, at least 3, and satisfying a further technial condition of being of "simple type". In this case they show that the generating function

$$f(\alpha) = \sum_d \frac{1}{d!}\, f_d(\alpha)$$

can be written as

$$f(\alpha) = e^{\alpha.\alpha/2} \sum_i a_i e^{\kappa_i.\alpha}\,, \tag{3}$$

where the κ_i run over a finite collection of "basic classes" in $H^2(X)$, and the a_i are numerical co-efficients. Kronheimer and Mrowka's proof was long and difficult, but their result lead to a radical advance in the field: it became possible to find all the invariants for many 4-manifolds whereas a few years before it was a major task to compute a single one.

Hard on the heels of this structural theorem came the renowned work of Seiberg and Witten (1994) which more-or-less brings our story up to date. All the work we have discussed so far falls within the realms of geometry, topology and global analysis. While some of the technicalities may be complicated, the foundations are perfectly secure. By contrast, Seiberg and Witten's work depended essentially on Quantum Field Theory. Witten had shown in 1989 [35] that the cohomology pairings could be obtained from functional integrals of the shape

$$\iint e^{-\|F(A)\|^2} \mathcal{F}(A,\phi)\mathcal{D}A\mathcal{D}\phi\,, \tag{4}$$

where the integral runs over the space of connections A and certain auxiliary fields ϕ. In 1994, Seibergy and Witten carried through a sophisticated analysis of this Quantum Field Theory invoking a new "S-duality" principle. A prominent feature of their theory is a complex parameter u, related to the auxiliary fields appearing in the functional integral, and S-duality goes over to modular properties with respect to the parameter u. Geometrically, u parametrises a family of elliptic curves, which degenerate when $u = \pm 1$. The analysis of the theory at the points ± 1 lead Seiberg and Witten to write down their celebrated equations; these are classical equations for a connection a on a $U(1)$ bundle $L \to X$ and a spinor field ψ over a 4-manifold, coupled to the line bundle L. The Seiberg–Witten equations have the shape:

$$D_a\psi = 0\,, \qquad F_a^+ = \psi^*\psi\,.$$

(The equations really only involve a Spin^c structure, so one does not need to restrict to Spin 4-manifolds.) Seiberg and Witten predicted that the basic classes κ_i of Kronheimer and Mrowka, and the co-efficients a_i, can be obtained in a more direct way from the moduli spaces of solutions to the Seiberg–Witten equations. But in any event, as Witten [36] and others quickly showed, the new equations could be used to obtain essentially all the results of the old theory in a much simpler way, and many new results besides. These developments, at least up to the spring of 1995, are surveyed in the article [13].

It is amusing to speculate about the other ways in which this story might have run. It would have certainly been quite possible for geometers to have written down the Seiberg–Witten equations, and discovered the invariants they lead to, at any time in the last decade — they fit in well with a large body of work on the differential geometry of Yang–Mills–Higgs theories, albeit that this has mainly emphasised the case of Riemann surfaces. Indeed one can imagine a story in which the Seiberg–Witten equations were discovered before the instanton theory in which case, going back to the dichotomy at the end of the previous section, it would have been very natural to search for formulae relating the two.

3. Elliptic Functions and Modular Forms

The structural formula (3) of Kronheimer and Mrowka pretty much completes the theory of the instanton invariants for a large class of 4-manifolds. But when one moves outside this class, for example to manifolds such as the complex projective plane with $b^+ < 3$, many interesting questions remain. At the time of writing great progress is being made on these questions by Gottsche and other [20, 21], the striking thing being the appearance of complicated formulae involving modular forms. The first formula of this kind was found in 1993 by Fintushel and Stern [16]. This deals with the relation between the invariants of a 4-manifold X and its "blow-up" $X' = X\#\overline{\mathbf{CP}}^2$. One can analyse a moduli space $M_{X'}$ of instantons over X' by considering a family of Riemannian metrics on X' in which the "neck" of the connected sum is pulled out into a long tube. This is the standard technique in

the Floer theory and, roughly speaking, it allows instantons over X' to be analysed in terms of solutions over the two summands. It is quite straightforward to obtain formulae in this way for low-dimensional moduli spaces $M_{X'}$, or more precisely when the "contribution" to the dimension from the $\overline{\mathbf{CP}}^2$ side is small. If we write e for the generator of $H_2(\overline{\mathbf{CP}}^2) \subset H_2(X')$ then one wants to compute

$$\langle\, \mu(e)^d \phi', M_{X'} \,\rangle\,, \tag{5}$$

where ϕ' is a cohomology class over the moduli space associated to homology classes in X. For general reasons one expects this to be expressed in the form

$$\Big\langle \sum b_{d,i} v^i \phi, M_X \Big\rangle\,, \tag{6}$$

where ϕ is the cohomology class over M_X defined in the same way as $\phi' \in H^*(M_{X'})$, $v \in H^4(M_X)$ is the 4-dimensional class $\mu(\text{point})$ and the $b_{d,i}$ are universal numerical co-efficients. The problem becomes harder as the exponent d increases: the difficulty is that the points at infinity in the compactifications of the moduli spaces play an essential role, and a direct attack — as in [25] — requires a detailed and laborious description of the ends of the moduli spaces. Fintushel and Stern found a trick which by-passes these difficulties. They considered the double blow-up $X'' = X\#\overline{\mathbf{CP}}^2\#\overline{\mathbf{CP}}^2$, with two cohomology classes $e_1, e_2, \in H_2(X'')$ corresponding to the two summands. The class e_1+e_2 is represented by a sphere of self-intersection -2, so X'' can also be regarded as a "generalised connected sum"

$$X'' = Y\#_{P^3}N\,, \tag{7}$$

where P^3 is a copy of the real projective 3-space embedded in X'', decomposing this 4-manifold into a neighbourhood N of the -2 sphere and its complement Y. Now the same neck-stretching analysis can be applied to this description of N, much as for connected sums (since P^3 is not very different from a 3-sphere). In particular we can analyse a pairing of the form

$$\langle\, \mu(e_1+e_2)^4\psi, M_{X''} \,\rangle\,,$$

where e_1+e_2 is the generator of $H_2(N)$ and ψ is a cohomology class associated to the other piece Y. This is similar to the analysis of the problem (5) on X' with the particular value $d=4$, which is small enough to be tractable, using the arguments like those of [18]. One gets a formula relating the pairings for three different moduli spaces: $M_{X''}, M^*_{X''}, M^{**}_{X''}$ say, over X'':

$$\langle\, \mu(e_1+e_2)^4\psi, M_{X''} \,\rangle = -4\langle\, \mu(e_1+e_2)^2 v\psi, M^*_{X''} \,\rangle - 4\langle\, \psi, M^{**}_{X''} \,\rangle\,,$$

where v is the 4-dimensional class, as before. In particular, Fintushel and Stern observed that this applies when ϕ is a power $\mu(e_1-e_2)^n$, since the class e_1-e_2 is supported in Y, so

$$\begin{aligned}\langle\, \mu(e_1+e_2)^4\mu(e_1-e_2)^n, M_{X''} \,\rangle = &-4\langle\, \mu(e_1+e_2)^2\mu(e_1-e_2)^n v, M^*_{X''} \,\rangle \\ &-4\langle\, \mu(e_1-e_2)^n, M^{**}_{X''} \,\rangle\,.\end{aligned}$$

On the other hand, the different terms can be expanded out using the blow-up formula (6) for the two summands, and the formula translates into a recursion relation for the co-efficients $b_{d,i}$. The key thing is that the exponent n is arbitrary — increasing n does not make the stretching analysis any harder in the decomposition (7) — and this gives information about the high values of d which appear intractable from the point of view of the original connected sum decomposition. (Actually, Fintushel and Stern's argument does rather more than we have said, since it also gives the existence of the general shape (6) of the formula, without having to assume this *a priori*.) In turn, by elementary manipulation, the recursion relation for the co-efficient $b_{d,i}$ can be expressed as a differential equation for the generating function

$$B(x,t) = \sum b_{d,i} x^i t^d \,.$$

The differential equation is

$$B^{(4)}B - 4B'''B' + 3(B'')^2 + 4x(B''B - (B')^2) + 2B^2 = 0\,,$$

where all derivatives are with respect to t, the variable x being thought of as a parameter. This can be integrated three times, using information about some low order terms to fix the constants of integration, to show that the function $y = -(\log B)'' - \frac{x}{3}$ satisfies the familiar equation

$$(y')^2 = 4y^3 - g_2 y - g_3\,,$$

defining the Weierstrasse $\wp$-function, where $g_2 = 4\left(\frac{x^2}{3} - 1\right)$, $g_3 = \frac{8x^3 - 36x}{27}$, and Finstushel and Stern deduce finally that

$$B(x,t) = e^{-t^2 x/6}\sigma_x(t)\,,$$

where $\sigma_x(t)$ is a particular Weierstrasse σ-function (a solution of $\left(\frac{\sigma'}{\sigma}\right)' = -\wp$) corresponding to these values of g_2, g_3.

While the geometric input into the argument of Fintushel and Stern is comparatively straightforward, the argument is indirect and one gets little insight into why the answer should turn out to involve these particular special functions. The work of Gottsche has a similar flavour. Thus what is lacking at the moment is a conceptual proof of these formulae. Elliptic curves and the associated functions have a central place in the Seiberg–Witten Quantum Field Theory analysis, so it does seem very likely that there are conceptual derivation from that point of view. On the other hand the Quantum Field Theory arguments of Seiberg and Witten lie at present a long way outside the boundary of standard mathematics: both with regard to the techniques and language, and probably also with regard to the underlying foundations (because of the notorious problems of defining functional integrals rigorously). So an exciting problem at the moment is to throw some kind of bridge across this gap. As well as the conceptual attraction of this problem, there are many

concrete open questions that one would like to answer. For example one would like to blend this blow-up formula of Fintushel and Stern into the general Floer theory, which applies to other kinds of decompositions of 4-manifolds — the equivariant Floer theory of Austin and Braam [3] is limited at present by precisely the same kind of dimensional restrictions which obstructed a direct attack on the blow-up formula.

The questions we have outlined in the previous paragraph, concluding our brief survey of this line of work on 4-manifolds and gauge theory, can perhaps be best thought of in a wider context. Over the last decade ideas from Quantum Field Theory have cut a wide swathe through many different fields of Geometry and Topology, often predicting results that are, at best, only proved later by more laborious, less perspicuous, techniques. So it is natural to expect that the line of work we have been discussing will flow into this larger stream, which probably makes up the dominant and most exciting line of research in geometry at the moment.

4. Stability and Kahler–Einstein Geometry

In this section we return to some of the themes from complex differential geometry, sketched in Section 1. The Hitchin–Kobayashi conjecture involves the solution of a nonlinear partial differential equation, and a certain amount of detailed analysis is inevitably involved. However the problem fits into a general framework, in large part going back to Atiyah and Bott [1], involving the geometry of infinite dimensional groups and spaces. Apart from anything else, this framework is useful as a guide to the analysis, and the differential-geometric calculations that arise. Let us briefly recall this "standard picture".

Suppose we have the following data:

(1) a complex manifold $\mathcal{C}$ with Kahler metric, giving a symplectic form Ω;
(2) a Lie group $\mathcal{G}$ which acts on $\mathcal{C}$ by holomorphic isometries;
(3) an equivariant *moment map* $\mu : \mathcal{C} \to \mathrm{Lie}(\mathcal{G})^*$;
(4) a complexification $\mathcal{G}^c$ of $\mathcal{G}$, and an extension of the action to a holomorphic action of $\mathcal{G}^c$ on $\mathcal{C}$

Then the general principle is that one has an identification of "symplectic" and complex quotients:

$$\mathcal{C}_s/\mathcal{G}^c = \mu^{-1}(0)/\mathcal{G}\,, \tag{8}$$

where $\mathcal{C}_s \subset \mathcal{C}$ is an open, $\mathcal{G}^c$-invariant, subset of "stable points". The important thing is that the notion of stability should depend only on the holomorphic geometry of the situation, while the symplectic quotient depends on the Hermitian geometry. To define the stable points it is useful to suppose that we have the additional data:

(5) a $\mathcal{G}^c$-equivariant holomorphic line bundle $\mathcal{L} \to \mathcal{C}$, with a $\mathcal{G}$-invariant unitary connection having curvature $i\Omega$ and with the action defined by the moment map μ. (This means that the infinitesimal action $\xi \in \mathrm{Lie}(\mathcal{G})$ on a section s of $\mathcal{L}$ is $\nabla_{v(\xi)}s + i\mu(\xi)s$, where $v(\xi)$ is the vector field on $\mathcal{C}$ defined by ξ.)

Then to test the stability of a point $x \in \mathcal{C}$ we consider the $\mathcal{G}^c$-orbit $\mathcal{O} \subset \mathcal{L}$ of any non-zero $\hat{x} \in \mathcal{L}$ lying over x. The point x is stable if the orbit $\mathcal{O}$ is a *closed* subspace of the total space of the line bundle. The point x satisfies $\mu(x) = 0$ if and only if $\hat{x}$ minimises the norm within its orbit $\mathcal{O}$: the content of the main principle (8) is that the norm has a minimum, unique up to the action of $\mathcal{G}$, in each closed orbit.

In finite dimensions the main example of this set-up is when $\mathcal{G}$ is a compact Lie group, $\mathcal{C}$ is a complex projective space and $\mathcal{L}$ is the tautological bundle. Thus we are dealing with linear actions of $\mathcal{G}$ and its complexification on $\mathbf{C}^n$, and the orbits in the tautological bundle can be identified with the orbits in $\mathbf{C}^n$. This is the setting for the Hilbert–Mumford Geometric Invariant Theory, which is the original source of these ideas [24].

There are a number of general features of this set-up which give interesting results in particular cases. For example, it is easy to prove in the general picture that if $x \in \mathcal{C}$ is a point with $\mu(x) = 0$ then the stabiliser of x in $\mathcal{G}^c$ is the complexification of the stabiliser in $\mathcal{G}$. Now, following an idea which apparently goes back to Cartan (see [22], page 196), and which was carried through by Richardson [26], consider a finite-dimensional complex vector space V with a fixed non-degenerate symmetric form and the action of $\mathcal{G}^c = O(V)$ on the subspace

$$\mathcal{C} \subset \mathbf{P}(\mathrm{Hom}(V \otimes V, V)),$$

representing Lie brackets (i.e. solutions of the Jacobi identity), whose Killing form is the given form, and so make V into a semi-simple Lie algebra. The subgroup $\mathcal{G}$ is the compact subgroup of maps which preserve in addition a fixed Hermitian form on V. Then, just as in the first part of [26], one shows that all the points in $\mathcal{C}$ are stable. Then one can apply the general principles above to find a norm-minimising representative $\hat{x}$ for any isomorphism class of semi-simple Lie algebra structure on V. The stabiliser of $\hat{x}$ in $\mathcal{G}^c$ is the Lie group corresponding to this Lie algebra, and one obtains the result that this group is the complexification of a compact subgroup — the stabiliser in $\mathcal{G}$. This point of view allows one to simplify the calculations in [26], as well as fitting them into a general setting.

The Hitchin–Kobayashi problem leads to a well-known infinite-dimensional example of this standard picture. Here one considers a C^∞ vector bundle E over a Kahler manifold X with a fixed Hermitian metric on E. The space $\mathcal{C}$ is the space of $\overline{\partial}$-operators on E, which can also be regarded as the space of unitary connections. The group $\mathcal{G}$ is the group of unitary automorphisms of E, which acts on $\mathcal{C}$ in the ordinary way by gauge transformations. The complexification $\mathcal{G}^c$ is the group of complex linear automorphisms of E which acts on $\mathcal{C}$ by conjugation of the $\overline{\partial}$-operators. There is a $\mathcal{G}^c$-invariant subspace $\mathcal{C}^{\mathrm{int}}$ of integrable $\overline{\partial}$-operators and the orbits in this subspace precisely parametrise the unitary connections compatible with a given holomorphic structure. The moment map μ can be identified with the contraction ΛF of the curvature of a connection. Provided the underlying manifold

X is algebraic, there is also a line bundle $\mathcal{L}$ over $\mathcal{C}$, which can be identified with a certain "determinant line bundle" [10]. In this setting the Kobayashi–Hitchin conjecture can be regarded as two statements:

(1) the identification $\mathcal{C}_s/\mathcal{G}^c = \mu^{-1}(0)/\mathcal{G}$ holds in this infinite-dimensional case,
(2) the notion of stability in the infinite-dimensional picture matches up with the notion of stability found earlier by algebraic-geometers (by embedding the bundle-classification problem in a finite-dimensional Geometric Invariant Theory picture of the above kind.)

Now we turn to the recent work of Tian [31]. This concerns the question: when does a compact complex manifold V admit a Kahler–Einstein metric? More precisely one considers some initial Kahler metric ω_0 and the set of metrices in the same cohomology class, which can be represented in the form $\omega = \omega_0 + i\overline{\partial}\partial\phi$, for a real-valued Kahler potential ϕ. One seeks a metric which satisfies the Einstein equation: $\mathrm{Ric} = \lambda g$. The cases when $\lambda \leq 0$ are well-understood through the renowned work of Calabi, Yau and others. When $\lambda > 0$ the problem is still not completely solved. It has been known for a long time that the there are obstructions to the existence of a solution coming from the holomorphic vector fields on the manifold: Tian finds a new kind of obstruction involving Geometric Invariant theory. An obvious condition for a solution with $\lambda > 0$ to exist is that the complex manifold V be a *Fano manifold* i.e. that the anticanonical bundle K_V^* be ample. Then for large s the sections of K_V^{-s} give a projective embedding $V \to \mathbf{P}^n$, and hence an orbit $[V]$ of the natural $SL(n+1)$ action on the Hilbert scheme parametrising subvarieties of $\mathbf{P}^n$. Tian proves that if V admits a Kahler–Einstein metric this orbit must be stable, and conjectures that the converse is true.

This conjecture of Tian, whose solution would represent a complete solution of the existence problem for Kahler–Einstein metrics, clearly falls into the same general pattern as the Kobayashi–Hitchin conjecture, so it is natural to ask whether the problem can be set up in the standard picture described above. We will now show that, with one reservation, this can indeed be done. (This builds on unpublished work of Atiyah and Quillen in the case of one complex dimension.)

Consider a compact symplectic manifold (M^{2n}, ω), and suppose for simplicity that $H^1(M) = 0$. Let $\mathcal{J}$ be the space of almost-complex structures on M which are compatible with ω. This is the space of sections of a bundle over M with fibre the Siegel upper half space $Sp(2n)/U(n)$, which has a $Sp(2n)$-invariant Kahler metric. This fibre-metric, together with the volume form on M, endows $\mathcal{J}$ with the structure of an infinite-dimensional Kahler manifold. More explicity, given one almost complex structure $J \in \mathcal{J}$ a variation of almost complex structure can be represented by a tensor $\mu \in \Omega_J^{0,1}(T)$, where we use the usual notation from complex geometry. The (1, 0) forms for the new almost-complex structure J' have the shape $\alpha + \mu(\alpha)$, where α is in $\Omega_J^{1,0}$. Using the hermitian metric defined by J and ω we can identify $\overline{T}^*$ with T, and so $\Omega^{0,1}(T)$ with the sections of $T \otimes T$. Then, to first order in μ, the condition that J' be compatible with ω is that μ is a

section of $s^2(T) \subset T \otimes T$. So the tangent space of $\mathcal{J}$ at J is the space of sections of the complex vector bundle $s^2(T)$, and this has a Hermitian metric defined by integration over M, in the usual way. It is not hard to see that this hermitian structure on $\mathcal{J}$ is Kahler — by Darboux's theorem one can essentially reduce to the case of $\mathrm{Maps}(M, Sp(2n)/U(n))$. We may also consider an infinite dimensional complex subvariety $\mathcal{J} \overset{\text{int}}{\subset} \mathcal{J}$ representing integrable almost complex structures.

Now let $\mathcal{G}$ be the identity component of the group of symplectomorphisms of (M, ω). The Lie algebra of $\mathcal{G}$ can be identified with the space C_0^∞ of functions on M with integral 0: a function H defines the usual Hamiltonian vector field X_H on M with

$$i_{X_H}(\omega) = dH\,.$$

We want to identify a moment map for the action of $\mathcal{G}$ on $\mathcal{J}$. To do this we review some (almost) complex differrential geometry. Suppose E is any complex vector bundle over M and ∇ is a covariant derivative on E. Then given an almost-complex structure $J \in \mathcal{J}$ we may decompose $\nabla = \nabla'_J + \nabla''_J$, where $\nabla'_J : \Omega^0(E) \to \Omega_J^{1,0}(E)$ and $\nabla''_J : \Omega^0(E) \to \Omega_J^{0,1}(E)$. We observe that if E has a hermitian metric there is a unique way to reconstruct a compatible covariant derivative ∇ from its (0, 1) component ∇''_J — the proof is just the same as in the familar, integrable, case. If E is the trivial bundle we write $\partial_J, \overline{\partial}_J$ for the (1, 0) and (0, 1) components of the derivative d. We may prolong these to the other differential forms, getting

$$\partial_J : \Omega_J^{p,q} \to \Omega_J^{p+1,q}\,,$$
$$\overline{\partial}_J : \Omega_J^{p,q} \to \Omega_J^{p.q+1}\,,$$

but it is not in general true that $d = \partial + \overline{\partial}$. This is where the *Nijenhius tensor* N — the obstruction to integrability — enters. The tensor N lies in $T \otimes \Lambda^{0,2}$ and one has

$$d = \partial + \overline{\partial} + i_N + i_{\overline{N}}\,,$$

where $i_N : \Omega_J^{p,q} \to \Omega_J^{p-1,q+2}, i_{\overline{N}} : \Omega_J^{p,q} \to \Omega_J^{p+2,q-1}$ are algebraic operators defined by contraction and wedge product with N.

For our immediate purposes, we just need to consider the operator $\overline{\partial}_J : \Omega_J^{1,0} \to \Omega_J^{1,1}$. This can be regarded as a ∇''-operator on the bundle T^*M, viewed as a complex vector bundle over M using the structure J. By the observation above there is therefore a preferred unitary connection ∇ on T^*M and hence a connection on the "canonical line bundle" $K_M = \Lambda_C^n T^*M$. Let ρ be $-i$ times the curvature form of this connection on K_M, so ρ is a real 2-form on M. Finally define a function S on M by

$$S\omega^n = n!\,\rho \wedge \omega^{n-1}\,.$$

We might call S the *Hermitian scalar curvature*: in the integrable case the connection above coincides with the Levi–Civita connection and, up to a factor,

S is the usual Riemannian scalar curvature. We can now state the result we have been driving at:

Propostion 9. *The map $J \mapsto S(J)$ is an equivariant moment map for the $\mathcal{G}$-action on $\mathcal{J}$, under the natural pairing*:

$$(S, H) \mapsto \int_M SH \frac{\omega^n}{n!}\,.$$

We proceed now with the proof of Proposition 9. Fix an almost-complex structure $J \in \mathcal{J}$; let

$$P : C_0^\infty \to \Gamma(S^2(T)) \subset \Omega_J^{0,1}$$

be the operator representing the infinitesimal action of $\mathcal{G}$ on $\mathcal{J}$ and

$$Q : \Gamma(s^2(T)) \to C_0^\infty$$

be the operator which represents the derivative of $J \mapsto S(J)$. By the definition of a moment map what we need to show is that, for all H and μ, $(P(H), J\mu) = (H, Q(\mu))$, where $(\ ,\ )$ denotes the usual (real) L^2 inner product. Expressed in terms of the formal adjoints this is $Q = P^* \circ J$. The operator P can be factored as $P = P_2 P_1$, where $P_1 : C_0^\infty \to \Gamma(T)$ maps H to the Hamiltonian vector field X_H and P_2 maps a vector field X to the infinitesimal variation in complex structure given by the Lie derivative $L_X J$.

Lemma 10. *$L_X J = \nabla''_J X - \overline{X}.(N)$, where ∇''_J is the operator on the complex tangent bundle induced from $\overline{\partial} J$ on the cotangent bundle,and the pairing in the second term is the contraction*

$$\overline{T} \otimes (\Lambda^{0,2} \otimes T) \to \Lambda^{0,1} \otimes T\,.$$

If the variation in almost-complex structure are identified with $\Omega^{0,1}(T)$, as above, then the action of $L_X J$ on a (1, 0) form α is just the (0,1) part of $L_X \alpha$. But $L_X \alpha = (d i_X - i_X d)\alpha$. The (0, 1) component of this is made up of the three terms: $\overline{\partial}(i_X \alpha) - i_X \overline{\partial}\alpha - i_{\overline{X}}(i_N \alpha)$. The first two terms give $(\nabla''_J X)\alpha$ by the definition of ∇'' on T and hence we get the formula stated in the Lemma.

We now turn to the other operator Q. This can also be written as a composite $Q = Q_1 Q_2$, factoring through the infinitesimal change in the connection on K_M, but in doing this we need to keep in mind that the line bundle K_M itself depends upon the almost complex structure. If μ is a deformation of almost complex structure to a new structure J' then the map $\alpha \mapsto \alpha + \mu(\alpha)$ gives an isomorphism between $\Omega_J^{1,0}$ and $\Omega_{J'}^{1,0}$ which is a Hermitian isometry to first order in μ, and with this identification fixed we can regard the connections $\nabla_J, \nabla_{J'}$ as two unitary connections on the same bundle over M, and similarly for the connections on the exterior power K_M.

The two connections on K_M differ by an imaginary 1-form $i\psi$ say over M, then we define

$$Q_2 : \Gamma(s^2(T)) \to \Gamma(T^*M)\,, \qquad Q_1 : \Gamma(T^*M) \to C^\infty(M)$$

by $Q_2(\mu) = \psi$ and $Q_1(\psi)\frac{\omega^n}{n!} = d\psi \wedge \omega^{n-1}$.

The main task remaining is to identify the infinitesimal change in the connection.

Lemma 11. *Under a change in almost complex structure μ, and using the bundle identification above, the change in the connection on T^*M is, to first order in μ,*

$$\partial\mu - N.\overline{\mu} \in \Omega^{1,1}(T)\,,$$

where we regard $\Omega^{1,1}(T)$ as identified with the 1-forms on M with values in the skew-adjoint endomorphisms of TM, by the map $\sigma \mapsto \sigma - \sigma^*$.

The proof of this Lemma is slightly confusing. To set the scene consider the case of a Hermitian bundle E over M. If we *fix* an almost complex structure J on M and vary the ∇''-operator on E by $\sigma \in \Omega^{0,1}(EndE)$ then the change in the corresponding connections is by $\sigma - \sigma^*$. On the other hand, suppose we fix a connection ∇ on E and vary the almost complex structure by μ then, under the identifications we have made, the new ∇'' operator is

$$\nabla''_{J'}s = \nabla''s + \mu\nabla's\,, \tag{12}$$

where μ acts on the $\Omega^{1,0}$ factor of $\nabla'(s) \in \Omega^{1,0}(E)$. Our case is complicated because both the connection and the almost complex structure are varying. We need to compute

$$\nabla_{J'} : \Omega^{1,0}_{J'} \to \Omega^{1,1}_{J'}\,.$$

We have identified $\Omega^{1,0}_J$ with $\Omega^{1,0}_{J'}$ by $\alpha \mapsto \alpha + \mu(\alpha)$, and similarly we can identify $\Omega^{1,1}_J$ with $\Omega^{1,1}_{J'}$ by, to first order in μ,

$$\theta \mapsto \theta + \mu(\theta) + \overline{\mu}(\theta)\,.$$

This means that if χ is a complex 2-form on M with components $\chi = \chi^{2,0}+\chi^{1,1}+\chi^{0,2}$ with respect to the J decomposition then, to first order in μ, the (1, 1) component of χ with respect to the J' decomposition is represented by

$$\chi^{1,1} - \mu(\chi^{2,0}) - \overline{\mu}(\chi^{0,2})\,.$$

Putting all this together, the operator $\overline{\partial}_{J'}$ maps a (1, 0) form α to the $\Omega^{1,1}_{J'}$ component of $d(\alpha + \mu(\alpha))$ which, to first order in μ, is

$$\overline{\partial}_J(\alpha) + \partial_J(\mu(\alpha)) - \mu(\partial\alpha) - \overline{\mu}(i_N(\alpha))\,. \tag{13}$$

Now let use temporarily write $E = T^*M$, so $\nabla'_J : E \to \Omega^{1,0}(E)$. We can write:

$$\partial_J(\mu(\alpha)) = \nabla'_J(\mu)\alpha + \mu.(\nabla'_J\alpha)\,,$$

where in the second term μ acts on the "bundle" factor E in $\Omega^{1,0}(E)$. This is not the same as the action in the second term of (12), where μ acts on the "cotangent" factor $\Omega^{1,0}$. (The confusing thing here is that E is the cotangent bundle!) The two differ by the action of μ on the antisymmetrisation of $\nabla'_J\alpha \in \Gamma(T^* \otimes T^*)$. This antisymmetrisation is just the tensor $\partial_J\alpha$, since the torsion of our connection is given by the Nijenhius tensor N, and has no component mapping T^* to $\Lambda^{2,0}(T^*)$. So the difference is precisely the other term $\mu(\partial\alpha)$, in (13). Thus we get

$$\nabla''_{J'}\alpha = (\nabla''_J + \mu\nabla'_J)\alpha + \sigma\alpha\,,$$

where

$$\sigma\alpha = (\nabla'\mu)\alpha - \overline{\mu}(i_N\alpha)\,.$$

The term $\nabla''_J + \mu\nabla'_J$ just represents the way that the ∇''-operator would vary with a fixed connection, so the change in connection is given by $\sigma - \sigma^*$ as required.

Using the Lemma above, we have

$$Q_2(\mu) = \mathrm{Re}\,(\mathrm{Tr}(\nabla'_J\mu - \overline{\mu}.N))\,. \tag{14}$$

Here the trace is taken using our identification of $\Omega^{1,1}(T)$ with the (0, 1)-forms with values in the skew-adjoint endomorphisms of the cotangent bundle. At this point we bring in the fact that μ lies in the *symmetric* part of $T \otimes T \cong \Omega^{0,1}(T)$. Thus the term $\nabla_J\mu$ can be regarded as a tensor in $T^* \otimes T \otimes T$ which is symmetric in the last two factors. This means that the trace of this term in (14) can be obtained equally via the contraction

$$\Lambda : \Omega^{1,1}(T) \to \Omega^0(T)\,,$$

and the standard identification of the tangent and cotangent bundles. Now the usual Kahler identity

$$\overline{\partial}^* = -i[\Lambda, \partial]$$

holds also in the almost-complex case, and in sum we get

$$Q_2(\mu) = (\nabla''_J)^*\mu - \mathrm{Re}\,(\mathrm{Tr}(\overline{\mu}.N))\,, \tag{15}$$

and we note that the first term is linear in μ while the second is anti-linear.

The proof of (9) is now just a matter of matching up the various pieces. (To fix the signs one needs to write things out a little more explicitly, since our notation is somewhat compressed.) Using the standard (metric) identification of tangent and cotangent bundles, we have

$$P_1^* = Q_1 \circ J$$
$$P_2^* = -J \circ Q_2 \circ J\,,$$

so $P^* = Q_* \circ J$ as required.

We will now discuss the relevance of this moment-map calculation to the Kahler–Einstein problem. First observe that for any J in $\mathcal{J}$ the integral of $S(J)$ over the manifold is a topological invariant,

$$\int_M S\,\frac{\omega^n}{n!} = \int_M \rho \wedge \omega^{n-1} = 2\pi c_1(M) \cup [\omega]^{n-1}[M]\,.$$

We fix d such that the integral of $S - d$ is zero, then $S - d$ is also a moment map and the relevant moment map equation is $S - d = 0$, i.e. constant Hermitian scalar curvature. In the integrable case one has the following standard argument. The Bianchi identity give $\overline{\partial} S = \overline{\partial}^* \rho$, where ρ is essentially the Ricci tensor, regarded as a $(1, 1)$ form. So if the scalar curvature is constant then the Ricci tensor is harmonic, and if $c_1(M)$ is a multiple of the Kahler class considered, we can conclude from the uniqueness of the harmonic representative that $\rho = \lambda\omega$. Thus, given the appropriate cohomological setting, our moment map equation does yield the Kahler–Einstein solutions.

The feature which prevents us from immediately fitting this set-up into the standard picture at the beginning of this section is the fact that the complexification of the group $\mathcal{G}$ does not exist. However we may certainly complexify the Lie algebra and this automatically acts on $\mathcal{J}$, since the structure on $\mathcal{J}$ is integrable. At each point $J \in \mathcal{J}$ we get a subspace of $T\mathcal{J}_J$ spanned by this complexified action and these subspaces form an integrable, holomorphic distribution on $\mathcal{J}$. Thus we get a foliation of $\mathcal{J}$ which plays the role of the "orbits" of the mythical group $\mathcal{G}^c$. (All of these remarks are to be taken rather formally, since we do not want to get involved here with the infinite-dimensional differential-topological aspects of the set-up). Moreover the subspace $\mathcal{J}^{\mathrm{int}}$ is invariant under this foliation, so we can restrict attention to this if we prefer. In this subspace, the setting for the classical Kahler-Einstein problem, the foliation has a straightforward geometric meaning. To see this we consider the infinitesimal action of a function iH, in the imaginary part of the complexified Lie algebra. By definition this is just the variation $JP(H)$ which is

$$\mu = J(\overline{\partial}_J X_H + \overline{X}_H.N) = \overline{\partial}_J(JX_H) + J\overline{X}_H.N\,.$$

In the integrable case, when N is zero, we have $\mu = \overline{\partial}_J(JX_H) = L_{JX_H}(J)$, which is just the natural action of the vector field JX_H on the complex structure. Thus the geometric effect of applying iH is the same as keeping the complex structure fixed and varying the symplectic form to $\omega' = \omega - L_{JX_H}\omega = \omega - dJdH = \omega - 2i\overline{\partial}\partial H$. This is just the familar parametrisation of Kahler forms by Kahler potentials. Conversely, suppose Y is a vector field on M such that the corresponding variation in complex structure $L_Y J = \overline{\partial} Y$ is tangent to $\mathcal{J}$, i.e. lies in s^2T. Let $\gamma \in \Omega^{0,1}$ correspond to Y under the isomorphism $T = \overline{T}^*$ given by the metric. Then one checks that the condition that $\overline{\partial} Y$ lies in s^2T is the same as saying that $\overline{\partial}\gamma \in \Omega^{0,2}$ vanishes. The Dolbeault cohomology group $H^{0,1}$ is zero since $H^1(M) = 0$, so we can write

$\gamma = \overline{\partial} f$ for some complex-valued function $f = H_1 + iH_2$ on M. Then this means that $Y = X_{H_1} + iX_{H_2}$. So we conclude that each leaf of the foliation of $\mathcal{J}^{\mathrm{int}}$ corresponds, up to the diffeomorphism of M, to the set of Kahler metrics, in the given cohomology class, for a fixed complex structure on M. In sum, we have fitted Tian's conjecture — which bears on the case when $c_1 = \lambda[\omega]$ — into the familar picture, to wit: there should be a notion of a stable "orbit" which on the one hand is equivalent to the algebraic geometers notion of stability via the Hilbert scheme and on the other hand admits the standard identification:

$$\mathcal{J}_s^{\mathrm{int}}/\mathcal{G}^c = \mu^{-1}(0)/\mathcal{G}\,.$$

Another facet of the theory that would be worth studying is the geometry of the equivariant line bundle $\mathcal{L}$ over $\mathcal{J}$: it is natural to expect that this can be identified with a determinant line bundle furnished with a metric via zeta-function regularisation, making contact with the theory developed by Quillen, Bismut and others.

As well as giving additional motivation for Tian's conjecture, the work above throws light on a number of known results in Kahler geometry. For example:

(1) Matsushima's Theorem that the holomorphic automorphism group of a compact Kahler–Einstein manifold is reductive: the complexification of its isometry group;
(2) the Mabuchi "K-energy" [5], which is just the "norm functional" in this context;
(3) Bando and Mabuchi's result [6] on the uniqueness of Kahler–Einstein metrics, which can be seen as a facet of general convexity property of the norm functional;
(4) the existence of a natural Kahler metric (the Weil–Peterson metric) on the moduli space of Kahler–Einstein structures.

In large part, these follow without further calculation, if one is familiar with the corresponding results in the standard picture. In addition, our set-up suggests that it may be worthwhile to study various extensions of Tian's conjecture. On the one hand one might expect to extend the ideas to constant scalar curvature Kahler metrics. In the case of surfaces there has been substantial work on these by Le Brun and others. On the other hand, we have seen that the integrability of the almost complex structure is not particularly relevant, at least to the formal picture, so one might seek extensions of the results in the realm of almost-complex geometry. In this case the geometrical meaning of the leaves of the foliation is less clear: one gets an equivalence relation on almost-Kahler structures generated by a modification of the notion of Kahler potential — two structures $(J_0, \omega_0), (J_1, \omega_1)$ are equivalent if they can be joined by a path (J_t, ω_t) for which there are functions H_t with

$$\frac{d}{dt}\omega_t = d(J_t(dH_t))\,, \qquad \frac{d}{dt}J_t = N_t.\overline{X}_t\,,$$

where X_t is the Hamiltonian vector field defined by H_t and the sympletic form ω_t.

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THE WORK OF M. H. FREEDMAN

by
JOHN MILNOR

Michael Freedman has not only proved the Poincaré hypothesis for 4-dimensional topological manifolds, thus characterizing the sphere S^4, but has also given us classification theorems, easy to state and to use but difficult to prove, for much more general 4-manifolds. The simple nature of his results in the topological case must be contrasted with the extreme complications which are now known to occur in the study of differentiable and piecewise linear 4-manifolds.

The "n-dimensional Poincaré hypothesis" is the conjecture that every topological n-manifold which has the same homology and the same fundamental group as an n-dimensional sphere must actually be homeomorphic to the n-dimensional sphere. The cases $n = 1, 2$ were known in the nineteenth century, while the cases $n \geq 5$ were proved by Smale, and independently by Stallings and Zeeman and by Wallace, in 1960–61. (The original proofs needed an extra hypothesis of differentiability or piecewise linearity, which was removed by Newman a few years later.) The 3- and 4-dimensional cases are much more difficult.

Freedman's 1982 proof of the 4-dimensional Poincaré hypothesis was an extraordinary tour de force. His methods were so sharp as to actually provide a complete classification of all compact simply connected topological 4-manifolds, yielding many previously unknown examples of such manifolds, and many previously unknown homeomorphisms between known manifolds. He showed that a compact simply connected 4-manifold M is characterized, up to homeomorphism, by two simple invariants. The first is the 2-dimensional homology group

$$H_2 = H_2(M;\ Z) \cong Z \oplus \cdots \oplus Z\,,$$

together with the symmetric bilinear intersection pairing

$$\omega : H_2 \otimes H_2 \to Z\,.$$

This pairing, which is defined as soon as we choose an orientation for M, must have determinant ± 1 Poincaré duality. The second is the Kirby–Siebenmann obstruction class, an element

$$\sigma \in H^4(M;\ Z/2) \cong Z/2$$

that vanishes if and only if M is stably smoothable. In other words, σ is zero if and only if the product $M \times R$ can be given a differentiable structure, or equivalently a piecewise linear structure. These two invariants ω and σ can be prescribed arbitrarily, except for a relation in one special case. If the form ω happens to be even, that is if $\omega(x, x) \equiv 0 \pmod 2$ for every $x \in H_2$, then the Kirby–Siebenmann obstruction must be equal to the Rohlin invariant:

$$\sigma \equiv \text{signature } (\omega)/8 \pmod 2\,.$$

(Freedman's original proof that these two invariants characterize M up to homemorphism required an extra hypothesis of "almost-differentiability," which was later removed by Quinn.)

If the intersection form $\omega \neq 0$ is indefinite or has rank at most eleven, then it follows from known results about quadratic forms that M can be built up (nonuniquely) as a connected sum of copies of four simple building blocks, each of which may be given either the standard or the reversed orientation. One needs the product $S^2 \times S^2$, the complex projective plane CP^2, and two exotic manifolds which were first constructed by Freedman. One of these is a nondifferentiable analogue of the complex projective plane, and the other is the unique manifold whose intersection form ω is positive definite and even of rank eight. (This ω can be identified with the lattice generated by the root vectors of the Lie group E_8. As noted by Rohlin in 1952, a 4-manifold with such an intersection form can never be differentiable.) By way of contrast, if we allow positive definite intersection forms, then the number of distinct simply connected manifolds grows more than exponentially with increasing middle Betti number.

Freedman's methods extend also to noncompact 4-manifolds. For example, he showed that the product $S^3 \times R$ can be given an exotic differentiable structure, which contains a smoothly embedded Poincaré homology 3-sphere and hence cannot be smoothly embedded in euclidean 4-sphere [**11**, **16**]. His methods apply also to many manifolds which are not simply connected [**22**]. For example, a "flat" 2-sphere in 4-space is unknotted if and only if its complement has free cyclic fundamental group; and a flat 1-sphere in S^3 has trivial Alexander polynomial if and only if it bounds a flat 2-disk in the unit 4-disk whose complement has free cyclic fundamental group.

The proofs of these results are extremely difficult. The basic idea, which had been used in low dimensions by Moebius and Poincaré, and in high dimensions by Smale and Wallace, is to build the given 4-manifold up inductively, starting with a 4-dimensional disk, by successively adding handles. The essential difficulty, which does not arise in higher dimensions, occurs when we try to control the fundamental group by inserting 2-dimensional handles, since a 2-dimensional disk immersed in a 4-manifold will usually have self-intersections. This problem was first attacked by Casson, who showed how to construct a generalized kind of 2-handle with prescribed

boundary within a given 4-manifold. Freedman's major technical tool is a theorem which asserts that every Casson handle is actually homeomorphic to the standard open handle, (closed 2-disk) $\times$ (open 2-disk). The proof involves a delicately controlled infinite repetition argument in the spirit of the Bing school of topology, and is nondifferentiable in an essential way.

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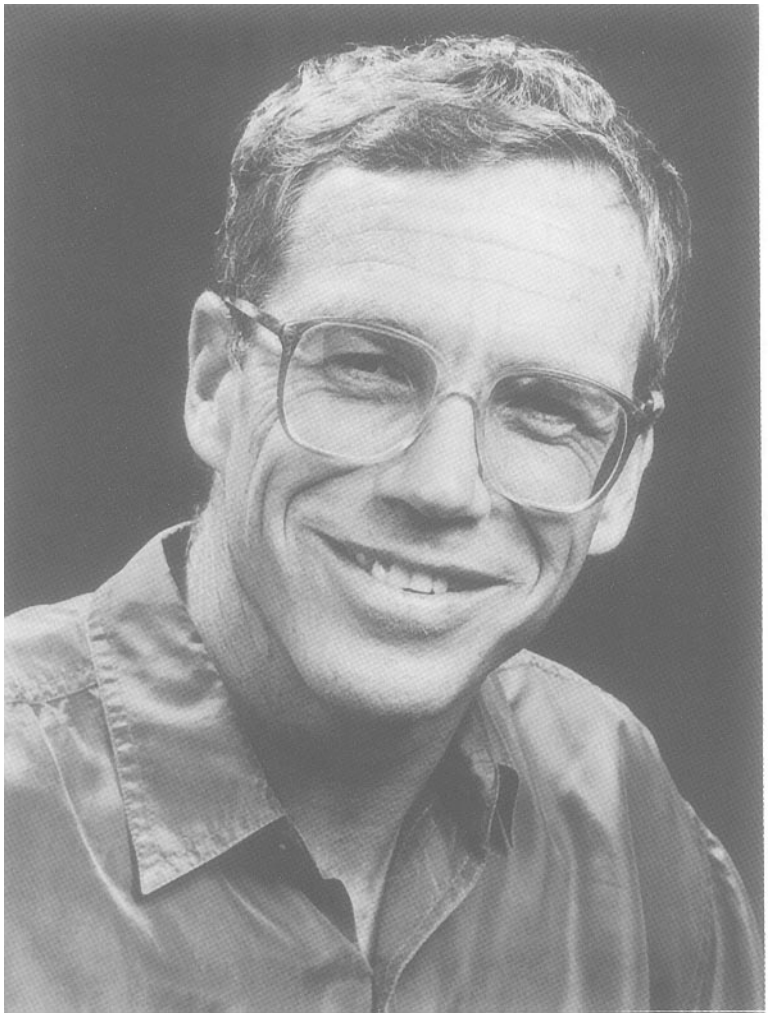

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BETTI NUMBER ESTIMATES FOR NILPOTENT GROUPS

by

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ABSTRACT. We prove an extension of the following result of Lubotzky and Magid on the rational cohomology of a nilpotent group G: if $b_1 < \infty$ and $G \otimes \mathbb{Q} \neq 0, \mathbb{Q}, \mathbb{Q}^2$ then $b_2 > b_1^2/4$. Here the b_i are the rational Betti numbers of G and $G \otimes \mathbb{Q}$ denotes the Malcev-completion of G. In the extension, the bound is improved when we know that all relations of G all have at least a certain commutator length. As an application of the refined inequality, we show that each closed oriented 3-manifold falls into exactly one of the following classes: it is a rational homology 3-sphere, or it is a rational homology $S^1 \times S^2$, or it has the rational homology of one of the oriented circle bundles over the torus (which are indexed by an Euler number $n \in \mathbb{Z}$, e.g. $n = 0$ corresponds to the 3-torus) or it is of *general type* by which we mean that the rational lower central series of the fundamental group does not stabilize. In particular, any 3-manifold group which allows a maximal torsion-free nilpotent quotient admits a rational homology isomorphism to a torsion-free nilpotent group.

1. The Main Results

We analyze the rational lower central series of 3-manifold groups by an extension of the following theorem of Lubotzky and Magid [15, (3.9)].

Theorem 1. *If G is a nilpotent group with $b_1(G) < \infty$ and $G \otimes \mathbb{Q} \neq \{0\}, \mathbb{Q}, \mathbb{Q}^2$, then*

$$b_2(G) > \frac{1}{4}\, b_1(G)^2 \,.$$

Here $b_i(G)$ denotes the ith rational Betti number of G and $G \otimes \mathbb{Q}$ the Malcev-completion of G. (We refer the reader to Appendix A, where we have collected the group theoretic definitions.)

This Theorem is an analogue of the Golod–Shafarevich Theorem (see [9] or [14, p. 186]), which states that if G is a finite p-group, then the inequality $r > d^2/4$

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holds, where d is the minimal number of generators in a presentation of G, and r is the number of relations in any presentation. It can be used to derive a result for finitely generated nilpotent groups similar to the one above, but with $b_2(G)$ replaced by the minimal number of relations in a presentation for G, see [6, p. 121].

We shall need a refined version of this result in the case where the relations of G are known to have a certain commutator length. In order to make this precise, let

$$H_2(G;\mathbb{Q}) =: \Phi_2^{\mathbb{Q}}(G) \supseteq \Phi_3^{\mathbb{Q}}(G) \supseteq \Phi_4^{\mathbb{Q}}(G) \supseteq \cdots$$

be the rational Dwyer filtration of $H_2(G;\mathbb{Q})$ (defined in Appendix A).

Theorem 2. *If G is a nilpotent group with $b_1(G) < \infty$, $H_2(G;\mathbb{Q}) = \Phi_r^{\mathbb{Q}}(G)$ and $G \otimes \mathbb{Q} \neq \{0\}, \mathbb{Q}, \mathbb{Q}^2$, then*

$$b_2(G) > \frac{(r-1)^{(r-1)}}{r^r}\, b_1(G)^r\,.$$

The statement that $H_2(G;\mathbb{Q}) = \Phi_r^{\mathbb{Q}}(G)$ is equivalent to the statement that in a minimal presentation of the Lie algebra of $G \otimes \mathbb{Q}$, all relations lie in the rth term of the lower central series of the free Lie algebra on the minimal generating set $H_1(G;\mathbb{Q})$. We know of no analogue of this result for p-groups, although there should be one. The result is a corollary of a more technical and stronger result, Proposition 5. Note that Theorem 1 is the special case of Theorem 2 with $r = 2$.

The intuitive idea behind these results is that $H_1(G)$ corresponds to generators of G and $H_2(G)$ to its relations. For example, if G is abelian, then the number of (primitive) relations ($\approx b_2$) grows quadratically in the number of generators ($\approx b_1$) because the commutator of a pair of generators has to be a consequence of the primitive relations. Similarly, if G is nilpotent, then the relations have to imply that for some r, all r-fold commutators in the generators vanish. If $H_2(G) = \Phi_r(G)$, then all relations are in fact r-fold commutators because, by Dwyer's Theorem (see Appendix A), this condition is equivalent to G/Γ_r being isomorphic to F/Γ_r where F is the free group on $b_1(G)$ generators. Thus b_2 should grow as the r-th power of b_1. But if some of the relations are shorter commutators, then they can imply a vast number of relations among the r-fold commutators. Therefore, one can only expect a lower order estimate in this case.

Example. Let $x_1, \ldots, x_4$ be generators of $H^1(\mathbb{Z}^4;\mathbb{Z})$. Then the central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow G \longrightarrow \mathbb{Z}^4 \longrightarrow 1$$

classified by the class $x_1x_2 + x_3x_4 \in H^2(\mathbb{Z}^4;\mathbb{Z})$ satisfies $b_1(G) = 4, b_2(G) = 5$. This shows that the lower bound for b_2 in Theorem 1 cannot be of the form $b_1(b_1-1)/2$ with equality in the abelian case.

This paper is organized as follows. In Section 2 we give applications of the above results to 3-dimensional manifolds. Section 3 explains how to derive the nilpotent classification of 3-manifolds from Theorem 2. In Section 4 we give further examples

and in Section 5 we prove Theorem 2 modulo two key lemmas. These are proven in Sections 6 and 7. Appendices A and B contain background information from group theory respectively rational homotopy theory.

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2. Applications to 3-Manifolds

We will look at a 3-manifold through nilpotent eyes, observing only the tower of nilpotent quotients of the fundamental group, but never the group itself. This point of view has a long history in the study of link complements and it arises naturally if one studies 3- and 4-dimensional manifolds together. For example, Stallings proved that for a link L in S^3 the nilpotent quotients $\pi_1(S^3 \setminus L)/\Gamma_r$ are invariants of the topological concordance class of the link L. These quotients contain the same information as Milnor's μ-invariants which are generalized linking numbers. For precise references about this area of research and the most recent applications to 4-manifolds see [7] and [8].

Let M be a closed oriented 3-manifold and $\{\Gamma_r^{\mathbb{Q}} \mid r \geq 1\}$ the rational lower central series (see Appendix A) of $\pi_1(M)$. Similarly to Stallings' result, the quotients $(\pi_1(M)/\Gamma_r^{\mathbb{Q}}) \otimes \mathbb{Q}$ are invariants of rational homology H-cobordism between such 3-manifolds.

Definition. A 3-manifold M is of *general type* if

$$\pi_1(M) = \Gamma_1^{\mathbb{Q}} \supsetneqq \Gamma_2^{\mathbb{Q}} \supsetneqq \Gamma_3^{\mathbb{Q}} \supsetneqq \cdots$$

and *special* if, for some $r > 0$, $\Gamma_r^{\mathbb{Q}} = \Gamma_{r+1}^{\mathbb{Q}}$. (Following the terminology used in group theory, the fundamental group of a special 3-manifold is called *rationally prenilpotent*, compare the Appendix A.)

Our nilpotent classification result reads as follows.

Theorem 3. *If a closed oriented* 3*-manifold* M *is special, then the maximal torsion-free nilpotent quotient of its fundamental group is isomorphic to exactly one of the groups*

$$\{1\}, \quad \mathbb{Z} \quad \textit{or} \quad H_n\,.$$

In particular, this quotient is a torsion-free nilpotent 3*-manifold group of nilpotency class* < 3.

Here the groups $H_n, n \geq 0$, are the central extensions

$$1 \longrightarrow \mathbb{Z} \longrightarrow H_n \longrightarrow \mathbb{Z}^2 \longrightarrow 1$$

classified by the Euler class $n \in H^2(\mathbb{Z}^2; \mathbb{Z}) \cong \mathbb{Z}$. This explains the last sentence in our theorem because H_n occurs as the fundamental group of a circle bundle over the 2-torus with Euler class n.

Since the groups above have nilpotency class < 3, it is very easy to recognize the class to which a given 3-manifold belongs; one simply has to compute $\pi_1(M)/\Gamma_3^{\mathbb{Q}}$. Note, in particular, that a 3-manifold M is automatically of general type if its first rational Betti number satisfies $b_1 M > 3$.

If $b_1 M = 0$, then M is a rational homology sphere, if $b_1 M = 1$, then it is a rational homology $S^1 \times S^2$. In the case $b_1 M = 2$, the cup-product between the two 1-dimensional cohomology classes vanishes and one can compute a Massey triple-product to obtain the integer $n \in \mathbb{Z}$ (note that $H^1(M;\mathbb{Z}) \cong \mathbb{Z}^2$ and thus one can do the computation integrally). It determines whether M is of general type ($n = 0$) or whether it belongs to one of the groups H_n, $n > 0$. Finally, if $b_1 M = 3$, then M is of general type if and only if the triple cup-product between the three 1-dimensional cohomology classes vanishes, otherwise it is equivalent to the 3-torus, i.e., to H_0.

One should compare the above result with the list of nilpotent 3-manifold groups from [22]:

finite	$\mathbb{Z}/k$	$Q_{2^n} \times \mathbb{Z}/(2k+1)$
infinite	$\mathbb{Z}$	H_k

In the case of finite groups, $k \geq 0$ is the order of the cyclic group $\mathbb{Z}/k$ and the quaternion group Q_k. In the infinite case, it is the Euler number that determines H_k. In Section 3 we will outline why no other nilpotent groups occur.

The reason why we used the rational version of the lower central series is that our Betti number estimates only give rational information. The integral version of Theorem 3 was in the meantime proven by the third author [21] using completely different methods.

The following result is the rational version of Turaev's Theorem 2 in [23].

Theorem 4. *A finitely generated nilpotent group G satifies*

$$G \otimes \mathbb{Q} \cong (\pi_1(M)/\Gamma_r) \otimes \mathbb{Q}$$

for a closed oriented 3-manifold M if and only if there exists a class $m \in H_3(G;\mathbb{Q})$ such that the composition

$$H^1(G;\mathbb{Q}) \xrightarrow{\cap m} H_2(G;\mathbb{Q}) \twoheadrightarrow H_2(G;\mathbb{Q})/\Phi_r^{\mathbb{Q}}(G)$$

is an epimorphism.

Here $\Phi_r^{\mathbb{Q}}(G)$ is the r-th term in the rational Dwyer filtration of $H_2(G;\mathbb{Q})$ (defined in the Appendix A).

3. Low-dimensional Surgery

In this section we first show how Theorems 4 and 1 imply Theorem 3. Then we recall the proof of Theorem 4 and finally we give a short discussion of nilpotent 3-manifold groups.

First note that by definition $\Phi_r^{\mathbb{Q}} = 0$ if $\Gamma_{r-1}^{\mathbb{Q}} = \{0\}$. So by Theorem 4, in order for a group G to be the maximal nilpotent quotient of a rationally prenilpotent 3-manifold group, it must possess the following property $\cap$ (here rational coefficients are to be understood):

$$\cap : \qquad \begin{cases} \text{There exits } m \in H_3(G) \text{ such that} \\ \cap m : H^1(G) \longrightarrow H_2(G) \text{ is an epimorphism.} \end{cases}$$

Theorem 3 now follows immediately from the following:

Proposition 1. *A finitely generated torsion-free nilpotent group satisfying property $\cap$ is isomorphic to exactly one of the groups*:

$$\{1\}, \quad \mathbb{Z} \quad \text{or} \quad H_n\,.$$

Observe that $H_0 \otimes \mathbb{Q} = \mathbb{Z}^3 \otimes \mathbb{Q} = \mathbb{Q}^3$. When $n > 0$ each of the groups $H_n \otimes \mathbb{Q}$ is isomorphic to the $\mathbb{Q}$ points of the Heisenberg group $H_{\mathbb{Q}}$. On the other hand, the groups $\mathbb{Z}^k$ are the only finitely generated torsion-free nilpotent groups whose Malcev completion is $\mathbb{Q}^k$ and, for $n > 0$, the groups H_n are the only finitely generated torsion-free nilpotent groups with Malcev completion $H_{\mathbb{Q}}$. Therefore, Proposition 1 follows from the following result and the fact that the Malcev completion of a finitely generated nilpotent group is a uniquely divisible nilpotent group with $b_1 < \infty$.

Proposition 2. *A uniquely divisible nilpotent group with $b_1 < \infty$ satisfying property $\cap$ is isomorphic to exactly one of the groups*:

$$\{0\}, \quad \mathbb{Q}, \quad \mathbb{Q}^3 \quad \text{or} \quad H_{\mathbb{Q}}\,.$$

Property $\cap$ implies $b_1 \geq b_2$. This, combined with the inequality $b_2 > b_1^2/4$ from Theorem 1, implies that Proposition 2 follows from the result below. The proof will be given in Section 4.

Proposition 3. *Suppose that G is a uniquely divisible nilpotent group with $b_1(G) < \infty$ satisfying property $\cap$. If $b_1 = b_2 = 2$, then G is isomorphic to $H_{\mathbb{Q}}$. If $b_1 = b_2 = 3$, then G is isomorphic to $\mathbb{Q}^3$.*

This finishes the outline of the proof of Theorem 3.

Proof of Theorem 4. Given a closed oriented 3-manifold M, we may take a classifying map $M \to K(\pi_1(M), 1)$ of the universal covering and compose with the projection $\pi_1(M) \twoheadrightarrow \pi_1(M)/\Gamma_r$ to get a map $u : M \to K(\pi_1(M)/\Gamma_r, 1)$ and a commutative diagram

$$\begin{array}{ccc} H^1(M) & \xrightarrow[\cong]{\cap\,[M]} & H_2(M) \\ {\scriptstyle u^*}\big\uparrow & & {\scriptstyle u_*}\big\downarrow \\ H^1(\pi_1(M)/\Gamma_r) & \xrightarrow{\cap\, u_*[M]} & H_2(\pi_1(M)/\Gamma_r) \end{array}$$

Clearly u induces an isomorphism on H_1, and therefore u^* is an isomorphism. Thus the "only if" part of our theorem follows directly from Dwyer's theorem (see the Appendix A). Here we could have worked integrally or rationally.

To prove the "if" part, we restrict to rational coefficients. Let $u : M \to K(G,1)$ be a map from a closed oriented 3-manifold with $u_*[M] = m$, the given class in $H_3(G)$. Such a map exists because rationally oriented bordism maps onto homology by the classical result of Thom. Now observe that we are done by Dwyer's theorem (and the above commutative diagram) if the map u induces an isomorphism on H_1. If this is not the case, we will change the map u (and the 3-manifold M) by surgeries until it is an isomorphism on H_1: We describe the surgeries as attaching 4-dimensional handles to the upper boundary of $M \times I$. Then M is the lower boundary of this 4-manifold and the upper boundary is denoted by M'. If the map u extends to the 4-manifold, then the image of the fundamental class of the new 3-manifold (M', u') is still the given class $m \in H_3(G)$.

First add 1-handles $D^1 \times D^3$ to $M \times I$ and extend u to map the new circles to the (finitely many) generators of G. This makes u' an epimorphism (on H_1). Then we want to add 2-handles $D^2 \times D^2$ to $M' \times I$ to kill the kernel K of

$$u'_* : H_1(M') \longrightarrow H_1(G)\,.$$

We can extend the map u' to the 4-manifold

$$W := M' \times I \cup \text{ 2-handles}$$

if we attach the handles to curves in the kernel of $u : \pi_1(M) \to G$.

Now observe that the new upper boundary M'' of W still maps onto $H_1(G)$ because one can obtain M' from M'' by attaching 2-handles $(2 + 2 = 4)$. But the problem is that $\mathrm{Ker}\{u''_* : H_1(M'') \to H_1(G)\}$ may contain new elements which are meridians to the circles c we are trying to kill. If c has a dual, i.e. if there is a surface in M' intersecting c in a point, then these meridians are null-homologous. But since we only work rationally we may assume all the classes in the kernel to have a dual and we are done.

The more involved integral case is explained in detail in [23] but we do not need it here. □

We finish this section by explaining briefly the table of nilpotent 3-manifold groups given in Section 2. The finite groups G in the table have representations into $SO(4)$ such that the corresponding action of G on the 3-sphere is free. Thus the corresponding 3-manifolds are homogenous spaces S^3/G.

To explain why only cyclic and quaternion groups can occur, first recall that a finite group is nilpotent if and only if it is the direct product of its p-Sylow subgroups [13]. Now it is well known [2] that the only p-groups with periodic cohomology are the cyclic groups and, for $p = 2$, the quaternion groups. The only fact about the group we use is that it acts freely on a homotopy 3-sphere, and thus has 4-periodic cohomology.

To understand why only the groups $\mathbb{Z}$ and H_n occur as infinite nilpotent 3-manifold groups, first notice that a nilpotent group is never a nontrivial free product. Except in the case $\pi_1(M) = \mathbb{Z}$, the sphere theorem implies that the universal cover of the corresponding 3-manifold must be contractible. In particular, the nilpotent fundamental groups ($\neq \mathbb{Z}$) of a 3-manifolds must have homological dimension 3. It is then easy to see that the groups H_n are the only such groups. For more details see [22] or [21].

4. Examples

In this section we prove Proposition 3 and give an example concerning the difference between integral and rational prenilpotence of 3-manifold groups.

Proof of Proposition 3. The key to both cases is the following commutative diagram (later applied with $r = 2, 3, 4$). Here all homology groups, as well as the groups Γ_r and Φ_r, have rational coefficients, i.e., we suppress the letter $\mathbb{Q}$ from the notation.

$$\begin{tikzcd}
\Phi_{r+1}(G) \arrow[r] & H_2(G) \arrow[r, "p_*"] & H_2(G/\Gamma_r) \arrow[r] & \Gamma_r(G)/\Gamma_{r+1}(G) \\
& H^1(G) \arrow[u, "\cap m", "\cong"'] & H^1(G/\Gamma_r) \arrow[l, "p^*", "\cong"'] \arrow[u, "\cap p_*(m)"'] &
\end{tikzcd}$$

The upper line is *short* exact by the 5-term exact sequence and the definition of Φ_{r+1}. Since we consider cases with $b_1 = b_2$, the map $\cap m$ is actually an isomorphism. Consider first the case $b_1(G) = b_2(G) = 3$ and $r = 2$ in the above diagram. Since $H_3(G/\Gamma_2) \cong H_3(\mathbb{Q}^3) \cong \mathbb{Q}$, there are only two cases to consider:

(i) $p_*(m) \neq 0$: Then, by Poincaré duality for $\mathbb{Q}^3$, the map $\cap p_*(m)$ is an isomorphism and thus $p_* : H_2(G) \to H_2(G/\Gamma_2)$ is also an isomorphism. By Stallings' Theorem this implies that $G \cong G/\Gamma_2 \cong \mathbb{Q}^3$.

(ii) $p_*(m) = 0$: Then $p_* : H_2(G) \to H_2(G/\Gamma_2)$ is the zero map and thus $\Phi_3(G) = \mathbb{Q}^3$. This contradicts Theorem 2 with $r = 3$, since $3 \not> 4$.

Now consider the case $b_1(G) = b_2(G) = 2$. If $p_* : H_2(G) \to H_2(G/\Gamma_2) \cong \mathbb{Q}$ is onto, then by Stallings' Theorem, G would be abelian and thus have $b_2(G) = 1$, a contradiction. Therefore, $H_2(G) = \Phi_3(G)$ and G/Γ_3 is the rational Heisenberg group $H_{\mathbb{Q}}$. This follows from Dwyer's Theorem by comparing G to the free group F on 2 generators and noting that $H_{\mathbb{Q}} \cong F/\Gamma_3 \otimes \mathbb{Q}$.

Now consider the above commutative diagram for $r = 3$. Since $H_3(H_{\mathbb{Q}}) \cong \mathbb{Q}$ there are again only two cases to consider:

(i) $p_*(m) \neq 0$: Then by Poincaré duality for $H_{\mathbb{Q}}$, the map $\cap p_*(m)$ is an isomorphism and thus $p_* : H_2(G) \to H_2(G/\Gamma_2)$ is also an isomorphism. By Stallings' Theorem this implies that $G \cong G/\Gamma_3 \cong H_{\mathbb{Q}}$.

(ii) $p_*(m) = 0$: Then $p_* : H_2(G) \to H_2(G/\Gamma_2)$ is the zero map and thus $H_2(G) = \Phi_4(G)$.

Unfortunately, this does not contradict Theorem 2 with $r = 4$ (since $2 > 27/16$) and we have to go one step further. Again by Dwyer's Theorem G/Γ_4 is isomorphic to $K := F/\Gamma_4 \otimes \mathbb{Q}$. One easily computes that the cap-product map

$$\cap : H_3(K) \otimes H^1(K) \longrightarrow H_2(K)$$

is identically zero and thus the above commutative diagram for $r = 4$ shows that $H_2(G) = \Phi_5(G)$. Now Theorem 2 does lead to the contradiction $2 > 8192/3125$.

□

We believe that the above proof is unnecessarily complicated because Theorem 2 does not give the best possible estimate. In fact, we conjecture that there is no nilpotent group with $b_1 = b_2 = 2$ and $\Phi_4 \cong \mathbb{Q}^2$, and that a nilpotent group with $b_1 = b_2 = 3$ is always rationally $\mathbb{Q}^3$ (without assuming property $\cap_{\mathbb{Q}}$). However, we found the following:

Example. Besides the Heisenberg group $H_{\mathbb{Q}}$, there are other nilpotent groups with $b_1 = b_2 = 2$. For example, take a nontrivial central extension

$$0 \longrightarrow \mathbb{Q} \longrightarrow G \longrightarrow H_{\mathbb{Q}} \longrightarrow 0.$$

By Proposition 3, G cannot satisfy property $\cap_{\mathbb{Q}}$, which can be checked directly.

We finish this section by discussing the 2-torus bundle over the circle with holonomy given by

$$(z_1, z_2) \mapsto (\bar{z}_1, \bar{z}_2), \;\; z_i \in S^1 \subset \mathbb{C}\,.$$

The fundamental group G of this 3-manifold is the semidirect product of $\mathbb{Z}$ with $\mathbb{Z}^2$, where a generator of the cyclic group $\mathbb{Z}$ acts as minus the identity on the normal subgroup $\mathbb{Z}^2$. One computes that

$$\Gamma_r(G) = 2^{r-1} \cdot \mathbb{Z}^2$$

and thus G is not prenilpotent. However, one sees that $\Gamma_r^{\mathbb{Q}}(G) = \mathbb{Z}^2$ for all r — i.e., G is rationally prenilpotent with maximal torsion-free nilpotent quotient $\mathbb{Z}$.

This example illustrates three phenomena. Firstly, the rational lower central series stabilizes more often than the integral one, even for 3-manifold groups. Secondly, going to the maximal torsion-free nilpotent quotient kills many details of the 3-manifold group which can be still seen in the nilpotent quotients. Finally, unlike in the geometric theory, even a 2-fold covering can alter the class (Theorem 3) to which a 3-manifold belongs.

5. The Proof of Theorem 2

The proof of the estimate in Theorem 2 consists of three steps which are parallel to those in Roquette's proof of the Golod–Shafarevich Theorem, as presented in [14]. We begin by recalling the structure of the proof:

1. The $\mathbb{F}_p$-homology of a p-group G can be calculated using a free $\mathbb{F}_p[G]$-resolution of the trivial module $\mathbb{F}_p$ in place of a free $\mathbb{Z}[G]$-resolution of $\mathbb{Z}$. This puts us in the realm of linear algebra.
2. There is a *minimal resolution* of $\mathbb{F}_p$ over $\mathbb{F}_p[G]$. This is an exact sequence
$$\cdots \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow \mathbb{F}_p \longrightarrow 0$$
of free $\mathbb{F}_p[G]$-modules C_i such that, for all $i \geq 0$, one has
$$\operatorname{rank}_{\mathbb{F}_p[G]} C_i = \dim_{\mathbb{F}_p} H_i(G; \mathbb{F}_p).$$
3. Let I be the kernel of the augmentation $\mathbb{F}_p[G] \to \mathbb{F}_p$. Define a generating function $d(t)$ by
$$d(t) := \sum_{k \geq 0} \dim_{\mathbb{F}_p}(I^k/I^{k+1})\, t^k\,.$$
This function is a polynomial which is positive on the unit interval. The proof is completed by filtering the first two pieces of a minimal resolution by powers of I. This leads to an inequality
$$(\text{a quadratic polynomial})d(t) \geq 1$$
for $t \in [0,1]$ from which one deduces the result by looking at the discriminant of the quadratic.

We now consider the analogues of these for a finitely generated torsion free nilpotent group G. We reduce to the case of a uniquely divisible group with $b_1 < \infty$ by replacing G by $G \otimes \mathbb{Q}$.

Ad(1) The $\mathbb{Q}$-homology of G can be clearly calculated using a free resolution of $\mathbb{Q}$ over the group ring $\mathbb{Q}[G]$ rather than over $\mathbb{Z}[G]$. More importantly, we can also resolve $\mathbb{Q}$ as a module over the I-adic completion of $\mathbb{Q}[G]$, where I is again the augmentation ideal. This step involves an Artin–Rees Lemma proved in this setting in [3]. Let A denote the completion $\mathbb{Q}[G]\widehat{\ }$.

Ad(2) There is a minimal resolution of $\mathbb{Q}$ over A in the same sense as above. More precisely, there is minimal resolution which takes into account the Dwyer filtration of $H_2(G)$. The construction of such a resolution will be given in Section 6. It is crucial that one uses the completed group ring A rather than $\mathbb{Q}[G]$.

Ad(3) Define a generating function $d(t)$ by
$$d(t) := \sum_{k \geq 0} \dim_{\mathbb{Q}}(I^k/I^{k+1})\, t^k\,.$$
The proof of the following result is at the end of this section.

Proposition 4. *With notation as above,*

$$d(t) = \prod_{r\geq 1} \frac{1}{(1-t^r)^{g_r}}$$

where g_r is the dimension of $\Gamma_r(G)/\Gamma_{r+1}(G)$. In particular, $d(t)$ is a rational function.

Define p_r to be the dimension of $\Phi_r^{\mathbb{Q}}(G)/\Phi_{r+1}^{\mathbb{Q}}(G)$. Recall that, roughly speaking, p_r is the number of relations which are r-fold commutators but not $(r+1)$-fold commutators. Let b_i be the ith rational Betti numbers of G. Set

$$p(t) := \sum_{r\geq 2} p_r\, t^r - b_1\, t + 1\,.$$

The following result will be proven in Section 7 using the resolution constructed in Section 6.

Proposition 5. *For all $0 < t < 1$ one has the inequality $p(t)\, d(t) \geq 1$.*

Now it is easy to prove Theorem 2: First note that when $0 < t < 1$, the generating function $d(t)$ is positive, and thus $p(t)$ is also positive in this interval. Now assume that $H_2(G;\mathbb{Q}) = \Phi_r^{\mathbb{Q}}(G)$ for some $r \geq 2$ and that $G \neq \{0\}, \mathbb{Q}$. Then $b_1 > 1, b_2 > 0$ and $p(t) \leq q(t)$ for each t in $0 < t < 1$, where

$$q(t) := b_2\, t^r - b_1\, t + 1\,.$$

The polynomial $q(t)$ has a minimum at

$$t_0 := \sqrt[r-1]{\frac{b_1}{r\, b_2}}\,.$$

The desired inequality follows from the inequality $q(t_0) > 0$ after a little algebraic manipulation. But in order to do this we need $t_0 < 1$, which is equivalent to the condition $b_1 < r\, b_2$.

We know that $0 \leq q(1) = b_2 - b_1 + 1$, and thus $b_2 \geq b_1 - 1$. Therefore, the only case where $t_0 < 1$ is not satisfied is $(b_1, b_2) = (2, 1)$ (and $r = 2$). We claim that $(b_1, b_2) = (2, 1)$ implies $G \cong \mathbb{Q}^2$.

Consider the projection $G \twoheadrightarrow G/\Gamma_2 \cong \mathbb{Q}^2$. If the induced map on $H_2(\ ;\mathbb{Q})$ is onto, then we are done by Stallings' Theorem. If not then we know that $H_2(G;\mathbb{Q}) = \Phi_3(G)$, and are in a case where $t_0 < 1$ since $r = 3$. But then $q(t_0) > 0$ leads to the contradiction $b_2 = 1 > 32/27$. $\square$

Proof of Proposition 4. Set

$$\mathrm{Gr}(G) = \bigoplus_{r\geq 1} \Gamma_r(G)/\Gamma_{r+1}(G)\,.$$

This is a graded Lie algebra. Observe that $\mathrm{Gr}(A) \cong \mathrm{Gr}(\mathbb{Q}[G])$. Then, by a theorem of Quillen [18], we have an isomorphism $\mathrm{Gr}(\mathbb{Q}[G]) \cong U(\mathrm{Gr}(G))$ of graded Hopf algebras. Here $U(\mathrm{Gr}(G))$ denotes the universal enveloping algebra of $\mathrm{Gr}(G)$. We shall write $\mathcal{G}$ for $\mathrm{Gr}(G)$ and $\mathcal{G}_r$ for $\Gamma_r(G)/\Gamma_{r+1}(G)$.

By the Poincaré–Birkhoff–Witt Theorem, there is a graded coalgebra isomorphism of $U\mathcal{G}$ with the symmetric coalgebra $S\mathcal{G}$ on $\mathcal{G}$. Note that we have the isomorphism

$$S\mathcal{G} \cong \bigotimes_{r\geq 1} S\mathcal{G}_r\,,$$

of graded vector spaces, where the tensor product on the right is finite as $\mathcal{G}$ is nilpotent. Since $\mathcal{G}_r$ has degree r, the generating function of the symmetric coalgebra $S\mathcal{G}_r$ is $1/(1-t^r)^{g_r}$. The result follows. □

6. The Minimal Resolution

In this section we use techniques from rational homotopy theory to prove the existence of the minimal resolutions needed in the proof of Theorem 2. We obtain the minimal resolution using Chen's method of formal power series connections, which provides a minimal associative algebra model of the loop space of a space. Chen's theory is briefly reviewed in Appendix B.

The precise statement of the main result is:

Theorem 5. *If G is a nilpotent group with $b_1(G) < \infty$, then there is a free $\mathbb{Q}[G]\widehat{\ }$ resolution*

$$\cdots \xrightarrow{\delta} C_2 \xrightarrow{\delta} C_1 \xrightarrow{\delta} C_0 \longrightarrow \mathbb{Q} \longrightarrow 0$$

of the trivial module $\mathbb{Q}$ with the properties:

1. *C_k is isomorphic to $H_k(G) \otimes \mathbb{Q}[G]\widehat{\ }$;*
2. *$\delta(\Phi_m^{\mathbb{Q}}) \subseteq H_1(G) \otimes I^{m-1}$, where I denotes the augmentation ideal of $\mathbb{Q}[G]\widehat{\ }$.*

Denote the nilpotent Lie algebra associated to G by $\mathfrak{g}$. This is a finite dimensional Lie algebra over $\mathbb{Q}$ as G is nilpotent and the $\mathbb{Q}$ Betti numbers of G are finite. Denote the standard (i.e., the Chevelley–Eilenberg) complex of cochains of $\mathfrak{g}$ by $\mathcal{C}^\bullet(\mathfrak{g})$. By Sullivan's theory of minimal models, this is the minimal model of the classifying space BG of G.

Let (ω, ∂) be a formal power series connection associated to $\mathcal{C}^\bullet(\mathfrak{g})$.[1] Note that there is a natural isomorphism between $\widetilde{H}_\bullet(\mathcal{C}^\bullet(\mathfrak{g}))$ and the reduced homology groups $\widetilde{H}_\bullet(G, \mathbb{Q})$. Set

$$A_\bullet(G) = (T(\widetilde{H}_\bullet(G)[1])\widehat{\ }, \partial)\,.$$

[1]The definition and notation can be found in Appendix B.

Since $A_\bullet(G)$ is complete, we may assume, after taking a suitable change of coordinates of the form

$$Y \mapsto Y + \text{ higher order terms}, \quad Y \in \widetilde{H}_\bullet(G)[1]\,,$$

that

$$\partial Y \in I^k \Longleftrightarrow k = \max\{m \geq 1 : \partial(Y + \text{ higher order terms}) \in I^m\}\,. \tag{1}$$

Since $\mathcal{C}^\bullet(\mathfrak{g})$ is a minimal algebra in the sense of Sullivan, we can apply the standard fact [1, (2.30)] (see also, [11, 2.6.2]) to deduce that $H^k(B(\mathcal{C}(\mathfrak{g})))$ vanishes when $k > 0$. It follows from Theorem B.2 that $H_k(A_\bullet(G))$ also vanishes when $k > 0$.

When $k = 0$, Chen's Theorem B.1 yields a complete Hopf algebra isomorphism

$$H_0(A_\bullet(G)) \cong U\widehat{\ }\mathfrak{g} \cong \mathbb{Q}[G]\widehat{\ }.$$

These two facts are key ingredients in the construction of the resolution. Another important ingredient is the Adams–Hilton construction:

Proposition 6. *There is a continuous differential δ of degree -1 on the free $A_\bullet(G)$-module*

$$W := H_\bullet(G) \otimes A_\bullet(G)$$

such that:

1. *the restriction of δ to $A_\bullet = 1 \otimes A_\bullet$ is ∂*;
2. *δ is a graded derivation with respect to the right $A_\bullet(X)$ action*;
3. *(W, δ) is acyclic.*

Proof. To construct the differential, it suffices to define δ on each $Y \otimes 1$, where $Y \in H_\bullet(G)$. To show that the tensor product is acyclic, we will construct a graded map $s : W \to W$ of degree 1, such that $s^2 = 0$ and

$$\delta s + s\delta = \mathrm{id} - \epsilon\,. \tag{2}$$

Here ϵ is the tensor product of the augmentations of $H_\bullet(G)$ and $A_\bullet(G)$. Define $s(1) := 0 =: \delta(1)$. For $Y \in \widetilde{H}_\bullet(G)$, we shall denote the corresponding element of $\widetilde{H}_\bullet(G)[1]$ by $\overline{Y}$. For $Y, Y_1, Y_2, \ldots, Y_k \in \widetilde{H}_\bullet(G)$, define

$$s(1 \otimes \overline{Y}_1\overline{Y}_2 \cdots \overline{Y}_k) := Y_1 \otimes \overline{Y}_2 \cdots \overline{Y}_k \text{ and } s(Y \otimes \overline{Y}_1\overline{Y}_2 \cdots \overline{Y}_k) := 0\,.$$

We shall simply write Y for $Y \otimes 1$ and $\overline{Y}$ for $1 \otimes \overline{Y}$. Since δ is a derivation, it suffices to define it on the $A_\bullet(G)$-module generators $1, Y, \overline{Y}$ of W. We already know how to define δ on the $\overline{Y}$'s. In order that (2) holds when applied to Y, we have to define

$$\delta Y := \overline{Y} - s\delta\overline{Y}\,.$$

Then (2) also holds when applied to $\overline{Y}$ and this implies $\delta^2 Y = 0$. One can now verify (2) by induction on degree using the fact that s is right $A_\bullet(G)$-linear. $\square$

The way to think of this result is that $A_\bullet(G)$ is a homological model of the loop space $\Omega_x BG$ of the classifying space of G. One can take $H_\bullet(G)$ with the trivial differential to be a homological model for BG. The complex (W, δ) is a homological model of the path-loops fibration whose total space is contractible. This picture should help motivate the next step, the construction of the minimal resolution.

Filter the complex (W, δ) by degree in $H_\bullet(G)$. This leads to a spectral sequence

$$E^1_{p,q} = H_p(G) \otimes H_q(A_\bullet(G)) \implies H_{p+q}(W, \delta)\,.$$

(This is the algebraic analogue of the Serre spectral sequence for the path-loops fibration.) But since $H_q(A_\bullet(G))$ vanishes when $q > 0$, the spectral sequence collapses at the E^1-term to a complex

$$\cdots \to H_2(G) \otimes H_0(A_\bullet(G)) \to H_1(G) \otimes H_0(A_\bullet(G)) \to H_0(A_\bullet(G)) \to \mathbb{Q} \to 0$$

which is acyclic as (W, δ) is. The existence of the resolution now follows from the fact that

$$H_0(A_\bullet(G)) \cong \mathbb{Q}[G]\widehat{\ }.$$

The $\mathbb{Q}[G]\widehat{\ }$ linearity of the differential follows directly from the second assertion of Proposition 6.

It remains to verify the condition satisfied by the differential. Observe that $H_0(A_\bullet(G))$ is the quotient of $T(H_1(G))\widehat{\ }$ by the closed ideal generated by the image of

$$\partial : H_2(G) \to T(H_1(G))\widehat{\ }.$$

Define a filtration

$$H_2(G; \mathbb{Q}) = L_2 \supseteq L_3 \supseteq \cdots$$

by $L_k = \partial^{-1} I^k$. It is not difficult to verify that in the complex $H_\bullet(G) \otimes H_0(A_\bullet(G))$ we have

$$\delta(L_k \otimes \mathbb{Q}) \subseteq H_1(G) \otimes I^{k-1}\,.$$

So, to complete the proof, we have to show that $\Phi_k^{\mathbb{Q}} = L_k$.

Consider the spectral sequence dual to the one in the proof of Theorem B.2. It has the property that $E^r_{-1,2}$ is a subspace of $H_2(G; \mathbb{Q})$ and $E^r_{-p,p}$ is a quotient of $H_1(G; \mathbb{Q})^{\otimes p}$. We thus have a filtration

$$H_2(G; \mathbb{Q}) = E^1_{-1,2} \supseteq E^2_{-1,2} \supseteq E^3_{-1,2} \supseteq \cdots$$

of $H_2(G; \mathbb{Q})$. From the standard description of the terms of the spectral sequence associated to a filtered complex, it is clear from condition (1) that when $r \geq 2$

$$L_r = E^{r-1}_{-1,2}\,.$$

We know from [4, (2.6.1)] that

$$E^p_{-p,p} = E^\infty_{-p,p} \cong I^p/I^{p+1}\,,$$

where I denotes the augmentation ideal of $\mathbb{Q}[\pi_1(G)]$. Consequently, a group homomorphism $G \to H$ induces an isomorphism

$$\mathbb{Q}[G]/I^k \to \mathbb{Q}[H]/I^k$$

if and only if the induced map $H_2(G;\mathbb{Q})/L_k \to H_2(H;\mathbb{Q})/L_k$ is an isomorphism. The proof is completed using the fact (cf. [11, (2.5.3)]) that a group homomorphism $G \to H$ induces an isomorphism $(G \otimes \mathbb{Q})/\Gamma_k \to (H \otimes \mathbb{Q})/\Gamma_k$ if and only if the induced map $\mathbb{Q}[G]/I^k \to \mathbb{Q}[H]/I^k$ is an isomorphism, [17].

7. The Inequality

In this section, we prove Proposition 5. We suppose that G is a nilpotent group with $b_1(G) < \infty$. Denote $\mathbb{Q}[G]\hat{}$ by A and its augmentation ideal by I. Recall that

$$d(t) = \sum_{l \geq 0} d_l\, t^l\,,$$

where $d_l = \dim I^l/I^{l+1}$, and that this is a rational function whose poles lie on the unit circle.

Consider the part

$$H_2(G) \otimes A \xrightarrow{\delta} H_1(G) \otimes A \to A \to \mathbb{Q} \to 0$$

of the resolution constructed in Section 6. Define a filtration

$$E^0 \supseteq E^1 \supseteq E^2 \supseteq \cdots$$

of $H_2(G) \otimes A$ by

$$E^l := \delta^{-1}\left(H_1(G) \otimes I^l\right)\,.$$

Then we have an exact sequence

$$0 \to E^l/E^{l+1} \to H_1(G) \otimes I^l/I^{l+1} \to I^{l+1}/I^{l+2} \to 0 \tag{3}$$

for all $l \geq 0$. Set $e_l := \dim E^l/E^{l+1}$ and

$$e(t) := \sum_{l \geq 0} e_l\, t^l \in \mathbb{R}[[t]]\,.$$

Proposition 7. *The series for $e(t)$ converges to the rational function*

$$\frac{b_1(G)\, t d(t) - d(t) + 1}{t}$$

all of whose poles lie on the unit circle.

Proof. Because the sequence (3) is exact, we have $e_l = b_1(G)\, d_l - d_{l+1}$. This implies that

$$te(t) = b_1(G)\, td(t) - d(t) + 1\,.$$

from which the result follows. □

Our final task is to bound the e_l. Our resolution has the property that

$$\delta(\Phi_m \otimes I^{l-m+2}) \subseteq H_1(G) \otimes I^{l+1}\,.$$

This implies that

$$\sum_{m\geq 2} \Phi_m \otimes I^{l-m+2} \subseteq E^{l+1}\,,$$

so that the linear map

$$H_2(G) \otimes A/\left(\sum_{m\geq 2} \Phi_m \otimes I^{l-m+2}\right) \to H_2(G) \otimes A/E^{l+1}$$

is surjective. This implies that

$$\dim\left(H_2(G) \otimes A/(\sum_{m\geq 2} \Phi_m \otimes I^{l-m+2})\right) \geq e_0 + e_1 + \cdots + e_l\,. \tag{4}$$

To compute the dimension of the left hand side, we apply the following elementary fact from linear algebra.

Proposition 8. *Suppose that*

$$B = B^0 \supseteq B^1 \supseteq B^2 \supseteq \cdots$$

and

$$C = C^0 \supseteq C^1 \supseteq C^2 \supseteq \cdots$$

are two filtered vector spaces. Define a filtration F of $B \otimes C$ by

$$F^k := \sum_{i+j=k} B^i \otimes C^j\,.$$

Then there is a canonical isomorphism

$$F^k/F^{k+1} \cong \bigoplus_{i+j=k} A^i/A^{i+1} \otimes B^j/B^{j+1}\,.$$ □

Applying this with $B = H_2(G)$ with the filtration $\Phi_\bullet$, and $C = A$ with the filtration $I^\bullet$, we deduce that

$$\dim\left(H_2(G) \otimes A/(\sum_{m\geq 2} \Phi_m \otimes I^{l-m+2})\right)$$
$$= \sum_{m+k\leq l+1} (\dim \Phi_m/\Phi_{m+1})(\dim I^k/I^{k+1}) \;=\; \sum_{m+k\leq l+1} p_m d_k$$

where $p_m = \dim \Phi_m$. Combined with (4), this implies that

$$\sum_{m+k \le l+1} p_m d_k \ge e_0 + \cdots + e_l \,.$$

Using geometric series, this can be assembled into the following inequality which holds when $0 < t < 1$:

$$\frac{(p_2\, t + p_3\, t^2 + \cdots + p_{l+1}\, t^l)\, d(t)}{1-t} \ge \frac{e(t)}{1-t} \,.$$

Plugging in the formula for $e(t)$ given in Proposition 7, we deduce the desired inequality

$$d(t) \left(\sum_{m \ge 2} p_m\, t^m - b_1(G)\, t + 1 \right) \ge 1$$

when $0 < t < 1$.

Appendix A

Here we collect the necessary group theoretic definitions. Let G be a group.

- $\Gamma_r(G)$ denotes the r-th term of the *lower central series* of G which is the subgroup of G generated by all r-fold commutators. We abbreviate $G/\Gamma_r(G)$ by G/Γ_r. We say G is *nilpotent* if $\Gamma_r(G) = \{1\}$ for some r and it is then said to have *nilpotency class* $< r$. For example, abelian groups have $\Gamma_2 = \{1\}$ and are of nilpotency class 1.
- G is *prenilpotent* if it's lower central series stabilizes at some term Γ_r, i.e., $\Gamma_r(G) = \Gamma_{r+1}(G)$. This happens if and only if G has a maximal nilpotent quotient (which is then isomorphic to G/Γ_r).

The main homological tool in dealing with nilpotent groups is the following result of Stallings:

Theorem A.1. [20] *If $f : G \to H$ is a homomorphism of groups inducing an isomorphism on H_1 and an epimorphism on H_2, then the induced maps $G/\Gamma_r \to H/\Gamma_r$ are isomorphisms for all $r \ge 1$.*

- The *Dwyer filtration* of $H_2(G;\mathbb{Z})$

$$H_2(G;\mathbb{Z}) =: \Phi_2(G) \supseteq \Phi_3(G) \supseteq \Phi_4(G) \supseteq \cdots$$

  is defined by (see [8] for a geometric definition of Φ_r using gropes)

$$\Phi_r(G) := \mathrm{Ker}(H_2(G;\mathbb{Z}) \longrightarrow H_2(G/\Gamma_{r-1};\mathbb{Z}))\,.$$

This filtration is used in Dwyer's extension of Stallings' Theorem:

Theorem A.2. [5] *If $f : G \longrightarrow H$ induces an isomorphism on $H_1(\ ;\mathbb{Z})$, then for $r \ge 2$ the following three conditions are equivalent:*

1. *f induces an epimorphism $H_2(G;\mathbb{Z})/\Phi_r(G) \to H_2(H;\mathbb{Z})/\Phi_r(H)$;*
2. *f induces an isomorphism $G/\Gamma_r \to H/\Gamma_r$;*
3. *f induces an isomorphism $H_2(G;\mathbb{Z})/\Phi_r(G) \to H_2(H;\mathbb{Z})/\Phi_r(H)$, and an injection $H_2(G;\mathbb{Z})/\Phi_{r+1}(G) \to H_2(H;\mathbb{Z})/\Phi_{r+1}(H)$.*

There are rational versions of the above definitions.

- $\Gamma_r^{\mathbb{Q}}$ denotes the r-th term of the *rational lower central series* of G, which is defined by

$$\Gamma_r^{\mathbb{Q}}(G) := \mathrm{Rad}(\Gamma_r(G)) := \{g \in G \mid g^n \in \Gamma_r \text{ for some } n \in \mathbb{Z}\}\,.$$

  It has the (defining) property that $G/\Gamma_r^{\mathbb{Q}} = (G/\Gamma_r)/\,\mathrm{Torsion}$. (Note that the torsion elements in a nilpotent group form a subgroup.) G is *rationally nilpotent* if $\Gamma_r^{\mathbb{Q}}(G) = \{1\}$ for some r.
- G is *rationally prenilpotent* if $\Gamma_r^{\mathbb{Q}}(G) = \Gamma_{r+1}^{\mathbb{Q}}(G)$ for some r. This happens if and only if G has a maximal torsion-free nilpotent quotient (which is then isomorphic to $G/\Gamma_r^{\mathbb{Q}}$).
- The *rational Dwyer filtration* of $H_2(G;\mathbb{Q})$

$$H_2(G;\mathbb{Q}) =: \Phi_2^{\mathbb{Q}}(G) \supseteq \Phi_3^{\mathbb{Q}}(G) \supseteq \Phi_4^{\mathbb{Q}}(G) \supseteq \dots$$

  is defined by

$$\Phi_r^{\mathbb{Q}}(G) := \Phi_r(G) \otimes \mathbb{Q} = \mathrm{Ker}(H_2(G;\mathbb{Q}) \longrightarrow H_2(G/\Gamma_{r-1}^{\mathbb{Q}};\mathbb{Q}))\,.$$

- The *Malcev completion* [16] $G \otimes \mathbb{Q}$ of a nilpotent group G may be defined inductively through the central extensions determining G as follows: If G is abelian then one takes the usual tensor product of abelian groups to define $G \otimes \mathbb{Q}$. It comes with a homomorphism $\epsilon : G \to G \otimes \mathbb{Q}$ which induces an isomorphism on rational cohomology. Using the fact that the cohomology group H^2 classifies central extensions, one can then define the Malcev completion for a group which is a central extension of an abelian group. It comes again with a map ϵ as above. The Serre spectral sequence then shows that ϵ induces an isomorphism on rational cohomology. Therefore, one can repeat the last step to define $(G \otimes \mathbb{Q}, \epsilon)$ for an arbitrary nilpotent group G.

  The map $\epsilon : G \to G \otimes \mathbb{Q}$ is universal for maps of G into uniquely divisible nilpotent groups and it is characterized by the following properties:

  1. $G \otimes \mathbb{Q}$ is a uniquely divisible nilpotent group.
  2. The kernel of ϵ is the torsion subgroup of G.
  3. For every element $x \in G \otimes \mathbb{Q}$ there is a number $n \in \mathbb{N}$ such that x^n is in the image of ϵ.

A version of Stallings' Theorem holds in the rational setting:

Theorem A.3. [20] *If $f : G \to H$ is a homomorphism of groups inducing an isomorphism on $H_1(\ ;\mathbb{Q})$ and an epimorphism on $H_2(\ ;\mathbb{Q})$, then the induced maps $(G/\Gamma_r) \otimes \mathbb{Q} \to (H/\Gamma_r) \otimes \mathbb{Q}$ are isomorphisms for all $r \geq 1$.*

A good example to keep in mind when trying to understand this theorem is the inclusion of the Heisenberg group H_1 into the Heisenberg group H_n. This induces an isomorphism on rational homology. Both groups are torsion free nilpotent of class 2.

There is also a rational analogue of Dwyer's theorem.

Theorem A.4. [5] *If $f : G \longrightarrow H$ induces an isomorphism on $H_1(\ ;\mathbb{Q})$, then for $r \geq 2$ the following three conditions are equivalent*:

1. *f induces an epimorphism $H_2(G;\mathbb{Q})/\Phi_r^{\mathbb{Q}}(G) \to H_2(H;\mathbb{Q})/\Phi_r^{\mathbb{Q}}(H)$;*
2. *f induces an isomorphism $(G/\Gamma_r) \otimes \mathbb{Q} \to (H/\Gamma_r) \otimes \mathbb{Q}$;*
3. *f induces an isomorphism $H_2(G;\mathbb{Q})/\Phi_r^{\mathbb{Q}}(G) \to H_2(H;\mathbb{Q})/\Phi_r^{\mathbb{Q}}(H)$, and an injection $H_2(G;\mathbb{Q})/\Phi_{r+1}^{\mathbb{Q}}(G) \to H_2(H;\mathbb{Q})/\Phi_{r+1}^{\mathbb{Q}}(H)$.*

Appendix B

In this appendix, we give a brief review of Chen's method of formal power series connections. Two relevant references are [4] and [10]. There is also an informal discussion of the ideas behind Chen's work in [12].

Fix a field F of characteristic zero. Suppose that $\mathcal{A}^\bullet$ is an augmented commutative d.g. algebra over F. Suppose in addition that $\mathcal{A}^\bullet$ is positively graded (i.e., $\mathcal{A}^k = 0$ when $k < 0$) and that $H^\bullet(\mathcal{A}^\bullet)$ is connected (i.e., $H^0(\mathcal{A}^\bullet) = F$). For simplicity, we suppose that each $H^k(\mathcal{A}^\bullet)$ is finite dimensional. Typical examples of such $\mathcal{A}^\bullet$ in this theory are the F (=$\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$) de Rham complex of a connected space with finite rational betti numbers, or the Sullivan minimal model of such an algebra.

Set

$$H_k(\mathcal{A}^\bullet) = \mathrm{Hom}_F(H^k(\mathcal{A}^\bullet), F)$$

and

$$\widetilde{H}_\bullet(\mathcal{A}^\bullet) = \sum_{k>0} H_k(\mathcal{A}^\bullet)\,.$$

Denote by $\widetilde{H}_\bullet(\mathcal{A}^\bullet)[1]$ the graded vector space obtained by taking $\widetilde{H}_\bullet(\mathcal{A}^\bullet)$ and lowering the degree of each of its elements by one.[2]

Denote the free associative algebra generated by the graded vector space V by $T(V)$. Assume that V_n is non-zero only when $n \geq 0$. Denote the ideal generated by V_0 by I_0. Let $T(V)\hat{}$ denote the I_0-adic completion

$$\lim_{\leftarrow} T(V)/I_0^m\,.$$

By defining each element of V to be primitive, we give $T(V)\hat{}$ the structure of a complete graded Hopf algebra. Recall that the set of primitive elements $PT(V)$ of

[2]Here we are using the algebraic geometers' notation that if V is a graded vectorspace, then $V[n]$ is the graded vector space with $V[n]_m = V_{m+n}$.

$T(V)$ is the free Lie algebra $\mathbb{L}(V)$, and the set of primitive elements of $T(V)\widehat{\ }$ is the closure $\mathbb{L}(V)\widehat{\ }$ of $\mathbb{L}(V)$ in $T(V)\widehat{\ }$.

Set $A_\bullet = T(\widetilde{H}_\bullet(\mathcal{A}^\bullet))\widehat{\ }$. Choose a basis $\{X_i\}$ of $\widetilde{H}_\bullet(\mathcal{A}^\bullet)$. Then $A_\bullet$ is the subalgebra of the non-commutative power series algebra generated by the X_i consisting of those power series with the property that each of its terms has a fixed degree. For each multi-index $I = (i_1, \ldots, i_s)$, we set

$$X_I = X_{i_1} X_{i_2} \cdots X_{i_s} \,.$$

If $X_i \in \widetilde{H}_{p_i}(\mathcal{A}^\bullet)$, we set $|X_I| = -s + p_1 + \cdots + p_s$.

A *formal power series connection* on $\mathcal{A}^\bullet$ is a pair (ω, ∂). The first part

$$\omega = \sum_I w_I \, X_I$$

is an element of the completed tensor product $\mathcal{A}^\bullet \widehat{\otimes} P A_\bullet$, where w_I is an element of $\mathcal{A}^\bullet$ of degree $1 + |X_I|$. The second is a graded derivation

$$\partial : A_\bullet \to A_\bullet$$

of degree -1 with square 0 and which satisfies the minimality condition $\partial(I A_\bullet) \subseteq I^2 A_\bullet$. Here $I^k A_\bullet$ denotes the kth power of the augmentation ideal of $A_\bullet$. These are required to satisfy two conditions. The first is the "integrability condition"

$$\partial\omega + d\omega + \frac{1}{2}[J\omega, \omega] = 0 \,.$$

Here $J : \mathcal{A}^\bullet \to \mathcal{A}^\bullet$ is the linear map $a \mapsto (-1)^{\deg a} a$. The value of the operators d, ∂ and J on ω are obtained by applying the operators to the appropriate coefficients of ω:

$$d\omega = \sum dw_I \, X_I, \quad \partial\omega = \sum w_I \, \partial X_I, \quad J\omega = \sum J w_I \, X_I \,.$$

The second is that if

$$\sum_i w_i \, X_i \tag{5}$$

is the reduction of ω mod $I^2 A_\bullet$, then each w_i is closed and the $[w_i]$ form a basis of $H^{>0}(\mathcal{A}^\bullet)$ dual to the basis $\{X_i\}$ of $\widetilde{H}_\bullet(\mathcal{A}_\bullet)$.

Such formal connections always exist in the situation we are describing. To justify the definition, we recall one of Chen's main theorems. It is the analogue of Sullivan's main theorem about minimal models.

Theorem B.1. *Suppose that X is a smooth manifold with finite betti numbers, x a fixed point of X, and suppose that $\mathcal{A}^\bullet$ is the F-de Rham complex of X, with the augmentation induced by x. If X is simply connected, the connection gives a natural Lie algebra isomorphism*

$$\pi_\bullet(X, x)[1] \otimes F \cong H_\bullet(P A_\bullet, \partial) \,.$$

If X is not simply connected, then the connection gives a Lie algebra isomorphism

$$\mathfrak{g}(X,x) \cong H_0(PA_\bullet, \partial)$$

and complete Hopf algebra isomorphisms

$$\mathbb{Q}[\pi_1(X,x)]\widehat{\ } \cong U\widehat{\ }\mathfrak{g}(X,x) \cong H_0(A_\bullet, \partial)\,.$$

Here $\mathfrak{g}(X,x)$ denotes the F form of the Malcev Lie algebra associated to $\pi_1(X,x)$ and $U\widehat{\ }\mathfrak{g}(X,x)$ is the completion of its enveloping algebra with respect to the powers of its augmentation ideal. □

We can apply the bar construction to the augmented algebra $\mathcal{A}^\bullet$ to obtain a commutative d.g. Hopf algebra $B(\mathcal{A}^\bullet)$. (We use the definition in [4].) One can define the *formal transport map* of such a formal connection. It is defined to be the element

$$T = 1 + \sum [w_I]\, X_I + \sum [w_I|w_J]\, X_I X_J + \sum [w_I|w_J|w_K]\, X_I X_J X_K + \cdots$$

of $B(\mathcal{A}^\bullet)\widehat{\otimes} A_\bullet$. It induces a linear map

$$\Theta : \mathrm{Hom}_F^{\mathrm{cts}}(A_\bullet, F) \to B(\mathcal{A}^\bullet)\,.$$

Here $\mathrm{Hom}_F^{\mathrm{cts}}(A_\bullet, F)$ denotes the continuous dual

$$\lim_{\rightarrow} \mathrm{Hom}_F(A_\bullet/I^k A_\bullet, F)$$

of $A_\bullet$. It is a commutative Hopf algebra. The map Θ takes the continuous functional ϕ to the result of applying it to the coefficients of T:

$$\Theta(\phi) = \\ 1 + \sum [w_I]\phi(X_I) + \sum [w_I|w_J]\phi(X_I X_J) + \sum [w_I|w_J|w_K]\phi(X_I X_J X_K) + \cdots$$

(Note that this is a finite sum as ϕ is continuous.) The properties of the formal connection imply that Θ is a d.g. Hopf algebra homomorphism (cf. [10, (6.17)]). The basic result we need is:

Theorem B.2. *The map Θ induces an isomorphism on homology.*

We conclude by giving a brief sketch of the proof. One can filter $A_\bullet$ by the powers of its augmentation ideal. This gives a dual filtration of $\mathrm{Hom}_F^{\mathrm{cts}}(A_\bullet, F)$. The corresponding spectral sequence has E_1 term

$$E_1^{-s,t} = [\widetilde{H}^\bullet(\mathcal{A}^\bullet)^{\otimes s}]^t\,.$$

On the other hand, one can filter $B(\mathcal{A}^\bullet)$ by the "bar filtration" to obtain a spectral sequence, also with this E_1 term. It is easy to check that Φ preserves the filtrations and therefore induces a map of spectral sequences. The condition on the w_i in (5) implies that the map on E_1 is an isomorphism. The result follows.

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THE WORK OF VAUGHAN F. R. JONES

by

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It gives me great pleasure that I have been asked to describe to you some of the very beautiful mathematics which resulted in the awarding of the Fields Medal to Vaughan F. R. Jones at ICM '90.

In 1984 Jones discovered an astonishing relationship between von Neumann algebras and geometric topology. As a result, he found a new polynomial invariant for knots and links in 3-space. His invariant had been missed completely by topologists, in spite of intense activity in closely related areas during the preceding 60 years, and it was a complete surprise. As time went on, it became clear that his discovery had to do in a bewildering variety of ways with widely separated areas of mathematics and physics, some of which are indicated in Figure 1. These included (in addition to knots and links) that part of statistical mechanics having to do with exactly solvable models, the very new area of quantum groups, and also Dynkin diagrams and the representation theory of simple Lie algebras. The central connecting link in all this mathematics was a tower of nested algebras which Jones had discovered some years earlier in the course of proving a theorem which is known as the "Index Theorem".

My plan is to begin by discussing the Index Theorem, and the tower of algebras which Jones constructed in the course of his proof. After that, I plan to return to the chart in Figure 1 in order to indicate how this tower of algebras served as a bridge between the diverse areas of mathematics which are shown on the chart. I will restrict my attention throughout to one very special example of the tower construction, and so also to one special example of the associated link invariants, in order to make it possible to survey a great deal of mathematics in a very short time. Even with the restriction to a single example, this is a very ambitious plan. On the other hand, it only begins to touch on Vaughan Jones' scholarly contributions.

1. The Index Theorem

Let $\mathbf{M}$ denote a von Neumann algebra. Thus $\mathbf{M}$ is an algebra of bounded operators acting on a Hilbert space $\mathcal{H}$. The algebra $\mathbf{M}$ is called a *factor* if its center consists

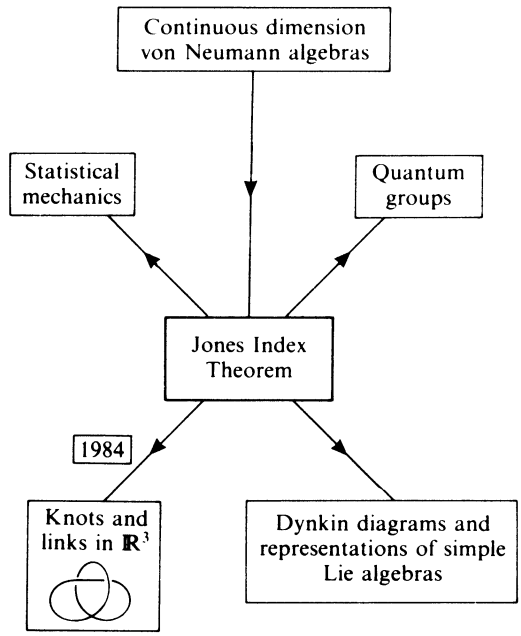

Fig. 1. The Jones Index Theorem

only of scalar multiples of the identity. The factor is *type* II_1 if it admits a linear functional, called a trace, $\mathrm{tr} : \mathbf{M} \to \mathbb{C}$, which satisfies the following three conditions:

$$\mathrm{tr}(\mathbf{xy}) = \mathrm{tr}(\mathbf{yx}) \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbf{M}$$

$$\mathrm{tr}(\mathbf{1}) = 1$$

$$\mathrm{tr}(\mathbf{xx}^*) > 0 \text{ for all } \mathbf{x} \in \mathbf{M}, \text{ where } \mathbf{x}^* \text{ is the adjoint of } \mathbf{x}$$

In this situation it is known that the trace is unique, in the sense that it is the only linear form satisfying the first two conditions. An old discovery of Murray and von Neumann was that factors of type II_1 provide a type of "scale" by which one can measure the dimension $\dim_{\mathbf{M}}(\mathcal{H})$ of $\mathcal{H}$. The notion of dimension which occurs here generalizes the familiar notion of integer-valued dimensions, because for appropriate $\mathbf{M}$ and $\mathcal{H}$ it can be any non-negative real number or ∞.

The starting point of Jones' work was the following question: if $\mathbf{M}_1$ is a type II_1 factor and if $\mathbf{M}_0 \subset \mathbf{M}_1$ is a *subfactor*, is there any restriction on the real numbers which occur as the ratio

$$\lambda = \dim_{\mathbf{M}_0}(\mathcal{H}) / \dim_{\mathbf{M}_1}(\mathcal{H}) \ ?$$

The question has the flavor of questions one studies in Galois theory. On the face of it, there was no reason to think that λ could not take on any value in $[1, \infty]$ so Jones' answer came as a complete surprise. He called λ the *index* $[\mathbf{M}_1 : \mathbf{M}_0]$ of

$\mathbf{M}_0$ in $\mathbf{M}_1$, and proved a type of rigidity theorem about type II_1 factors and their subfactors.

The Jones Index Theorem. *If* $\mathbf{M}_1$ *is a* II_1 *factor and* $\mathbf{M}_0$ *a subfactor, then the possible values of the index* $[\mathbf{M}_1 : \mathbf{M}_0]$ *are restricted to*:

$$[4, \infty] \cup \{4\cos^2(\pi/p)\,, \quad \textit{where } p \in \mathbb{N},\, p \geq 3\}\,.$$

Moreover, each real number in the continuous part of the spectrum $[4, \infty]$ *and also in the discrete part* $\{4\cos^2(\pi/p),\, p \in \mathbb{N},\, p \geq 3\}$ *is realized.*

We now sketch the idea of the proof, which is to be found in [Jo1]. Jones begins with the type II_1 factor $\mathbf{M}_1$ and the subfactor $\mathbf{M}_0$. There is also a tiny bit of additional structure: In this setting there exists a map $\mathbf{e}_1 : \mathbf{M}_1 \to \mathbf{M}_0$, known as the *conditional expectation* of $\mathbf{M}_1$ on $\mathbf{M}_0$. The map $\mathbf{e}_1$ is a *projection*, i.e. $(\mathbf{e}_1)^2 = \mathbf{e}_1$.

His first step is to prove that the ratio λ is independent of the choice of the Hilbert space $\mathcal{H}$. This allows him to choose an appropriate $\mathcal{H}$ so that the algebra $\mathbf{M}_2 = \langle \mathbf{M}_1, \mathbf{e}_1 \rangle$ generated by $\mathbf{M}_1$ and $\mathbf{e}_1$ makes sense. He then investigates $\mathbf{M}_2$ and proves that it is another type II_1 factor, which contains $\mathbf{M}_1$ as a subfactor, moreover the index $|\mathbf{M}_2 : \mathbf{M}_1|$ is equal to the index $|\mathbf{M}_1 : \mathbf{M}_0|$, i.e. to λ. Having in hand another II_1 factor $\mathbf{M}_2$ and its subfactor $\mathbf{M}_1$, there is also a trace on $\mathbf{M}_2$ which (by the uniqueness of the trace) coincides with the trace on $\mathbf{M}_1$ when it is restricted to $\mathbf{M}_1$, and another conditional expectation $\mathbf{e}_2 : \mathbf{M}_2 \to \mathbf{M}_1$. This allows Jones to iterate the construction, to build algebras $\mathbf{M}_1, \mathbf{M}_2 \ldots$ and from them a family of algebras:

$$\mathbf{J}_n = \{\mathbf{1}, \mathbf{e}_1, \ldots, \mathbf{e}_{n-1}\} \subset \mathbf{M}_n\,, \quad n = 1, 2, 3, \ldots\,.$$

Rewriting history a little bit in order to make the subsequent connection with knots a little more transparent, we now replace the $\mathbf{e}_k$'s by a new set of generators which are units, defining:

$$\mathbf{g}_k = q\mathbf{e}_k - (1 - \mathbf{e}_k)\,,$$

where

$$(1 - q)(1 - q^{-1}) = 1/\lambda\,.$$

The $\mathbf{g}_k$'s generate $\mathbf{J}_n$, because the $\mathbf{e}_k$'s do, and we can solve for the $\mathbf{e}_k$'s in terms of the $\mathbf{g}_k$'s. So

$$\mathbf{J}_n = \mathbf{J}_n(q) = \{\mathbf{1}, \mathbf{g}_1, \ldots, \mathbf{g}_{n-1}\}\,,$$

and we have a *tower of algebras*, ordered by inclusion:

$$\mathbf{J}_1(q) \subset \mathbf{J}_2(q) \subset \mathbf{J}_3(q) \subset \ldots\,.$$

The parameter q, which replaces the index λ, is the quantity now under investigation.

The parameter q is woven into the construction of the tower. First, defining relations in $\mathbf{J}_n(q)$ depend upon q:

$$\mathbf{g}_i\mathbf{g}_k = \mathbf{g}_k\mathbf{g}_i \quad \text{if } |i-k| \geq 2\,, \tag{1}$$

$$\mathbf{g}_i\mathbf{g}_{i+1}\mathbf{g}_i = \mathbf{g}_{i+1}\mathbf{g}_i\mathbf{g}_{i+1}\,, \tag{2}$$

$$\mathbf{g}_i^2 = (q-1)\mathbf{g}_i + q\,, \tag{3q}$$

$$\mathbf{1} + \mathbf{g}_i + \mathbf{g}_{i+1} + \mathbf{g}_i\mathbf{g}_{i+1} + \mathbf{g}_{i+1}\mathbf{g}_i + \quad + \mathbf{g}_i\mathbf{g}_{i+1}\mathbf{g}_i = 0\,. \tag{4}$$

A second way in which q enters into the structure involves the trace. Recall that since $\mathbf{M}_n$ is type II_1 it supports a unique trace, and since $\mathbf{J}_n$ is a subalgebra it does too, by restriction. This trace is known as a *Markov trace*, i.e. it satisfies the important property:

$$\mathrm{tr}(\mathbf{w}\mathbf{g}_n) = \tau(q)\ \mathrm{tr}(\mathbf{w}) \quad \text{if } \mathbf{w} \in \mathbf{J}_n\,, \tag{5q}$$

where $\tau(q)$ is a fixed function of q. Thus, for each fixed value of q the trace is multiplied by a fixed scalar when one passes from one stage of the tower to the next, by multiplying an arbitrary element of $\mathbf{J}_n$ by the new generator $\mathbf{g}_n$ of $\mathbf{J}_{n+1}$.

Relations (1) and (2) above have an interesting geometric meaning, familiar to topologists. They are defining relations for the *n-string braid group*, $\mathbf{B}_n$, discovered by Emil Artin [Ar] in a foundational paper written in 1923. We pause to discuss braids.

An n-braid may be visualized by a weaving pattern of strings in 3-space which join n points on a horizontal plane to n corresponding points on a parallel plane, as illustrated in the example in Figure 2, where $n = 4$. In the case $n = 3$, the familiar braid in a person's hair gives another example. The strings are allowed to be stretched and deformed, the key features being that strings cannot pass through one-another and always proceed directly downward in their travels from the upper plane to the lower one. The equivalence class of weaving patterns under such deformations

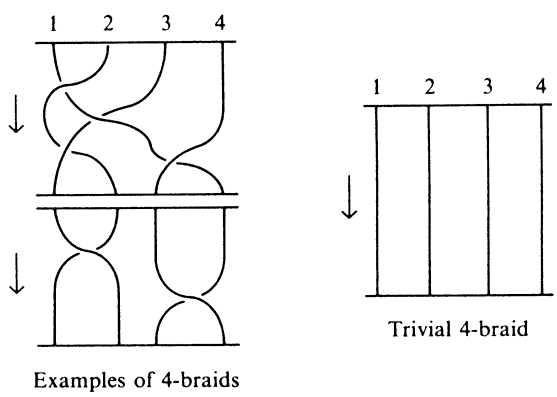

Fig. 2.

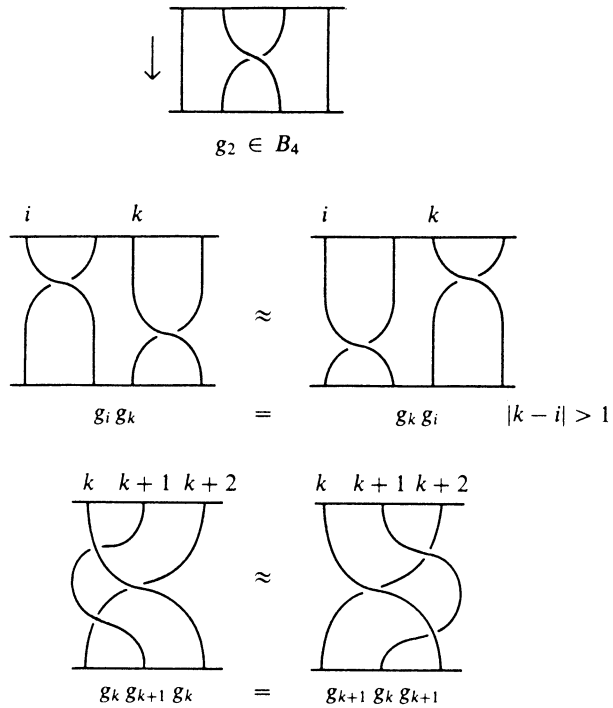

Fig. 3. Generators and defining relations in $\mathbf{B}_n$

is an *n-braid*. One multiplies braids by concatenation and erasure of the middle plane. This multiplication makes them into a group, the n-string braid group $\mathbf{B}_n$. The identity is a braid which, when pulled taut, goes over to n straight lines. Generators are the $n-1$ elementary braids which (by an abuse of notation) we continue to call $\mathbf{g}_1, \ldots, \mathbf{g}_{n-1}$. The pictures in Figure 3 show that relations (1) and (2) hold between the generators of $\mathbf{B}_n$. In fact, Artin proved they are *defining relations* for $\mathbf{B}_n$. Thus for each n there is a homomorphism from the n-string braid group $\mathbf{B}_n$ into the Jones algebra $\mathbf{J}_n(q)$, and from the group algebra $\mathbb{C}\mathbf{B}_n$ onto $\mathbf{J}_n(q)$.

Returning to the business at hand, i.e. the proof of the Index Theorem, Jones next shows that properties (1), (2), (3_{q}) and (5_{q}) suffice for the calculation of the trace of an arbitrary element $\mathbf{x} \in \mathbf{J}_n(q)$. It turns out that trace($\mathbf{x}$) is an *integer polynomial* in $(\sqrt{q})^{\pm 1}$. (We will meet it again in a few moments as the Jones polynomial associated to $\mathbf{x}$.) Jones proof of the Index Theorem is concluded when he shows that the infinite sequence of algebras $\mathbf{J}_n(q)$, with the given trace, could not exist if q did not satisfy the restrictions of the Index Theorem.

2. Knots and Links

We have already seen hints of topological meaning in $\mathbf{J}_n(q)$ via braids. There is more to come. Knots and links are obtained from braids by identifying the initial points and end points of a braid in a circle, as illustrated in Figure 4. It was proved

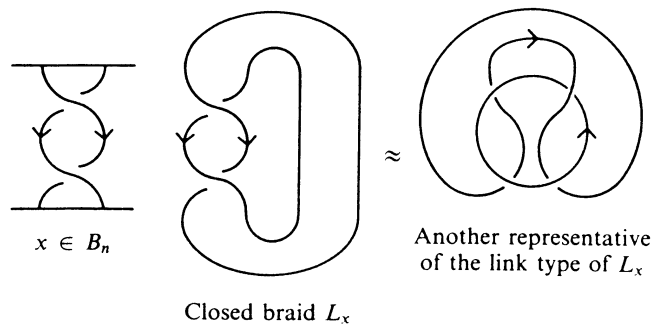

Fig. 4. Braids determine links

by J. W. Alexander in 1928 that every knot or link arises in this way. Earlier we described an equivalence relation on weaving patterns which yields braids, and there is a similar (but less restrictive) equivalence relation on knots, i.e. a knot or link *type* is its equivalence class under isotopy in 3-space. Note that isotopy in 3-space which takes one closed braid representative of a link to another closed braid representative will pass through a sequence of representatives which are not closed braids in an obvious way. For example see the 2-component link which is illustrated in Figure 4. The left picture is an obvious closed braid representative, whereas the right is not.

Let $\mathbf{B}_\infty$ denote the disjoint union of all of the braid groups $\mathbf{B}_n$, $n = 1, 2, 3, \ldots$. In 1935 the mathematician A. A. Markov proposed the equivalence relation on $\mathbf{B}_\infty$ which corresponds to link equivalence [M]. Remarkably, the properties of the trace, or more particularly the facts that $\mathrm{tr}(\mathbf{xy})=\mathrm{tr}(\mathbf{yx})$ together with property (5_q), were exactly what was needed to make the trace polynomial into an invariant on Markov's equivalence classes! Using Markov's proposed equivalence relation (which was proved to be the correct one in 1972 [Bi]), Jones proved, with almost no additional work beyond results already established in [Jo1], the following theorem:

Theorem [Jo3]. *If* $\mathbf{w} \in \mathbf{B}_\infty$, *then* (*after multiplication by an appropriate scalar, which depends upon the braid index* n) *the trace of the image of* $\mathbf{w}$ *in* $\mathbf{J}_n(q)$ *is a polynomial in* $(\sqrt{q})^{\pm 1}$ *which is an invariant of the link type defined by the closed braid* $\mathbf{L}_\mathbf{w}$.

The invariant of Jones' theorem is the one-variable *Jones polynomial* $\mathbf{V}_x(q)$. Notice that the independent "variable" in this polynomial is essentially the index of a type II_1 subfactor in a type II_1 factor! It's discovery opened a new chapter in knot and link theory.

3. Statistical Mechanics

We promised to discuss other ways in which the work of Jones was related to yet other areas of mathematics and physics, and begin to do so now. As it turned

out, when Jones did his work the family of algebras $\mathbf{J}_n(q)$ were already known to physicists who were concerned with *exactly solvable models* in Statistical Mechanics. (For an excellent introduction to this topic, see R. Baxter's article in these Proceedings.) One of the simplest examples in this area is known as the *Potts model*. In that model one considers an array of "atoms" arranged at the vertices of a planar lattice with m rows and n columns as in Figure 5. Each "atom" in the system has various possible spins associated to it, and in the simplest case, known as the Ising model, there are two choices, "+" for spin up and "−" for spin down. We have indicated one of the 2^{nm} choices in Figure 5, determining a *state* of the system. The goal is to compute the free energy of the system, averaged over all possible states.

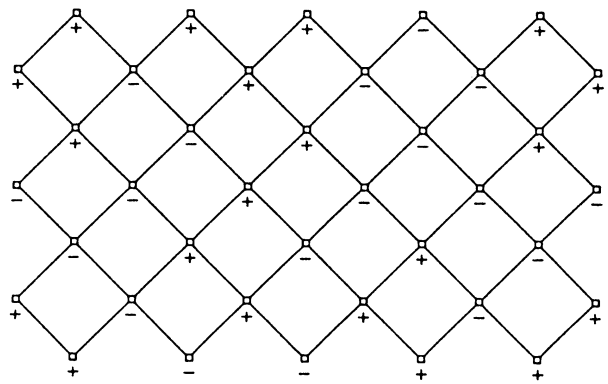

Fig. 5.

Letting σ_i denote the spin at site i, we note that an edge e contributes an *energy* $E_e(\sigma_i, \sigma_j)$, where σ_i and σ_j are the states of the endpoints of e. Let E be the collection of lattice edges. Let β be a parameter which depends upon the temperature. Then the Gibbs *partition function* $\mathbf{Z}$ is given by the formula:

$$\mathbf{Z} = \sum_{\sigma_1,\ldots,\sigma_{mn}} \prod_{e\in E} \exp(-\beta E_e(\sigma_i, \sigma_j)) \,.$$

All of this is microscopic, nevertheless the major macroscopic thermodynamic quantities are functions of the partition function. In particular, the free energy $\mathbf{F}$, the object of interest to us at this time, is given by $\mathbf{Z} = \exp(-\beta\mathbf{F})$.

To compute the manner in which the atoms in one row of the lattice interact with atoms in the next, physicists set up the *transfer matrix* $\mathbf{T}$, which expresses the row-to-row interactions. It turned out that, in order to be able to calculate the free energy, the transfer matrices must satisfy conditions known as the *Yang-Baxter equations* and (to the great surprise of everyone) they turned out to be the braid relations (1) and (2) in disguise! (Remark: before Jones' work, to the best of our

knowledge, it was not known that the Yang-Baxter equation was related to braids or knots.) Even more, the algebra which the transfer matrices generate in the Ising model, known to physicists as the *Temperley-Lieb algebra*, is our algebra $\mathbf{J}_n(q)$. The partition function $\mathbf{Z}$ is related in a very simple way to the transfer matrix:

$$\mathbf{Z} = \text{ trace}(\mathbf{T})^m .$$

In fact, it is closely related to the Jones trace.

The initial discovery of a relationship between the Potts model and links was reported on in [Jo3]. It opened a new chapter in the flow of ideas between mathematics and physics. We give an explicit example of a way in which the relationship of Jones' work to physics led to new insight into mathematics. Learning that the partition function was a sum over states of the system, Louis Kauffman was led to seek a decomposition of the Jones polynomial into a related sum over "states" of knot diagram, and arrived in [K1] at an elegant "states model" for the Jones polynomial. The Jones polynomial, and Kauffman's states model for it, were later seen to generalize to other polynomial invariants with associated states models, for links in S^3 and eventually into invariants for 3-manifolds M^3 and links in 3-manifolds. The full story is not known at this writing, however we refer the reader to V. Turaev's article in these *Proceedings* for an excellent account of it, as of August 1990.

4. Quantum Groups and Representations of Lie Algebras

We begin by explaining the structure of the algebra $\mathbf{J}_n(q)$. It will be convenient to begin with another algebra $\mathbf{H}_n(q)$, which is generated by symbols $\mathbf{g}_1, \ldots, \mathbf{g}_{n-1}$ which now have a third meaning), with defining relations (1), (2) and (3_{q}). The algebra $\mathbf{H}_n(q)$ is very well-known to mathematicians. It's the *Iwahori-Hecke algebra*, also known as the *Hecke algebra of the symmetric group* [Bo]. Its relationship with the symmetric group is simple to describe and beautiful. Notice that when $q = 1$, relation (3_{q}) simplifies to $(\mathbf{g}_k)^2 = 1$. One recognizes (1), (2) and (3_1) as defining relations for the group algebra $\mathbb{C}\mathbf{S}_n$ of the symmetric group $\mathbf{S}_n$. Here $\mathbf{g}_k$ is to be re-interpreted as a transposition which exchanges the symbols $\mathbf{k}$ and $\mathbf{k{+}1}$. In this way we may view $\mathbf{H}_n(q)$ as a "q-deformation" of the complex group algebra $\mathbb{C}\mathbf{S}_n = \mathbf{H}_n(1)$.

The algebra $\mathbb{C}\mathbf{S}_n$ is *rigid*, that is if one deforms it in this way its irreducible summands continue to be irreducible summands of the same dimension, in fact $\mathbf{H}_n(q)$ is actually algebra-isomorphic to $\mathbb{C}\mathbf{S}_n$ for generic q. Thus $\mathbf{H}_n(q)$ is direct sum of finite dimensional matrix algebras, its irreducible summands being in one-to-one correspondence with the irreducible representations of the symmetric group $\mathbf{S}_n$. In this setting, Jones showed in [Jo2] that for generic q the algebra $\mathbf{J}_n(q)$ may be interpreted as the algebra associtated to the q-deformations of those irreducible representations of $\mathbf{S}_n$ which have Young diagrams with at most two rows.

We now explain how $\mathbf{H}_n(q)$ is related to quantum groups. It will be helpful to recall the classical picture. The fundamental representation of the Lie group GL_n acts on $\mathbb{C}^n$, and so its k-fold tensor product acts naturally on $(\mathbb{C}^n)^{\otimes k}$. The symmetric group $\mathbf{S}_k$ also acts naturally on $(\mathbb{C}^n)^{\otimes k}$, permuting factors. (Remark: In this latter action, the representations of $\mathbf{S}_k$ which are relevant are those whose Young diagrams have $\leq n$ rows.) As is well known, the actions of GL_n and $\mathbf{S}_k$ are each other's commutants in the full group of linear transformations of $(\mathbb{C}^n)^{\otimes k}$. If one replaces GL_n and $\mathbb{C}\mathbf{S}_k$ by the quantum group $U_q(GL_n)$ and the Hecke algebra $\mathbf{H}_k(q)$ respectively, then the remarkable fact is that $U_q(GL_n)$ and $\mathbf{H}_k(q)$ are still each other's commutants [Ji]. The corresponding picture for $\mathbf{J}_n(q)$ is obtained by restricting to GL_2 and to representations of $\mathbf{S}_k$ having Young diagrams with at most 2 rows.

We remark that these are not isolated instances of algebraic accidents, but rather special cases of a phenomenon which relates a large part of the mathematics of quantum groups to finite dimensional matrix representations of the group algebra $\mathbb{C}\mathbf{B}_n$ which support a Markov trace (e.g. see [BW]).

5. Dynkin Diagrams

Dynkin diagrams arise in the tower construction which we described in § 1 via the inclusions of the algebras $\mathbf{J}_n(q)$ in the Jones tower. The inclusions for the Jones tower are very simple, and correspond to the Dynkin diagram of type A_n. However, other, more complicated towers may be obtained by replacing the II_1 factor $\mathbf{M}_1$ in the tower construction of § 1 above by $\mathbf{M}_1 \cap (\mathbf{M}_0)'$, where $(\mathbf{M}_0)'$ is the commutant of $\mathbf{M}_0$ in $\mathbf{M}_1$. We refer the reader to [GHJ] for an introduction to this topic and a discussion of the "derived tower" and the Dynkin diagrams which occur. The connections with the representations of simple Lie algebras can be guessed at from our discussion in § 4 above.

6. Concluding Remarks

I hope I have succeeded in showing you some of the ways in which Jones' work created bridges between the areas of mathematics which were illustrated in Figure 1. To conclude, I want to indicate very briefly some of the ways in which those bridges have changed the mathematics which many of us are doing.

There is another link polynomial in the picture, the famous *Alexander polynomial*. It was discovered in 1928, and was of fundamental importance to knot theory, both in the classical case of knots in S^3 and in higher dimensional knotting. Shortly after Jones' 1984 discovery, it was learned that in fact both the Alexander and Jones polynomial were specializations of the 2-variable Jones polynomial. That discovery was made simultaneously by five separate groups of authors: Freyd and Yetter, Hoste, Lickorish and Millett, Ocneanu, and Przytycki and Traczyk, a simple version of the proof of Ocneanu being given in [Jo3]. One of the techniques used

in the proof was the combinatorics of link diagrams, and that technique led to the discovery of yet another polynomial, by Louis Kauffman [K2].

From the point of view of algebra, the Jones polynomial comes from a trace function on $\mathbf{J}_n(q)$, and the 2-variable Jones polynomial from a similar trace on the full Hecke algebra $\mathbf{H}_n(q)$. Beyond that, there is another algebra, the so-called Birman–Wenzl algebra [BW], and Kauffman's polynomial is a trace on it. Even more, physicists who had worked with solutions to the Yang Baxter equation, realized that they knew of still other Markov traces, so they began to grind out still other polynomials, in initially bewildering confusion. That picture is fairly well understood at this moment, however the work of Witten [W] indicates there are still other, related, link invariants. The generalizations are vast, with much work to be done.

There is also a different and very direct way in which Jones had had equal influence. His style of working is informal, and one which encourages the free and open interchange of ideas. During the past few years Jones wrote letters to various people which described his important new discoveries at an early stage, when he did not yet feel ready to submit them for journal publication because he had much more work to do. He nevertheless asked that his letters be shared, and so they were widely circulated. It was not surprising that they then served as a rich source of ideas for the work of others. As it has turned out, there has been more than enough credit to go around. His openness and generosity in this regard have been in the best tradition and spirit of mathematics.

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A POLYNOMIAL INVARIANT FOR KNOTS VIA VON NEUMANN ALGEBRAS[1]

by

VAUGHAN F. R. JONES[2]

A theorem of J. Alexander [**1**] asserts that any tame oriented link in 3-space may be represented by a pair (b, n), where b is an element of the n-string braid group B_n. The link L is obtained by closing b, i.e., tying the top end of each string to the same position on the bottom of the braid as shown in Figure 1. The closed braid will be denoted $b^\wedge$.

Thus, the trivial link with n components is represented by the pair $(1, n)$, and the unknot is represented by $(s_1, s_2 \cdots s_{n-1}, n)$ for any n, where $s_1, s_2, \ldots, s_{n-1}$ are the usual generators for B_n.

The second example shows that the correspondence of (b, n) with $b^\wedge$ is many-to-one, and a theorem of A. Markov [**15**] answers, in theory, the question of when two braids represent the same link. A Markov move of type 1 is the replacement of (b, n) by (gbg^{-1}, n) for any element g in B_n, and a Markov move of type 2 is the replacement of (b, n) by $(bs_n^{\pm 1}, n+1)$. Markov's theorem asserts that (b, n) and (c, m) represent the same closed braid (up to link isotopy) if and only if they are equivalent for the equivalence relation generated by Markov moves of types 1 and 2 on the *disjoint* union of the braid groups. Unforunately, although the conjugacy problem has been solved by F. Garside [**8**] within each braid group, there is no known algorithm to decide when (b, n) and (c, m) are equivalent. For a proof of Markov's theorem see J. Birman's book [**4**].

The difficulty of applying Markov's theorem has made it difficult to use braids to study links. The main evidence that they might be useful was the existence of a representation of dimension $n-1$ of B_n discovered by W. Burau in [**5**]. The representation has a parameter t, and it turns out that the determinant of 1-(Burau matrix) gives the Alexander polynomial of the closed braid. Even so, the Alexander polynomial occurs with a normalization which seemed difficult to predict.

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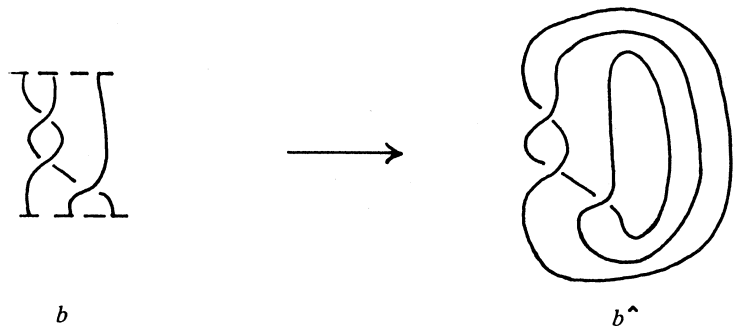

Figure 1

In this note we introduce a polynomial invariant for tame oriented links via certain representations of the braid group. That the invariant depends only on the closed braid is a direct consequence of Markov's theorem and a certain trace formula, which was discovered because of the uniqueness of the trace on certain von Neumann algebras called type II_1 factors.

Notation. In this paper the Alexander polynomial Δ will always be normalized so that it is symmetric in t and t^{-1} and satisfies $\Delta(1) = 1$ as in Conway's tables in [**6**].

While investigating the index of a subfactor of a type II_1 factor, the author was led to analyze certain *finite-dimensional* von Neumann algebras A_n generated by an identity 1 and n projections, which we shall call $e_1, e_2, \ldots, e_n$. They satisfy the relations

(I) $e_i^2 = e_i, e_i^* = e_i$,
(II) $e_i e_{i_{\pm 1}} e_i = t/(1+t)^2 e_i$,
(III) $e_i e_j = e_j e_i$ if $|i-j| \geq 2$.

Here t is a complex number. It has been shown by H. Wenzl [**24**] that an arbitrarily large family of such projections can only exist if t is either real and positive or $e^{\pm 2\pi i/k}$ for some $k = 3, 4, 5, \ldots$. When t is one of these numbers, there exists such an algebra for all n possessing a trace tr: $A_n \to \mathbf{C}$ completely determined by the normalization tr(1)=1 and

(IV) $\mathrm{tr}(ab) = \mathrm{tr}(ba)$,
(V) $\mathrm{tr}(we_{n+1}) = t/(1+t)^2\ \mathrm{tr}(w)$ if w is in A_n,
(VI) $\mathrm{tr}(a^*a) > 0$ if $a \neq 0$

(note $A_0 = \mathbf{C}$).

Conditions (I)–(VI) determine the structure of A up to $*$-isomorphism. This fact was proved in [**9**], and a more detailed description appears in [**10**]. Remember

that a finite-dimensional von Neumann algebra is just a product of matrix algebras, the $*$ operation being conjugate-transpose.

For real t, D. Evans pointed out that an explicit representation of A_n on $\mathbf{C}^{2^n+2}$ was discovered by H. Temperley and E. Lieb [**23**], who used it to show the equivalence of the Potts and ice-type models of statistical mechanics. A readable account of this can be found in R. Baxter's book [**2**]. This representation was rediscovered in the von Neumann algebra context by M. Pimsner and S. Popa [**18**], who also found that the trace tr is given by the restriction of the Powers state with $t = \lambda$ (see [**18**]).

For the roots of unity the algebras A_n are intimately connected with Coxeter groups in a way that is far from understood.

The similarity between relations (II) and (III) and Artin's presentation of the n-string braid group,

$$\{s_1, s_2, \ldots, s_n \,:\, s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},\ s_i s_j = s_j s_i \text{ if } |i-j| \geq 2\}\,,$$

was first pointed out by D. Hatt and P. de la Harpe. It transpires that if one defines $g_i = \sqrt{t}(te_i - (1-e_i))$, the g_i satisfy the correct relations, and one obtains representations r_t of B_n by sending s_i to g_i.

Theorem 1. *The number* $(-(t+1)/\sqrt{t})^{n-1}\ \mathrm{tr}(r_t(b))$ *for* b *in* B_n *depends only on the isotopy class of the closed braid* $b^\wedge$.

Definition. If L is a tame oriented classical link, the *trace invariant* $V_L(t)$ is defined by

$$V_L(t) = (-(t+1)/\sqrt{t})^{n-1}\ \mathrm{tr}(r_t(b))$$

for any (b, n) such that $b^\wedge = L$.

The Hecke algebra approach shows the following.

Theorem 2. *If the link* L *has an odd number of components,* $V_L(t)$ *is a Laurent polynomial over the integers. If the number of components is even,* $V_L(t)$ *is* $\sqrt{t}$ *times a Laurent polynomial.*

The reader may have observed that the von Neumann algebra structure (i.e., the $*$ operation) and condition (VI) are redundant for the definition of $V_L(t)$. This explains why V_L can be extended to all values of t except 0. However, it must be pointed out that for positive t and the relevant roots of unity, the presence of *positivity* gives a powerful method of proof.

The trace invariant depends on the oriented link but not on the chosen orientation. Let $L^\sim$ denote the mirror image link of L.

Theorem 3. $V_{L^\sim}(t) = V_L(1/t)$.

Thus, the trace invariant can be used to detect a lack of amphicheirality. It seems to be very good at this. A glance at Table 1 shows that it distinguishes the

trefoil knot from its mirror image and hence, via Theorem 6, it distinguishes the two granny knots and the square knot.

Conjecture 4. If L is not amphicheiral, $V_{L^\sim} \neq V_L$.

There is some evidence for this conjecture, but only \$10 hangs on it. In this direction we have the following result, where b is in B_n, b_+ is the sum of the positive exponents of b, and b_- is the (unsigned) sum of the negative ones in some expression for b as a word on the usual generators.

Theorem 5. *If $b_+ - 3b_- - n + 1$ is positive, then $b^\wedge$ is not amphicheiral.*

For $b_- = 0$, i.e., positive braids, this follows from a recent result of L. Rudolf [**21**]. Also, if the condition of the theorem holds, we conclude that $b^\wedge$ is not the unknot. This is similar in kind to a recent result of D. Bennequin [**3**].

The connected sum of two links can be handled in the braid group provided one pays proper attention to the components being joined. Let us ignore the subtleties and state the following (where # denotes the connected sum).

Theorem 6. $V_{L_1 \# L_2} = V_{L_1} V_{L_2}$.

As evidence for the power of the trace invariant, let us answer two questions posed in [**4**]. Both proofs are motivated by the fact, shown in [**10**], that $r_t(B_n)$ is sometimes finite.

Theorem 7. *For every n there are infinitely many words in B_{n+1} which give close braids inequivalent to closed braids coming from elements of the form $Us_n^{-1}Vs_n$, where U and V are in B_n.*

Explicit examples are easy to find; e.g., all but a finite number of powers of $s_n^{-1}s_2s_3$ will do.

Theorem 8 (see [**4** p. 217, q. 8]). *If b is in B_n and there is an integer k greater than 3 for which $b \in \ker r_t$, $t = e^{2\pi i/k}$, then $b^\wedge$ has braid index n.*

Here the braid index of a link L is the smallest n for which there is a pair (b, n) with $b^\wedge = L$. The kernel of r_t is not hard to get into for these values of t.

Corollary 9. *If the greatest common divisor of the exponents of $b \in B_n$ is more than 1, then the braid index of $b^\wedge$ is n.*

More interesting examples can be obtained by using generators and relations for certain finite groups; e.g., the finite simple group of order 25,920 (see [**10**, **7**]). In

general, the trace invariant can probably be used to determine the braid index in a great many cases.

Note also that the trace invariant detects the kernel of r_t.

Theorem 10. *For* $t = e^{2\pi i/k}, k = 3, 4, 5, \ldots,$ $V_{b^\wedge}(t) = (-2\cos\pi/k)^{n-1}$ *if and only if* $b \in \ker r_t$ *(for* $b \in B_n$*).*

Corollary 11. *For transcendental* t, $b \in \ker r_t$ *if and only if* $V_{b^\wedge}(t) = (-(t+1)/\sqrt{t})^{n-1}$.

For transcendental t, r_t is very likely to be faithful.

There is an alternate way to calculate V_L without first converting L into a closed braid. In [**6**] Conway describes a method for rapidly computing the Alexander polynomials of links inductively. In fact, his first identity suffices in principle — see [**11**]. This identity is as follows.

Let L^+, L^-, and L be links related as in Figure 2, the rest of the links being identical. Then $\Delta_{L^+} - \Delta_{L^-} = (\sqrt{t} - 1/\sqrt{t})\Delta_L$.

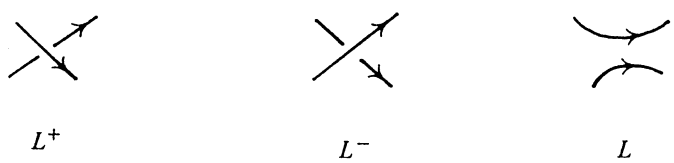

Figure 2

For the trace invariant we have

Theorem 12. $1/tV_{L^+} - tV_{L^-} = (\sqrt{t} - 1/\sqrt{t})V_L$.

Corollary 13. *For any link* $L, V_L(-1) = \Delta_L(-1)$.

That the trace invariant may always be calculated by using Theorem 12 follows from the proof of the same thing for the Alexander polynomial. We urge the reader to try this method on, say, the trefoil knot.

The special nature of the algebras A_n when t is a relevant root of unity can be exploited to give information about V_L at these values.

Theorem 14. *If* K *is a knot then* $V_K(e^{2\pi i/3}) = 1$.

Theorem 15. $V_L(1) = (-2)^{p-1}$, *where* p *is the number of components of* L.

A more subtle analysis at $t = 1$ via the Temperley–Lieb–Pimsner–Popa representation gives the next result.

Theorem 16. *If K is a knot then $d/dtV_K(1) = 0$.*

It is thus sensible to simplify the trace invariant for knots as follows.

Definition 17. If K is a knot, define W_K to be the Laurent polynomial

$$W_K(t) = (1 - V_K(t))/(1 - t^3)(1 - t)\,.$$

Amphicheirality is less obvious for W. In fact, $W_{K^\sim}(t) = 1/t^4 W_K(1/t)$. It is amusing that for W the unknot is 0 and the trefoil is 1. The connected sum is also less easy to see in the W picture. For the record the formula is

$$W_{K_1 \# K_2} = W_{K_1} + W_{K_2} - (1 - t)(1 - t^3) W_{K_1} W_{K_2}\,.$$

Corollary 18. $\Delta_K(-1) \equiv 1$ *or* $5 \pmod 8$.

When $t = i$ the algebras A_n are the complex Clifford algebras. This together with a recent result of J. Lannes [**13**] allows one to show the following.

Theorem 19. *If K is a knot the Arf invariant of K is $W_K(i)$.*

Corollary 20. $\Delta_K(-1) = 1$ *or* $5 \pmod 8$ *when the Arf invariant is* 0 *or* 1, *respectively.*

This is an alternate proof of a result in Levine [**14**]; also see [**11**, p. 155]. Note also that Corollary 20 allows one to define an Arf invariant for links as $V(i)$. It may be zero and is always plus or minus a power of two otherwise.

The values of V at $e^{\pi i/3}$ are also of considerable interest, as the algebra A_n is then related to a kind of cubic Clifford algebra. Also, in this case, $r_t(B_n)$ is always a finite group, so one can obtain a rapid method for calculating $V(t)$ without knowing V completely. We have included this value of V in the tables. Note that it is always in $1 + 2\mathbf{Z}(e^{i\pi/3})$.

There is yet a third way to calculate the trace invariant. The decomposition of A_n as a direct sum of matrix algebras is known [**10**], and H. Wenzl has explicit formulae for the (irreducible) representations of the braid group in each direct summand. So in principle this method could always be used. This brings in the Burau representation as a direct summand of r_t. For 3 and 4 braids this allows one to deduce some powerful relations with the Alexander polynomial. An application of Theorem 16 allows one to determine the normalization of the Alexander polynomial in the Burau matrix for proper knots, and one has the following formulae.

Theorem 21. *If b in B_3 has exponent sum e, and $b^\wedge$ is a knot, then*

$$V_{b^\wedge}(t) = t^{e/2}(1 + t^e + t + 1/t - t^{e/2-1}(1 + t + t^2)\Delta_{b^\wedge}(t))\,.$$

Theorem 22. *If b in B_4 has exponent sum e, and $b^\wedge$ is a knot, then*

$$t^{-e}V(t) + t^e V(1/t) = (t^{-3/2} + t^{-1/2} + t^{1/2} + t^{3/2})(t^{e/2} + t^{-e/2}) \\ -(t^{-2} + t^{-1} + 2 + t + t^2)\Delta(t)$$

(where $V = V_{b^\wedge}$ and $\Delta = \Delta_{b^\wedge}$).

These formulas have many interesting consequences. They show that, except in special cases, e is a knot invariant. They also give many obstructions to being closed 3 and 4 braids.

Corollary 23. *If K is a knot and $|\Delta_K(i)| > 3$, then K cannot be represented as a closed 3 braid.*

Of the 59 knots with 9 crossings or less which are known not to be closed 3 braids, this simple criterion establishes the result for 43 of them, at a glance.

Corollary 24. *If K is a knot and $\Delta_K(e^{2\pi i/5}) > 6.5$, then K cannot be represented as a closed 4 braid.*

For $n > 4$ there should be no simple relation with the Alexander polynomial, since the other direct summands of r_t look less and less like Burau representations.

In conclusion, we would like to point out that the q-state Potts model could be solved if one understood enough about the trace invariant for braids resembling certain braids discovered by sailors and known variously as the "French sinnet" (sennit) or the "tresse anglaise", depending on the nationality of the sailor. See [**21**, p. 90].

The author would like to thank Joan Birman. It was because of a long discussion with her that the relation between condition (V) and Markov's theorem became clear.

Tables. A single example should serve to explain how to read the tables. The knot 8_8 has trace invariant

$$t^{-3}(-1 + 2t - 3t^2 + 5t^3 - 4t^4 + 4t^5 - 3t^6 + 2t^7 - t^8)\,.$$

Its W invariant is

$$t^{-3}(1 - t + 2t^2 - t^3 + t^4)\,.$$

A braid representation for it is

$$s_1^{-1}s_2s_1^2s_3^{-1}s_2^2s_3^{-2} \text{ in } B_4\,.$$

Also note that $w = e^{\pi i/3}$.

Table 1. The trace invariant for prime knots to 8 crossings.

knot	braid rep.	p_0	pol (v)	v(w)	p_0	pol (w)
0_1	1	0	1	1	0	0
3_1	1^3	1	$101-1$	$i\sqrt{3}$	0	1
4_1	$12^{-1}12^{-1}$	-2	$1-11-11$	-1	-2	-1
5_1	1^5	2	$101-11-1$	-1	0	1101
5_2	$2^21^{-1}21^2$	1	$1-12-11-1$	-1	0	101
6_1	$12^{-1}13^{-1}23^{-1}2^{-1}$	-4	$1-11-22-11$	$i\sqrt{3}$	-4	$-10-1$
6_2	$1^{-1}21^{-1}2^3$	-1	$1-12-22-21$	1	-1	$-11-1$
6_3	$1^{-1}2^21^{-2}2$	-3	$-12-23-22-1$	1	-3	$1-11$
7_1	1^7	3	$101-11-11-1$	-1	0	1111101
7_2	$1^{-1}3^321^23^{-1}2$	1	$1-12-22-11-1$	1	0	10101
7_3	$2^512^21^{-2}$	2	$1-12-23-21-1$	1	0	110201
7_4	$3^21^{-1}23^{-1}21^22$	1	$1-23-23-21-1$	-1	0	10201
7_5	$2^312^41^{-2}$	2	$1-13-33-32-1$	-1	0	$1102-11$
7_6	$3^{-1}1^{-1}2113^{-1}2^{-3}$	-6	$-12-34-33-21$	-1	-6	$1-12-1$
7_7	$13^{-1}23^{-1}21^{-1}23^{-1}2$	-3	$-13-34-43-21$	$-i\sqrt{3}$	-3	$1-21-1$
8_1	$12^{-1}3^{-1}214^{-2}.$ $3^{-1}2^{-1}4$	-6	$1-11-22-22-11$	1	-6	$-10-10-1$
8_2	$2^51^{-1}21^{-1}$	0	$1-12-23-32-21$	-1	1	$1-11-1$
8_3	$123^{-1}4^{-1}3^{-1}2.$ $1^{-1}3^22^43^{-1}2^{-2}$	-4	$1-12-33-32-11$	-3	-4	$-10-20-1$
8_4	$11132^{-1}3^{-2}12^{-1}$	-3	$1-12-33-33-21$	-1	-3	$-10-21-1$
8_5	$1112^{-1}1112^{-1}$	0	$1-13-33-43-21$	$i\sqrt{3}$	1	$1-21-1$
8_6	$3^{-2}12^{-1}132^{-3}$	-7	$1-23-44-43-11$	1	-7	$-11-21-1$
8_7	$2^{-2}12^{-1}1^4$	-2	$-12-24-44-32-1$	1	-2	$1-12-11$
8_8	$1^{-1}2113^{-1}223^{-2}$	-3	$-12-35-44-32-1$	1	-3	$1-12-11$
8_9	$2^31^{-1}21^{-3}$	-4	$1-23-45-43-21$	1	-4	$-11-21-1$
8_{10}	$221^{-2}2^31^{-1}$	-2	$-12-35-45-42-1$	$i\sqrt{3}$	-2	$1-13-11$
8_{11}	$12^{-2}32^{-1}3^{-2}12^{-1}$	-7	$1-23-55-44-21$	$-i\sqrt{3}$	-7	$-11-22-1$
8_{12}	$12^{-1}34^{-1}34^{-1}.$ $213^{-1}2^{-1}$	-4	$1-24-55-54-21$	-1	-4	$-11-31-1$
8_{13}	$1123^{-1}21^{-1}3^{-2}2$	-3	$13-45-55-32-1$	-1	-3	$1-22-11$
8_{14}	$11221^{-1}3^{-1}23^{-1}2$	-1	$1-24-56-54-31$	-1	-1	$-12-22-1$
8_{15}	$111231^{-1}2^332^{-1}$	2	$1-25-56-64-31$	$-i\sqrt{3}$	0	$1103-22-1$
8_{16}	$112^{-1}112^{-1}12^{-1}$	-2	$-13-46-66-53-1$	1	-2	$1-23-21$
8_{17}	$21^{-1}21^{-1}221^{-2}$	-4	$1-35-67-65-31$	1	-4	$-12-32-1$
8_{18}	$(12^{-1})^3$	-4	$1-46-79-76-41$	3	-4	$-13-33-1$
8_{19}	121212^21	3	$10100-1$	$-i\sqrt{3}$	0	11111
8_{20}	21^321^{-3}	-1	$-12-12-11-1$	$i\sqrt{3}$	-1	101
8_{21}	$2^312^21^{-2}21^{-1}$	1	$2-23-32-21$	$i\sqrt{3}$	0	$1-11-1$

Table 2. The trace invariant for some divers knots and links.

link	p_0	pol(V)	v(w)	p_0	pol(w)	braid rep.
$10_{141}{}^{(1)}$	-2	$1-23-34-32-21$	$i\sqrt{3}$	-2	$-11-11-1$	$2^{-4}112221$
$KT^{(2)}$	-4	$-12-22001-22-21$	1	-4	$1-110-11-1$	$1113323^{-1}1^{-2}2.$ $1^{-1}3^{-1}2^{-1}$
$C^{(3)}$	-4	$-12-22001-22-21$	1	-4	$1-110-11-1$	$22213^{-1}2^{-2}1.$ $2^{-1}13^{-1}$
2_1^2	$1/2$	$-10-1$	$-i$			1^2
$4_1^{2(4)}$	$3/2$	$-10-11-1$	$-i$			1^4
$4_1^{2(4)}$	$1/2$	$-11-10-1$	i			$12^{-1}122$
5_1^2	$-7/2$	$1-21-21-1$	i			$12^{-1}12^{-2}$
6_1^2	$5/2$	$-10-11-11-1$	$\sqrt{3}$			1^6
6_2^2	$3/2$	$-11-22-21-1$	$-i$			2221121^{-1}
6_3^2	$-3/2$	$-12-22-31-1$	$\sqrt{3}$			$21^{-1}23^{-1}2123^{-1}$
$H_1{}^{(5)}$	$1/2$	$-11-10-1$	i			$12^{-1}122$
$H_2{}^{(5)}$	$-3/2$	$1-10-10-1$	$-i$			$12^{-3}122$
$W^{(6)}$	$-3/2$	$-11-21-21$	$-i$			$11221^{-1}2^{-2}$
6_1^3	-1	$1-13-13-21$	$i\sqrt{3}$			$221^{-1}221^{-1}$
6_2^3	-3	$-13-24-23-1$	1			$12^{-1}12^{-1}12^{-1}$
6_3^3	2	10102	1			122122
$A^{(7)}$	$5/2$	$-10-32-34-22-1$	$3\,i$			11222333
$B^{(7)}$	$5/2$	$-10-32-34-22-1$	$3\,i$			11122333

Table Notes

(1) Compare 8_5 which has the same Alexander polynomial.

(2) The Kinoshita–Terasaka knot with 11 crossings. See [**12**].

(3) This is Conway's knot with trivia Alexander polynomial. See [**20**].

(4) Same link, different orientation.

(5) These links have homeomorphic complements.

(6) The Whitehead link.

(7) Two composite links with the same trace invariant.

Added in Proof. The similarity between the relation of Theorem 12 and Conway's relation has led several authors to a two-variable generalization of V_L. This has been done (independently) by Lickorish and Millett, Ocneanu, Freyd and Yetter, and Hoste.

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INDEX FOR SUBFACTORS

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1. Introduction

One of the first things Murray and von Neumann did with their theory of continuous dimension for subspaces affiliated with a type II_1 factor was to define an invariant for its action on a Hilbert space. This invariant has come to be known as the coupling constant and it measures the relative mobility of the factor and its commutant. To be precise, if M is a II_1 factor on $\mathcal{H}$ and M' is its commutant, the coupling constant C_M is infinite if M' is an infinite factor and if M' is finite, one takes any non-zero vector ξ in $\mathcal{H}$ and one considers the closed subspaces $\overline{M\xi}$ and $\overline{M'\xi}$ affiliated with M' and M, respectively. It is a nontrivial result of Murray and von Neumann that the ratio $C_M = \dim_{M'}(\overline{M\xi})/\dim_M(\overline{M'\xi})$ does not depend on ξ. This real number between 0 and ∞ is called the coupling constant.

From a more modern point of view, the coupling constant measures the dimension of a Hilbert space on which M acts and may be defined in terms of intertwining maps in the category of normal M-modules. This leads to the notation $C_M = \dim_M(\mathcal{H})$. This notation is more natural and has been useful in the study of the von Neumann algebra of a foliation where the dimensions of certain geometric Hilbert spaces are measured by a von Neumann algebra (see [8]). We will use this more suggestive notation while keeping the term "coupling constant". Two normal representations of M are unitarily equivalent iff they have the same coupling constant.

It is an easy observation that the coupling constant can be used to define a conjugacy invariant for subfactors of II_1 factors. We call this invariant the index since if the subfactor comes from a subgroup in the group constructions of II_1 factors, the conjugacy invariant is the index of the subgroup. The index is defined in general as $\dim_N(L^2(M, \mathrm{tr}))$ where N is the subfactor and tr is the trace on M. This definition was probably noticed by Murray and von Neumann and appears more or less explicitly in works by Goldman [14], Suzuki [26] and

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others. In fact Goldman proves that if a subfactor has index 2 then the whole factor may be expressed as the crossed product of the subfactor by a $\mathbb{Z}_2$ action. This is analogous to the fact that any subgroup of index 2 of a group is normal. It may be combined with Connes' classification of periodic automorphisms of the hyperfinite II_1 factor R ([6]) to yield the pleasing positive result that there is, up to conjugacy, only one subfactor of index 2 of R.

The question immediately arises: what possible values can the index take? The difficulty in answering this question is that there is very little to play with given an arbitrary subfactor of a II_1 factor. The only general result available is the existence of a conditional expectation onto the subfactor shown by Umegaki in [30]. In fact this tool will allow us to determine completely the possible values of the index for subfactors of R. Experience with dimension in II_1 factors suggests that the index will take on a continuum of values. This is indeed true but surprisingly one cannot turn on this effect until index 4 and even then it involves the fundamental group. It seems entirely plausible that for II_1 factors without full fundamental group, whose existence was shown by Connes in [9], the index may take only countably many values.

What, then, are the possible values of the index for subfactors of R? The answer is $\{4\cos^2 \pi/n | n = 3, 4, \ldots\} \cup \{r \in \mathbb{R} | r \geqq 4\} \cup \{\infty\}$. The value ∞ and the real values $\geqq 4$ are easily obtained (2.1.19 and 2.2.5). The situation between 1 and 4 is more difficult. The basic idea of the analysis for index < 4 is as follows: Let $N \subseteq M$ be II_1 factors. One represents M on $L^2(M, \mathrm{tr})$ and considers the extension e_N to $L^2(M, \mathrm{tr})$ of the conditional expectation onto N. One defines $\langle M, e_N \rangle$ to be the II_1 factor generated by M and e_N on $L^2(M, \mathrm{tr})$. (This construction appears also in [5], [24]).) The crucial observation at this point is that the index of M in $\langle M, e_N \rangle$ is the same as that of N in M. Thus one may iterate this extension process and one obtains a sequence of II_1 factors, each one obtained from the previous one by adding a projection. The inductive limit gives a II_1 factor and if the projections in the construction are numbered $e_1, e_2, \ldots$, then they satisfy $e_i e_{i\pm 1} e_i = \tau e_i,\ e_i e_j = e_j e_i$ if $|i - j| \geqq 2$ and $\mathrm{tr}(e_{i_1} e_{i_2} \ldots e_{i_n}) = \tau^n$ if $|i_j - i_k| \geqq 2$ for $j \neq k$. Here τ is the reciprocal of the index of N in M and tr denotes the trace on the inductive limit. Analysis of the algebra generated by the e_i's yields that if $\tau > 1/4$ it can only be $\frac{1}{4}\sec^2 \pi/n$ for $n = 3, 4, \ldots$. A large bonus of the analysis is that it shows how to construct subfactors with these values as τ.

It seem strange that the set of values contains a discrete and a continuous part, but this may yet be understood by the fact that, if the index is less than 4, the relative commutant is trivial while the constructions available for the continuous part all have nontrivial relative commutant. We have very little information on what happens to the values of the index if the relative commutant is required to be trivial, but note that a result of Popa in [23], together with Connes' result on injective factors [7] shows that any II_1 factor has a subfactor of infinite index with trivial relative commutant.

Before closing the introduction, I would like to pose 4 problems which I hope will lead to a more profound understanding of subfactors.

Problem 1 (due to Connes). What are the possible values of the index for subfactors of R with trivial relative commutant? Or, what is $\mathcal{C}_R$?

Problem 2. For each $n = 3, 4, 5, \ldots$ are there only finitely many subfactors of R up to conjugacy with index $4\cos^2 \pi/n$?

Problem 3. If N is a subfactor of R, is N conjugate to $N \otimes R$ in a decomposition $R \cong R \otimes R$? ($[R : N] < \infty$)

Problem 4. If N is a subfactor of M which is regular and has trivial relative commutant, is M the crossed product of N by a group action? (True if $M = R$ by a result of Ocneanu [22], see [17].)

It would be impossible to thank everyone who helped me with this paper. I have tried to mention individual contributions in the text, but let me also thank especially B. Baker, A. Connes, F. Goodman, R. Powers, M. Takesaki, and A. Wassermann for many fruitful conversations. This paper is an extended version of the Comptes Rendus note [18].

2. Generalities

2.1. *The global index*

If M is a finite factor acting on a Hilbert space $\mathcal{H}$ with finite commutant M', the coupling constant $\dim_M(\mathcal{H})$ of M is defined as $\mathrm{tr}_M(E_\xi^{M'})/\mathrm{tr}_{M'}(E_\xi^M)$ where ξ is a non-zero vector in $\mathcal{H}$, tr_A denotes the normalized trace and E_ξ^A is the projection onto the closure of the subspace $A\xi$. This definition, due to Murray and von Neumann in [20], is independent of ξ.

We recall some rules of calculation associated with $\dim_M$. (See [10, p. 263].)

$$\dim_M(\mathcal{H}) > 0 \tag{2.1.1}$$

$$\dim_M(\mathcal{H}) = (\dim_{M'}(\mathcal{H}))^{-1} \tag{2.1.2}$$

$$\text{If } e \text{ is a projection in } M',\ \dim_{M_e}(e\mathcal{H}) = \mathrm{tr}_{M'}(\mathrm{e})\ \dim_M(\mathcal{H}) \tag{2.1.3}$$

$$\text{If } E \text{ is a projection in } M,\ \dim_{M_e}(e\mathcal{H}) = (\mathrm{tr}_M(e))^{-1}\dim_M(\mathcal{H}) \tag{2.1.4}$$

$$\text{If } M \otimes 1 \text{ is the amplification of } M \text{ on } \mathcal{H} \otimes \mathcal{K},$$

$$\dim_M(\mathcal{H} \otimes \mathcal{K}) = \dim_{\mathbb{C}}(\mathcal{K})\dim_M(\mathcal{H}) \tag{2.1.5}$$

$$\dim_M(\mathcal{H}) = 1 \qquad \text{iff } M \text{ is standard on } \mathcal{H}, \text{ i.e. there is a cyclic trace vector for } M. \tag{2.1.6}$$

Agree to put $\dim_M(\mathcal{H}) = \infty$ if M' is infinite.

Proposition 2.1.7. *Let M be as above and N be a subfactor. The number $\dim_N(\mathcal{H})/\dim_M(\mathcal{H})$ is independent of $\mathcal{H}$ provided M' is finite.*

Proof. Any two such representations differ by an amplification and an induction. By (2.1.3) and (2.1.5), both $\dim_M$ and $\dim_N$ are multiplied by the same constants in this process. □

Definition. If N is a subfactor of M, the number $\dim_N(\mathcal{H})/\dim_M(\mathcal{H})$ defined in 2.1.7 is called the (global) *index* of N in M and written $[M : N]$. Note that $[M : N] = \infty$ means that N' is infinite for any normal representation of M.

By (2.1.7) and (2.1.6), $[M : N] = \dim_N(L^2(M, \mathrm{tr}))$. Thus $[M : N]$ is a conjugacy invariant for N as a subfactor of M.

The rules of calculation for $\dim_M$ give some rules for $[M : N]$.

Proposition 2.1.8. *If $P \subseteq Q \subseteq M$ are II_1 factors then*

$$[M : M] = 1 \tag{2.1.9}$$

$$[M : P] \geqq 1 \tag{2.1.10}$$

$$[M : P] = [M : Q]\,[Q : P] \tag{2.1.11}$$

$$[M : P] \geqq [M : Q] \tag{2.1.12}$$

$$[M : P] = [M : Q] \qquad \textit{implies } P = Q \tag{2.1.13}$$

$$[M : P] = [P' : M'] \qquad \textit{if } P' \textit{ is finite}. \tag{2.1.14}$$

Proof. The only nontrivial property is (2.1.13). To prove it first note that by (2.1.11) and (2.1.9) we may suppose that $M = Q$ and that M acts on $L^2(M, \mathrm{tr})$ with cyclic trace vector ξ. Then $P\xi$ is dense in $\mathcal{H}$ by hypothesis. But then for any $a \in M$ there is a net b_n of elements of P with $b_n\xi \to a\xi$ in $\mathcal{H}$, i.e. $\|b_n - a_n\|_2 \to 0$. By [10, Lemma 1, p. 270], this implies $a \in P$. □

We next examine how the index behaves under tensor products.

Proposition 2.1.15. *Let N_1 and N_2 be subfactors of the finite factors M_1 and M_2, respectively. Then $N_1 \otimes N_2$ is a subfactor of $M_1 \otimes M_2$ and $[M_1 \otimes M_2 : N_1 \otimes N_2] = [M_1 : N_1]\,[M_2 : N_2]$.*

Proof. Let M_1 and M_2 act with cyclic trace vectors $\xi_1 \in \mathcal{H}_1$ and $\xi_2 \in \mathcal{H}_2$ respectively. Then if e_1 and e_2 are the projections onto $\overline{N_1\xi_1}$ and $\overline{N_2\xi_2}$, $e_1 \otimes e_2$ is the projection onto $\overline{N_1 \otimes N_2(\xi_1 \otimes \xi_2)}$. Moreover $M_1 \otimes M_2$ is standard on $\mathcal{H}_1 \otimes \mathcal{H}_2$ and $(N_1 \otimes N_2)' = N_1' \otimes N_2'$. Thus $\mathrm{tr}_{(N_1 \otimes N_2)'}(e_1 \otimes e_2) = \mathrm{tr}_{N_1'}(e_1)\mathrm{tr}_{N_2'}(e_2)$ which gives the desired result. □

Proposition 2.1.16. *Let N_i, M_i be as above and suppose $N_i' \cap M_i = \mathbb{C}$, $i = 1, 2$. Then $(N_1 \otimes N_2)' \cap (M_1 \otimes M_2) = \mathbb{C}$.*

Proof. $(N_1 \otimes N_2)' \cap (M_1 \otimes M_2) = (N_1' \otimes N_2') \cap M_1 \otimes M_2$ (see [28, p. 227]) and $(N_1' \otimes N_2') \cap (M_1 \otimes M_2) = (N_1' \cap M_1) \otimes (N_2' \cap M_2)$. □

We now introduce two isomorphism invariants for II_1 factors.

Definition. If M is a finite factor let

$$\begin{aligned}\mathcal{I}_M = \{r \in \mathbb{R} \cup \{\infty\} |\ & \text{there is a } \mathrm{II}_1 \text{ subfactor } N \text{ of } M \\ & \text{with } [M : N] = r\} \\ \mathcal{C}_M = \{r \in \mathbb{R} \cup \{\infty\} |\ & \text{there is a } \mathrm{II}_1 \text{ subfactor } N \text{ of } M \\ & \text{with } [M : N] = r \text{ and } N' \cap M = \mathbb{C}\}\,.\end{aligned}$$

The determination of $\mathcal{I}_M$ and $\mathcal{C}_M$ is in general rather difficult. We shall gather some immediate results about them.

Proposition 2.1.17. *If $M \cong M \otimes M$ then both $\mathcal{I}_M$ and $\mathcal{C}_M$ are subsemigroups (with 1) of $\{r \in \mathbb{R} | r \geqq 1\} \cup \{\infty\}$ under multiplication.*

Proof. This follows from 2.1.15 and 2.1.16. □

Lemma 2.1.18. *If N is a hyperfinite subfactor of M then $[M : N] < \infty$ implies that M is hyperfinite.*

Proof. If M acts in such a way that N' is finite, then N' is hyperfinite (e.g. [29]) and by [7], M' and so M is hyperfinite. □

Corollary 2.1.19. *For any II_1 factor M, $\infty \in \mathcal{I}_M$.*

Proof. If $M \cong R$ then $R \otimes 1 \subseteq R \otimes R$ is of infinite index. Otherwise by [21] there is a hyperfinite subfactor of M which is of infinite index by 2.1.18. □

Proposition 2.1.20. *For any separable II_1 factor M, $\infty \in \mathcal{C}_M$.*

Proof. A result of Popa in [23] says that we can find a maximal abelian subalgebra A and a unitary u with $u\,A\,u^* = A$ such that u and A generate a subfactor N of M, isomorphic to R. Since N contains a maximal abelian subalgebra, $N' \cap M = \mathbb{C}$.

So if M is not hyperfinite, combining this result with 2.1.18 gives the required subfactor. If $M = R$, see [17] or Sec. 2.3. □

In this paper we will show that

$$\mathcal{I}_M \cap [1,4) = \mathcal{C}_M \cap [1,4) \subseteq \{4\cos^2 \pi/n | n = 3,4,\ldots\}$$

and that

$$\mathcal{I}_R = \{4\cos^2 \pi/n | n = 3,4,\ldots\} \cup \{r \in \mathbb{R} | r \geqq 4\} \cup \{\infty\}\,.$$

2.2. *The local index*

If the global index of a subfactor is finite, one may define a finer invariant, which I call the local index, obtained by restricting the trace on N' to $N' \cap M$. I would like to thank A. Connes for suggesting this approach.

Definition. Let $N \subseteq M$ be II_1 factors and let $p \in N' \cap M$ be a projection. The *index of N at p* will be $[M_p : N_p] = [M : N]_p$.

Lemma 2.2.1. *The index at p and the global index are related by the formula*

$$[M : N]_p = [M : N]\ \mathrm{tr}_M(p)\ \mathrm{tr}_{N'}(p)\,.$$

Proof. If M begins in standard form on $\mathcal{H}$ then by 2.1.4, $\dim_{M_p}(p\mathcal{H}) = \mathrm{tr}_M(p)^{-1}$. Also $\dim_N(\mathcal{H}) = [M : N]$ so by 2.1.3, $\dim_{N_p}(p\mathcal{H}) = [M : N]\ \mathrm{tr}_{N'}(p)$. Thus $[M : N]_p = \dim_{N_p}(p\mathcal{H})/\dim_{M_p}(p\mathcal{H}) = [M : N]\ \mathrm{tr}_M(p)\ \mathrm{tr}_{N'}(p)$. □

Lemma 2.2.2. *If $\{p_i\}$ is a partition of unity in $N' \cap M$ then*

$$[M : N] = \sum_i \mathrm{tr}_M(p_i)^{-1}[M : N]_{p_i}\,.$$

Proof. For each i, $[M : N]\ \mathrm{tr}_{N'}(p_i) = \mathrm{tr}_M(p_i)^{-1}[M : N]_{p_i}$ and summing over i gives the result. □

Corollary 2.2.3. *If $[M : N] < \infty$ then $N' \cap M$ is finite dimensional.*

Proof. If $N' \cap M$ were infinite dimensional we could find arbitrarily large partitions of unity and by 2.2.2 $[M : N] = \infty$. □

In fact with a little more care one may obtain the bound $[M : N] \geqq \dim_{\mathbb{C}}(N' \cap M)$.

Corollary 2.2.4. *If $[M : N] < 4$, $N' \cap M = \mathbb{C}$.*

Proof. If $N' \cap M \neq \mathbb{C}$, then it contains two mutually orthogonal non-zero projections and since $[M : N]_{p_i} \geqq 1$, $[M : N] \geqq \mathrm{tr}(p_1)^{-1} + \mathrm{tr}(p_2)^{-1} \geqq 4$. □

Corollary 2.2.5. *If M has fundamental group$=\mathbb{R}$, $\mathcal{I}_M$ contains $\{r \in \mathbb{R} | r \geqq 4\}$.*

Proof. We must exhibit for any $r \geqq 4$ a subfactor of index r. Choose $d \in (0,1)$ with $1/d + 1/(1-d) = r$ and choose a projection p with $\mathrm{tr}_M(p) = d$. Then M_p and M_{1-p} are isomorphic so choose some isomorphism $\theta : M_p \to M_{1-p}$ and let N be the subfactor $\{x + \theta(x) | x \in M_p\}$. Then $N_p = M_p$ and $N_{1-p} = M_{1-p}$ so by 2.2.2, $[M : N] = 1/d + 1/(1-d) = r$. □

2.3. *Examples*

We give three examples of subfactors and their indices.

Example 2.3.1. Suppose $M = N \otimes P$ where P is a type I_n factor. Then $[M : N \otimes 1] = n^2$. This follows from 2.1.5.

Thus for any II_1 factor M, $\mathcal{I}_M$ always contains $\{n^2 | n \in \mathbb{Z} - \{0\}\} \cup \{\infty\}$. It is not inconceivable that there are II_1 factors for which $\mathcal{I}_M$ is no larger than this set.

Example 2.3.2. Let A be a von Neumann algebra and G a countable discrete group of automorphisms for which the crossed product $A \rtimes G$ is a finite factor. If H is a subgroup of G such that $A \rtimes H$ is a finite factor then $[A \rtimes G : A \rtimes H] = [G : H]$.

Proof. Write G as a disjoint union of cosets, $G = \coprod_{i \in I} Hg_i$. Then let $V_i = \overline{(A \rtimes H)\, u_{g_i}}$. The V_i are subspaces of $L^2(A \rtimes G, \mathrm{tr})$ affiliated with $(A \rtimes H)'$ which are mutually orthogonal, where tr is the extension to $A \rtimes G$ of a faithful normal G-invariant trace on A and the u_g's are the implementing unitaries of the crossed product. Moreover if p_i is the projection onto V_i then $\sum_{i \in I} p_i = 1$ and the p_i are mutually equivalent in $(A \rtimes H)'$ since $V_i = J u_{g_i} J V_0$ where $V_0 = \overline{A \rtimes H}$ and J is the involution on $L^2(A \rtimes G, \mathrm{tr})$. This is because $J u_{g_i} J \in (A \rtimes G)' \subseteq (A \rtimes H)'$.

Thus if $[G : H] = \infty$, $(A \rtimes H)'$ is infinite so $[A \rtimes G : A \rtimes H] = \infty$. If $[G : H] < \infty$, $(A \rtimes H)'$ is finite since reduction by p_0 puts $A \rtimes H$ in standard form. So $[G : H]\, \mathrm{tr}_{(A \rtimes H)'}(p_0) = 1$ which establishes that $[A \rtimes G : A \rtimes H] = [G : H]$. □

This example is the justification for the name "index".

Example 2.3.3. If M is a II_1 factor and G is a finite group of outer automorphisms of M with fixed point algebra M^G, $[M : M^G] = |G|$.

Proof. This result could be established using 2.3.2 but I shall give a different proof which brings in the basic construction of Chapter 3.

Let M act on $L^2(M, \mathrm{tr})$ and let u_g be the unitaries extending the action of G on M. Then the u_g's act also on M' and it is established in [1] that $(M^G)'$ is isomorphic in the obvious way to $M' \rtimes G$. The projection onto $\overline{M^G}$ is $|G|^{-1} \sum_{g \in G} u_g$ and by the isomorphism with the crossed product its trace is $|G|^{-1}$. Thus $[M : M^G] = |G|$. □

3. The Basic Construction

3.1. *Extending finite von Neumann algebras by subalgebras*

Let M be a finite von Neumann algebra with faithful normal normalized trace tr and let N be a von Neumann subalgebra. By [30] there is a conditional expectation $E_N : M \to N$ defined by the relation $\mathrm{tr}(E_N(x)y) = \mathrm{tr}(xy)$ for $x \in M,\ y \in N$. The map E_N is normal and has the following properties:

$$E_N(axb) = aE_N(x)b \quad \text{for } x \in M,\ a, b \in N \text{ (the bimodule property)} \tag{3.1.1}$$

$$E_N(x^*) = E_N(x)^* \quad \text{for all } x \in M \tag{3.1.2}$$

$$E_N(x^*)E_N(x) \leqq E_N(x^*x) \quad \text{and} \quad E_N(x^*x) = 0 \quad \text{implies } x = 0\,. \tag{3.1.3}$$

Let ξ be the canonical cyclic trace vector in $L^2(M, \mathrm{tr})$. Identify M with the algebra of left multiplication operators on $L^2(M, \mathrm{tr})$. The conditional expectation E_N extends to a projection e_N on $\mathcal{H}$ via $e_N(x\xi) = E_N(x)\xi$. Let J be the involution $x\xi \mapsto x^*\xi$.

Proposition 3.1.4.

(i) *For $x \in M,\ e_N x e_N = E_N(x)e_N$.*
(ii) *If $x \in M$ then $x \in N$ iff $e_N x = x e_N$.*
(iii) *$N' = \{M' \cup \{e_N\}\}''$.*
(iv) *J commutes with e_N.*

Proof. (i) If y is an arbitrary element of M then $e_N x e_N(y\xi) = e_N x(E_N(y)\xi)$ $= E_N(x)E_N(y)\xi$ by 3.1.1, and $E_N(x)e_N(y\xi) = E_N(x)E_N(y)\xi$. But the vectors $y\xi$ are dense in $L^2(M, \mathrm{tr})$.

(ii) Relation 3.1.1 shows that e_N commutes with N as in (i). Moreover if $x \in M$ and $e_N x = x e_N$ then $(e_N x)\xi = E_N(x)\xi = (xe)\xi = x\xi$. Since ξ is separating, $x = E_N(x)$.
(iii) It suffices to show that $\{M' \cup \{e_N\}\}' = N$. This follows from (i).
(iv) This follows from 3.1.2. □

These calculations lead to the following.

Definition. Let $\langle M, e_N\rangle$ be the von Neumann algebra on $L^2(M, \mathrm{tr})$ generated by M and e_N. This is the basic construction.

Proposition 3.1.5. (i) $\langle M, e_N\rangle = JN'J$.

(ii) *Operators of the form $a_0 + \sum_{i=1}^{n} a_i e_N b_i$ with $a_i,\ b_i \in M$, give a dense $*$-subalgebra of $\langle M, e_N\rangle$.*
(iii) *$x \mapsto x e_N$ is an isomorphism of N onto $e_N\langle M, e_N\rangle e_N$.*

(iv) *The central support of* e_N *in* $\langle M, e_N\rangle$ *is* 1.
(v) $\langle M, e_N\rangle$ *is a factor iff* N *is.*
(vi) $\langle M, e_N\rangle$ *is a finite iff* N' *is.*

Proof. (i) and (ii) follow immediately from 3.1.4. For (iii), to show that $e_N\langle M, e_N\rangle e_N \subseteq Ne_N$ it suffices by (ii) to show that $e_N(ae_Nb)e_N \in Ne_N$. This follows from (i) of 3.1.4. Moreover if $xe_N = 0$ then $xe_N\xi = x\xi = 0$ and ξ is separating. Thus $x \mapsto xe_N$ is injective. Affirmations (iv), (v) and (vi) are now easy. □

We want to consider special traces on $\langle M, e_N\rangle$.

Definition. If P is a subalgebra of $\langle M, e_N\rangle$, a trace Tr on $\langle M, e_N\rangle$ is called a (τ, P) trace if Tr extends tr and $\mathrm{Tr}(e_Nx) = \tau\ \mathrm{tr}(x)$ for $x \in P$.

Lemma 3.1.6. *A* (τ, N) *trace is a* (τ, M) *trace.*

Proof. If $x \in M$, $\mathrm{Tr}(xe_N) = \mathrm{Tr}(e_Nxe_N) = \mathrm{Tr}(E_N(x)e_N) = \tau\ \mathrm{tr}(E_N(x)) = \tau\ \mathrm{tr}(x)$. □

We shall now concentrate on the case where M and N are factors.

Proposition 3.1.7. *If* M *and* N *are factors then* $[M : N] < \infty$ *iff* $\langle M, e_N\rangle$ *is finite and in this case the canonical trace* Tr *on* $\langle M, e_N\rangle$ *is a* (τ, M) *trace where* $\tau = [M : N]^{-1}$. *In particular* $\mathrm{Tr}(e_N) = [M : N]^{-1}$. *Also* $[\langle M, e_N\rangle : M] = [M : N]$.

Proof. By 3.1.6 it suffices to show that Tr is a (τ, N) trace. But consider the map $y \to \mathrm{Tr}(e_Ny)$ defined on N. This is a trace by (ii) of 3.1.4 so since N is a factor there is a constant K such that $\mathrm{Tr}(e_Ny) = K\ \mathrm{tr}(y)$. Moreover the trace of e_N in N' is by definition τ so by (iv) of 3.1.4, (i) of 3.1.5 and uniqueness of the trace, this is the same as $\mathrm{Tr}(e_N)$. Thus $K = \tau$, and Tr is a (τ, N) trace.

To prove this last assertion note that

$$\begin{aligned}[\langle M, e_N\rangle : M] &= \dim_{M'}(L^2(M, \mathrm{tr}))/\dim_{N'}(L^2(M, \mathrm{tr})) = [\dim_{N'}(L^2(M, \mathrm{tr}))]^{-1} \\ &= \dim_N(L^2(M, \mathrm{tr})) = [M : N]\,. \end{aligned}$$ □

For some general results about projections onto finite subalgebras see Skau's paper [24].

Finally in this section I show that the basic construction is generic for subfactors of finite index in the sense that all such subfactors are of the form $M \subseteq \langle M, e_N\rangle$, although not canonically.

Lemma 3.1.8. *Let* N *be a* II_1 *factor acting on* $L^2(N, \mathrm{tr})$ *and let* M *be a* II_1 *factor containing* N. *Then* $[M : N]$ *is finite and there is a subfactor* P *of* N *such that* $M = \langle N, e_P\rangle$.

Proof. Since M' is a II_1 factor, $[M:N]<\infty$. Let $P=JM'J$. Then $[N:P]=[M:P]$ as in 3.1.7 and $\langle N,e_p\rangle$ is a subfactor of M with $[M:\langle N,e_p\rangle]=1$. Thus by 2.1.13, $M=\langle N,e_p\rangle$. □

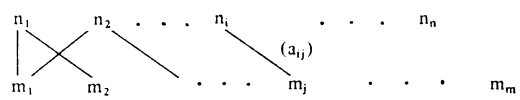
Corollary 3.1.9. *Let $N\subseteq M$ be II_1 factors with $[M:N]<\infty$ then there is a subfactor P of N and an isomorphism $\theta:\ M\to\langle N,e_P\rangle$ with $\theta|_N$=*id.

Proof. Represent M on $L^2(M,\mathrm{tr})$. Choose a projection $p\in M'$ with $\mathrm{tr}(p)=[M:N]^{-1}$. Then by [20], on $pL^2(M,\mathrm{tr})$, M and N act with N in standard form. By 3.1.8 we are through. □

3.2. *Inclusions of complex semisimple algebras*

Let $N\subseteq M$ be finite dimensional complex semisimple algebras and $N=\bigoplus_{i=1}^{n}N_i$, $M=\bigoplus_{j=1}^{m}M_j$ be their canonical decompositions as direct sums of simple algebras, $N_i\cong M_{n_i}(C)$, $M_j\cong M_{m_j}(C)$. The inclusion of N in M is specified up to conjugacy by an $n\times m$ matrix $A_N^M=(a_{ij})$ where (a_{ij}) is the number of simple components of a simple M_j module viewed as an N_i module. This may be zero or any positive integer. If $\{p_i\}$ are the central idempotents for the N_i and q_j those for the M_j, $a_{ij}=0$ iff $p_iq_j=0$. We will call the matrix A_N^M the inclusion matrix.

The inclusion can also be described diagramatically as follows:

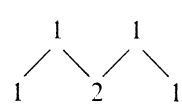


Here there are a_{ij} lines between n_i and m_j. This diagram will be called the Bratteli diagram after [4].

If the identity of M is the same as that of N we have the obvious relation $m_j=\sum_{i=1}^{n}a_{ij}n_i$ which we shall write as

$$\vec{m}=\vec{n}A_N^M\,. \tag{3.2.1}$$

A concrete example is



which is the diagram for the inclusion of $\mathbb{C}S_2$ in $\mathbb{C}S_3$.

If $N\subseteq M\subseteq P$, note the formula

$$A_N^P=A_N^M\,A_M^P\,. \tag{3.2.2}$$

If V is a faithful M-module, the centres of M and M'' are identical so that (if 3.2.1 holds) in the decompositions of M' and N' as simple algebras we may write $N' = \bigoplus_{i=1}^{n} N'_i$ and $M' = \bigoplus_{j=1}^{m} M'_j$. The following formula is in [3, Sec. 5, ex. 17].

$$A_{M'}^{N'} = (A_N^M)^T \,. \tag{3.2.3}$$

Since there is only one normalized trace on $M_n(\mathbb{C})$, a trace on $M = \bigoplus_j M_j$ may be specified by a column vector $\vec{t} = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{pmatrix}$ where t_j is the trace of a minimal idempotent in M_j. The trace of the identity is the product $\vec{m} \cdot \vec{t}$, where $\vec{m} = (m_1, m_2, \ldots, m_m)$.

If $\vec{t}$ defines a trace on M whose restriction to N is defined by the vector $\vec{s}$ then the following relation is immediate.

$$\vec{s} = A_N^M \vec{t} \,. \tag{3.2.4}$$

Conversely if $\vec{s}$ and $\vec{t}$ define traces on N and M, respectively, then they agree on N if 3.2.4 holds.

3.3. *Finite dimensional C^*-algebras*

A finite dimensional C^*-algebra is of course semisimple so the discussion and notation of Sec. 3.2 applies. The presence of the $*$-operation allows us to perform the basic construction of Sec. 3.1. The main question to be answered in this section is: given a faithful trace tr on the finite dimensional C^*-algebra M, when does there exist a (faithful) (τ, M) trace on $\langle M, e_N \rangle$? All traces in this section will be positive, i.e. $\vec{t}$ is a positive vector in the natural ordering of $\mathbb{R}^n$.

We begin with a lemma giving more information on the basic construction in finite dimensions.

Lemma 3.3.1. *Let $N \subseteq M$ be finite dimensional C^*-algebras and let* tr *be a faithful positive normalized trace on M. Let e_N and $\langle M, e_N \rangle$ be as in Sec.* 3.1. *Suppose $\{p_i | i = 1, 2, \ldots, n\}$ are the minimal central projections of N. Then*

(i) *Jp_iJ are the minimal central projections of $\langle M, e_N \rangle$*
(ii) *$A_M^{\langle M, e_N \rangle} = (A_N^M)^T$ (with the obvious identification of the indices, $p_i \leftrightarrow Jp_iJ$)*
(iii) *$e_N Jp_iJ = e_N p_i$*
(iv) *$x \to e_N x Jp_iJ$ is an isomorphism from p_iN onto $(e_N Jp_iJ)\langle M, e_N \rangle (e_N Jp_iJ)$.*

Proof. (i) The p_i are the minimal central projections of N' and $\langle M, e_N \rangle = JN'J$.

(ii) This follows from 3.2.3. and (i).

(iii) If $x \in M$, $(e_N J p_i J)(x\xi) = e_N(xp_i\xi) = E_N(x)p_i\xi$, and

$$(e_N p_i)(x\xi) = e_N(p_i x\xi) = p_i E_N(x)\xi = E_N(x)p_i\xi\,.$$

(iv) Injectivity follows from (iii), and (iii) of 3.1.5. If $y \in p_i e_N \langle M, e_N\rangle p_i e_N$ then $y = e_N z$ for $z \in N$ and $p_i z = z$ by (iii) of 3.1.5 so $y = e_N p_i z$. □

Theorem 3.3.2. *Let M, N and* tr *be as above and let* tr *be given on M by the vector $\vec{t}$ and on N by the vector $\vec{s}$. Then there is a (τ, M) trace* Tr *on $\langle M, e_N\rangle$ iff*

(i) $A^T A\vec{t} = (1/\tau)\vec{t}$

(ii) $AA^T\vec{s} = (1/\tau)\vec{s}$

where $A = A_N^M$.

Proof. ($\Rightarrow$) Let Tr be given on $\langle M, e_N\rangle$ by $\vec{r}$, r_i being the trace of a minimal projection in $J_{P_i}J\langle M, e_N\rangle$. By (iv) of 3.3.1, such a minimal projection may be chosen of the form $e_N q$, q being a minimal projection in $p_i N$. Since Tr is a (τ, M) trace, $r_i = \tau s_i$ so $\vec{r} = \tau\vec{s}$. But by (ii) of 3.3.1, 3.2.2 and 3.2.4, $\vec{s} = AA^T\vec{r}$ $= \tau AA^T\vec{s}$, i.e. $AA^T\vec{s} = (1/\tau)\vec{s}$. Also $\vec{t} = A^T\vec{r} = \tau A^T AA^T\vec{r} = \tau A^T A\vec{t}$.

($\Leftarrow$) Define a trace Tr on $\langle M, e_N\rangle$ by the vector $\vec{r} = \tau\vec{s}$. It is a faithful positive trace since tr is. By 3.1.6 it suffices to show that it extends tr and that it is a (τ, N) trace. For the former, by 3.2.4 we need $A^T\vec{r} = \vec{t}$. But $A\vec{t} = \vec{s}$ and by (i), $A^T\vec{s} = (1/\tau)\vec{t}$ so $A^T\vec{r} = \vec{t}$. For the latter let q be a minimal projection in $p_i N$. Then $e_N q$ is a minimal projection in $J_{P_i}J\langle M, e_N\rangle$ and by definition $\mathrm{Tr}(e_N q)$ $= \tau\,\mathrm{tr}(q)$. The map $x \to \mathrm{Tr}(e_N x)$ is a trace on $p_i N$ so by uniqueness and linearity, $\mathrm{Tr}(e_N x) = \tau\,\mathrm{tr}(x)$ for all $x \in N$. □

3.4. *Two extensions*; *Goldman's theorem*

Let N be a proper von Neumann subalgebra of the finite von Neumann algebra M with faithful normal normalized trace tr. Suppose there is a faithful normal (τ, M) trace Tr on $\langle M, e_N\rangle$. Then we may form the extension $\langle\langle M, e_N\rangle, e_M\rangle$.

Proposition 3.4.1.

(i) $e_M e_N e_M = \tau e_M$

(ii) $e_N e_M e_N = \tau e_N$

(iii) $e_M \wedge e_N = e_M \wedge e_N^\perp = e_M^\perp \wedge e_N = 0$.

Proof. (i) By (i) of 3.1.4, $e_M e_N e_M = E_M(e_N)e_M$ and since Tr is a (τ, M) trace, $E_M(e_N) = \tau$.

(ii) By (ii) of 3.1.5 it suffices to verify the relation on vectors of the form $a\xi$ and $ae_N b\xi$ where a, $b \in M$. But

$$e_N e_M e_N(a\xi) = e_N(E_M(e_N a)\xi) = \tau e_N(a\xi)\,,$$

and

$$e_N e_M e_N(ae_N b\xi) = e_N e_M(E_N(a)e_N b\xi) = e_N(\tau E_N(a)b\xi)$$

while $\tau e_N(ae_N b\xi) = \tau e_N E_N(a)b\xi$.

(iii) $e_N \wedge e_M = s - \lim_{n\to\infty}(e_N e_M e_N)^n = 0$ since $\tau < 1$.

The other relations follow from $(1-\tau) < 1$. □

Now suppose there is a faithful $(\tau, \langle M, e_N\rangle)$ trace Tr on $\langle\langle M, e_N\rangle, e_M\rangle$.

Corollary 3.4.2. *If $\tau \neq 1/2$, the von Neumann algebra generated by e_N and e_M is isomorphic to $M_2(\mathbb{C}) \oplus \mathbb{C}$. If $\tau = 1/2$ it is isomorphic to $M_2(\mathbb{C})$ and we have the relation $e_M + e_N - e_M e_N - e_N e_M = 1/2$.*

Proof. The relations of 3.4.1 show immediately that the von Neumann algebra generated by e_N and e_M has dimension at most 5 and is not abelian. But $\mathrm{Tr}(e_M \vee e_N) = 2\tau - \mathrm{Tr}(e_M \wedge e_N) = 2\tau$. This is enough to prove the affirmations about the structure of the algebra. An easy calculation shows that $p = (1-\tau)^{-1}(e_M + e_N - e_M e_N - e_N e_M)$ is a projection with $pe_N = e_N$ and $pe_M = e_M$. Thus $p = e_1 \vee e_2$ and if $\tau = 1/2$, $p = 1$. □

Corollary 3.4.3. (Goldman's theorem [14]). *Let N be a subfactor of the II_1 factor M with $[M : N] = 2$. Then M decomposes as the crossed product of N by an outer action of $\mathbb{Z}_2$.*

Proof. By 3.1.9 we know that M is of the form $\langle N, e_p\rangle$ for a subfactor of index 2 of N. Thus N is generated by N and $u = 2e_p - 1$. Moreover since tr is a $(1/2, N)$ trace on M, $\mathrm{tr}(ux) = 0$ for $x \in N$, and the relationship of 3.4.2 implies that, if $v = 2e_N - 1$, $uv = -vu$. Thus for $x \in N$, $vuxu^* = uxu^*v$. So uxu^* commutes with e_N and by (ii) of 3.1.4, $uxu^* \in N$. Thus by [25], M is the crossed product of N by a $\mathbb{Z}_2$ action which is necessarily outer since M is a factor. □

4. Possible Values of the Index

4.1. *Certain algebras generated by projections*

Chapter 4 will be largely devoted to proving the following result.

Theorem 4.1.1. *Let M be a von Neumann algebra with faithful normal normalized trace* tr. *Let $\{e_i | i = 1, 2, \ldots\}$ be projections in M satisfying*

(a) $e_i e_{i\pm1} e_i = \tau e_i$ *for some* $\tau \leqq 1$
(b) $e_i e_j = e_j e_i$ *for* $|i - j| \geqq 2$
(c) $\mathrm{tr}(we_i) = \tau\,\mathrm{tr}(w)$ *if w is a word on* $1, e_1, e_2, \ldots, e_{i-1}$

Then if P denotes the von Neumann algebra generated by the e_i's,

(i) $P \cong R$ (*the hyperfinite* II_1 *factor*)
(ii) $P_\tau = \{e_2, e_3, \ldots\}''$ *is a subfactor of* P *with* $[P : P_\tau] = \tau^{-1}$.
(iii) $\tau \leqq 1/4$ *or* $\tau = \frac{1}{4}\sec^2 \pi/n$, $n = 3, 4, \ldots$.

In this section we will prove (i) and (ii). We begin the proof with some notation.

Definition. Let $A_{m,n}$ be the $*$-algebra generated by 1, e_m, $e_{m+1}, \ldots$, e_n for $1 \leqq m \leqq n \leqq \infty$. Let A_n be $A_{1,n}$, $A_0 = \mathbb{C}$. Thus $A_\infty = A_{1,\infty}$ is the $*$-algebra generated by 1 and the e_i's.

Next some combinatorial results. If w is an (associative) word on the e_i's, call it *reduced* if it is of minimal length for the grammatical rules $e_i e_{i\pm 1} e_i \leftrightarrow e_i$, $e_i e_j \leftrightarrow e_j e_i$ for $|i - j| \geqq 2$, $e_i^2 \leftrightarrow e_i$.

Lemma 4.1.2. *Let* $e_{i_1} e_{i_2} \ldots e_{i_k}$ *be a reduced word. Then if* $m = \max\{i_1, i_2, \ldots, i_k\}$, m *occurs only once in the list* $i_1, i_2, \ldots, i_k$.

Proof. By induction on the length of a reduced word. It is trivial for words of length $\leqq 1$. Suppose true for words of length $\leqq n$ and let w be a reduced word of length $n + 1$. Suppose $w = w_1 e_m w_2 e_m w_3$ where m is the maximum index and w_2 does not contain e_m. Then there are 2 possibilities.

(a) w_2 does not contain e_{m-1}. In this case e_m commutes with all the e_i's in w_2 so the length of w may be shortened using $e_m^2 \to e_m$.
(b) w_2 contains e_{m-1}. Then since w is reduced, so is w_2 and by induction $w_2 = v_1 e_{m-1} v_2$ where v_1 and v_2 are words on e_1, e_2, $\ldots$, e_{m-2}. But then e_m commutes with v_1 and v_2 so that the length of w may be reduced using $e_m e_{m-1} e_m \to e_m$. □

It is clear that in the algebra A, any word on the e_i's is proportional to a reduced word.

Corollary 4.1.3. (i) A_n *is finite dimensional.*

(ii) *For* $x \in A_n$, $e_{n+1} x e_{n+1} = E_{A_{n-1}}(x) e_{n+1}$ (*here* $E_{A_{n-1}} : A_n \to A_{n-1}$ *is with respect to the restriction of* tr).
(iii) $x \mapsto x e_{n+1}$ *is an isomorphism of* A_{n-1} *onto* $e_{n+1} A_{n+1} e_{n+1}$.

Proof. (i) If there are only finitely many reduced words, A_n is finite dimensional. But this follows immediately by induction from 4.1.2.

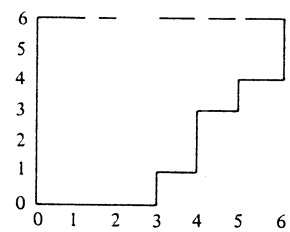
(ii) First of all $\mathrm{tr}(x e_{n+1}) = \tau\, \mathrm{tr}(x)$ for $x \in A_n$ follows immediately from 4.1.1(c) so that $E_{A_{n-1}}(e_n) = \tau$. By 4.1.2 and linearity it suffices to consider x of the form $w e_n w'$ with w and w' in A_{n-1}. But then $e_{n+1} x e_{n+1} = \tau w w' e_{n+1}$ by 4.1.1 (a) and (b), and $E_{A_{n-1}}(x) e_{n+1} = \tau w w' e_{n+1}$ by the bimodule property of $E_{A_{n-1}}$.

(iii) If w is a reduced word on $e_1, e_2, \ldots, e_{n+1}$ write $w = xe_{n+1}y$ with $x, y \in A_n$. Then by (ii), $e_{n+1}\, we_{n+1} = te_{n+1}$ for $t \in A_{n-1}$. Thus $e_{n+1}A_{n+1}e_{n+1} \subseteq A_{n-1}e_{n+1}$. To show that the map is an isomorphism, suppose $xe_{n+1} = 0,\ x \in A_{n-1}$. Then $xx^*e_{n+1} = 0$ so $\mathrm{tr}(xx^*) = 0$, i.e. $x = 0$. □

Aside 4.1.4. In fact it is possible to uniquely order reduced words by pushing $e_{\max}$ to the right as far as possible. It is easy to show that such an ordered reduced word is of the form

$$(e_{j_1}e_{j_1-1}\ldots e_{k_1})(e_{j_2}e_{j_2-1}\ldots e_{k_2})\ldots(e_{j_p}e_{j_p-1}\ldots e_{k_p})$$

where j_p is the maximum index, $j_i \geqq k_i$ and $j_{i+1} > j_i,\ k_{i+1} > k_i$. To each such word we may associate an increasing path on the integer lattice between (0,0) and $(n+1,\, n+1)$, which does not cross the diagonal. For instance $(e_3e_2e_1)(e_4e_3)(e_5e_4)$ in A_5 would correspond to the path (I owe this observation to H. Wilf):



It is well known that such paths are counted by the Catalan numbers $1/(n+2)\binom{2(n+1)}{n+1}$ so we obtain $\dim\ A_n \leqq 1/(n+2)\binom{2(n+1)}{n+1}$. Uniqueness and linear independence of ordered reduced words would follow from $\dim A_n = 1/(n+2)\binom{2(n+1)}{n+1}$ which we will prove in Sec. 5.1 for $\tau \leqq 1/4$. See Aside 5.1.1.

We now want to study traces on A_∞. For this define a *totally reduced word* to be a reduced word on the e_i's where we also allow cyclic permutations. The following result is obvious.

Remark 4.1.5. Any trace on A_n is determined by its effect on totally reduced words.

Lemma 4.1.6. Any totally reduced word is of the form $w = e_{i_1}e_{i_2}\ldots e_{i_n}$ with $|i_j - i_k| \geqq 2,\ j \neq k$, and $\mathrm{tr}(w) = \tau^n$.

Proof. The last assertion is immediate from 4.1.1(c). We prove the first assertion by induction on the length of a totally reduced word. It suffices to prove that if $m = \max\{i_1, i_2, \ldots, i_n\}$ in a totally reduced word $e_{i_1}e_{i_2}\ldots e_{i_n}$, then e_{m-1} does not occur. For this note that by a cyclic permutation we may suppose the word is of the form $e_m w$ and that e_{m-1} occurs at most once in w (since a totally reduced

word is reduced). But then we may proceed to $e_m w e_m$ using $e_m^2 = e_m$ and then eliminate e_{m-1} from w using (a) and (b) of 4.1.1. □

We shall now show that any normal normalized trace on P is equal to tr. For this it suffices to show that it equals tr on completely reduced words. For each subset $I \subseteq N$ with $|i-j| \geqq 2$ whenever $i,\ j \in I,\ i \neq j$, define A_I to be the algebra generated by $\{e_i | i \in I\}$.

Lemma 4.1.7. *For any finite permutation σ of I, there is a unitary $u \in A_\infty$ such that $ue_k u^* = e_{\sigma(k)}$ for all $k \in I$.*

Proof. It suffices to show that any transposition $e_i \leftrightarrow e_j (j \geqq i)$ can be effected by a unitary. We may even suppose that no k strictly between i and j is in I. If we can find a unitary $u \in A_{i,j}$ with $ue_i u^* = e_j$ and $ue_j u^* = e_i$ then this u will do since it commutes with all the other e_k's, $k \in I$. But for this it suffices to show that e_i and e_j are equivalent in $A_{i,j}$. And $e_i e_{i+1} \ldots e_j$ is a multiple of a partial isometry v with $vv^* = e_i,\ v^*v = e_j$. □

Corollary 4.1.8. *Any normal normalized trace on P is equal to* tr *on A_I''.*

Proof. Since all the e_i's are independent for tr, A_I'' may be identified with an infinite tensor product Bernoulli shift algebra. By 4.1.7 the normalizer induces the obvious action of S_∞ on A_I''. This action is well known to be ergodic. Hence any invariant measure which is absolutely continuous with respect to tr is proportional to tr. □

Corollary 4.1.9. *P is a II_1 factor isomorphic to R.*

Proof. By 4.1.4, 4.1.5 and 4.1.8 there is only one normal normalized trace on P. Thus P is a factor. By [21] and (a) of 4.1.3, $P \cong R$. □

Note that our proof that P is a factor follows similar lines to the scheme laid out in [27].

Corollary 4.1.10. *P_τ is a subfactor of P.*

Proof. Writing $f_i = e_{i+1}$, the f_i satisfy the same relations as the e_i. □

Lemma 4.1.11. *For each n, the map $e_i \to e_{n-i}$ extends to a* tr*-preserving $*$-automorphism σ_n of A_n.*

Proof. The map obviously extends to a $*$-automorphism of the free involutive monoid on the self-adjoint e_i. It thus suffices to know that if w is a word on $e_1,\ e_2, \ldots,\ e_n$ then $\mathrm{tr}(\sigma_n(w)) = \mathrm{tr}(w)$. For then if $x = \sum_i c_i w_i$ for $c_i \in \mathbb{C}$,

$$\operatorname{tr}(xx^*) = \operatorname{tr}\left(\sum_{i,j} c_i\bar{c}_j w_i w_j^*\right) = \operatorname{tr}\left(\sum_{i,j} c_i\bar{c}_j \sigma(w_i)\sigma(w_j^*)\right)$$

so that $\sum_i c_i w_i \to \sum_i c_i\sigma(w_i)$ is a well defined isometry for the definite hermitian scalar product defined by tr. But the trace of w is determined by 4.1.1 (a) and (b) and the formula of 4.1.6, all of which are invariant under the interchange $e_i \leftrightarrow e_{n-1}$. □

Corollary 4.1.12.

(i) $E_{A_{2,n}}(e_1) = \tau$
(ii) *for* $x \in A_{2,n}$, $e_1xe_1 = E_{A_{3,n}}(x)e_1$
(iii) $E_{P_\tau}(e_1) = \tau$, $e_1xe_1 = E_{A''_{3,\infty}}(x)e_1$ *for* $x \in P_\tau$.

Proof. (i) and (ii) follow from σ_n applied to (ii) of 4.1.3, and (iii) is just the limit as $n \to \infty$ of (i) and (ii). □

Proof of (ii) of 4.1.1 (calculation of $[P : P_\tau]$)**.** Do the basic construction to obtain $\langle P, e_{P_\tau}\rangle$. Then $e_{P_\tau}e_1e_{P_\tau} = \tau e_{P_\tau}$ follows from (i) of 4.1.12. We further claim that $e_1e_{P_\tau}e_1 = \tau e_1$. By (iii) of 4.1.12, elements of the form $a_0 + \sum_{i=1}^{n} a_ie_1b_i$ with a_i, $b_i \in P_\tau$ are dense in P so it suffices to verify $e_1e_{P_\tau}e_1 = \tau e_1$ on $x\xi$ and $xe_1y\xi$ with $x, y \in P_\tau$. But $e_1e_{P_\tau}e_1(x\xi) = \tau e_1x\xi = (\tau e_1)(x\xi)$, and

$$e_1e_{P_\tau}e_1(xe_1y\xi) = e_1(E_{P_\tau}(e_1E_{A''_{3,\infty}}(x)y)\xi) = \tau e_1E_{A''_{3,\infty}}(x)y\xi = \tau e_1(xe_1y\xi)\,.$$

These two relations imply that e_1 and e_{P_τ} are equivalent in $\langle P, e_{P_\tau}\rangle$. Now by (iii) of 3.1.5, e_{P_τ} is a finite projection and e_1 is in a II_1 factor so that $\langle P, e_{P_\tau}\rangle$ is necessarily a finite factor, i.e. $[P : P_\tau] < \infty$. But we know that $\operatorname{tr}(e_1) = \tau$ so $[P : P_\tau] = \operatorname{tr}(e_{P_\tau})^{-1} = \tau^{-1}$. □

4.2. *Restrictions on τ*

We shall now prove (iii) of 4.1.1. We keep the notation of Sec. 4.1. Also define $s_n = e_1 \vee e_2 \vee \ldots \vee e_n$.

Lemma 4.2.1. *If* $1 - s_n \neq 0$ *then it is a minimal projection in* A_n *which belongs to* $Z(A_n)$.

Proof. If w is a word on $e_1, e_2, \ldots, e_n$ then

$$(1 - e_1 \vee e_2 \vee \ldots \vee e_n)w = 0\,.$$ □

Lemma 4.2.2. *If* $s_n \neq 1$ *then* $e_{n+1} \wedge s_n = e_{n+1}s_{n-1}$.

Proof. By 4.1.3(ii), $e_{n+1}s_ne_{n+1} = E_{A_{n-1}}(s_n)e_{n+1}$. But by 4.2.1 and the bimodule property of $E_{A_{n-1}}$, $E_{A_{n-1}}(s_n) \in Z(A_{n-1})$. Let $p_0, p_1, \ldots, p_k$ be the

minimal projections in $Z(A_{n-1})$ with $p_0 = 1 - s_{n-1}$, and write $E_{A_{n-1}}(s_n) = \sum_{i=0}^{k} \lambda_i p_i$. Then since $e_{n+1} \wedge s_n \geqq e_{n+1}s_{n-1}$ and $\lim_{m\to\infty} (e_{n+1}s_n e_{n+1})^m = e_{n+1} \wedge s_n$, and by (iii) of 4.1.3, $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 1$. Since $s_n \neq 1$, $E_{A_n}(s_n) \neq 1$ so by (iii) of 4.1.3, $\lambda_k < 1$. Thus $e_{n+1} \wedge s_n = e_{n+1}s_{n-1}$. □

Corollary 4.2.3. *If* $\operatorname{tr}(1 - s_k) > 0$, $\operatorname{tr}(1 - s_{k+1}) = \operatorname{tr}(1 - s_k) - \tau \operatorname{tr}(1 - s_{k-1})$. *And* $\operatorname{tr}(1 - s_1) = 1 - \tau$, $\operatorname{tr}(1 - s_2) = 1 - 2\tau$.

Proof. The trace tr satisfies $\operatorname{tr}(p \vee q) = \operatorname{tr}(p) + \operatorname{tr}(q) - \operatorname{tr}(p \wedge q)$. So by 4.2.2 and (c) of 4.1.1, $\operatorname{tr}(1 - s_{k+1}) = \operatorname{tr}(1 - s_k) - \tau \operatorname{tr}(1 - s_{k-1})$. □

For this reason we define the polynomials $P_n(x)$ by $P_0(x) = 0$, $P_1(x) = 1$ and $P_{n+1} = P_n - xP_{n-1}$, so that if $P_{k+2}(\tau) > 0\ \forall k \leqq n$ then $P_{k+2}(\tau) = \operatorname{tr}(1 - s_k)$.

Lemma 4.2.4. *Let* $\sigma = \frac{1+\sqrt{1-4x}}{2}$, $\widetilde{\sigma} = \frac{1-\sqrt{1-4x}}{2}$. *Then*

(i) $P_n(x) = \frac{\sigma^n - \widetilde{\sigma}^n}{\sigma - \widetilde{\sigma}}$
(ii) $P_n(\frac{1}{4}\sec^2\theta) = \sin n\theta / (2^{n-1}\cos^{n-1}\theta \sin\theta)$
(iii) $\deg P_n = \left[\frac{n-1}{2}\right]$.

Proof. (i) The general solution to the difference equation is $P_n = A\sigma^n + B\tilde{\sigma}^n$. The initial conditions give $A + B = 0$, $A(\sigma - \widetilde{\sigma}) = 1$.

(ii) Putting $\sigma = re^{i\theta}$, $\widetilde{\sigma} = re^{-i\theta}$, $x = \frac{1}{4}\sec^2\theta$, $r = \frac{1}{2}\sec\theta$, $\sigma^n - \widetilde{\sigma}^n = 2\,ir^n \sin n\theta, \sigma - \widetilde{\sigma} = 2\,ir \sin\theta$.
(iii) Follows easily by induction from the difference equation. □

Corollary 4.2.5. (i) *The smallest root of* P_n *is* $\frac{1}{4}\sec^2\frac{\pi}{n}$.

(ii) $P_n(\tau) > 0$ *for* $\tau < \frac{1}{4}\sec^2\frac{\pi}{n}$.
(iii) $P_{n+1}(\tau) < 0$ *for* τ *between* $\frac{1}{4}\sec^2\frac{\pi}{n+1}$ *and* $\frac{1}{4}\sec^2\frac{\pi}{n}$.

Proof. (i) By counting the number of distinct values of $\frac{1}{4}\sec^2\frac{m\pi}{n}$ (which are roots of P_n by 4.2.4) we find that all roots of P_n are real and they are the numbers $\frac{1}{4}\sec^2\frac{m\pi}{n}$ with $\frac{m\pi}{n} < \frac{\pi}{2}$. The smallest is $\frac{1}{4}\sec^2\frac{\pi}{n}$.

(ii) By induction the coefficient of $x^{[(n-1)/2]}$ in $P_n(x)$ is positive when $\left[\frac{n-1}{2}\right]$ is even and negative when $\left[\frac{n-1}{2}\right]$ is odd.
(iii) $P_{n+1}(\tau)$ must be negative between its first and second real roots, and $\sec^2\pi/(n+1) < \sec^2\pi/n < \sec^2 2\pi/(n+1)$. □

Proof of (iii) of 4.1.1. Suppose $\tau > 1/4$ and $\tau \neq \frac{1}{4}\sec^2\frac{\pi}{n},\ n = 3, 4, 5, \ldots$. Then there is a $k \geqq 3$ with $\frac{1}{4}\sec^2\pi/(k+1) < \tau < \frac{1}{4}\sec^2\pi/k$. But then $P_n(\tau) > 0$ for all $n \leqq k$ so $P_{n+1}(\tau) = \mathrm{tr}(1 - s_{n-1})$ by 4.2.3. But by (iii) of 4.2.5, $P_{n+1}(\tau) < 0$ which is impossible since $1 - s_{n-1}$ is a projection. □

4.3. *Values of the index*

Theorem 4.3.1. *If N is a subfactor of the* II_1 *factor M then either* $[M : N] \geqq 4$ *or* $[M : N] = 4\cos^2\pi/n$ *for some* $n \geqq 3$.

Proof. If $[M : N] < \infty$, define the increasing sequence $M_i,\ i = 0, 1, 2, \ldots$ of II_1 factors by the relations $M_0 = M,\ M_1 = \langle M, e_n\rangle,\ M_{i+1} = \langle M_i, e_{M_{i-1}}\rangle$ for $i \geqq 1$. The inductive limit becomes a II_1 factor with faithful normal normalized trace tr (by uniqueness of the trace — see also [19]). Moreover if $\tau = [M : N]^{-1}$ and $e_i = e_{M_i}$ then the e_i satisfy the conditions of 4.1.1 by 3.4.2 and 3.1.7. By Theorem 4.1.1 either $[M : N] \geqq 4$ or $[M : N] = 4\cos^2\pi/n$ for some $n \in \mathbb{Z},\ n \geqq 3$.□

Theorem 4.3.2. *For each $n = 3, 4, \ldots$ there is a subfactor P_τ of R with* $[R : P_\tau] = 4\cos^2\pi/n$ *and* $P' \cap R = \mathbb{C}$. *For each $r \geqq 4,\ r \in \mathbb{R}$, there is a subfactor P of R with* $[R : P] = r$.

Proof. The existence of subfactors with index $r \geqq 4$ was shown in 2.2.5 and the assertion about the relative commutant when $[R : P] = 4\cos^2\pi/n$ follows from 2.2.4. Thus we only need to construct subfactors with index $4\cos^2\pi/n$.

To do this note that the conditions of 3.3.2 for finite dimensional C^*-algebras $N \subseteq M$ together with (ii) of 3.3.1 show, by interchanging A and A^T, $\vec{s}$ and $\vec{t}$, that if there is a (τ, M) trace on $\langle M, e_N\rangle$ then there is a $(\tau, \langle M, e_N\rangle)$ trace on $\langle\langle M, e_N\rangle, e_M\rangle$ and so on. Thus we may iterate the basic construction once we have started it. Once the construction has been iterated, the inductive limit has a faithful normal normalized trace on it and the e_i's resulting from the iteration satisfy the conditions of 4.1.1 so by the result of 4.1.1 we may choose P_τ as the subfactor of R.

Thus it suffices to find $N \subseteq M$, finite dimensional C^*-algebras with inclusion matrix A and positive vectors $\vec{s}$ and $\vec{t}$ with $A^T A\vec{t} = (4\cos^2\pi/n)\vec{t}$ and $AA^T\vec{s} = (4\cos^2\pi/n)\vec{s}$. In fact the matrix A is enough since the subalgebra N can then be taken as the direct sum of as many copies of $\mathbb{C}$ as there are rows in the matrix.

Let A be the square $n \times n$ matrix (a_{ij}) with $a_{ij} = 1$ if $|i - j| = 1$ and 0 otherwise, e.g. $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. We leave it to the reader to check the linear algebra. This choice of A was suggested by F. Goodman. □

Remark 4.3.3. If we had started with any $m \times n$ non-negative integer valued matrix we could have made the construction of 4.3.2. Thus 4.1.1 implies that for such a matrix, if $\|A\| \leqq 2$ then $\|A\| = 2\cos\pi/n,\ n = 3, 4, \ldots$. This can also be proved using the well-known result of Kronecker which asserts that if z is an

algebraic integer all of whose conjugates have absolute value equal to 1, then z is a root of unity.

5. The Bratteli Diagram; the Relative Commutant

5.1. *The Bratteli Diagram when* $\tau \leqq 1/4$

Let $\{e_1, e_2, \ldots\}$, M and tr be as in Sec. 4.1. Let $A_n = \{e_1, e_2, \ldots, e_n\}''$ and B_n be the algebra generated by $\{e_1, e_2, \ldots, e_n\}$ (without 1). We know that A_n and B_n are finite dimensional C^*-algebras. We want to determine the Bratteli diagram for them and the value of tr on minimal projections. In this section we suppose $\tau \leqq 1/4$. This means that

$$P_{n+2}(\tau) = \mathrm{tr}(1 - e_1 \vee e_2 \vee \ldots \vee e_n) > 0 \quad \text{for all } n \geqq 1$$

(where P_n is as in Sec. 4.2).

Let $\left\{ \begin{matrix} n \\ b \end{matrix} \right\} = \binom{n}{b} - \binom{n}{b-1}$ $\left(\text{ordinary binomial symbols with the convention } \binom{n}{-1} = 0\right)$. We shall show by induction that

(a) $A_n = \bigoplus_{k=0}^{[\frac{n+1}{2}]} Q_k^n$ where $Q_k^n \cong M_{\left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\}}(\mathbb{C})$.

(b) $Q_0^n = (1 - e_1 \vee e_2 \vee \ldots \vee e_n)\mathbb{C}$ so that

(c) $B_n = \bigoplus_{k=1}^{[\frac{n+1}{2}]} Q_k^n$

(d) The trace of a minimal projection in Q_k^n is $\tau^k P_{n+2-2k}(\tau)$ for $k = 0, 1, \ldots, \left[\frac{n+1}{2}\right]$.

(e) The inclusion matrix of A_{n-1} in A_n is

(i) When n is even

$$A = (a_{ij}) \text{ with } a_{ij} = \begin{cases} 1 \text{ if } j = i \text{ or } i+1 \\ 0 \text{ otherwise,} \end{cases}$$

$$i, j = 0, 1, \ldots, \left[\frac{n+1}{2}\right]$$

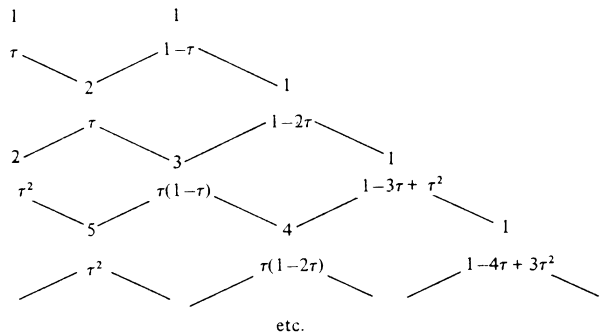
(here the indices i and j refer to the subscript of Q).

(ii) when n is odd

$$A = (a_{ij})\ a_{ij} = \begin{cases} 1 \text{ if } j = i \text{ or } i+1 \\ 0 = \text{ otherwise} \end{cases}$$

$$i = 0,., \ldots, (n+1)/2;\ j = 0, 1, \ldots, (n+3)/2.$$

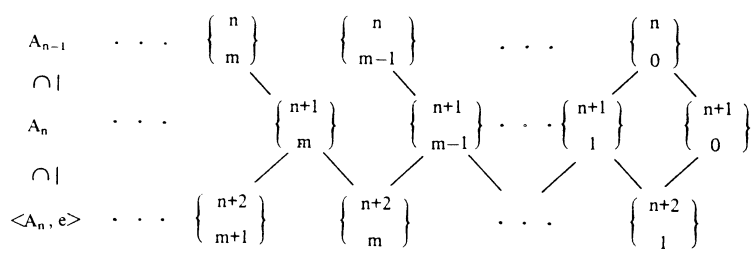
All this information is summed up by the following diagram (which also appears on p. 118 of [31]).



The numbers are the $\left\{ {n+1 \atop k} \right\}$ and the polynomials, which give the trace on minimal projections in the corresponding matrix algebra, change by multiplication by τ each step down a vertical column.

Proof. The proof will use the basic construction for the inclusion $A_{n-1} \subseteq A_n$ to obtain an almost faithful representation of A_{n+1}.

The truth of assertions $(a) \to (e)$ for $n = 2$ follows immediately from Sec. 3.4. For the inductive step we shall treat only the case $n = 2m$, the odd case being essentially identical. Suppose $(a) \to (e)$ are true for all $k \leqq n$ and apply the basic construction of Sec. 3.1 with $M = A_n$, $N = A_{n-1}$ with respect to tr. Let $E = E_{A_{n-1}}$, $e_{A_{n-1}} = e$. By induction and (ii) of 3.3.1 we know that we have the following Bratteli diagram:



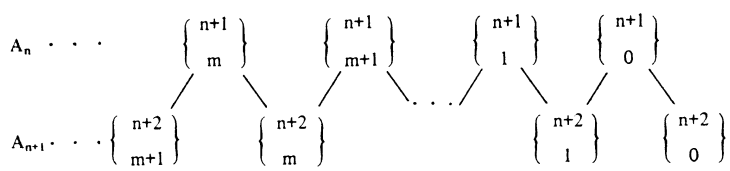
The numbers on the bottom line follow from 3.7.1, (ii) of 3.3.1 and the identities

$$\left\{ {a \atop b} \right\} + \left\{ {a \atop b-1} \right\} = \left\{ {a+1 \atop b} \right\}, \qquad \left\{ {2m+2 \atop m+1} \right\} = \left\{ {2m+1 \atop m} \right\}.$$

By assertion (c), the algebra generated by $\{e_1, e_2, \ldots, e_n\}$ is the direct sum of the first m terms on the A_n line so if we define the faithful (non-normalized) trace Tr on $\langle A_n, e\rangle$ by the rule $\mathrm{Tr}(ep_k) = \tau\ \mathrm{tr}(p_k)$ where p_k is a minimal projection in Q_{k-1}^{n-1} $(k = 1, 2, \ldots, m+1)$, the identity $P_j = P_{j+1} + \tau P_{j-1}$ ensures that Tr agrees with tr on B_n. Moreover as in 3.3.2, uniqueness of the trace and linearity show that $\mathrm{Tr}(ex) = \tau\ \mathrm{tr}(x)$ for all x in A_{n-1} and hence all x in A_n. Also Tr is a faithful positive trace since all the polynomials $P_n(\tau)$ are positive.

But now consider the algebras B_{n+1} and $\langle A_n, e\rangle$. B_{n+1} is generated by B_n and e_{n+1} so that any element can be written $a_0 + \sum_{i=1}^{i} a_i e_{n+1} b_i$ with $a_0 \in B_n,\ a_i,\ b_i \in A_n$ for $i = 1, 2, \ldots, k$. Multiplication is defined by $e_{n+1} x e_{n+1} = E(x) e_{n+1}$ for $x \in A_n$ (see 4.1.3) and the faithful trace tr defined by $\mathrm{tr}(x e_{n+1}) = \tau\ \mathrm{tr}(x)$ for $x \in A_n$. In $\langle A_n, e\rangle$, sums of the form $a_0 + \sum_{i=1}^{j} a_i e b_i$ with $a_0 \in B_n,\ a_i,\ b_i \in A_n$ for $i \geqq 1$, form a 2-sided ideal. Since the central support of e is 1 ((iv) of 3.1.5), any element of $\langle A_n, e\rangle$ can be written in this form. Also $exe = E(x)e$ for $x \in A_n$ and the faithful trace Tr on $\langle A_n, e\rangle$ satisfies $\mathrm{Tr}(aeb) = \mathrm{tr}(aeb)$ for $a,\ b \in A_n$ and $\mathrm{Tr}(b) = \mathrm{tr}(b)$ for $b \in B_n$. Thus we may define a map from B_{n+1} to $\langle A_n, e\rangle$ by $a_0 + \sum_i a_i e_{n+1} b_i \mapsto a_0 + \sum_i a_i e b_i$ which is a surjective isometry for the definite hermitian scalar products defined by tr and Tr (and hence is well defined).

At this stage (†) we have obtained assertion (c) for $n+1$ and the values of tr on the minimal projections in $Q_1^{n+1},\ Q_2^{n+1},\ \ldots,\ Q_{m+1}^{n+1}$. But A_{n+1} is just $\{B_{n+1} \cup \{1\}\}''$ and since $\mathrm{tr}(1 - e_1 \vee \ldots \vee e_{n+1}) = P_{n+3}(\tau) > 0$, A_{n+1} is $B_{n+1} \oplus (1 - e_1 \vee e_2 \vee \ldots \vee e_{n+1})\mathbb{C}$. Moreover $x(1 - e_1 \vee e_2 \vee \ldots \vee e_{n+1}) = 0$ for any $x \in B_n$ and $(1 - e_1 \vee \ldots \vee e_{n+1})(1 - e_1 \vee \ldots \vee e_n) \neq 0$ so the Bratteli diagram for $A_n \subseteq A_{n+1}$ is forced to be



This proves assertions (a) and (e) for $n+1$, and assertion (d) follows from $\mathrm{tr}(1 - e_1 \vee \ldots \vee e_{n+1}) = P_{n+3}(\tau)$. This ends the proof. □

Aside 5.1.1. The binomial identity

$$\sum_{i=0}^{\left[\frac{n+1}{2}\right]} \binom{n+1}{i}^2 = \frac{1}{n+2} \binom{2(n+1)}{n+1}$$

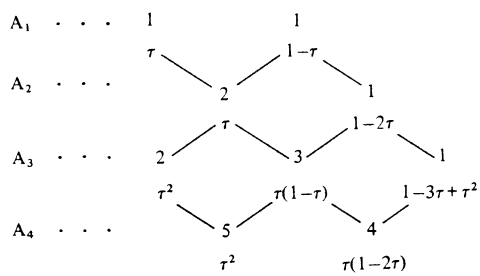
follows from [13, p. 63]. This shows that $\dim A_n$ is the same as the number of ordered reduced words which are thus linearly independent. See 4.1.4.

5.2. *The Bratteli Diagrams* $\tau = \frac{1}{4}\sec^2 \frac{\pi}{n}$

If $\tau = \frac{1}{4}\sec^2 \pi/(n+2)$ then $P_k(\tau) > 0$ for all $k \leqq n+1$ so the inductive argument of Sec. 5.1 goes through until the point marked (†) for assertions $(a) \to (e)$ up to step n. At this stage we find that $e_1 \vee e_2 \vee \ldots \vee e_n = 1$ so $B_n = A_n$. From this point on the basic construction for the pair $A_k \subseteq A_{k+1}$ will give an isometric

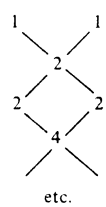
(so faithful) surjective representation of A_{k+2} and the Bratteli diagram will not grow any wider.

To convince the reader of these assertions without boring him with the details, we treat the case $n = 4$. The argument of Sec. 5.1 shows that we have the following diagram

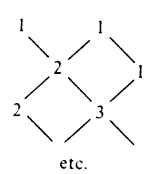


Now if $\vec{s} = \binom{\tau}{1-2\tau}$, $\vec{t} = \begin{pmatrix} \tau^2 \\ \tau(1-\tau) \\ 1-3\tau+\tau^2 \end{pmatrix}$, and $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\vec{s}$ is an eigenvector for $A^T A$ with eigenvalue $3 = \tau^{-1} = 4\cos^2 \frac{\pi}{6}$ and $\vec{t}$ is an eigenvector for AA^T with the same eigenvalue. They give normalized traces on A_2 and A_3 so by Sec. 3.3 the basic construction will continue to give faithful representations of the A_i by the same argument as Sec. 5.1, since $A_n = B_n$ for $n \geqq 4$.

The Bratteli diagrams will be:
For $n = 2(\tau = 1/2)$

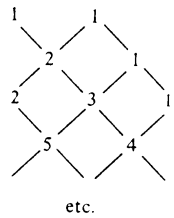


Note that this is the same as the complex Clifford algebras (see [16, p. 148]).
For $n = 3(\tau = 1/\varphi^2,\ \varphi$=golden ratio)



This diagram already appears in [4], [11].

For $n = 4$



The pattern is now clear. The traces on minimal projections are the same as for the case $\tau < 1/4$ where this makes sense.

5.3. *The relative commutant when* $\tau \leqq 1/4$

One of the main motivations of this paper was to decide whether there is a continuum of values of the index realized by subfactors with trivial relative commutant (see problem 1). It was originally thought that the subfactors $P_\tau,\ \tau \leqq 1/4$, had trivial relative commutant. In this section we shall show that this is only true when $\tau = 1/4$. The calculations were first finished by A. Wassermann to whom the author is grateful. The difficulty is the absence of an orthonormal basis which makes it impossible to exhibit an element of the relative commutant.

The case $\tau < 1/4$. Let us adopt the notation of Sec. 4.1 so R is presented as the algebra generated by the e_i's. We intend to show that $E_{R\cap P'_\tau}(e_1)$ is not a scalar, i.e. $E_{R\cap P'_\tau}(e_1) \neq \tau$. For this we will show that $\|E_{R\cap P'_\tau}(e_1) - \tau\|_1 \neq 0$ (remember $\|x\|_1 = \mathrm{tr}(|x|)$). We shall need some lemmas.

Lemma 5.3.1. $E_{R\cap P'_\tau}(x) = s - \lim\limits_{n\to\infty} E_{R\cap A'_{2,n}}(x)$ *for* $x \in R$.

Proof. Since $P_\tau = \left(\bigcup\limits_{n=1}^{\infty} A_{2,n}\right)''$, the algebras $R\cap A'_{2,n}$ are a decreasing sequence of von Neumann algebras with intersection $P'_\tau \cap R$. The rest is well known (e.g. it is true in $L^2(R, \mathrm{tr})$). $\square$

Lemma 5.3.2. $\|E_{A'_{2,n+1\cap R}}(e_1) - \tau\|_1 = \|E_{A'_n\cap A_{n+1}}(e_{n+1}) - \tau\|_1$.

Proof. Let du denote Haar measure on the unitary group of a finite dimensional C^*-algebra. Then

$$\|E_{A'_{2,n+1\cap R}}(e_1 - \tau)\|_1 = \left\| \int\limits_{U(A_{2,n+1})} (ue_1u^* - \tau)\, du \right\|_1 .$$

Applying the isomorphism σ_{n+1} of 4.1.11 we find that this expression equals

$$\left\| \int_{U(A_n)} (ue_{n+1}u^* - \tau)\, du \right\|_1$$

which equals $\|E_{A'_n \cap A_{n+1}}(e_{n+1}) - \tau\|_1$. □

Thus to show that $P'_\tau \cap R \neq C$, it suffices to show that

$$\lim_{n\to\infty} \|E_{A'_n \cap A_{n+1}}(e_{n+1}) - \tau\|_1 \neq 0\,.$$

Since we know that the limit exists, it suffices to consider n odd; say $n = 2m-1$. From Sec. 5.1 let $p_0, p_1, \ldots, p_{m-1}$ be the minimal central projections in A_{n-1} corresponding to $Q_0^{n-1},\ Q_1^{n-1},\ \ldots,\ Q_{m-1}^{n-1}$, similarly $q_0,\ q_1,\ \ldots,\ q_m$ for A_n and $r_0,\ r_1,\ \ldots,\ r_m$ for A_{n+1}. So $p_0 = 1 - e_1 \vee e_2 \vee \ldots \vee e_{n-1},\ q_0 = 1 - e_1 \vee e_2 \vee \ldots \vee e_n$ and $r_0 = 1 - e_1 \vee e_2 \vee \ldots \vee e_{n+1}$. We also know from Sec. 5.1 and (iii) of 3.3.1 that

$$e_{n+1} r_i = e_{n+1}\, p_{i-1} \quad \text{for} \quad i \geqq 1\,. \tag{5.3.3}$$

Since all the embeddings on the Bratteli diagram are of multiplicity at most one, the relative commutant of A_n in A_{n+1} is the abelian algebra generated by the mutually orthogonal projections $q_0r_0, q_0r_1, q_1r_0, \ldots, q_mr_m$. Thus

$$\begin{aligned} E_{A'_n \cap A_{n+1}}(e_1) = \frac{\operatorname{tr}(q_0r_0e_{n+1})}{\operatorname{tr}(q_0r_0)} q_0r_0 &+ \frac{\operatorname{tr}(q_0r_1e_{n+1})}{\operatorname{tr}(q_0r_1)} q_0r_0 + \cdots \\ &+ \frac{\operatorname{tr}(q_mr_me_{n+1})}{\operatorname{tr}(q_mr_m)} q_mr_m \end{aligned}$$

so that

$$\begin{aligned} \|E_{A'_n \cap A_{n+1}}(e_{n+1}) - \tau\|_1 &= \sum_{i=0}^{m} |\operatorname{tr}(q_ir_ie_{n+1}) - \tau \operatorname{tr}(q_ir_i)| \\ &\quad + \sum_{i=0}^{m-1} |\operatorname{tr}(q_ir_{i+1}e_{n+1} - \tau \operatorname{tr}(q_ir_i)| \\ &\geqq \sum_{i=1}^{m} \tau |\operatorname{tr}(q_ip_{i-1}) - \operatorname{tr}(q_ir_i)| \quad \text{by } 5.3.3 \\ &\geqq \tau \left| \sum_{i=1}^{m} \operatorname{tr}(q_ip_{i-1}) - \operatorname{tr}(q_ir_i) \right|\,. \end{aligned}$$

But from the Bratteli diagram, $q_i p_{i-1}$ is a projection of rank $\left\{ \begin{matrix} 2m-1 \\ i-1 \end{matrix} \right\}$ in the matrix algebra $Q_i^{2m-1} (= q_i A_n)$ and $q_i r_i$ is of rank $\left\{ \begin{matrix} 2m \\ i \end{matrix} \right\}$ in $Q_i^{2m} (= r_i A_{n+1})$. So the last sum may be written

$$L = \tau \left| \sum_{i=1}^{m} \left(\left\{ \begin{matrix} 2m-1 \\ i-1 \end{matrix} \right\} \tau^i \, P_{2(m-i)+1}(\tau) - \left\{ \begin{matrix} 2m \\ i \end{matrix} \right\} \tau^i \, P_{2(m-i)+2}(\tau) \right) \right| .$$

We saw in 4.2.4 that $P_n(\tau) = (\sigma^n - \widetilde{\sigma}^n)/(\sigma - \widetilde{\sigma})$ where $\sigma = (1 + \sqrt{1-4\tau})/2$ and $\widetilde{\sigma} = (1 - \sqrt{1-4\tau})/2$. Note that for $\tau < 1/4$, $\sigma > 1/2$, $\widetilde{\sigma} < 1/2$ and $\widetilde{\sigma} + (\tau/\widetilde{\sigma}) = 1$, $\sigma + (\tau/\sigma) = 1$. Let us now calculate the relevant limits.

Lemma 5.3.6. (a) $\displaystyle \lim_{m\to\infty} \left(\sum_{i=1}^{m} \left\{ \begin{matrix} 2m-1 \\ i-1 \end{matrix} \right\} \widetilde{\sigma}^{2(m-i)+1} \tau^i \right) = 0$

(b) $\displaystyle \lim_{m\to\infty} \left(\sum_{i=1}^{m} \left\{ \begin{matrix} 2m \\ i \end{matrix} \right\} \widetilde{\sigma}^{2(m-i)+2} \tau^i \right) = 0$

(c) $\displaystyle \lim_{m\to\infty} \left(\sum_{i=1}^{m} \left\{ \begin{matrix} 2m-1 \\ i-1 \end{matrix} \right\} \sigma^{2(m-i)+1} \tau^i \right) = \tau(1 - \tau/\sigma^2)$

(d) $\displaystyle \lim_{m\to\infty} \left(\sum_{i=1}^{m} \left\{ \begin{matrix} 2m \\ i \end{matrix} \right\} \sigma^{2(m-i)+2} \tau^i \right) = \sigma^2(1 - \tau/\sigma^2).$

Proof. (a) and (c). Note that $\sigma^{2(m-i)+1}\tau^i = \tau(\tau/\sigma)^{i-1}\sigma^{2m-1}$ so if $x = \sigma$ or $\widetilde{\sigma}$, in both cases we have to eveluate

$$\lim_{m\to\infty} \tau \left(\sum_{i=1}^{m} \binom{2m-1}{i-1} \left(\frac{\tau}{x}\right)^{i-1} x^{2m-1} - \frac{\tau}{x^2} \sum_{i=1}^{m} \binom{2m-1}{i-2} \left(\frac{\tau}{x}\right)^{i-2} x^{2m-i+1} \right).$$

Since $x + \tau/x = 1$, we recognize the probability of $\geqq m$ successes in $2m-1$ Bernoulli trials. If $x = \sigma$, the probability of success is $> 1/2$ so by the de Moivre–Laplace central limit theorem the limit is $\tau(1 - \tau/\sigma^2)$ and if $x = \widetilde{\sigma}$, the probability of success is $< 1/2$ so the limit is 0.

(b) and (d) are proved in the same way with the substitution $x^{2(m-i)+2}\tau^i = x^2(\tau/x)^i x^{2m-i}$. □

Expanding L and using 4.3.5 we see that

$$(\sigma - \widetilde{\sigma})/L = \tau|(1 - \tau/\sigma^2)(\tau - \sigma^2)| \neq 0 \quad \text{for } \tau \neq 1/4 .$$

This shows that $P'_\tau \cap R \neq \mathbb{C}$ for $\tau < 1/4$.

The case $\tau = 1/4$. In this case we contend that $P'_\tau \cap R = \mathbb{C}$. Luckily we can use another model of $P_{1/4}$. Let R be realized as the closure of the Fermion algebra $\bigotimes_{i=1}^{\infty}(M_2(\mathbb{C}))_i$ with respect to the trace tr. The fixed point algebra of the obvious infinite product action of $U(2)$ is generated by the representation of S_∞ coming from interchanging the tensor product components. The transpositions between successive components may be written $2e_i - 1$ with $\mathrm{tr}(e_i) = 1/4$ and it is a matter of calculation to show that the e_i's satisfy $e_i e_{i\pm 1} e_i = \frac{1}{4} e_i$, $e_i e_j = e_j e_i$ for $|i-j| \geqq 2$. Thus the e_i algebra is $R^{U(2)}$ and the subfactor $P_{1/4}$ is $M_2(C)'_1 \cap R^{U(2)}$. But it is shown in [31] that $(R^{U(2)})' \cap R = \mathbb{C}$ so that $P'_{1/4} \cap R = M_2(\mathbb{C})_1$ and $M_2(\mathbb{C})_1 \cap R^{U(2)} = \mathbb{C}$. Thus $P'_{1/4} \cap R^{SU(2)} = \mathbb{C}$. For $R^{U(2)}$ see also [12], [15], [31], [2]. □

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THE WORK OF SHIGEFUMI MORI

by

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The most profound and exciting development in algebraic geometry during the last decade or so was the *Minimal Model Program* or *Mori's Program* in connection with the classification problems of algebraic varieties of dimension three. Shigefumi Mori initiated the program with a decisively new and powerful technique, guided the general research direction with some good collaborators along the way, and finally finished up the program by himself overcoming the last difficulty. The program was constructive and the end result was more than an existence theorem of minimal models. Even just the existence theorem by itself was the most fundamental result toward the classification of general algebraic varieties in dimension 3 up to birational transformations. The constructive nature of the program, moreover, provided a way of factoring a general birational transformation of threefolds into elementary transformations (divisorial contractions, flips and flops) that could be explicitly describable in principle. Mori's theorems on algebraic threefolds were stunning and beautiful by the totally new features unimaginable by those algebraic geometers who had been working, probably very hard too, only in the traditional world of algebraic or complex-analytic surfaces. Three in dimension was in fact a quantum jump from two in algebraic geometry.

Historically, to classify algebraic varieties has always been a fundamental problem of algebraic geometry and even an ultimate dream of algebraic geometers. During the early decades of this century, many new discoveries were made on the new features in classifying algebraic surfaces, unimaginable from the case of curves. They were mainly done by the so-called Italian school of algebraic geometers, such as Guido Castelnuovo (1865–1952), Federigo Enriques (1871–1946), Francesco Severi (1889–1961) and many others. Since then, there have seen several important modernization, precision, reconstruction with rigor, extensions, in the theory of surfaces. The most notable among those were the works by Oscar Zariski during the late 1950s (especially, Castelnuovo's criterion for rational surfaces and minimal models of surfaces) and those by Kunihiko Kodaira during the early 1960s (especially, detailed study on elliptic surfaces and complex-analytic extensions,

especially non-algebraic). In particular, Kodaira and his younger colleague S. Iitaka have produced many talented followers and collaborators. Up to and after the "Algebraic Surfaces" by I. R. Shafarevich, Yu I. Manin, B. G. Moishezon, et al. (1965), the Russian school of algebraic geometers advanced the study of some important algebraic surfaces and their deformations, especially the study of Torelli problems. As for the extensions to positive characteristics, outstanding and remarkable were the works of D. Mumford and E. Bombieri, and their followers during the late 1960s and 1970s. There have been so many other important contributors to the theory of surfaces that I would not try here to list all the important names and works in the theory of surfaces. There were also several unique works on classification problems of non-complete surfaces or complete surfaces with specified divisors, whose studies were instigated as being "2.5 dimensional" by S. Iitaka around 1977.

As for the higher dimensional algebraic varieties, the decade of 1970s saw three lines of new progress, just to name a few. One was the discovery, in the early 1970s, of the gap between the *rationality* and *unirationality*. The second was an attempt to classify *Fano* 3*-folds*, of which the birational classification was begun by Yu. I. Manin (1971), V. A. Iskovskih (1971, 1977–78), and followed by V. V. Shokurov (1979). However the biregular classification of Fano 3-folds in case $B_2 > 1$, with existence and number of moduli, had to wait a few more years to be achieved by Mori, jointly with S. Mukai, after a new powerful technique was invented by Mori in 1980. (The biregular classification in case $B_2 = 1$ was recently completed by S. Mukai using moduli of vector bundles on $K3$ surfaces.) The third line of progress was an early version of serious attempts toward *higher-dimensional classification* problems, largely inspired by S. Iitaka's bold conjectures proposed around 1970, and especially pressed hard during the latter half of the 1970s, by the Tokyo school of complex-algebraic geometers. The essence of their results was reported by H. Esnault in her Bourbaki talk, exp. 568 (1981).

Notably, K. Ueno produced some structure theorems in higher dimensions during 1977–79, which appeared at that time to boost the program of Iitaka et al. However, the scope of their achievements were limited toward the original goal of classifying general algebraic 3-folds in a style of extending the beautiful old theory of classifying surfaces. Their crucial limitation was the lack of a good higher-dimensional analog to Zariski's theory of minimal models. Technically, as it later became clear, the drawback was the absence of Mori's stunning success in analyzing birational transformations from the point of view of extremal contractions in dimensions higher than 2. In short, a new insight had to be injected into the classification program and it came from the technique of extremal rays inaugurated by Mori which inspired M. Reid, J. Kollár, Y. Kawamata et al. to begin working in earnest in search of minimal models around the turn of the decade.

Early in 1979, Mori brought to algebraic geometry a completely new excitement, that was his proof of *Hartshorne's conjecture*, proposed in 1970, which said that the projective spaces are the only smooth complete algebraic varieties with ample tangent bundles. It was also exciting news to differential geometers, such as Y.-T.

Siu and S.-T. Yau who subsequently found an independent proof for Frankel's conjecture of 1961, which was implied by Hartshorne's. It is not clear that Hartshorne's conjecture can actually be proven by the differential-geometric method of Siu and Yau. In all approaches, the most important step was to show the existence of *rational curves* in the manifold in question. Mori's idea was simple and natural as good ones were always so, while the proof was not. The idea was that, under some numerical conditions, rational curves should be obtained by deforming a given curve inside the manifold and degenerating it into a bunch of curves of lower genera. But the difficulty was proving such plentifulness of deformations. There, Mori's ingenuity was to overcome this difficulty first in the cases of *positive characteristics*, where the Frobenius maps did a miracle, and then deduce the complex case from them.

Mori extended and reformulated his new and powerful technique of finding rational curves, which was referred to as *extremel rays* in the *cone of curves* by himself in another monumental paper of his on "Threefolds whose canonical bundles are not nef" (nef=numerically effective). The cone of curves in a projective 3-fold was defined to be the closure of the real convex hull generated by the numerical equivalence classes (or classes in the second homology group over the real numbers) of irreducible algebraic curves, and an extremal ray was to be an edge of the cone within the region where the canonical divisor takes negative intersection number. Mori's discovery was that the cone was locally finitely generated in the canonically negative region and that each extremal ray was generated by a *rational curve*. In the above "not nef" paper, moreover, Mori established an outstanding theorem which, in dimension three, completely classified the geometric structures of the family of rational curves corresponding to an extremal ray. The union of such curves was either an irreducible divisor (either a smooth projective line bundle, or a projective plane, or a quadric in a projective 3-space) or the entire 3-fold. The former led to a birational blowing down while the latter to a fibration map (either conic bundle, or Del Pezzo fiber space or Fano 3-fold). In each case, the description was precise and the target variety (blown-down birational model or the base of fibration) was again *projective*. This result was absolutely stunning to anybody who have had experiences with non-projective or even non-algebraic examples of birational transformations, and it was so beautiful as to encourage many algebraic geometers once again to look into the birational problems in dimension 3. Subsequently, Mori completed the *classification of Fano* 3*-folds*, jointly with S. Mukai in 1981, and he established a *criterion for uniruledness*, jointly with Y. Miyaoka in 1985. The Fano 3-folds and uniruled 3-folds were those to be investigated separately in all details and there are still left open some important problems about those special 3-folds. Excluding these, the most important general problem was to prove the *existence of minimal models* in which the canonical bundles were nef.

Mori's "not nef" paper showed, for the first time in the history of algebraic geometry, that a general birational transformation between smooth algebraic 3-folds was not completely untouchable and in fact "finitely manageable" in some sense. Naturally the dimension 3 was far more complicated than the dimension 2,

and moreover some sort of singularities (seemingly finitely classifiable after Mori) were created in the process of factoring it into elementary transformations. Excited with Mori's discoveries, several algebraic geometers began clustering to him, this time with much clearer vision and better hope than ever, to work on factorization of birational transformations, and more importantly to work on defining what *minimal models* should be and how to obtain them, notably M. Reid, V. V. Shokurov, Y. Kawamata, J. Kollár and of course S. Mori himself. *Mori's Program*, named by J. Kollár and meant to imply the process of obtaining minimal models, was as follows: Start from any projective smooth variety X, which is not birationally uniruled, and find a finite succession of "elementary" birational transformations by which X is transformed to a "minimal model". Of course, the central questions here were the meanings of "elementary transformation" and "minimal model". Firstly, the inverse of a blowing-up with a smooth center in a smooth 3-fold (even in a 3-fold with a "mildly" singular point) was definitely elementary. It was called a *divisorial contraction*. This certainly decreased the Néron–Severi number and hence it should make the variety closer to its minimal model. Suppose we could not make a divisorial contraction any more and the Néron–Severi number reached its minimum. Is the variety then good enough to be called a minimal model? The answer was clearly *no*, although the singularities created by those divisorial contractions were quite acceptable. Unlike the 2-dimensional cases, the resulting 3-fold can still have *extremal rays* which cause unpleasant behaviors of the canonical and pluricanonical bundles. The importance of these bundles had been clearly recognized even from the time of Castelnuovo and Enriques in the theory of classification problems as well as in the theory of deformations. Thus, to *eliminate an extremal ray*, a completely new type of elementary transformation was needed. This was the one later called *flip*, which was, roughly speaking, to take out extremal (canonically negative, or with negative intersection number with the canonical divisor) rational curves and put some rational curves back in with a new imbedding type so that new curves are canonically positive. Such a surgery type operation is unique if it exists. The *existence*, however, turned out to be extremely delicate and hard to prove.

As early as in 1981, encouraged by Mori's "not nef" paper, Miles Reid made fairly clear the idea of what minimal models should be and what elementary transformations would be, by publishing a paper on "Minimal models of canonical 3-folds", which was an expansion of his lecture "Canonical 3-folds" in the Journées de Géométrie Algébrique d'Angers 1980. It then became absolutely clear (mildly suggested by Mori's work before) that some special kind of singularities must be permitted in the good notion of minimal models. M. Reid introduced the notions of *canonical singularity* and *terminal singularity*, which turned out to be very useful in the minimal model program of which he was the first to conjecture in literature. (See Miles Reid "Decomposition of Toric morphisms" 1983.) The former was, as the name suggests, the kind of singularity that appeared in the canonical models and could be quite horrible algebraically and geometrically. The latter was the kind that behaved much better than the former and which people hoped to be the only

kind to appear in the minimal models. In any case, Reid's definition was simple and clear and some basic theorems were proved by himself, in which he showed that terminal singularities were closely related to the deformations of the classic singularities of Du Val.

Historically, P. Du Val in 1934 systematically studied the singularities that did not affect the condition of adjunction, that is, in the language of Reid, the canonical singularities in dimension 2. In 1966, M. Artin extended and modernized the classification of what he called *rational singularities*. Du Val singularities were exactly rational singularities of multiplicity 2, or *rational double points*. After M. Reid's introduction of canonical and terminal singularities, rational singularities and their deformations were studied once again. Extensive and direct studies on terminal singularities followed M. Reid's works and by 1987 a complete classification of terminal singularities in dimension 3 was achieved with technically useful lemmas by a combination of works by several mathematicians, V. I. Danilov, D. Morrison, G. Stevens, S. Mori, J. Kollár, N. Shepherd–Barron, and others. Some experimental works were done about higher dimensional terminal singularities, such as the one in dimension 4 by S. Mori, D. Morrison and I. Morrison which seemed to indicate far more complexity than the case of dimension 3.

Back to the minimal model problem, Mori's technique of *extremal ray contraction* had to be generalized to singular varieties, at least to the varieties with only terminal singularities, which were even "rationally factorial". It does not look completely hopeless that eventually Mori's all characteristics method can be modified and extended to such singular cases. (See J. Kollár work that is to appear in the proceedings of the Algebraic Geometry Satellite Conference at Tokyo Metropolitan University, 1990.) However it then was not done in Mori's way. Instead, the generalization was obtained first by generalizing Kodaira's Vanishing Theorem and then by making an ingenious use of this generalization. Here, and subsequently, the contributions of Y. Kawamata were big to the minimal model program.

The celebrated *Vanishing Theorem* was proven by K. Kodaira in 1953. Numerous generalizations, in special cases or in general, were done especially during the 1970s and 80s mostly in an effort to investigate the nicety of the structure of the canonical ring of a projective variety. Here the *canonical ring* means the graded algebra generated by sections of pluricanonical bundles (say, on a desingularized model). As for the nicety, they looked for the property of *being finitely generated* of the canonical algebra or "rationality or better" properties of the singularities of the canonical model. Y. Kawamata had a very clear objective in generalizing and using the Vanishing Theorem, that was to generalize Mori's theory of extremal ray contractions and then verify the minimal model program.

The operation called *flip* is clearly directed, i.e., it changes "canonically negative" to "canonically positive". There is a similar operation for a rational curve with zero intersection number with the canonical divisor, which is called a *flop*. Unlike flips, the inverse of flops are again flops. A flop is symmetric in this sense. A flip changes a variety into another, birational to and better than the original, better

in singularities, better in terms of pluricanonical bundles, and so on. Historically, there had been known many examples of flops. The simplest flop, between smooth 3-folds, was known and used even by earlier algebraic geometers. Some flops were shown to be useful in studying degeneration of $K3$ surfaces by V. Kulikov in 1977. In contrast, examples of flips are not so easy to find because 3-folds had to be singular in order to have singularities improved. P. Francia gave an explicit example in his paper published in 1980. Other examples were seen in a paper of M. Reid published in 1983, here with prototypes of minimal models. In 1983, V. V. Shokurov published the *Non-vanishing Theorem*, which was proven by using the Vanishing Theorem. His theorem implied that flips cannot be done infinitely many times.

For Mori's Program, therefore, not only the divisorial contractions are finite but also the flips are finite in any sequence. The very final problem was hence to prove the *existence* of flips for given extremal rational curves, after all the previous works by S. Mori, M. Reid, J. Kollár, V. V. Shokurov and Y. Kawamata, just to name some of the most important contributors to the Program. At any rate, Y. Kawamata reduced the problem of the flip to the existence of a "nice" doubly anticanonical divisor *globally* in a neighborhood of a given extremal rational curve, while the existence had been only proven by M. Reid *locally* about each point of the curve. This seemingly small gap between *global* and *local* was actually enormous. Mori finally overcame this gap by checking cases after cases with very delicate and intricate investigations and established the final existence theorem of algebraic flips by reducing them to sequences of simpler analytic flips and smaller contractions. This monumental paper of Mori was published in the very first issue of the new Journal of the American Mathematical Society, establishing a constructive existence theorem of minimal models which had been shown to have many important consequences.

We need much more of Mori's originality to break the stubborn prejudice that it is a Herculean task to extend the classical classification theorems to all dimensions.

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BIRATIONAL CLASSIFICATION OF ALGEBRAIC THREEFOLDS

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1. Introduction

Let us begin by explaining the background of the birational classification. We will work over the field $\mathbb{C}$ of complex numbers unless otherwise mentioned.

Let X be a non-singular projective variety of dimension r. The canonical divisor class K_X is the only divisor class (up to multiples) naturally defined on an arbitrary X. Its sheaf $\mathcal{O}_X(K_X)$ is the sheaf of holomorphic r-forms. An alternative description is $K_X = -c_1(X)$, where $c_1(X)$ is the first Chern class of X. Therefore it is natural to expect some role of K_X in the classification of algebraic varieties.

The classification of non-singular projective curves C is classical, and summarized in the following table, where $g(C)$ is the genus (the number of holes) of $C, H = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$ and Γ is a subgroup of $SL_2(\mathbb{R})$:

(1.1)

$g(C)$	0	1	≥ 2
$\deg K_C$	-2	0	$2g(C)-2$
C	$\mathbb{P}^1$	$\mathbb{C}/(\text{lattice})$	H/Γ

Here we see three different situations. For instance, everything is explicit if $g(C) = 0$; the moduli (to parametrize curves) is the main interest if $g(C) \geq 2$.

Our interest is in generalizing this to higher dimensions. The first difficulty which arises in surface case is that there are too many varieties for genuine classification (*biregular classification*).

(1.2) For a non-singular projective surface X and an arbitrary point $x \in X$, there is a birational morphism $\pi : B_x X \to X$ from a non-singular projective surface $B_x X$ such that $E = \pi^{-1}(x)$ is isomorphic to $\mathbb{P}^1$ (E is called a (-1)-*curve*) and π induces an isomorphism $B_x X - E \simeq X - x$.

In view of (1.2), it is impractical to distinguish X from $B_x X, B_y B_x X, \ldots$ if we want a reasonable classification list. More generally, we say that two algebraic varieties X and Y are *birationally equivalent* and we write $X \sim Y$ if there is a

birational mapping $X \cdots \to Y$ or equivalently if their rational function fields $\mathbb{C}(X)$ and $\mathbb{C}(Y)$ are isomorphic function fields over $\mathbb{C}$. We did not face this phenomenon in curve case, since $X \simeq Y$ iff $X \sim Y$ for curves X and Y.

In view of the list (1.1) for curves, we need to divide the varieties into several classes to formulate more precise problems. This is why the *Kodaira dimension* $\kappa(X)$ of a non-singular projective variety X was introduced by [Iitaka1] and [Moishezon].

(1.3) Let $H^0(X, \mathcal{O}(\nu K_X))$ be the space of global ν-ple holomorphic r-forms ($\nu \geq 0, r = \dim X$), and $\phi_0, \ldots, \phi_N$ be its basis. If $N \geq 0$, then

$$\Phi_{\nu K_X} : X \cdots \to \mathbb{P}^N \text{ given by } \Phi_{\nu K}(x) = (\phi_0(x) : \cdots : \phi_N(x))$$

is a rational map. We set $P_\nu(X) = N + 1$. It is important that $H^0(X, \mathcal{O}(\nu K_X))$ and $\Phi_{\nu K}$ are birational invariants, that is $X \sim Y$ induces $H^0(X, \mathcal{O}(\nu K_X)) = H^0(Y, \mathcal{O}(\nu K_Y))$ for $\nu > 0$. We set $\kappa(X) = -\infty$ if $P_\nu(X) = 0$ for all $\nu > 0$. If $P_e(X) > 0$ for some $e > 0$, then

$$\kappa(X) := \mathrm{Max}\{\dim \Phi_{\nu K_X}(X) \mid \nu > 0\}\,.$$

In particular, $P_\nu(X)$ and $\kappa(X)$ are birational invariants of X.

We remark that $\kappa(X) \in \{-\infty, 0, 1, \ldots, \dim X\}$, and that X with $\kappa(X) = \dim X$ is said to be *of general type*. We have the following table for curves.

(1.4)

$g(C)$	0	1	≥ 2
$\kappa(C)$	$-\infty$	0	1

To have some idea on higher dimension, we can use the easy result $\kappa(X \times Y) = \kappa(X) + \kappa(Y)$. In particular,

(1.5) case ($\kappa(X) = -\infty$) $\qquad \kappa(\mathbb{P}^1 \times Y) = -\infty\,,$

(1.6) case ($0 < \kappa(X) < \dim X$)

$$\kappa(\underbrace{E \times \cdots \times E}_{a \text{ times}} \times \underbrace{C \times \cdots \times C}_{b \text{ times}}) = b \text{ if } g(E) = 1 \text{ and } g(C) \geq 2\,.$$

The case $0 < \kappa(X) < \dim X$ is studied by the Iitaka fibration.

(1.7) **Iitaka Fibering Theorem** [Iitaka2]. *Let X be a non-singular projective variety with $0 < \kappa(X) < \dim X$. Then there is a morphism $f : X' \to Y'$ of non-singular projective varieties with connected fibers such that $X' \sim X$, $\dim Y' = \kappa(X)$ and $\kappa(f^{-1}(y)) = 0$ for a sufficiently general point $y \in Y'$.*

In (1.7), we cannot expect $\kappa(Y') = \dim Y'$ or even $\kappa(Y') \geq 0$. Therefore X' is not so simple as (1.6). Nevertheless (1.7) reduces the case $0 < \kappa(X) < \dim X$ to the cases $\kappa(X) = -\infty, 0, \dim X$. Thus we can explain the birational classification as in (1.1) for higher dimensions.

2. Birational Classification

For a non-singular projective variety X, we define a graded ring (called the *canonical ring*)

$$R(X) = \oplus_{\nu \geq 0} H^0(X, \mathcal{O}(\nu K_X)) .$$

If $\kappa(X) \geq 0$, the ν-canonical image $\Phi_{\nu K}(X)$ is a birational invariant of X. The existence of stable canonical image is interpreted in terms of $R(X)$ by the following easy proposition.

(2.1) **Proposition.** *Let X be a non-singular projective variety of $\kappa(X) \geq 0$. Then $\Phi_{\nu K}(X)$ for sufficiently divisible $\nu > 0$ are all naturally isomorphic iff $R(X)$ is a finitely generated $\mathbb{C}$-algebra.*

(2.2) For X of general type, constructing moduli spaces is one of our main interests. One standard way is to try to find a uniform ν such that $\Phi_{\nu K} : X \cdots \to \Phi_{\nu K}(X)$ is birational for all X and classify the image. One can expect nice properties of the image (canonical model to be explained later) if there is a stable canonical image. Therefore we would like to ask whether the canonical ring is finitely generated for X of general type (2.1).

(2.3) The case $\kappa(X) = \dim X$ suggests to reduce the birational classification of all varieties to the biregular classification of standard models (like $\Phi_{\nu K}(X)$ for sufficiently divisible ν). However, when $\kappa(X) < \dim X$, there are no obvious candidates for the standard models. For $0 \leq \kappa(X)$, we can ask to find some "standard" models.

We only say the following for $\kappa(X) \leq 0$ at this point.

(2.4) For X with $\kappa(X) = 0$, we would like to find some "standard" model $Y \sim X$ and to classify all such Y.

(2.5) For many X with $\kappa(X) = -\infty$, there exist infinitely many "standard" models $\sim X$. To study the relation among these models is a role of birational geometry. We would like to have a structure theorem of such models. One general problem is to see if all such X are *uniruled*, i.e. there exists a rational curve through an arbitrary point of X, or equivalently there is a dominating rational map $\mathbb{P}^1 \times Y \cdots \to X$ for some Y of dimension $n-1$. (It is easy to see that uniruled varieties have $\kappa = -\infty$ as in (1.5).)

Since we use the formulation by Iitaka and Moishezon, one basic problem will be the deformation invariance of κ.

(2.6) **Conjecture** [Iitaka1, Moishezon]. *Let $f : X \to Y$ be a smooth projective morphism with connected fibers and connected Y. Then $\kappa(f^{-1}(y))$ and $P_\nu(f^{-1}(y))$ $(\nu \geq 1)$ are independent of $y \in Y$.*

3. Surface Case

We review a few classical results on surfaces which may help the reader to understand the results for 3-folds.

The basic result is the inverse process of (1.2).

(3.1) **Castelnuovo-Enriques.** *Let E be a curve on a non-singular projective surface X'. Then E is a (-1)-curve (i.e. $X' = B_x X$ and E is the inverse image of x for some non-singular projective surface X and $x \in X$) iff $E \simeq \mathbb{P}^1$ and $(E \cdot K_{X'}) = -1$. We write $\mathrm{cont}_E : X' \to X$ and call it the contraction of the (-1)-curve E.*

Finding a (-1)-curve in every exceptional set, we have the following:

(3.2) **Factorization of Birational Morphisms.** *Let $f : X \to Y$ be a birational morphism of non-singular projective surfaces. Then f is a composition of a finite number of contractions of (-1)-curves.*

Starting with a non-singular projective surface X, we can keep contracting (-1)-curves if there are any. After a finite number of contractions, we get a non-singular projective surface $Y(\sim X)$ with no (-1)-curves. Depending on whether K_Y is *nef* ($(K_Y \cdot C) \geq 0$ for all curves C), $\kappa(X)$ takes different values.

(3.3) Case where K_Y is nef. Then Y is the only non-singular projective surface $\sim X$ with no (-1)-curves. To be precise, if Y' is a such surface, then the composite $Y \cdots \to X \cdots \to Y'$ is an isomorphism. This Y is called the *minimal model* of X and denoted by $X_{\min}$. In this case, we have $\kappa(X) \geq 0$.

(3.4) Case where K_Y is not nef. Then an arbitrary Y' (including Y) which is birational to X and has no (-1)-curves is isomorphic to either $\mathbb{P}^2$ or a $\mathbb{P}^1$-bundle over some non-singular curve. In this case, X has no minimal models and we have $\kappa(X) = -\infty$ by (1.5).

The above (3.3) together with (3.4) says that the birational classification of X with $\kappa \geq 0$ is equivalent to the biregular classification of minimal models.

Based on (3.3) and (3.4), the canonical model is defined.

(3.5) Let X be a non-singular projective surface of general type. Then there exists exactly one normal projective surface $Z(\sim X)$ such that Z has only Du Val (rational double) point and K_Z is ample, where Du Val points are defined by one of the following list.

$$
\begin{aligned}
&A_n : xy + z^{n+1} = 0 \ (n \geq 0)\,,\\
&D_n : x^2 + y^2 z + z^{n-1} = 0 \ (n \geq 4)\,,\\
&E_6 : x^2 + y^3 + z^4 = 0\,,\\
&E_7 : x^2 + y^3 + yz^3 = 0\,,\\
&E_8 : x^2 + y^3 + z^5 = 0\,.
\end{aligned}
$$

Such Z is called the *canonical model* of X and denoted by X_{can}. The natural map $X_{\min} \cdots \to X_{\text{can}}$ is a morphism which contracts all the rational curves C with $(C \cdot K_{X_{\min}}) = 0$ into Du Val points and is isomorphic elsewhere.

(3.5.1) **Remark.** This X_{can} can also be obtained as $\Phi_{\nu K}(X_{\min}) = \Phi_{\nu K}(X)$ for an arbitrary $\nu \geq 5$ (Bombieri).

(3.6) Let X be a non-singular projective minimal surface with $\kappa = 0$. Thus X has torsion K_X, i.e. some non-zero multiple of it is trivial. There is a precise classification of all such X.

(3.7) The deformation invariance of $\kappa(x)$ and $P_\nu(X)$ was done by [Iitaka3] using the classification of surfaces. [Levine] gave a simple proof without using classification.

4. The Extremal Ray Theory (The Minimal Model Theory)

The first problem in generalizing the results in Sec. 3 to higher dimensions is to find some class of varieties in which there is a reasonable contraction theorem because there is no immediate generalization of (3.1) to 3-folds, since the contraction process inevitably introduces singularities [Mori1]. To define the necessary class of singularities, the first important step was taken by Reid [Reid1,3].

(4.1) **Definition** [Reid3]. Let (X, P) be a normal germ of an algebraic variety (or an analytic space) which is normal. We say that (X, P) has *terminal singularities* (resp. *canonical singularities*) iff

(i) K_X is a $\mathbb{Q}$-*Cartier* divisor, i.e. rK_X is Cartier for some positive integer r (minimal such r is called the *index* of (X, P)), and

(ii) for some (or equivalently, every) resolution $\pi : Y \to (X, P)$, we have $a_i > 0$ (resp. $a_i \geq 0$) for all i in the expression:

$$rK_Y = \pi^*(rK_Y) + \sum a_i E_i \,,$$

where E_i are all the exceptional divisors and $a_i \in \mathbb{Z}$.

For surfaces, a terminal (resp. canonical) singularity is smooth (resp. a Du Val point). We note that, for projective varieties X with only canonical singularities, the same definitions of $P_\nu(X)$, $\Phi_{\nu K}$ and $\kappa(X)$ work and these are still birational invariants. We can also talk about the ampleness of K_X and the intersection number $(K_X \cdot C) \in \mathbb{Q}$ for such X.

The idea of the cone of curves which is the core of the extremal ray theory was first introduced in Hironaka's thesis [Hironaka].

(4.2) **Definition.** Let X be a projective n-fold. A 1-cycle $\sum a_C C$ is a formal finite sum of irreducible curves C on X with coefficients $a_C \in \mathbb{Z}$. For a 1-cycle Z and a $\mathbb{Q}$-Cartier divisor D, the intersection number $(Z \cdot D) \in \mathbb{Q}$ is defined. Then

$$N_1(X)_{\mathbb{Z}} = \{\text{ 1-cycles }\}/\{\text{ 1-cycles } Z \mid (Z \cdot D) = 0 \text{ for all } D\}$$

is a free abelian group of finite rank $\rho(X) < \infty$. Thus $N_1(X) = N_1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$ is a finite dimensional Euclidean space. The classes $[C]$ of all the irreducible curves C span a convex cone $NE(X)$ in $N_1(X)$. Taking the closure for the metric topology, we have a closed convex cone $\overline{NE}(X)$. Then

(4.3) **Cone Theorem.** *If X has only canonical singularities, then there exist countably many half lines $R_i \subset NE(X)$ such that*

(i) $\overline{NE}(X) = \sum_i R_i + \{z \in \overline{NE}(X) \mid (z \cdot K_X) \geq 0\}$,
(ii) *for an arbitrary ample divisor H of X and arbitrary $\varepsilon > 0$, there are only finitely many R_i's contained in*

$$\{z \in \overline{NE}(X) \mid (z \cdot K_X) \leq -\varepsilon(z \cdot H)\}\,.$$

Such an R_i is called an *extremal ray* of X if it cannot be ommitted in (i) of (4.3). We note that an extremal ray exists on X iff K_X is not nef. Each extremal ray R_i defines a contraction of X.

(4.4) **Contraction Theorem.** *Let R be an extremal ray of a projective n-fold X with only canonical singularities. Then there exists a morphism $f : X \to Y$ to a projective variety Y (unique up to isomorphism) such that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and an irreducible curve $C \subset X$ is sent to a point by f iff $[C] \in R$. Furthermore* $\mathrm{Pic}Y = \mathrm{Ker}[(C\cdot) : \mathrm{Pic}X \to \mathbb{Z}]$ *for such a contracted curve C. This f is called the contraction of R and denoted by* cont_R.

The contraction of an extremal ray is not always birational.

(4.5) *Let X be a smooth projective surface with an extremal ray R. Then* cont_R *is one of the following.*

(i) *the contraction of a (-1)-curve,*
(ii) *a $\mathbb{P}^1$-bundle structure $X \to C$ over a non-singular curve,*
(iii) *a morphism to one point, when $X \simeq \mathbb{P}^2$.*

The description of all the possible contractions for a nonsingular projective 3-fold X is given in [Mori1]. Here we only remark that $\mathrm{cont}_R X$ can have a terminal singularity $\mathbb{C}^3/<\sigma>$ of index 2, where σ is an involution $\sigma(x, y, z) = (-x, -y, -z)$.

(4.6) The category of varieties in which we play the game of the minimal model program is the category $\mathcal{C}$ of projective varieties with only terminal singularities which are $\mathbb{Q}$ -*factorial* (i.e. every Weil divisor is $\mathbb{Q}$-Cartier). The goal of the game is to get a *minimal* (resp. *canonical*) model, i.e. a projective n-fold X with only terminal (resp. canonical) singularities such that K_X is nef (resp. ample). Let us first state the minimal model program which involves two conjectures.

(4.7) Let X be an n-fold $\in \mathcal{C}$. If K_X is nef, then X is a minimal model and we are done. Otherwise, X has an extremal ray R. Then $\mathrm{cont}_R : X \to X'$ satisfies one of the following.

(4.7.1) Case where $\dim X' < \dim X$. Then cont_R is a surjective morphism with connected fibers of dimension > 0 and relatively ample $-K_X$ (like $\mathbb{P}^1$-bundle), and X is uniruled ([Miyaoka-Mori]). This is the case where we can never get a minimal model, and we stop the game since we have the global structure of X, $\mathrm{cont}_R : X \to X'$.

(4.7.2) Case where $\mathrm{cont}_R : X \to X'$ is birational and contracts a divisor. This cont_R is called a *divisorial contraction*. In this case $X' \in \mathcal{C}$ and $\rho(X') < \rho(X)$. Therefore we can work on X' instead of X.

(4.7.3) Case where $\mathrm{cont}_R : X \to X'$ is birational and contracts no divisors. In this case, $K_{X'}$ is not $\mathbb{Q}$-Cartier and $X' \notin \mathcal{C}$. So we cannot continue the game with X'. This is the new phenomenon in dimension ≥ 3.

To get around the trouble in (4.7.3) and to continue the game, Reid proposed the following.

(4.8) **Conjecture (Existence of Flips).** *In the situation of* (4.7.3), *there is an n-fold $X^+ \in \mathcal{C}$ with a birational morphism $f^+ : X^+ \to X'$ which contracts no divisors and such that K_{X^+} is f^+-ample. The map $X \cdots \to X^+$ is called a flip.*

Since $\rho(X^+) = \rho(X)$ in (4.8), the divisorial contraction will not occur for infinitely many times. Therefore the following will guarantee that the game will be over after finitely many steps.

(4.9) **Conjecture (Termination of Flips).** *There does not exist an infinite sequence of flips $X_1 \cdots \to X_2 \cdots \to \cdots$.*

Therefore the minimal model program is completed only when the conjectures (4.8) and (4.9) are settled affirmatively.

The conjecture (4.9) was settled affirmatively by [Shokurov1] for 3-folds and by Kawamata-Matsuda-Matsuki [KMM] for 4-folds. (4.8) was first done by [Tsunoda], [Shokurov2], [Mori3] and [Kawamata6] in a special but important case. Finally (4.8) was done for 3-folds by [Mori5] using the work of [Kawamata6] mentioned above.

(4.10) Thus for 3-folds, we can operate divisorial contractions and flips for a finite number of times and get either a minimal model $\in \mathcal{C}$ or an $X \in \mathcal{C}$ which has an extremal ray R of type (4.7.1). Thus we can get 3-fold analogues of results in Sec. 3.

(4.11) For simplicity of the exposition, we did not state the results in the strongest form and we even omitted various results. Therefore we would like to mention names and give a quick review.

After the prototype of the extremal ray theory was given in [Mori1], the theory has been generalized to the relative setting with a larger class of singularities (toward the conjectures of Reid [Reid3,4]) by Kawamata, Benveniste, Reid, Shokurov and Kollár (in the historical order) and perhaps some others. First through the works of [Benveniste] and [Kawamata2], Kawamata introduced a technique [Kawamata3] which was an ingenious application of the Kawamata-Viehweg vanishing ([Kawamata1] and [Viehweg2]). Based on the works by [Shokurov1] (Non-vanishing theorem) and [Reid2] (Rationality theorem), [Kawamata4] developed the technique to prove Base point freeness theorem (and others) in arbitrary dimensions. The discreteness of the extremal rays was later done by [Kollár1]. As for this section, we refer the reader to the talk of Kawamata.

5. Applications of the Minimal Model Program (MMP) to 3-Folds

Considering MMP in relative setting, one has the factorization generalizing (3.2):

(5.1) **Theorem.** *Let $f : X \to Y$ be a birational morphism of projective 3-folds with only $\mathbb{Q}$-factorial terminal singularities. Then f is a composition of divisorial contractions and flips.*

Since minimal 3-folds have $\kappa \geq 0$ by the hard result of Miyaoka [Miyaoka1-3], one has the following (cf. (3.3) and (3.4)).

(5.2) **Thoerem.** *A 3-fold X has a minimal model iff $\kappa(X) \geq 0$.*

Unlike the surface case, the minimal model of a 3-fold X is not unique; it is unique only in codimension 1. If we are given a $\mathbb{Q}$-factorial minimal model $X_{\min}$, every other $\mathbb{Q}$-factorial minimal models of X are obtained from $X_{\min}$ by operating a simple operation called a *flop* for a finite number of times ([Kawamata6], [Kollár4]). Many important invariants computed by minimal models do not depend on the choice of the minimal model. We refer the reader to the talk of Kollár.

(5.3) **Theorem.** *For a 3-fold X, the following are equivalent.*

(i) $\kappa(X) = -\infty$,
(ii) *X is uniruled,*
(iii) *X is birational to a projective 3-fold Y with only $\mathbb{Q}$-factorial terminal singularities which has an extremal ray of type* (4.7.1).

It will be an important but difficult problem to classify all the possible Y in (iii) of (5.3). There are only finitely many families of such Y with $\rho(Y) = 1$ ([Kawamata7]).

Since a canonical model exists if a minimal model does ([Benveniste] and [Kawamata2]), one has the following (cf. (3.5)).

(5.4) **Theorem.** *If X is a 3-fold of general type, then X has a canonical model and the canonical ring $R(X)$ is a finitely generated $\mathbb{C}$-algebra.*

The argument for (5.4) can be considered as a generalization of the argument for (3.5.1). However the effective part "$\nu \geq 5$" of (3.5.1) has not yet been generalized to dimension ≥ 3.

To study varieties X with $\kappa \geq 0$, [Kawamata4] posed the following.

(5.5) **Conjecture (Abundance Conjecture).** *If X is a minimal variety, then rK_X is base point free for some $r > 0$.*

For 3-folds, there are works by [Kawamata4] and [Miyaoka4] (cf. [KMM]). However the torsionness of K for minimal 3-folds with $\kappa = 0$ is the unsolved, and it remains to prove:

(5.6) **Problem.** *Let X be a minimal 3-fold with H an ample divisor such that $(K_X^3) = 0$ and $(K_X^2 \cdot H) > 0$. Then prove that $\kappa(X) = 2$.*

(5.7) **Remark** ($\kappa = 0$). The 3-folds X with $\kappa(X) = 0$ and $H^1(X, \mathcal{O}_X) \neq 0$ were classified by [Viehweg1] and (5.5) holds for these. This was based on Viehweg's solution of the addition conjecture for 3-folds, and we refer the reader to [Iitaka4]. However not much is known about the 3-folds X with $\kappa(X) = 0$ (or even K_X torsion) and $H^1(X, \mathcal{O}_X) = 0$: so far many examples have been constructed and it is not known if there are only finitely many families. There is a conjecture of [Reid6] in this direction.

By studying the flips more closely, [Kollár-Mori] proved the deformation invariance of κ and P_ν (cf. (3.7)):

(5.8) **Theorem.** *Let $f : X \to \Delta$ (unit disk) be a projective morphism whose fibers are connected 3-folds with only $\mathbb{Q}$-factorial terminal singularities. Then*

(i) *$\kappa(X_t)$ is independent of $t \in \Delta$, where $X_t = f^{-1}(t)$,*
(ii) *$P_\nu(X_t)$ is independent of $t \in \Delta$ for all $\nu \geq 0$ if $\kappa(X_0) \neq 0$.*

Indeed for such a family X/Δ, the simultaneous minimal model program is proved and the (modified) work of [Levine] is used to prove (5.8). We cannot drop the condition "$\kappa(X_0) \neq 0$" at present since the abundance conjecture is not completely solved for 3-folds.

As for other applications (e.g. addition conjecture, deformation space of quotient surface singularities, birational moduli), we refer the reader to [KMM] and [Kollár6].

6. Comments on the Proofs for 3-Folds

Many results on 3-folds are proved by using only the formal definitions of terminal singularities. However some results on 3-folds rely on the classification of

3-fold terminal singularities [Reid3], [Danilov], [Morrison-Stevens], [Mori2] and [KSB] (cf. Reid's survey [Reid5] and [Stevens].) The existence of flips and flops rely heavily on it. Thus generalizing their proofs to higher dimension seems hopeless. At present, there is no evidence for the existence of flips in higher dimensions except that they fit in the MMP beautifully. I myself would accept them as working hypotheses. A more practical problem will be to complete the log-version of the minimal model program for 3-folds [KMM]. This is related to the birational classification of open 3-folds and n-folds with $\kappa = 3$. Since log-terminal singularities have no explicit classification, this might be a good place to get some idea on higher dimension. Shokurov made some progress in this direction [Shokurov3].

There are two other results relying on the classification.

(6.1) **Theorem** [Mori4]. *Every* 3*-dimensional termal singularity deforms to a finite sum of cyclic quotient terminal singularities* (*i.e. points of the form* $\mathbb{C}^3/\mathbb{Z}_r(1,-1,a)$ *for some relatively prime positive integers* a *and* r).

This was used in the Barlow-Fletcher-Reid plurigenus formula for 3-folds [Fletcher] and [Reid5] (cf. also [Kawamata5]). Given a 3-fold X with only terminal singularities, each singularity of X can be deformed to a sum of cyclic quotient singularities $\mathbb{C}^3/\mathbb{Z}_r(1,-1,a)$. Let $S(X)$ be the set of all such (counted with multiplicity). For each $P = \mathbb{C}^3/\mathbb{Z}_r(1,-1,a) \in S(X)$, we let

$$\phi_P(m) = (m - \{m\}_r)\frac{r^2-1}{12r} + \sum_{j=0}^{\{m\}_r - 1} \frac{\{aj\}_r(r - \{aj\}_r)}{2r}\,,$$

where $\{m\}_r$ is the integer $s \in [0, r-1]$ such that $s \equiv m (\mathrm{mod}\ r)$. For a line bundle L on X, let $\chi(L) = \sum_j (-1)^j \dim H^j(X, L)$ and let $c_2(X)$ be the second Chern class of X, which is well-defined since X has only isolated singularities. Then the formula is stated as the following.

(6.2) **The Barlow-Fletcher-Reid Plurigenus Formula.**

$$\chi(\mathcal{O}_X(mK_X)) = \frac{m(m-1)(2m-1)}{12}(K_X^3) + (1-2m)\chi(\mathcal{O}_X) + \sum_{P \in S(X)} \phi_P(m)\,,$$

$$\chi(\mathcal{O}_X) = -\frac{1}{24}(K_X \cdot c_2(X)) + \sum_{P \in S(X)} \frac{r^2-1}{24r}\,.$$

This is important for effective results on 3-folds (cf. Sec. 7).

(6.3) **Theorem** ([KSB]). *A small deformation of a* 3*-dimensional terminal singularity is terminal.*

This is indispensable in the construction of birational moduli. An open problem in this direction is

(6.4) **Problem.** *Is every small deformation of a* 3*-dimensional canonical singularity canonical?*

Since this remains unsolved, we cannot put an algebraic structure on

$$\{\text{canonical 3-folds}\}/\text{isomorphisms}\,.$$

7. Related Results

I would like to list some of the directions, which I could not mention in the previous sections. This is by no means exhaustive. For instance, I could not mention the birational automorphism groups (cf. [Iskovskih] for the works before 1983) due to my lack of knowledge.

(7.1) *Effective Classification.* The Kodaira dimension κ is not a simple invariant. For instance, we know that $\kappa(X) = -\infty$ iff $P_\nu(X) = 0 (\forall \nu > 0)$. Therefore $P_{12}(X) = 0$ was an effective criterion for a surface X to be ruled, while $\kappa(X) = -\infty$ was not. The 3-dimensional analogue is not known yet.

There are results by Kollár [Kollár2] in the case dim $H^1(X, \mathcal{O}_X) \geq 3$ (cf. [Mori4]). The Barlow-Fletcher-Reid plurigenus formula (6.2) is applied for instance to get $aK_X \sim 0$ with some effectively given $a > 0$ for 3-folds X with numerically trivial K_X by [Kawamata5] and [Morrison], and to get $P_{12}(X) > 0$ for canonical 3-folds X with $\chi(\mathcal{O}_X) \leq 1$ by [Fletcher].

(7.2) *Differential Geometry.* As shown by [Yau], there are differential geometric results (especially when K is positive) which seem out of reach of algebraic geometry. Therefore we welcome differential geometric approaches. In this direction is Tsuji's construction of Kähler-Einstein metrics on canonical 3-folds [Tsuji].

(7.3) *Characteristic p.* [Kollár5] generalized [Mori1] (extremal rays of smooth projective 3-folds over $\mathbb{C}$) to char p. This suggests the possibility of little use of vanishing theorems in MMP for 3-folds. A goal will be the MMP for 3-folds in char p. However even the classification of terminal singularities is open.

(7.4) *Mixed Characteristic Case.* One can ask about the extremal rays (and so on) for arithmetic 3-folds X/S. The methods of [Shokurov2] and [Tsunoda] might work, if X/S is semistable. In the general case, I do not know any results in this direction.

(7.5) *Analytic or Non-projective* 3*-Folds.* Studying analytic or non-projective 3-folds will require a substitute for the cone of curves modulo numerical equivalence. However analytic or non-projective minimal 3-folds can be handled by the flop [Kollár4]. There is a work of [Kollár6].

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THE WORK OF E. WITTEN

by
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It is a duty of the chairman of the Fields Medal Committee to appoint the speakers, who describe the work of the winners at this session. Professor M. Atiyah was asked by me to speak about Witten. He told me that he would not be able to come but was ready to prepare a written address. So it was decided that I shall make an exposition of his address adding my own comments. The full text of Atiyah's address is published separately.

Let me begin by the statement that Witten's award is in the field of Mathematical Physics.

Physics was always a source of stimulus and inspiration for Mathematics so that Mathematical Physics is a legitimate part of Mathematics. In classical time its connection with Pure Mathematics was mostly via Analysis, in particular through Partial Differential Equations. However quantum era gradually brought a new life. Now Algebra, Geometry and Topology, Complex Analysis and Algebraic Geometry enter naturally into Mathematical Physics and get new insights from it. And[1]

In all this large and exciting field, which involves many of the leading physicists and mathematicians in the world, Edward Witten stands out clearly as the most influential and dominating figure. Although he is definitely a physicist (as his list of publications clearly shows) his command of mathematics is rivalled by few mathematicians, and his ability to interpret physical ideas in mathematical form is quite unique. Time and again he has surprised the mathematical community by a brilliant application of physical insight leading to new and deep mathematical theorems.

Now I come to description of the main achievements of Witten. In Atiyah's text many references are given to Feynman Integral, so that I begin with a short and rather schematic reminding of this object.

In quantum physics the exact answers for dynamical problems can be expressed in a formal way as follows:

$$Z = \int e^{iA} \prod_x d\,\mu$$

[1]Small print type here and after refers to Atiyah's text.

where A is an action functional of local fields — functions of time and space variables x, running through some manifold M. The integration measure is a product of local measures for values of fields in a point x over all M. The result of integration Z could be a number or function of parameters defining the problem — coupling constants, boundary or asymptotical conditions, etc.

In spite of being an ill-defined object from the point of view of rigorous mathematics, Feynman functional integral proved to be a powerful tool in quantum physics. It was gradually realized that it is also a very convenient mathematical means. Indeed the geometrical objects such as loops, connections, metrics are natural candidates for local fields and geometry produces for them interesting action functionals. The Feynman integral then leads to important geometrical or topological invariants.

Although this point of view was expressed and exemplified by several people (e.g., A. Schvarz used 1-forms ω on a three-dimensional manifold with action $A = \int \omega d\omega$ to describe the Ray–Singer torsion) it was Witten who elaborated this idea to a full extent and showed the flexibility and universality of Feynman integral.

Now let me follow Atiyah in description of the main achievements of Witten in this direction.

1. Morse Theory

His paper [2] on supersymmetry and Morse theory is obligatory reading for geometers interested in understanding modern quantum field theory. It also contains a brilliant proof of the classic Morse inequalities, relating critical points to homology. The main point is that homology is defined via Hodge's harmonic forms and critical points enter via stationary phase approximation to quantum mechanics. Witten explains that "supersymmetric quantum mechanics" is just Hodge–de Rham theory. The real aim of the paper is however to prepare the ground for supersymmetric quantum field theory as the Hodge–de Rham theory of infinite-dimensional manifolds. It is a measure of Witten's mastery of the field that he has been able to make intelligent and skilful use of this difficult point of view in much of his subsequent work.

Even the purely classical part of his paper has been very influential and has led to new results in parallel fields, such as complex analysis and number theory.

2. Index Theorem

One of Witten's best known ideas is that the index theorem for the Dirac operator on compact manifolds should emerge by a formally exact functional integral on the loop space. This idea (very much in the spirit of his Morse theory paper) stimulated an extensive development by Alvarez–Gaumé, Getzler, Bismut and others which amply justified Witten's view-point.

3. Rigidity Theorems

Witten [7] produced an infinite sequence of such equations which arise naturally in the physics of string theories, for which the Feynman path integral provides a heuristic explanation of rigidity. As usual Witten's work, which was very precise and detailed in its formal aspects, stimulated great activity in this area, culminating in rigorous proofs of these new rigidity theorems by Bott and Taubes [1]. A noteworthy aspect of these proofs is that they involve elliptic function theory and deal with the infinite sequence of

operators simultaneously rather than term by term. This is entirely natural from Witten's view-point, based on the Feynman integral.

4. Knots

Witten has shown that the Jones invariants of knots can be interpreted as Feynman integrals for a 3-dimensional gauge theory [11]. As Lagrangian, Witten uses the Chern–Simons function, which is well-known in this subject but had previously been used as an addition to the standard Yang–Mills Lagrangian. Witten's theory is a major breakthrough, since it is the only intrinsically 3-dimensional interpretation of the Jones invariants: all previous definitions employ a presentation of a knot by a plane diagram or by a braid.

Although the Feynman integral is at present only a heuristic tool it does lead, in this case, to a rigorous development from the Hamiltonian point of view. Moreover, Witten's approach immediately shows how to extend the Jones theory from knots in the 3-sphere to knots in arbitrary 3-manifolds. This generalization (which includes as a specially interesting case the empty knot) had previously eluded all other efforts, and Witten's formulas have now been taken as a basis for a rigorous algorithmic definition, on general 3–manifolds, by Reshetikin and Turaev.

Now I turn to another beautiful result of Witten — proof of positivity of energy in Einstein's Theory of Gravitation.

Hamiltonian approach to this theory proposed by Dirac in the beginning of the fifties and developed further by many people has led to a natural definition of energy. In this approach a metric γ and external curvature h on a space-like initial surface $S^{(3)}$ embedded in space-time $M^{(4)}$ are used as parameters in the corresponding phase space. These data are not independent. They satisfy Gauss–Codazzi constraints – highly nonlinear PDE. The energy H in the asymptotically flat case is given as an integral of indefinite quadratic form of $\nabla\gamma$ and h. Thus it is not manifestly positive. The important statement that it is nevertheless positive may be proved only by taking into account the constraints — a formidable problem solved by Yau and Schoen in the late seventies and as Atiyah mentions, "leading in part to Yau's Fields Medal at the Warsaw Congress".

Witten proposed an alternative expression for energy in terms of solution of a linear PDE with the coefficients expressed through γ and h. This equation is

$$\mathcal{D}^{(3)}\psi = 0$$

where $\mathcal{D}^{(3)}$ is the Dirac operator induced on $S^{(3)}$ by the full Dirac operator on $M^{(4)}$. Witten's formula somewhat schematically can be written as follows:

$$H(\psi_0, \psi_0) = \int (|\nabla\psi|^2 + \psi^* G\psi) d\, S$$

where ψ_0 is the asymptotic boundary value for ψ and G is proportional to the Einstein tensor $R_{ik} - \frac{1}{2} g_{ik} R$. Due to the equation of motion $G = T$, where T is the energy-momentum tensor of matter and thus manifestly positive. So the positivity of H follows.

This unexpected and simple proof shows another ability of Witten — to solve a concrete difficult problem by specific elegant means.

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THE WORK OF EDWARD WITTEN

by
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1. General

The past decade has seen a remarkable renaissance in the interaction between mathematics and physics. This has been mainly due to the increasingly sophisticated mathematical models employed by elementary particle physicists, and the consequent need to use the appropriate mathematical machinery. In particular, because of the strongly non-linear nature of the theories involved, topological ideas and methods have played a prominent part.

The mathematical community has benefited from this interaction in two ways. First, and more conventionally, mathematicians have been spurred into learning some of the relevant physics and collaborating with colleagues in theoretical physics. Second, and more surprisingly, many of the ideas emanating from physics have led to significant new insights in purely mathematical problems, and remarkable discoveries have been made in consequence. The main input from physics has come from quantum field theory. While the analytical foundations of quantum field theory have been intensively studied by mathematicians for many years the new stimulus has involved the more formal (algebraic, geometric, topological) aspects.

In all this large and exciting field, which involves many of the leading physicists and mathematicians in the world, Edward Witten stands out clearly as the most influential and dominating figure. Although he is definitely a physicist (as his list of publications clearly shows) his command of mathematics is rivalled by few mathematicians, and his ability to interpret physical ideas in mathematical form is quite unique. Time and again he has surprised the mathematical community by a brilliant application of physical insight leading to new and deep mathematical theorems.

Witten's output is remarkable both for its quantity and quality. His list of over 120 publications indicates the scope of his research and it should be noted that many of these papers are substantial works indeed.

In what follows I shall ignore the bulk of his publications, which deal with specifically physical topics. This will give a very one-sided view of his contribution, but it is the side which is relevant for the Fields Medal. Witten's standing as a physicist is for others to assess.

Let me begin by trying to describe some of Witten's more influential ideas and papers before moving on to describe three specific mathematical achievements.

2. Influential Papers

His paper [2] on supersymmetry and Morse theory is obligatory reading for geometers interested in understanding modern quantum field theory. It also contains a brilliant proof of the classic Morse inequalities, relating critical points to homology. The main point is that homology is defined via Hodge's harmonic forms and critical points enter via stationary phase approximation to quantum mechanics. Witten explains that "supersymmetric quantum mechanics" is just Hodge–de Rham theory. The real aim of the paper is however to prepare the ground for supersymmetric quantum field theory as the Hodge–de Rham theory of infinite-dimensional manifolds. It is a measure of Witten's mastery of the field that he has been able to make intelligent and skilful use of this difficult point of view in much of his subsequent work.

Even the purely classical part of this paper has been very influential and has led to new results in parallel fields, such as complex analysis and number theory.

Many of Witten's papers deal with the topic of "Anomalies". This refers to classical symmetries or conservation laws which are violated at the quantum level. Their investigation is of fundamental importance for physical models and the mathematical aspects are also extremely interesting. The topic has been extensively written about (mainly by physicists) but Witten's contributions have been deep and incisive. For example, he pointed out and investigated "global" anomalies [3], which cannot be studied in the traditional perturbative manner. He also made the important observation that the η-invariant of Dirac operators (introduced by Atiyah, Patodi and Singer) is related to the adiabatic limit of a certain anomaly [4]. This was subsequently given a rigorous proof by Bismut and Freed.

One of Witten's best known ideas is that the index theorem for the Dirac operator on compact manifolds should emerge by a formally exact functional integral on the loop space. This idea (very much in the spirit of his Morse theory paper) stimulated an extensive development by Alvarez–Gaumé, Getzler, Bismut and others which amply justified Witten's view-point.

Also concerned with the Dirac operator is a beautiful joint paper with Vafa [5] which is remarkable for the fact that it produces sharp uniform bounds for eigenvalues by an essentially topological argument. For the Dirac operator on an odd-dimensional compact manifold, coupled to a background gauge potential, Witten and Vafa prove that there is a constant C (depending on the metric, but

independent of the potential) such that *every interval of length C contains an eigenvalue*. This is not true for Laplace operators or in even dimensions, and is a very refined and unusual result.

3. The Positive Mass Conjecture

In General Relativity the positive mass conjecture asserts that (under appropriate hypotheses) the total energy of a gravitating system is positive and can only be zero for flat Minkowski space. It implies that Minkowski space is a stable ground state. The conjecture has attracted much attention over the years and was established in various special cases before being finally proved by Schoen and Yau in 1979. The proof involved non-linear P. D. E. through the use of minimal surfaces and was a major achievement (leading in part to Yau's Fields Medal at the Warsaw Congress). It was therefore a considerable surprise when Witten outlined in [6] a much simpler proof of the positive mass conjecture based on linear P. D. E. Specifically Witten introduced spinors and studied the Dirac operator. His approach had its origin in some earlier ideas of supergravity and it is typical of Witten's insight and technical skill that he eventually emerged with a simple and quite classical proof. Witten's paper stimulated both mathematicians and physicists in various directions, demonstrating the fruitfulness of his ideas.

4. Rigidity Theorems

The space of solutions of an elliptic differential equation on a compact manifold is naturally acted on by any group of symmetries of the equation. All representations of compact connected Lie groups occur this way. However, for very special equations, these representations are trivial. Notably this happens for the spaces of harmonic forms, since these represent cohomology (which is homotopy invariant). A less obvious case arises from harmonic spinors (solutions of the Dirac equation), although the relevant space here is the "index" (virtual difference of solutions of D and D^*). This was proved by Atiyah and Hirzebruch in 1970. Witten raised the question whether such "rigidity theorems" might be true for other equations of interest in mathematical physics, notably the Rarita–Schwinger equation. This stimulated Landweber and Stong to investigate the question topologically and eventually Witten [7] produced an infinite sequence of such equations which arise naturally in the physics of string theories, for which the Feynman path integral provides a heuristic explanation of rigidity. As usual Witten's work, which was very precise and detailed in its formal aspects, stimulated great activity in this area, culminating in rigorous proofs of these new rigidity theorems by Bott and Taubes [1]. A noteworthy aspect of these proofs is that they involve elliptic function theory and deal with the infinite sequence of operators simultaneously rather than term by term. This is entirely natural from Witten's view-point, based on the Feynman integral.

5. Topological Quantum Field Theories

One of the remarkable aspects of the Geometry/Physics interaction of recent years has been the impact of quantum field theory on low-dimensional geometry (of 2, 3 and 4 dimensions). Witten has systematized this whole areas by showing that there are, in these dimensions, interesting *topological* quantum field theories [8], [9], [10]. These theories have all the formal structure of quantum field theories but they are purely topological and have no dynamics (i.e. the Hamiltonian is zero). Typically the Hilbert spaces are finite-dimensional and various traces give well-defined invariants. For example, the Donaldson theory in 4 dimensions fits into this framework, showing how rich such structures can be.

A more recent example, and in some ways a more surprising one, is the theory of Vaughan Jones related to knot invariants, which has just been reported on by Joan Birman. Witten has shown that the Jones invariants of knots can be interpreted as Feynman integrals for a 3-dimensional gauge theory [11]. As Lagrangian, Witten uses the Chern–Simons function, which is well-known in this subject but had previously been used as an addition to the standard Yang–Mills Lagrangian. Witten's theory is a major breakthrough, since it is the only intrinsically 3-dimensional interpretation of the Jones invariants: all previous defintions employ a presentation of a knot by a plane diagram or by a braid.

Although the Feynman integral is at present only a heuristic tool it does lead, in this case, to a rigorous development from the Hamiltonian point of view. Moreover, Witten's approach immediately shows how to extend the Jones theory from knots in the 3-sphere to knots in arbitrary 3-manifolds. This generalization (which includes as a specially interesting case the empty knot) had previously eluded all other efforts, and Witten's formulas have now been taken as a basis for a rigorous algorithmic definition, on general 3-manifolds, by Reshetikin and Turaev.

Moreover, Witten's approach is extremely powerful and flexible, suggesting a number of important generalizations of the theory which are currently being studied and may prove to be important.

One of the most exciting recent developments in theoretical physics in the past year has been the theory of 2-dimensional quantum gravity. Remarkably this theory appears to have close relations with the topological quantum field theories that have been developed by Witten [12]. Detailed reports on these recent ideas will probably be presented by various speakers at this congress.

6. Conclusion

From this very brief summary of Witten's achievements it should be clear that he has made a profound impact on contemporary mathematics. In his hands physics is once again providing a rich source of inspiration and insight in mathematics. Of course physical insight does not always lead to immediately rigorous mathematical proofs but it frequently leads one in the right direction, and technically correct proofs can then hopefully be found. This is the case with Witten's work. So far

his insight has never let him down and rigorous proofs, of the standard we mathematicians rightly expect, have always been forthcoming. There is therefore no doubt that contributions to mathematics of this order are fully worthy of a Fields Medal.

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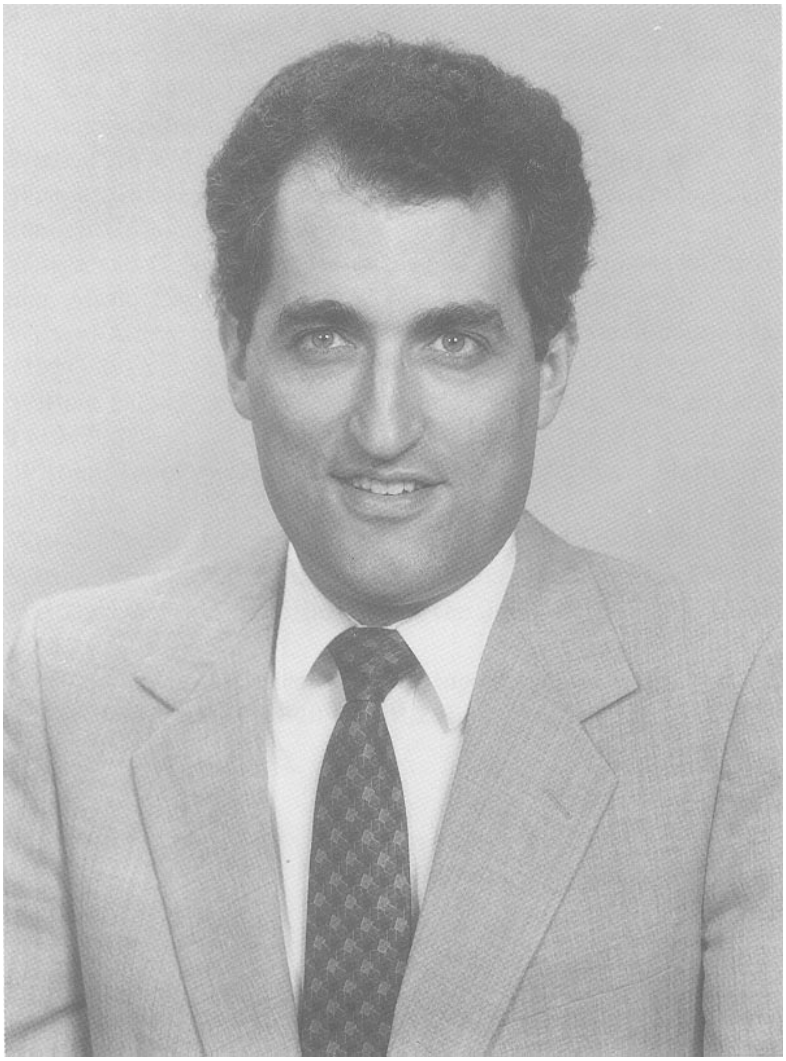

Edward Witten

Edward Witten

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Education:	BA Brandeis University	1971
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	Einstein Medal, Einstein Society of Berne, Switzerland	1985
	Award For Physical and Mathematical Sciences, New York Academy of Sciences	1985
	Dirac Medal, International Center For Theoretical Physics	1985
	Alan T. Waterman Award, National Science Foundation	1986
	Fields Medal, International Union of Mathematicians	1990
	Madison Medal, Princeton University	1992
	Member of the Board, Americans For Peace Now	Feb. 1992–present
	Fellow, American Academy of Arts and Sciences	1984
	Fellow, American Physical Society	1984
	Fellow, National Academy of Sciences	1988
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GEOMETRY AND QUANTUM FIELD THEORY

by
EDWARD WITTEN

1. Introduction. First of all, I would like to thank the American Mathematical Society for inviting me to lecture here on this occasion, and to thank the organizers for arranging such a stimulating meeting. And I would like to echo the sentiments of some previous speakers, who expressed the wish that we will all meet here in good health on the 150th anniversary of the American Mathematical Society, to hear the younger mathematicians explain the solutions of some of the unsolved problems posed this week.

It is a challenge to try to speak about the relation of quantum field theory to geometry in just one hour, because there are certainly many things that one might wish to say. The relationship between theoretical physics and geometry is in many ways very different today than it was just ten or fifteen years ago. It used to be that when one thought of geometry in physics, one thought chiefly of classical physics — and in particular of general relativity — rather than quantum physics. Geometrical ideas seemed (except perhaps to some visionaries) to be far removed from quantum physics — that is, from the bulk of contemporary physics. Of course, quantum physics had from the beginning a marked influence in many areas of mathematics — functional analysis and representation theory, just to mention two. But it would probably be fair to say that twenty years ago the day to day preoccupations of most practicing theoretical elementary particle physicists were far removed from considerations of geometry.

Several important influences have brought about a change in this situation. One of the principal influences was the recognition — clearly established by the middle 1970s — of the central role of nonabelian gauge theory in elementary particle physics. The other main influence came from the emerging study of supersymmetry and string theory. Of course, these different influences are inter-related, since nonabelian gauge theories have elegant supersymmetric generalizations, and in string theory these appear in a fascinating new light. Bit by bit, the study of nonabelian gauge theories, supersymmetry, and string theory have brought new questions to the fore, and encouraged new ways of thinking.

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An important early development in this process came in the period 1976–77 with the recognition that the Atiyah–Singer index theorem was the proper context for understanding some then current developments in the theory of strong interactions. (In particular, the solution by Gerard't Hooft [**1**] of the "U(1) problem," a notorious paradox in strong interaction theory, involved Yang–Mills instantons, originally introduced in [**2**], and "fermion zero modes" whose proper elucidation involves the index theorem.) Influenced by this and related developments, physicists gradually learned to think about quantum field theory in more geometrical terms. As a bonus, ideas coming at least in part from physics shed new light on some mathematical problems. In the first stage of this process, the purely mathematical problems that arose (at least, those that had motivations independent of quantum field theory, and in which progress could be made) involved "classical" mathematical concepts — partial differential equations, index theory, etc. — where physical considerations suggested new questions or a new point of view.

In the talk just before mine, Karen Uhlenbeck described some purely mathematical developments that at least roughly might be classified in this area. She described advances in geometry that have been achieved through the study of systems of nonlinear partial differential equations. Among other things, she sketched some aspects of Simon Donaldson's work on the geometry of four-manifolds [**3**], in which dramatic advances have been made by studying the moduli spaces of instantons — solutions, that is, of a certain nonlinear system of partial differential equations, the self-dual Yang–Mills equations, which were originally introduced by physicists in the context of quantum field theory [**2**].

If "classical" objects (such as instantons) that arise in quantum field theory could be so interesting mathematically, one might well suspect that mathematicians will soon find the quantum field theories themselves, and not only the "classical" objects that they give rise to, to be of interest. Such a question was indeed raised by Karen Uhlenbeck at the end of her talk, and is much in line with the perspective offered by Michael Atiyah in [**4**], which was the starting point for many of my own efforts.

I will talk today about three areas of recent interest where quantum field theory seems to be the right framework for thinking about a problem in geometry:

(1) Our first problem will be to explain the unexpected occurrence of modular forms in the theory of affine Lie algebras. This problem, which was described the other day by Victor Kac, has two close cousins — to explain "monstrous moonshine" in the theory of the Fischer–Griess monster group [**5, 6**], and to account for the surprising role of modular forms in algebraic topology [**7**], about which Raoul Bott spoke briefly at the end of his talk. Quantum field theory supplies a more or less common explanation for these three phenomena, but the first requires the least preliminary explanation, and it is the one that I will focus on.

(2) The second problem is to give a geometrical definition of the new knot polynomials — the Jones polynomial and its generalizations — that have been

discovered in recent years. The essential properties of the Jones polynomial have been described to us the other day by Vaughn Jones.

(3) The third problem is to get a more general insight into Donaldson theory of four-manifolds — which was sketched in the last hour by Karen Uhlenbeck — and the closely related Floer groups of three-manifolds. Here again there are lower dimensional cousins, namely the Casson invariant of three-manifolds, Gromov's theory of maps of a Riemann surface to a symplectic manifold, and Floer's closely related work on fixed points of symplectic diffeomorphisms. But among these formally rather analogous subjects, I will concentrate on Donaldson/Floer theory.

2. Physical Hilbert Spaces and Transition Amplitudes. Let us sketch these three problems in a little more detail. In the first problem, one considers the group $\mathcal{L}G$ of maps $\mathbf{S}^1 \to G$, where G is a finite-dimensional compact Lie group, and $\mathbf{S}^1$ is the ordinary circle. The representations of $\mathcal{L}G$ with "good" properties, analogous to the representations of compact finite-dimensional groups, are the so-called integrable highest weight representations (see [**8, 9**] for introductions.) These representations are rigid (no infinitesimal deformations). From this it follows that any connected group of outer automorphisms of $\mathcal{L}G$ must act at least projectively on any integrable highest weight representation $\mathcal{R}$ of $\mathcal{L}G$. In fact, the group diff $\mathbf{S}^1$ of diffeomorphisms of $\mathbf{S}^1$ acts on $\mathcal{L}G$ by outer automorphisms and acts projectively on the integrable highest weight representations. Thus, in particular, the vector field $d/d\theta$ that generates an ordinary rotation of $\mathbf{S}^1$ is represented on $\mathcal{R}$ by some operator H.

One computes in such a representation the "character"

$$F_{\mathcal{R}}(q) = \mathrm{Tr}_{\mathcal{R}}\, q^H \tag{2.1}$$

(here q is a complex number with $|q| < 1$), and one finds this to be a modular function with a simple transformation law under a suitable congruence subgroup of the modular group. Setting $q = \exp(2\pi i\tau)$, the modular group is of course the group PSL(2,$\mathbb{Z}$) of fractional linear transformations

$$\tau \to \frac{a\tau + b}{c\tau + d}\,, \tag{2.2}$$

of the upper half-plane, with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. I will not enter here into the complicated question of exactly what kind of modular functions the characters (2.1) are. (One simple, general statement, which from one point of view is the statement that comes most directly from quantum field theory, is that the $F_{\mathcal{R}}(q)$, with $\mathcal{R}$ running over all highest weight representations of fixed "level," transform as a unitary representation of the full modular group PSL(2,$\mathbb{Z}$).)

To understand the significance of the modularity of the characters $F_{\mathcal{R}}(q)$, let us recall that the group SL(2,$\mathbb{Z}$) has a natural interpretation as the (orientation preserving) mapping class group of a two-dimensional torus $\mathbf{T}^2$. Thus, we interpret $\mathbf{T}^2$ as the quotient of the $x-y$ plane by the equivalence relations $(x,y) \sim (x+1,y)$ and $(x,y) \sim (x,y+1)$. Clearly, if a,b,c and d are integers such that $ad-bc=1$, the formula $(x,y) \to (ax+by, cx+dy)$ gives a diffeomorphism of $\mathbf{T}^2$ to itself, and every orientation preserving diffeomorphism of $\mathbf{T}^2$ is isotopic to a unique one of these. Thus, $\mathbf{SL}(2,\mathbb{Z})$ can be considered in this sense to arise as a group of diffeomorphisms of a two-dimensional surface.

Thus, while it is natural that the one-dimensional symmetry group diff $\mathbf{S}^1$ plays a role in the representation theory of the loop group $\mathcal{L}G$, the appearance of SL(2,$\mathbb{Z}$) means that in fact a kind of *two-dimensional symmetry* appears in this theory. Our first problem — modular moonshine in the theory of affine Lie algebras — is the problem of explaining the origin of this two-dimensional symmetry.

Now we move on to our second problem. A *braid* is a time dependent history of n points in $\mathbb{R}^2$, which are required, up to a permutation, to end where they begin (Figure 1(a)). Braids with n strands form a group, the Artin braid group $\mathcal{B}_n$, with an evident law of composition, sketched in Figure 1(b). From a braid, one can make a knot (or in general a link) by gluing together the top and bottom as in Figure 1(c). Although every braid gives in this way a unique link, the converse is not so; the same link may arise from many different braids. The crucial difference between braids and links is the following. Braids are classified up to time dependent diffeomorphisms of $\mathbb{R}^2$ (that is, up to diffeomorphisms of $\mathbb{R}^3$ that leave fixed one of the coordinates, the "time" t), while links are classified up to full three-dimensional diffeomorphisms.

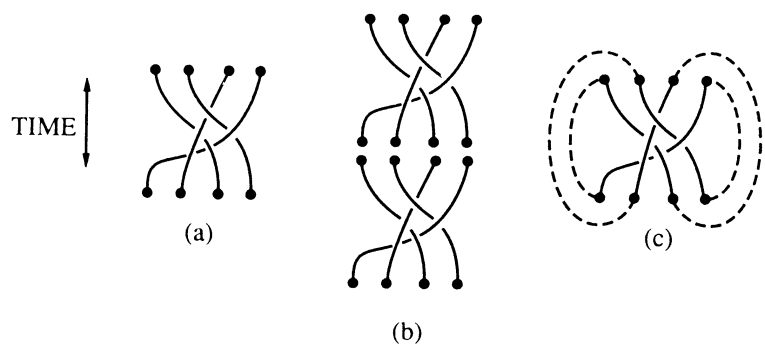

Figure 1. (a) A braid; (b) composition of two braids; (c) making a braid into a link.

If one is given a representation $\mathcal{S}$ of the braid group $\mathcal{B}_n$, and a braid $B \in \mathcal{B}_n$, then $\mathrm{Tr}_{\mathcal{S}} B$ (the trace of the matrix that represents B in the representation $\mathcal{S}$) is an invariant of the *braid* B (and depends in fact only on its conjugacy class in $\mathcal{B}_n$),

but there is no reason for it to be an invariant of the *link* that is obtained by joining the ends of the braid B according to the recipe in the figure.

Nevertheless, Vaughn Jones found a special class of representations of the braid group with the magic property that suitable linear combinations of the braid traces are in fact knot invariants and not just braid invariants. These knot invariants can be combined into the Jones polynomial some of whose remarkable properties were described in Jones's lecture the other day. The discovery of the Jones polynomial stimulated in short order the discovery of some related knot polynomials — the HOMFLY and Kauffman polynomials — whose logical status is rather similar. The challenge of understanding the Jones polynomial is to explain why the Jones braid representations, which obviously have two-dimensional symmetry, should really have three-dimensional symmetry.

Thus, we have two examples where one studies a *group representation* that obviously has d-dimensional symmetry, for some d, but turns out to have $(d+1) = D$-dimensional symmetry, for reasons that might look mysterious. In our first example, the group is $\mathcal{L}G$, $d = 1$, and $D = 2$. In the second example, the group is the braid group, $d = 2$, and $D = 3$.

Our third example, Donaldson/Floer theory, is of a somewhat different nature. In this case, $d = 3$ and $D = 4$, but unlike our previous examples, Donaldson/Floer theory began historically not in the lower dimension but in the upper dimension. The mathematical theory begins in this case with Donaldson's invariants of a closed, oriented four-manifold M. In Donaldson's original considerations, it was important that the boundary of M should vanish. The attempt to generalize the Donaldson invariants to the case that $\partial M = Y \neq \phi$ led to the introduction of the "Floer homology groups" which are vector spaces $\mathrm{HF}^*(Y)$ canonically associated with an oriented three-manifold Y. Though these vector spaces did not originate as group representations, their formal role is just like that of the group representations that entered in our first two examples.

In our first two problems of understanding modular moonshine and the Jones polynomial, the crucial question is to explain why $(d+1)$-dimensional symmetry is present in a construction that appears to only have d-dimensional symmetry. At least from a historical point of view, Donaldson theory is of a completely different nature, since the four-dimensional symmetry has been built in from the beginning. Nevertheless, the logical structure of Donaldson/Floer theory is of a similar nature to that of the first two examples.

In each of our three examples, a pair of dimensions, d and $D = d + 1$, plays a key role. With the lower dimension we associate a vector space (the representations of $\mathcal{L}G$ or $\mathcal{B}_n$ or the Floer groups) and with the upper dimension we associate an invariant (the characters $\mathrm{Tr}_{\mathcal{R}} q^H$, the knot polynomials, or the Donaldson invariants of four-manifolds). The facts are summarized in Table 1.

Table 1

Theory	Dimensions	Vector space in lower dimension	Invariant in upper dimensions
Modular moonshine	1, 2	Representations of loop groups	Modular forms $\operatorname{Tr} q^H$
Jones polynomial	2, 3	Jones representations of braid group	Invariants of knots and three-manifolds
Donaldson/Floer theory	3, 4	Floer Homology groups	Donaldson invariants of four-manifolds

3. Axioms of Quantum Field Theory. Let us now formalize the precise relationship between the vector spaces that appear in dimension d and the invariants in dimension $d+1$. (In the physical context, d is called the dimension of space, and $d+1$ is the dimension of space-time.) In formalizing this relationship, we will follow axioms originally proposed (in the context of conformal field theory, essentially our first example) by Graeme Segal [**10**]. (In addition, Michael Atiyah has adapted those axioms for the topological context that is relevant to our second and third examples [**11**], with considerably more precision than I will attempt here.)

So we will consider quantum field theory in space-time dimension $D = d + 1$. The manifolds that we consider will be smooth manifolds possibly endowed with some additional structure. The type of additional structure considered will be characteristic of the theory. For instance, in the case of modular moonshine, this additional structure is a conformal structure; quantum field theories requiring such a structure (but not requiring a choice of Riemannian metric) are called conformal field theories. In the case of Donaldson/Floer theory, the extra structure consists of an orientation; in the case of the Jones polynomial, one requires an orientation and "framing" of tangent bundles (in a suitable stable sense). Theories that require structure of such a purely topological kind may be called topological quantum field theories. The "ordinary" quantum field theories most extensively studied by physicists require metrics on all manifolds considered.

The first notion is that to every d-dimensional manifold X, without boundary, and perhaps with some additional structure characteristic of the particular theory, one associates a vector space $\mathcal{H}_X$. A quantum field theory is said to be "unitary" if these vector spaces actually carry a Hilbert space structure; this is so in the theories of modular moonshine and the Jones polynomial, but not in the case of Donaldson/Floer theory. In the case of the Jones polynomial and Donaldson/Floer theory, the vector spaces $\mathcal{H}_X$ are finite dimensional, and a morphism of vector spaces is taken to mean an arbitrary linear transformation (preserving the unitary

structure in the case of the Jones polynomial); in the theory of modular moonshine, the $\mathcal{H}_X$ are infinite dimensional, and it is necessary to be more precise about what is meant by a morphism among these spaces.

In Segal's language, the association $X \to \mathcal{H}_X$ is to be a functor from the category of d-dimensional manifolds with additional structure (and diffeomorphisms preserving the specified structures) to the category of vector spaces (and linear transformations of the appropriate kind).

Certain additional restrictions are imposed. The empty d-manifold ϕ is permitted, and one requires that $\mathcal{H}_\phi = \mathbb{C}$ ($\mathbb{C}$ here being a one-dimensional vector space with a preferred generator which we call "1"). If $X \amalg Y$ denotes the disjoint union of two d-dimensional manifolds X and Y, then one requires $\mathcal{H}_{X \amalg Y} = \mathcal{H}_X \otimes \mathcal{H}_Y$. If $-X$ is X with opposite orientation, and $*$ denotes the dual of a vector space, one requires $\mathcal{H}_{-X} = \mathcal{H}_X^*$.

Since the late 1920s, the spaces $\mathcal{H}_X$ have been known to physicists as the "physical Hilbert spaces" (of the particular quantum field theory under consideration). The association $X \to \mathcal{H}_X$ is roughly half of the basic structure considered in quantum field theory. The second half corresponds in physical terminology to the "transition amplitudes."

To introduce the transition amplitudes, we consider (Figure 2(a) on the next page) a cobordism of oriented (and possibly disconnected or empty) d-dimensional manifolds. Such a cobordism is defined by an oriented $(d+1)$-dimensional manifold W whose boundary is, say, $\partial W = X \cup (-Y)$, where X and Y are oriented d-dimensional manifolds (whose orientations respectively agree or disagree with that induced from W). It is required that whatever structure (conformal structure, framing, metric, etc.) has been introduced on X and Y is extended over W. Such a cobordism is regarded as a morphism from X to Y. To every such morphism of manifolds, a quantum field theory associates a morphism of vector spaces

$$\Phi_W : \mathcal{H}_X \to \mathcal{H}_Y \,. \tag{3.1}$$

Of course, this association $W \to \Phi_W$ should be natural, invariant under any diffeomorphism of W that preserves the relevant structures. Regarding $-W$ as a morphism from $-Y$ to $-X$, one requires that $\Phi_{(-W)} : \mathcal{H}_{(-Y)} \to \mathcal{H}_{(-X)}$ should be the dual linear transformation to Φ_W. And if $W = W_1 \cup W_2$ is a composition of cobordisms (Figure 2(b) on the next page), one requires that

$$\Phi_W = \Phi_{W_2} \circ \Phi_{W_1} \,. \tag{3.2}$$

These requirements correspond physically to relativity, locality, and causality.

A very important special case of this is the case in which W is a closed $D = (d+1)$-dimensional manifold without boundary. Such a W can be regarded as a morphism from the empty d-dimensional manifold ϕ to itself. Since $\mathcal{H}_\phi = \mathbb{C}$, the

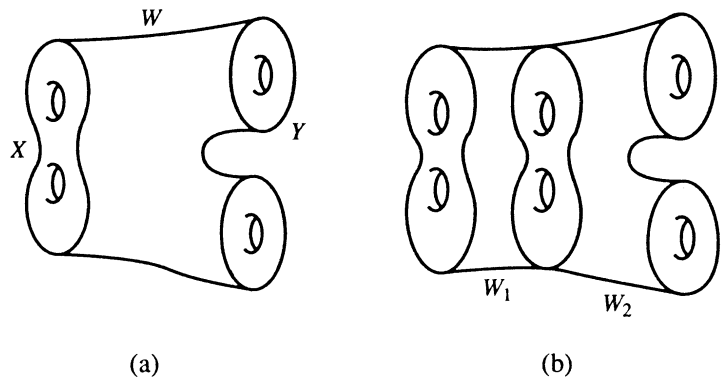

Figure 2. (a) A cobordism of oriented d-dimensional manifolds; (b) a composition of of such cobordisms.

associated morphism $\Phi_W : \mathcal{H}_\phi \to \mathcal{H}_\phi$ is simply a number, which for physicists is often called the partition function of W and denoted $Z(W)$. This partition function is the fundamental invariant in quantum field theory; for different choices of theory, one gets the invariants of D-dimensional manifolds indicated in the last column in Table 1.

For $Z(W)$ to be defined in a given quantum field theory, W must of course be endowed with the structure appropriate to the particular quantum field theory in question. For instance, in the case of modular moonshine, W must be a Riemann surface with a conformal structure. In genus one, this means that W is an elliptic curve, which can be represented by a point in the upper half-plane subject to the action of the mapping class group. The naturality of the association $W \to Z(W)$ means that $Z(W)$ can depend only on the equivalence class of the conformal structure of W, and it is this which leads to modular forms. In our other two examples, no metric or conformal structure is present, so we are dealing with topological invariants. In our second example of the Jones polynomial, the invariant $Z(W)$ is an invariant of oriented three-manifolds which is an analog for three-manifolds of the Jones polynomials for knots in $\mathbf{S}^3$. (The actual knot invariants can be obtained by an elaboration of the quantum field theory structure.) In our third example of Donaldson theory, the invariant $Z(W)$ is the prototype of the invariants that appear in the celebrated Donaldson polynomials of oriented four-manifolds.

It is built into the axioms of quantum field theory that the fundamental invariants $Z(W)$ can be computed from a decomposition of the type that is known in the case of three-manifolds as a Heegaard splitting. This means a realization of W as $W = W_1 \cup W_2$, where W_1 and W_2 are D-manifolds joined together along their common boundary Σ. In this case the morphism W from the empty manifold ϕ to itself factorizes as a morphism W_2 from ϕ to Σ composed with a morphism W_1 from Σ to ϕ, i.e.,

$$\Phi_W = \Phi_{W_1} \circ \Phi_{W_2}\,. \tag{3.3}$$

If 1 is the canonical generator of $\mathcal{H}_\phi$, we then have

$$Z(W) = (1, \Phi_W \ \cdot 1) = (1, \Phi_{W_1} \circ \Phi_{W_2} \cdot 1)\,. \tag{3.4}$$

Let $v \in \mathcal{H}_\Sigma$ be the vector $v = \Phi_{W_2}(1)$. Also, think of $-W_1$ as a morphism from ϕ to $-\Sigma$, and let and $w \in \mathcal{H}_{-\Sigma}$ be the vector $w = \Phi_{-W_1}(1)$. Then (3.4) amounts to

$$Z(W) = (w, v)\,. \tag{3.5}$$

This ability to calculate via Heegaard splittings is part of the conventional *definition* of the Casson invariant (which has a quantum field theory interpretation analogous to that of Donaldson theory), and is essential in the calculability of the three-manifold invariants that are related to the Jones polynomial. Likewise, in the case of modular moonshine, the decomposition (3.5) is the key to the fact that the partition function $Z(W)$ can be written as the character $\mathrm{Tr}_{\mathcal{R}} q^H$ of equation (2.1).

4. Construction of Quantum Field Theories. The question arises, of course, of how these quantum field theories are to be constructed. About this enormous subject it is possible only to say a few words here.

The starting point is always the choice of an appropriate *Lagrangian*, which is the integral of a local functional of appropriate fields. For instance, if one is interested in understanding the Jones polynomial, one picks a finite-dimensional compact simple group G and one considers a connection A on a G-bundle E over a three-manifold M. Let $F = dA + A \wedge A$ denote the curvature of this connection. On the Lie algebra $\mathcal{G}$ of a compact group G, there is an invariant quadratic form which we denote by the symbol Tr (that is, we write $(a, b) = \mathrm{Tr}(ab)$).[1] For the Lagrangian, we take the Chern–Simons invariant of the connection A:

$$\mathcal{L} = \frac{k}{4\pi} \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)\,. \tag{4.1}$$

(Here k is a positive integer, a fact that is required so that the argument $e^{i\mathcal{L}}$ in the Feynman path integral is gauge invariant.) To construct a quantum field theory from this Lagrangian, there are two basic requirements. First, we must construct a functor from Riemann surfaces Σ to Hilbert spaces $\mathcal{H}_\Sigma$; and second, for every cobordism W from Σ to Σ', we must construct a morphism $\Phi_W : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}$.

For the first step, one proceeds as follows. Given the surface Σ, we consider the Lagrangian (4.1) on the three-manifold $\Sigma \times \mathbb{R}$. The space of critical points of the Lagrangian, up to gauge transformations, is known in classical mechanics as the "phase space" of the system under investigation. Let us call this phase space

[1] The quadratic form is to be normalized so that the characteristic class $\frac{1}{4\pi}\mathrm{Tr} F \wedge F$ has periods that are multiples of 2π.

$\mathcal{M}_\Sigma$. In the case at hand, the Euler–Lagrange equation for a critical point of the Lagrangian (4.1) is the equation $F = 0$, where $F = dA + A \wedge A$ is the curvature of the connection A. (That is, (4.1) is invariant to first order under variations of the connection of compact support if and only if $F = 0$.) A flat connection on $\Sigma \times \mathbb{R}$ defines a homomorphism of the fundamental group $\pi_1(\Sigma \times \mathbb{R})$ into G. Of course, this is the same as a homomorphism of $\pi_1(\Sigma)$ into G. The classical phase space $\mathcal{M}_\Sigma$ associated with the Lagrangian (4.1) is simply the moduli space of homomorphisms of $\pi_1(\Sigma) \to G$, up to conjugation by G.

Now, it is a general fact in the calculus of variations that the phase space associated with a Lagrangian such as (4.1) is always endowed with a canonical symplectic structure ω. Indeed, this is how symplectic structures originally appeared in classical mechanics, and as such it was the starting point of symplectic geometry as a mathematical subject also. In the case at hand, the symplectic structure thus obtained on $\mathcal{M}_\Sigma$ is known [**13**], and has been studied very fruitfully from the point of view of two-dimensional gauge theory [**14**], but my point is that this symplectic structure on $\mathcal{M}_\Sigma$ can be considered to arise from a *three-dimensional* variational problem. This elementary fact is an important starting point for understanding the mysterious three-dimensionality of the Jones polynomial.

Once the appropriate phase space $\mathcal{M}_\Sigma$ is identified, the association $\Sigma \to \mathcal{H}_\Sigma$ is made by "quantizing" $\mathcal{M}_\Sigma$ to obtain a Hilbert space $\mathcal{H}_\Sigma$. Geometric quantization is not sufficiently well developed to make quantization straightforward in general (or perhaps this is actually impossible in general), but in the case at hand quantization can be carried out by choosing a complex polarization of $\mathcal{M}$. This is accomplished by picking a complex structure J on Σ and using the Narasimhan–Seshadri theorem to identify $\mathcal{M}$ with the moduli space of stable holomorphic $G_\mathbb{C}$ bundles over Σ. This moduli space is then quantized by defining $\mathcal{H}_\Sigma$ to be the space of holomorphic sections of a certain line bundle over $\mathcal{M}$. This space is independent of J (up to a projective factor) because of its interpretation in terms of quantization of the underlying classical phase space $\mathcal{M}$. The association $\Sigma \to \mathcal{H}_\Sigma$ is the geometric origin of the Jones braid representations (or rather their analog for the mapping class group in genus g).

Once the association $\Sigma \to \mathcal{H}_\Sigma$ is understood, it remains to define for every cobordism W from Σ to Σ', a corresponding morphism $\Phi_W : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}$. The key notion here is the "Feynman path integral," that is, Feynman's concept of integration over the whole function space $\mathcal{V}$ of connections on (the given G-bundle E over) W. Roughly speaking, the function space integral with prescribed boundary conditions gives the kernel of the morphism Φ_W. I have tried to explain heuristically the role of functional integrals in [**12**], and I will not repeat those observations here.

In conclusion, let me point out that if G is a *compact* group, then, as I have argued in [**15**], the quantum field theory associated with the Lagrangian (4.1) is related to the Jones polynomial and its generalizations. However, (4.1) makes sense for any gauge group G with an invariant quadratic form "Tr" on the Lie algebra. It is natural to ask what mathematical constructions are related to the quantum

field theories so obtained. One case that can be conveniently analyzed is the case in which one replaces the compact group G by a group $TG = \mathcal{G} \ltimes G$; here $\mathcal{G} \ltimes G$ denotes the semidirect product of G with its own Lie algebra $\mathcal{G}$, the latter regarded as an abelian group acted on by G. It turns out [**16**] that with this choice of gauge group, the quantum field theory derived from (4.1) (with a certain choice of "Tr") is related to recent work of D. Johnson on Reidemeister torsion [**17**], while if instead one considers a certain super-group whose bosonic part is TG then (4.1) (again with a certain choice of "Tr") is related to the Casson invariant of three-manifolds. It is also very interesting to take G to be a semisimple but noncompact Lie group. The corresponding quantum field theories are very little understood, but there are indications that they should be very rich. In fact, it appears [**18**] that the theories based on $\mathrm{SL}(2,\mathbb{R})$ and especially $\mathrm{SL}(2,\mathbb{C})$ must be intimately connected with the theory of hyperbolic structures on three-manifolds, as surveyed the other day by Thurston.

5. Conclusion. In attending this meeting, I have found it striking how many of the lectures were concerned with questions that are associated with quantum field theory — and in many cases questions that might be characterized as questions about quantum field theory. In time we will hopefully gain a clearer picture of the scope of some of these newly emerging relations between geometry and physics. It is not too much to anticipate that many important constructions relating quantum field theory to topology and differential geometry remain to be discovered. Harder to foresee is whether — by the time of the one hundred fiftieth anniversary of the American Mathematical Society — the influence of quantum field theory will also extend to other areas of mathematics, such as algebraic geometry and number theory, which superficially might appear to be comparatively immune. Let me recall that in his lecture earlier this week, Dick Gross concluded by urging physicists and mathematicians to find a quantum field theory explanation for the appearance of modular forms in the study of the L-functions of elliptic curves. (The question was, of course, motivated by the relation of quantum field theory to the different kinds of modular moonshine.) Perhaps this challenge, or analogous ones about which one might speculate, will be met. Hints today concerning quantum field theoretic insights about number theory are probably no more compelling than hints of quantum field theoretic insight about differential geometry were ten years ago.

What significance might the emerging links between quantum field theory and geometry have for physics? It is very noticeable that the aspects of quantum field theory that are most useful in understanding the geometrical problems that I have been talking about are pretty close to the slightly specialized aspects of quantum field theory that appear in string theory. Modular invariance in the theory of affine Lie algebras is certainly a familiar story to string theorists. The Jones polynomial and its generalizations are related to the "rational conformal field theories" which are one of our main tools for finding exact classical solutions in string theory. The constructions that enter in formulating Donaldson theory as a quantum field theory

are also very similar to what string theorists are accustomed to (in the use of world-sheet BRST operators).

Apart from being at least loosely connected with all of the geometrical problems that we have been discussing, string theory seems to be the center of some geometrical questions of central physical interest. The towering puzzle in contemporary theoretical physics is — at least from my standpoint — the puzzle of finding the geometrical context in which string theory should be properly formulated and understood. I am sure many physicists would share this judgment. With our present concepts, this problem (to which I attempted a thumbnail introduction in [**12**]) seems well out of reach. Perhaps it is not too far-fetched to hope that some insight in this central mystery can be obtained from the further study of geometrical questions arising in quantum field theory.

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THE WORK OF JEAN BOURGAIN

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Introduction

Bourgain's work touches on several central topics of mathematical analysis: the geometry of Banach spaces, convexity in high dimensions, harmonic analysis, ergodic theory, and finally, nonlinear partial differential equations (P.D.E.'s) from mathematical physics. In all of these areas, he made spectacular inroads into questions where progress had been blocked for a long time.

This he did by simultaneously bringing into play different areas of mathematics: number theory, combinatorics, probability, and showing their relevance to the problem in a previously unforeseen fashion.

To give a flavor of his work, I have concentrated on his recent research, of about the last ten years.

The solution of the Λ_p problem

A great part of the work of Bourgain, in the study of the geometry of Banach spaces, concentrated on the question: Given a Banach space of finite dimension n, how large a section can we find that resembles a Hilbert subspace?

Maybe his most relevant paper in this field is his solution of the $\Lambda(p)$ problem: Given a subset Λ of the set of characters of a compact Abelian group. Λ is a p-set ($p > 2$) if the L^p and L^2 norms are equivalent in the subspace of $L^p(G)$ generated by Λ.

The longstanding question was: Do $\Lambda(p)$ and $\Lambda(q)$ sets coincide?

Bourgain answers this problem in the negative with the following sharp estimates:

Among n given characters there is a subset of optimal size $[n^{2/p}]$ for which

$$\Big\| \sum a_i \varphi_i \Big\|_{L^p} \leq C(p) \left(\sum |a_i|^2 \right)^{1/2} . \qquad (*)$$

Through a lacunary argument, one can construct a $\Lambda(p)$ set, which is not $\Lambda(q)$ for any $q > p$.

The converse of Santalo's inequality

Another product of Bourgain's studies is his proof of Santalo's inequality:

Given K the unit ball of a norm on R^n and K^* its dual, Bourgain and Milman prove

$$\mathrm{vol}(K)\mathrm{vol}(K^*) \geq C^n |B|^2$$

for some absolute constant $0 < c < 1$.

This has applications to number theory and computer sciences.

Ergodic theory

In ergodic theory, Bourgain developed a completely new theory, where averages under very sparse (polynomial) families of iterations are studied (and shown to converge).

The basic theorem, from which the general setting follows by a well-known transformation, due to Calderon, is the maximal theorem for $\ell^2(\mathbb{Z})$.

Theorem. *Let $f \in \ell^2(\mathbb{Z})$, and ℓ be a positive integer; let*

$$Mf(n) = \sup_{N>0} \left| \frac{1}{N} \sum_{k=1}^{N} f(n+k^\ell) \right| .$$

Then

$$\|Mf\|_{\ell^2} \leq C\|f\|_{\ell^2}$$

i.e., the maximal function of the partial averages of the k^ℓ iterations of the 1-translation is bounded in ℓ^2.

Oscillatory integrals

An important family of ideas introduced by Stein in harmonic analysis concerns the study of the restriction of classical operators (maximal functions, Hilbert and Fourier transforms) to curves in space (parabolics, circles) that have special relevance to the study of partial differential equations (singular integrals of parabolic type, spherical averages related to the wave equation, etc.).

In this area I'll mention two fundamental contributions of Bourgain:

The circle maximal function

$$Mf(y) = \sup_r \frac{1}{A(S_r)} \int\limits_{x \in S_r(0)} f(y+x)dA$$

was shown by Stein to be a bounded operator in L^p for some range $p(n)$ for $n \geq 3$.

The two-dimensional case ($p \geq 2$) remained open for a long time until Bourgain closed the gap.

As the "solid" maximal function in any dimension can be written as an average of spherical maximal functions in a fixed low dimension, this allows us in particular to prove bounds for the "solid" maximal function independent of dimensions.

The second contribution refers to the restriction of Fourier transforms to spheres, or, related to it, the properties of the characteristic function of the ball as a Fourier multiplier (a natural generalization of taking partial sums of Fourier series).

In two dimensions, these problems were well understood (C. Fefferman).

In higher dimensions some range of continuity is expected around L^2, and a series of results was obtained by Tomas and Stein.

Bourgain considerably sharpened these results, but what is more important than the exact ranges, is the fact that, in doing so, he introduced completely new techniques, where instead of relying on L^2 theory (i.e., decomposing functions and operators in "L^2 pieces") he sharpened the geometric understanding of Besicovitch-type maximal operators and Kakeya sets.

Nonlinear partial differential equations

Bourgain's contributions to nonlinear partial differential equations are very recent, and it is somewhat difficult to decide where to stop in this presentation because results from him and many others in the field (Kenig, Klainerman, Machedon, Ponce, Vega) have been pouring in, in good part thanks to the revitalization of the field brought in by Bourgain's approach.

Let us say that he obtained very sharp results for the well-posedness of the nonlinear Schrödinger equation,

$$iu_t + \Delta u + u|u|^\alpha = 0$$

for non smooth data.

Previous to his work, there was mainly local well-posedness in H^s for large enough s.

By introducing new, suitable space-time functional spaces, Bourgain started a new, more deep and elegant way of treating dispersive equations.

In closing, let me reiterate that some of the outstanding qualities of Bourgain are his power to use whatever it takes — number theory, probabilistic methods, covering techniques, sharp decompositions — to understand the problem at hand, and his versatility, which allowed him to deeply touch so many areas in such a short period of time.

Jean Bourgain

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HAMILTONIAN METHODS IN NONLINEAR EVOLUTION EQUATIONS

by
JEAN BOURGAIN*

1. Introduction

One of the main problems in the theory of nonlinear Hamiltonian evolution equations is the behavior of the solutions for time $t \to \infty$. We have in mind the general (not integrable) case, with only few conserved quantities at our disposal. Here one may nevertheless obtain certain information on the longtime behavior of the flows, using the Hamiltonian structure of the equation. Recently, there has been progress along these lines in various directions and we do not intend to try to give a complete survey. In this report, we will discuss the following aspects

(i) Use of methods of statistical mechanics and symplectic geometry
(ii) Use of methods of dynamical systems

The main theme, regarding the first topic, is to establish the existence of a well defined global flow on the support of the Gibbs measure $e^{-\beta H(\phi)} \prod d\phi$ and show its invariance under this flow. This leads thus to the existence of an invariant measure. This measure lives on functions with poor regularity or fields. Proving well-posedness of the Cauchy problem for such data turns out to require a rather delicate analysis of an independent interest. The well-posedness is usually only local in time and a global result is obtained using the Gibbs-measure as a substitute for a conservation law. In this way, global solutions are constructed for data of a regularity considerably below what may be shown by purely PDE methods. We will carry out our discussion here for the nonlinear Schrödinger equation (NLS) in dimensions 1 and 2 and some related equations, completing a line of investigation initiated in the paper of [L-R-S]. Other symplectic invariants, called symplectic capacities, originating from M. Gromov's pioneering work [Grom], allow us to study "squeezing properties" and energy transitions in the symplectically normalized phase space. These normalizations are however such that the resulting theories deal with low regularity solutions and consequently, as a first step, require again to study the flow for such data.

The second topic concerns the persistency of periodic or quasi-periodic (in time) solutions of linear or integrable equations under small Hamiltonian perturbation. A natural approach to such questions is to try to adapt the standard KAM technology from classical mechanics. The problem is thus the persistency of finite dimensional tori in an infinite dimensional phase space. In this direction, important

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contributions are due to S. Kuksin [Kuk$_1$]. His work gives satisfactory results for $1D$ problems with Dirichlet boundary conditions. The shortcoming of the standard KAM approach is the fact that it seems unable to deal with multiplicities or clusters in the normal frequencies; those appear in $1D$ under periodic boundary conditions and in dimension ≥ 2. A different approach, based on the Liapounov–Schmidt decomposition scheme avoids this limitation of the KAM method. It has been elaborated by W. Craig, E. Wayne [C-W$_{1,2}$] for $1D$ time-periodic solutions and more recently by the author. The method is more flexible than KAM and depends less on Hamiltonian structure. Research on its applications is still in progress. Presently, we may deal with time periodic problems in any dimension and time quasi-periodic solutions in $1D$, $2D$. The second part of the report is devoted to a sketch of the underlying ideas. We mainly restrict ourselves to nonlinear perturbations of linear equations here. The included reference list is strictly for the purpose of this exposition.

2. Nonlinear Schrödinger Equations and Invariant Gibbs Measures

We consider the nonlinear Schrödinger equation (NLS)

$$iu_t + \Delta u \pm u|u|^{p-2} = 0 \tag{2.1}$$

with periodic boundary conditions. Thus u is a complex function on $\mathbf{T}^d \times I$ (local) or $\mathbf{T}^d \times \mathbf{R}$ (global). The equation may be rewritten in Hamiltonian format as

$$u_t = i\,\frac{\partial H}{\partial\,\overline{u}} \tag{2.2}$$

where $H(\phi) = \frac{1}{2}\int_{\mathbf{T}^d} |\nabla\phi|^2 \mp \frac{1}{p}\int_{\mathbf{T}^d} |\phi|^p$. Both the Hamiltonian $H(\phi)$ and the L^2-norm $\int |\phi|^2$ are preserved under the flow. The $1D$ case $p = 4$ is special ($1D$ cubic NLS) because integrable and there are many invariants of motion. This aspect will however play no role in the present discussion. The possible sign choice $\pm$ in (2.1) corresponds to the focusing (resp. defocusing) case. In the focusing case, the Hamiltonian may be unbounded from below and blowup phenomena may occur (for $p \geq 2 + \frac{4}{d}$). The canonical coordinates are $(\mathrm{Re}\,\phi, \mathrm{Im}\phi)$ or alternatively $(\mathrm{Re}\widehat{\phi}, \mathrm{Im}\widehat{\phi})$. The formal Gibbs measure on this infinite dimensional phase space is given by

$$d\gamma_\beta = e^{-\beta H(\phi)} \prod_x d\phi(x) = e^{\pm\frac{\beta}{p}\int |\phi|^p} \cdot e^{-\frac{\beta}{2}\int |\nabla\phi|^2} \prod_x d\phi(x) \tag{2.3}$$

($\beta > 0$ is the reciprocal temperature and we may take $\beta = 1$ in this discussion). From Liouville's theorem, (2.3) defines an invariant measure for the flow of (2.1). Making this statement precise requires one to clarify the following two issues

(i) The rigorous construction (normalization) of the measure (2.3)
(ii) The existence problem for the flow of (2.1) on the support of the measure.

The first issue is well understood in the defocusing case. The case $D = 1$ is trivial, the case $D = 2$, p even integer is based on the Wick-ordering procedure (see [G-J]) and the normalization for $D = 3$, $p = 4$ is due to Jaffe [Ja]. In the focusing case,

only the case $D = 1$ is understood [L-R-S] and normalization of the measure is possible for $p \leq 6$, restricting ϕ to an appropriate ball in $L^2(\mathbf{T})$.

The construction of a flow is clearly a PDE issue. The author succeeded in this in the $D = 1$ and $D = 2$, $p = 4$ cases ([B$_1$], [B$_2$]). For $D = 2$, $p = 4$ there is a natural PDE counterpart of the Wick-ordering procedure and equation (2.1) has to be suitably modified (this modification seems physically inessential however). We may summarize the results as follows.

Theorem 2.4. ($D = 1$)

(i) *In the defocusing case, the measure* (2.3) *appears as a weighted Wiener measure, the density being given by the first factor. The same statement is true in the focusing case for* $p \leq 6$, *provided one restricts the measure to an* L^2*-ball* $[\|\phi\|_2 \leq B]$. *The choice of* B *is arbitrary for* $p < 6$ *and* B *has to be sufficiently small if* $p = 6$.

(ii) *Assuming the measure exists, the corresponding* $1D$ *equation* (2.1) *is globally well-posed on a* K_σ *set* A *of data,* $A \subset \cap_{s<\frac{1}{2}} H^s(\mathbf{T})$, *carrying the Gibbs measure* γ_β. *The set* A *and the Gibbs measure* γ_β *are invariant under the flow.*

(Recall that the first part of the theorem is due to [L-R-S]).

Remarks.

(i) In dimension 1, the L^2-restriction is acceptable, since L^2 is a conserved quantity and a typical ϕ in the support of the Wiener measure is a function in $H^s(\mathbf{T})$, for all $s < \frac{1}{2}$. Instead of restricting to an L^2-ball, one may alternatively multiply with a weight function with a suitable exponential decay in $\|\phi\|_2$.

(ii) Let for each $N = 1, 2, \ldots$

$$P_N\phi = \phi_N = \sum_{|n|\leq N} \widehat{\phi}(n)\, e^{i\langle n,x\rangle} \tag{2.5}$$

be the restriction operator to the N first Fourier modes. Finite dimensional versions of the PDE model are obtained considering "truncated" equations

$$\begin{cases} iu_t^N + u_{xx}^N \pm P_N\left(u^N|u^N|^{p-2}\right) = 0 \\ u^N(0) = P_N\phi\,. \end{cases} \tag{2.6}$$

It is proved that for typical ϕ, the solutions u^N of (3.6) converge in the space $C_{H^s(\mathbf{T})}[0,T]$ for all time T and $s < \frac{1}{2}$ to a (strong) solution of

$$\begin{cases} iu_t + u_{xx} \pm P\left(u|u|^{p-2}\right) = 0 \\ u(0) = \phi\,. \end{cases} \tag{2.7}$$

Theorem 2.8. ($D = 2$, $p = 4$)

(i) *Denote $\widetilde{H}_N$ to be the Wick-ordered Hamiltonians, obtained by replacing*

$$|\phi_N|^4 \quad \text{by} \quad |\phi_N|^4 - 4a_N|\phi_N|^2 + 2|a_N|^2 \qquad \left(a_N = \sum_{|n|\le N} \frac{1}{|n|^2+\rho} \sim \log N \right).$$

The corresponding measures $e^{-\beta \widetilde{H}_N(\phi)} \prod d\phi$ converge for $N \to \infty$ to a weighted 2-dimensional Wiener measure whose density belongs to all L^p-spaces. Denote $\widetilde{\gamma}_\beta$ to be this "Wick-ordered" Gibbs measure.

(ii) *The measure $\widetilde{\gamma}_\beta$ is invariant under the flow of the "Wick-ordered" equation*

$$iu_t + \Delta u - \left(u|u|^2 - 2u \int |u|^2 \right) = 0 \tag{2.9}$$

which is well-defined. More precisely, denoting u^N the solutions of

$$\begin{cases} iu_t^N + \Delta u^N - P_N \left(u^N|u^N|^2 - 2u^N \int |u^N|^2 \right) = 0 \\ u^N(0) = P_N\phi \end{cases} \tag{2.10}$$

the sequence

$$u^N(t) - \sum_{|n|\le N} \widehat{\phi}(n)\ e^{i(\langle n,x\rangle + |n|^2 t)} \tag{2.11}$$

converges for typical ϕ in $C_{H^s(\mathbf{T}^2)}[0,T]$ for some $s > 0$, all time T, to

$$u(t) - \sum \widehat{\phi}(n)\ e^{i(\langle n,x\rangle + |n|^2 t)}\,. \tag{2.12}$$

Remarks.

(i) We repeat that the novelty of Theorem 2.8 lies in the second statement on the existence of a flow. The first statement is a classical result (see G-J]).

(ii) The second terms in (2.11), (2.12) are the solutions to the linear problem

$$\begin{cases} iu_t + \Delta u = 0 \\ u(0) = \phi\,. \end{cases} \tag{2.13}$$

Here a typical ϕ is a distribution, not a function. However the difference (2.12) between solutions of linear and nonlinear equation is an H^s-function for some $s > 0$, which is a rather remarkable fact.

(iii) The failure in $D = 2$ of typical ϕ to be an L^2-function makes the [L-R-S] construction for $D = 1$ inadequate to deal with the $D = 2$ focusing case. Some recent work on this issue is due to A. Jaffe, but for cubic nonlinearities in the Hamiltonian only. The problem for $D = 2$, $p = 4$ in the focusing

case is open and intimately related to blowup phenomena ($p = 4$ is critical in $2D$). Observe that in the preceding the invariance of the (Wick-ordered) Gibbs-measure implies the "quasi-invariance" of the free measure $\prod d^2\phi$. I. Gelfand* proposed to investigate this fact directly. Such approach would avoid normalization problems. For instance, in $1D$, one may consider the focusing equation $iu_t + u_{xx} + u|u|^{p-2} = 0$ with $p \geq 6$ and prove this quasi-invariance local in time.

(iv) One may prove an analogue of Theorem 2.8 for $2D$ wave equations, with defocusing polynomial nonlinearity

$$u_{tt} - \Delta u + \int u + f'(u) = 0 \quad (u \text{ real } \int > 0), \tag{2.14}$$

$$f(u) = u^{2k} = 0 + (\text{lower order}) \tag{2.15}$$

and replacing $f(u)$ by its Wick ordering: $f(u)$: (cf. [G-J]). However, in this case there is no conservation of the L^2-norm and the result needs to be formulated in terms of the truncated equation. Observe that the (optimal) PDE result on local well-posedness deals with data $\phi \in H^{1/4}$ (see [L-S]).

The $1D$ cubic NLSE appears as the limit of the $1D$ Zakharov model (ZE)

$$\begin{cases} iu_t = -u_{xx} + nu \\ n_{tt} - c^2 n_{xx} = c^2 \left(|u|^2\right)_{xx} \end{cases} \tag{2.16}$$

when $c \to \infty$. The physical meaning of u, n, c are resp. the electrostatic envelope field, the ion density fluctuation field and the ion sound speed. This model is discussed in [L-R-S]. Defining an auxiliary field $V(x,t)$ by

$$\begin{cases} n_t = -c^2\ V_x \\ V_t = -n_x - |u|_x^2 \end{cases} \tag{2.17}$$

we may write (2.16) in a Hamiltonian way, where

$$H = \tfrac{1}{2} \int \left[|u_x|^2 + \tfrac{1}{2}\ (n^2 + c^2\ V^2) + n|u|^2\right]\ dx \tag{2.18}$$

and $(\operatorname{Re} u,\ \operatorname{Im} u)$, $\left(\widetilde{n}, \widetilde{V}\right)$ with $\widetilde{n} = 2^{-1/2} n$, $\widetilde{V} = 2^{-1/2} \int^x V$ as pairs of conjugate variables. Considering the associated Gibbs measure

$$e^{-\beta H} \cdot \chi_{\{\int |u|^2 dx \leq B\}} \prod_x d^2u(x)\ d\widetilde{n}(x)\ d\widetilde{V}(x) \tag{2.19}$$

one gets the $1D$ cubic NLS Gibbs measure as marginal distribution of the u-field.

Theorem 2.20. [B$_3$] *The $1D$ (ZE) is globally well-posed for almost all data $(u_0, \widetilde{n}_0, \widetilde{V}_0)$ in the support of the Gibbs measure which is invariant under the resulting flow.*

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Remarks.

(i) In the study of invariant Gibbs measures, it suffices to establish local well-posedness of the IVP for typical data in the support of the measure. One may then exploit the invariance of the measure as a conservation law and generate a global flow. For instance, for the $1D$ NLS $iu_t + u_{xx} \pm u|u|^{p-2} = 0$, there is for $p = 4$ a global well-posedness result for L^2-data (L^2 is conserved). However, for $p > 4$, we only dispose presently of a local result (in the periodic case) for data ϕ satisfying

$$\begin{cases} \phi \in H^s \ , \ s > 0 & (p \leq 6) \\ \phi \in H^s \ , \ s > s_* \, , \quad p = 2 + \frac{4}{1-2s_*} & (p > 6) \end{cases} \tag{2.21}$$

and a global flow is established from the invariant measure considerations.

(ii) There has been other investigations in $1D$ on invariant measures, mostly by more probabilistic arguments. In this respect, we mention the works of McKean–Vaninski and in particular Mckean [McK] on the $1D$ cubic NLS. These methods are more general but give less information on the flow.

3. Symplectic Capabilities, Squeezing and Growth of Higher Derivatives

The works of Gromov and Ekeland, Hofer, Zehnder, Viterbo lead to new finite dimensional symplectic invariants, different from Liouville measure on the phase space. Let us recall the following construction of a symplectic capacity for open domains O in $\mathbf{R}^n \times \mathbf{R}^n$, $dp \wedge dq$. Call a smooth function f m-admissible ($m > 0$) if $f = 1$ on a neighborhood of O and $f = 0$ on a nonempty subdomain of O. Denote V_f the associated Hamiltonian vector field $\left(\frac{\partial f}{\partial p} \ , \ -\frac{\partial f}{\partial q}\right)$. Define the symplectic invariant

$$\begin{aligned} c_{2n}(O) = \inf \{ m > 0 \mid V_f \text{ has nontrivial periodic orbit of period } \leq \ 1 \, , \\ \text{whenever } f \text{ is } m\text{-admissible for } O \} \, . \end{aligned} \tag{3.1}$$

Then $c_{2n}(\cdot)$ is monotonic and translation invariant and scales as $c_{2n}(\tau O) = \tau^2 c_{2n}(O)$. The main property is that

$$c_{2n}(B_\rho) = \pi \rho^2 = c_{2n}\left(\Pi_\rho\right) \tag{3.2}$$

where B_ρ is the ball $B_\rho = \left\{|p|^2 + |q|^2 < \rho^2\right\}$ and $\prod_\rho$ a cylinder, say $\prod_\rho = \{p_1^2 + q_1^2 < \rho^2\}$. As a corollary, there is no symplectic squeezing of a ρ-ball in a cylinder of width ρ', $\rho' < \rho$.

Exploiting such invariant in Hamiltonian PDE requires an infinite dimensional setting. Notice that although the theory described above is finite dimensional,

a conclusion such as (3.2) is dimension free. An appropriate "finite dimensional approximation" appears to be possible if the flow S_t of the considered equation is of the form

$$\text{linear operator} \ + \ \text{“smooth compact operator”} \tag{3.3}$$

or, more generally, if the evolution of individual Fourier modes on a finite time interval is approximately the same as in a truncated model $\dot{v} = J\nabla H(v, x, t)$, $v = P_N v$. Here the cutoff N should only depend on the required approximation, the time interval $[0, T]$ and the size of the initial data in phase space. Here and also in (3.3), the phase space has to be defined in a specific way, corresponding to the finite dimensional normalizations. Hence, the flow properties derived this way relate to a specific "symplectic Hilbert space", for instance

L^2 for nonlinear Schrödinger equations (in any dimension)
$H^{1/2} \times H^{1/2}$ for nonlinear wave equations (in any dimension)
$H^{-1/2}$ for KdV type equations

and "non-squeezing" refers to that particular space.

Theorem 3.4. ([B_6], [Kuk_2])

There is non-squeezing of balls in cylinders of smaller width

(i) *for nonlinear wave equations* $u_{tt} = \nabla u + p(u; t, x)$ *with smooth nonlinearity of arbitrary polynomial growth in* u *in dimension* 1 *and polynomial in* u *of degree* ≤ 4 (*resp.* ≤ 2) *in dimension* 2 (*resp.* 3, 4).
(ii) *for certain* $1D$ *nonlinear Schrödinger equations.*

The interest of the squeezing or non-squeezing properties lies in its relevance to the energy transition to higher modes, more precisely whether for instance part of the energy may leave a given Fourier mode, which would correspond to squeezing in a small cylinder. The non-squeezing implies also the lack of uniform asymptotic stability of bounded solutions, i.e. $\operatorname{diam} \mathrm{S_t}(\mathrm{B}_\rho)$ does not tend to 0 for $t \to \infty$ if $\rho > 0$.

The drawback of those results is that they do not relate to properties of the flow in a classical sense, because of the phase space topology. On the other hand, S. Kuksin showed recently that in fact certain squeezing of balls in cylinders may occur in spaces of higher smoothness, if one considers for instance a nonlinear wave equation $u_{tt} = \rho\Delta u + p(u)$ where ρ is a small parameter (small dispersion) ([Kuk_3]). The squeezing phenomena appear in some finite time and are stronger when $\rho \to 0$.

As far as the behaviour of individual smooth solutions are concerned, some examples are obtained in [B_6] of Hamiltonian PDE (in NLS or KdV form) defined as a smooth perturbation of a linear equation, showing in particular that higher derivatives of solutions $u(t)$ for smooth data $u(0) = \phi$ need not be bounded in time. For instance

Proposition 3.5. *There is a Hamiltonian* NLSE *with smooth and local nonlinearity such that* $S_t(B^s(\delta)), t > 0$ *is not a bounded subset of* H^{s_0}, *for any* $s < \infty, \delta > 0$. *Here* $B^s(\delta)$ *denote* $\{\varphi \in H^s \mid \|\varphi\|_s < \delta\}$ *and* s_0 *is some fixed number.*

Another example, closely related to the discussion in the next section is the following. Considering a linear Schrödinger equation

$$-iu_t = -u_{xx} + V(x)u \tag{3.6}$$

where $V(x)$ is a real smooth periodic potential and the periodic spectrum $\{\lambda_k\}$ of $-\frac{d^2}{dx^2} + V$ satisfies a "near resonnance" property

$$\text{dist } (\lambda_{n_j}, \mathbf{Z}\lambda_{n_0}) \to 0 \quad \text{rapidly for} \quad j \to \infty \tag{3.7}$$

for some subsequence $\{n_j\}$. We construct a Hamiltonian perturbation $\Gamma(u) = \frac{\partial}{\partial \overline{u}} G$ such that the solution $u_{\varepsilon,q}$ of the IVP

$$\begin{cases} -iu_t = -u_{xx} + V(x)u + \varepsilon\, \Gamma(u) \\ u(0) = q \end{cases} \tag{3.8}$$

satisfies

$$\inf_{q \in O} \sup_t \|u_{\varepsilon,q}(t)\|_{H^{s_0}} \to \infty \quad \text{for} \quad \varepsilon \to 0\,. \tag{3.9}$$

Here s_0 is again some positive integer and O is some nonempty open subset of $H^{s_0}(\mathbf{T})$.

The general procedure of constructing global solutions piecing together local solutions, using conserved quantities of low smoothness, permit to bound higher derivatives as $C^{|t|}$ for $|t| \to \infty$. The exponential estimate was more recently improved to a polynomial one, provided the Cauchy problem may be solved subcritically with respect to the H^1-norm, for which we assume an *a priori* bound from conservation of the Hamiltonian. The question what is, generically speaking, the "true" order of growth seems an important open problem.

4. Persistency of Periodic and Quasi-periodic Solutions under Perturbation

One of the most exciting recent developments in nonlinear PDE is the use of the classical KAM-type techniques to construct time quasi-periodic solutions of Hamiltonian equations obtained by perturbation of a linear or integrable PDE. This subject is rapidly developing.

In this brief discussion, we only consider perturbations of linear equations. We work in the real analytic category. Important contributions are due to S. Kuksin [Kuk$_1$], using the standard KAM scheme and more precisely infinite dimensional versions of Melnikov's theorem on the persistency of n-dimensional tori in systems with $N > n$ degrees of freedom. His work yields a rather general theory and

we mention only some typical examples of applications to $1D$ nonlinear wave or Schrödinger equations

$$w_{tt} = \left(\frac{\partial^2}{\partial x^2} - V(x;a)\right) w - \varepsilon \, \frac{\partial \varphi}{\partial w} \, (x,\, w;\, a) \tag{4.1}$$

$$-iu_t = -u_{xx} + V(x,a)u + \varepsilon \, \frac{\partial \varphi}{\partial |u|^2} \, \left(x, |u|^2; a\right) u \, . \tag{4.2}$$

Here $V(x,a)$ is a real periodic smooth potential, depending on n outer parameters $a = (a_1, \ldots, a_n)$. Denote $\{\lambda_j(a)\}$ the Dirichlet spectrum of the Sturm–Liouville operator $-\frac{d^2}{dx^2} + V(x,a)$. Thus $\lambda_j(a) = \pi^2 j^2 + 0(1)$ and we assume the following nondegeneracy condition

$$\det \{\partial \lambda_j(a)/\partial a_k \mid 1 \le j,k \le n\} \neq 0 \tag{4.3}$$

(this condition is a substitute for the classical "twist" condition). Denoting $\{\varphi_j\}$ the corresponding eigenfunctions, the $2n$-dimensional linear space

$$Z^0 = \text{span } \{\varphi_j, i\varphi_j \mid 1 \le j \le n\} \tag{4.4}$$

is invariant under the flow of equation (4.2) for $\varepsilon = 0$ and foliated into invariant n-tori

$$T^n(I) = \left\{ \sum_{j=1}^{n} (x_j^+ + ix_j^-)\varphi_j \mid (x_j^+)^2 + (x_j^-)^2 = 2I_j, \; j = 1, \ldots, n \right\} \tag{4.5}$$

which are filled with quasi-periodic solutions of (4.2) for $\varepsilon = 0$. A typical result from [Kuk$_1$] is that under assumption (4.3), for most parameter values of a there is an invariant torus $\sum_{a,I}^{\varepsilon}(\mathbf{T}^n)$ nearby the unperturbed torus $\sum_{a,I}^{0}$ given by (4.5) and filled with quasi-periodic solutions of (4.2). The frequency vector ω_ε of a perturbed solution will be $c\varepsilon$ close to $\omega = (\lambda_1, \ldots, \lambda_n)$ of the unperturbed one.

The methods in [K$_1$] leave out the case of periodic boundary conditions, because of certain limitations of the KAM method (second Melnikov condition) excluding multiplicities in the normal frequencies. A different approach has been recently used by W. Craig and C. E. Wayne [C-W$_{1,2}$], based on the Liapounov–Schmidt decomposition and leading to time periodic solutions of perturbed equations under periodic boundary conditions. This method consists in splitting the problem in a (finite dimensional) resonnant part (Q-equation) and an infinite dimensional non-resonnant part (P-equation). In the PDE-case (contrary to the case of a finite dimensional phase space), small divisor problems appear when solving the P-equation by a Newton iteration method, also in the time periodic case. Writing u in the form

$$u = \sum_{m,k} \widehat{u}(m,k) \; e^{im\lambda t} \; \varphi_k(x) \tag{4.6}$$

and letting the linearized operator act on the Fourier coefficients $\widehat{u}(m,k)$, one gets operators of the form

$$(m\lambda - \lambda_k) + \varepsilon\, T \tag{4.7}$$

where the first term is diagonal and T is essentially given by Toeplitz operators with exponentially decreasing matrix elements. The main task is then to obtain reasonable bounds on their inverses. The problem is closely related to a line of research around localization in the Anderson model with quasi-periodic potential. In this case, the operator T in (4.7) is replaced by $-\Delta$, $\Delta =$ lattice Laplacian, and the first term plays the role of the potential. Of primary importance is the structure of "singular sites", i.e. the pairs $(m,\, k)$ such that $|m\lambda - \lambda_k|$ is small. If we think of λ_k as k^2, it is clear that these sites have separation $\to \infty$ for $k \to \infty$. Let us consider the higher dimensional problem for time periodic solutions. We assume the potential $V(x)$ of the form

$$V(x) = V_1(x_1) + \cdots + V_d(x_d) \tag{4.8}$$

in order to avoid certain problems (the dependence of spectrum and eigenfunctions on V) appearing for $d \geq 2$, other than the small divisor issue that is our primary concern here. Alternatively, one may replace in (4.1), (4.2) the term $V.u$ by a Fourier multiplier to introduce a frequency parameter. The eigenfunctions are then simply exponentials. Replacing λ_k by $|k|^2 = k_1^2 + \cdots + k_d^2$, it turns out that the singular sites may still be partitioned into distant clusters, of diameter and mutual distance $\to \infty$ for $|k| \to \infty$. This enables one to large extent to reproduce the arguments from [C-W$_{1,2}$] to deal with that case also.

We now pass to the quasi-periodic problem. The singular sites are now the pairs $(m,\, k)$ satisfying

$$|\langle m,\, \lambda\rangle - \lambda_k| < 1 \text{ where } m \in \mathbb{Z}^b,\, b > 1\,. \tag{4.9}$$

The structure of those is clearly different here and already in case of a finite dimensional phase space (i.e. finitely many k's), new arguments are needed. There is resemblance with the works of Fröhlich, Spencer, Wittwer [F-S-W] and Surace [Sur] on localization for a quasi-periodic potential. In particular, one relies on a multi-scale analysis. The first step in these investigations is to recover the KAM and Melnikov results using this Liapounov–Schmidt technique. Next, one considers the PDE applications, where the phase space is infinite dimensional. In [B$_4$], we discuss NLS and NLW in $1D$ obtained by perturbing a linear equation. In [B$_5$], $2D$-problems are considered. The presence of spectral clusters of unbounded size leads to certain difficulties we may deal with in $d = 2$ but so far not for larger dimension $d > 2$ (except for time periodic solutions). One may essentially formulate the main model result as follows.

Theorem 4.10. (see [B$_5$]) *Consider a perturbed Schrödinger equation*

$$iu_t - \Delta u + V(x)u + \varepsilon \frac{\partial H}{\partial \overline{u}}(u,\, \overline{u},\, x) = 0 \tag{4.11}$$

or

$$iu_t - \Delta u + (u * V) + \varepsilon \frac{\partial H}{\partial \overline{u}}(u, \overline{u}, x) = 0 \tag{4.12}$$

where $x \in \prod^2, V(x) = V_1(x_1) + V_2(x_2)$ *a real periodic potential in* (4.11) *and* $u * V$ *a real Fourier multiplier in* (4.12). *Denote* $\{\mu_n\}$ *the periodic spectrum of* $-\Delta + V$ *and* $\{\varphi_n\}$ *the corresponding eigenfunctions. Fix indices* $n_1, n_2, \ldots, n_b$ *and consider* $\lambda_1 = \mu_{n_1}, \ldots, \lambda_b = \mu_{n_b}$ *as parameters* (*letting* V *vary in an appropriate way*). *We assume following non-resonnance property* (1^e *Melnikov condition*, cf. [Kuk$_1$])

$$|\langle m, \lambda\rangle + \mu_n| > N_0^{-c} \tag{4.13}$$

for $|m| \leq N_0$, $|n| \leq N_0$ *and* $n \notin \{n_1, \ldots, n_b\}$.

(*The* μ_n *may have a weak dependence on* λ).

Consider the solution to the unperturbed equation $\varepsilon = 0$

$$u_0(x, t) = \sum_{j=1}^{b} a_j e^{i\lambda_j t} \varphi_{n_j}(x) \tag{4.14}$$

with $\{a_j\}$ *fixed. For* ε *small enough* (*depending in particular on* N_0), *there is a* (*Cantor*) *set* Δ *of frequencies* λ (*depending on* $|a_j|$ $(j = 1, \ldots, b)$) *with* $\mathrm{mes}(C\Delta) \to 0$ for $\varepsilon \to 0$ *in the parameter set, such that if* $\lambda \in \Delta$, *there is a perturbed frequency*

$$|\lambda' - \lambda| < C\varepsilon \tag{4.15}$$

and a perturbed solution

$$u_\varepsilon(x, t) = \sum_n \hat{u}_\varepsilon(n)(t)\varphi_n(x) \tag{4.16}$$

$$\hat{u}_\varepsilon(n)(t) = \sum_m \hat{u}_\varepsilon(n, m) e^{i\langle \lambda', m\rangle t} \tag{4.17}$$

with

$$\hat{u}_\varepsilon(n_j, e_j) = a_j \qquad (e_j = j \text{ unit vector in } \mathbb{Z}^b) \tag{4.18}$$

$$\sum_{(n, m) \notin S} e^{(|m|+|n|)^c} |\hat{u}_\varepsilon(n, m)| < \varepsilon^{1/2} \tag{4.19}$$

with

$$S = \{(n_j, e_j) | j = 1, \ldots, b\}.$$

Equations (4.11), (4.12) are parameter dependent, since $V = V(x, \lambda)$. This may sometimes be avoided, relying on amplitude-frequency modulation based on the nonlinear term satisfying an appropriate twist condition (see [C-W$_{1,2}$]). It is our aim to investigate also perturbations of Birkhoff integrable systems, following the same

approach. Written in an appropriate normal form, the unperturbed Hamiltonian may be given the form

$$H(p,\, q,\, y) = h(p) + \frac{1}{2}\langle A(p)y,\, y\rangle + 0(|y|^3) \tag{4.20}$$

(cf. [Kuk$_1$]). Here the phase-space appears as $Z = Z_0 \oplus Y$ and $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ are action-angle variables for Z_0. Specifying $p = a$, write $h(p) = h(a) + \langle \frac{\partial h}{\partial p}(a),\, p - a\rangle + 0(|p-a|^2)$. Then the λ-frequency vector is given by $\frac{\partial h}{\partial p}(a)$. Thus the usual twist condition

$$\det(\operatorname{Hess} H) \neq 0 \tag{4.21}$$

corresponds to (4.3). At this stage, the problem appears essentially as perturbation of a linear system.

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THE WORK OF PIERRE-LOUIS LIONS

by

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Pierre-Louis Lions has made unique contributions over the last fifteen years to mathematics. His contributions cover a variety of areas, from probability theory to partial differential equations (PDEs). Within the PDE area he has done several beautiful things in nonlinear equations. The choice of his problems has always been motivated by applications. Many of the problems in physics, engineering and economics when formulated in mathematical terms lead to nonlinear PDEs; these problems are often very hard. The nonlinearity makes each equation different. The work of Lions is important because he has developed techniques that, with variations, can be applied to classes of such problems. To say that something is nonlinear does not mean much; in fact it could even be linear. The entire class of nonlinear PDEs is therefore very extensive and one does not expect an all-inclusive theory. On the other hand, one does not want to treat each example differently and have a collection of unrelated techniques. It is thus extremely important to identify large classes that admit a unified treatment.

In dealing with nonlinear PDEs one has to allow for nonclassical or nonsmooth solutions. Unlike the linear case one cannot use the theory of distributions to define the notion of a weak solution. One has to invent the appropriate notion of a generalized solution and hope that this will cover a wide class and be sufficient to yield a complete theory of existence, uniqueness, and stability for the class.

Due to the very limited time that is available, I shall focus on three areas within nonlinear PDE where Lions has made major contributions. The first is the so called "viscosity method". This development is a long story that started with some work in collaboration with Crandall. Over many years, in partial collaboration with others (besides Crandall, Evans and Ishii), Lions has developed the method, which is applicable to the large class of nonlinear PDEs known as fully nonlinear second order degenerate elliptic PDEs. The class contains very many important subclasses that arise in different contexts.

By solving a nonlinear PDE one is trying to solve an equation involving an unknown function and its derivatives. Let u be a function in a region G in some

R^n and let $Du, D^2u, \ldots, D^ku$ be its derivatives of order up to k. A nonlinear PDE is an equation of the form

$$F[x, u(x), (Du)(x), (D^2u)(x), \ldots, (D^k(u)(x))] = 0 \text{ in } G$$

with some boundary conditions on ∂G. Such a PDE is said to be nonlinear and of order k. The viscosity method applies in cases where $k = 2$ and $F(x, u, p, H)$ has certain monotonicity properties in the arguments u and H. More precisely, it is nondecreasing in u and nonincreasing in H. Here u is a scalar and H is a symmetric matrix of size $n \times n$ with the natural partial ordering for symmetric matrices.

Some of the many examples of such functions are described below.

Linear elliptic equations:

$$-\sum_{i,j} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + f(x) = 0$$

where the matrix $a^{ij}(x)$ is uniformly positive definite.

In this case the function F is given by

$$F(x, u, p, H) = -\text{Trace } [a(x)H] + f(x)\,.$$

First order equations:

$$f(x, u(x),\ (Du)(x)) = 0$$

These include Hamilton–Jacobi equations where it all started. One added a term of the form $\epsilon\Delta$ to the equation and constructed the solution in the limit as ϵ went to zero. The theory owes its name to its early origins.

If one has a family F_α of such functions one can generate a new one by defining

$$F = \sup_\alpha\ F_\alpha\,.$$

If one has a two-parameter family $F_{\alpha\beta}$ of such functions one can generate a new one by defining

$$F = \sup_\alpha\ \inf_\beta\ F_{\alpha\beta}\,.$$

Such examples arise naturally in control theory and game theory and are referred to as Hamilton–Jacobi–Bellman and Isaacs equations.

In order to understand the notion of a generalized solution it is convenient to talk about supersolutions and subsolutions. Suppose u is a subsolution, i.e.

$$F(x, u(x),\ (Du)(x),\ (D^2u(x))) \le 0$$

and we have another function ϕ, which is smooth, such that $u - \phi$ has a maximum at some point $\hat{x}$. Then by calculus $Du(\hat{x}) = D\phi(\hat{x})$ and $D^2(u)(\hat{x}) \le D^2(\phi)(\hat{x})$. From the monotonicity properties of F it follows that

$$F(\hat{x}, u(\hat{x}),\ (Du)(\hat{x}),\ (D^2u(\hat{x}))) \ge F(\hat{x}, u(\hat{x}),\ (D\phi)(\hat{x}),\ (D^2\phi(\hat{x})))\,.$$

Therefore

$$F(\hat{x}, u(\hat{x}),\ (D\phi)(\hat{x}),\ (D^2\phi(\hat{x}))) \le 0\,.$$

The last inequality makes sense without any smoothness assumption on u. We can try to define a nonsmooth subsolution as a u that satisfies the above for arbitrary smooth ϕ and $\hat{x}$ provided $u - \phi$ has a maximum at $\hat{x}$. The definition of a supersolution is similar, and a solution is one that is simultaneously a super and a subsolution.

Let us consider first a Dirichlet boundary value problem where we want to find a u that solves our equation and has boundary value zero.

The main step is to establish the key comparison theorem (with a long history that began with the work of Crandall and Lions and saw an important contribution from Jensen) that if u is a subsolution and if v is a supersolution in a bounded domain G and if $u \leq v$ on the boundary ∂G then $u \leq v$ in $G \cup \partial G$. This requires some mild regularity conditions on F as well as some nondegeneracy conditions. After all, we have not ruled out $F \equiv 0$. Once such conditions are imposed one can establish the key comparison theorem. From this point on, the theory proceeds in a way similar to the classical Perron's method for solving the Dirichlet problem. Assuming that there is at least one subsolution $\bar{u}$ and at least one supersolution $\bar{v}$ with the given boundary value, one establishes that

$$W(x) = \sup \{w(x) : \bar{u} \leq w \leq \bar{v},\ w \text{ is a subsolution}\}$$

is a solution. The comparison theorem is of course enough to guarantee uniqueness. The constructibility of $\bar{u}$ and $\bar{v}$ depends on the circumstances and is relatively easy to establish.

The richness of the theory is in its flexibility. One can prove stability results of various kinds and the validity of various approximation schemes. One can modify the theory to include Neumann boundary conditions. This is tricky because one has to interpret the normal derivative suitably for a function that has no smoothness requirements and the boundary condition can be nonlinear as well. Treating parabolic equations is not any different. One can just consider t as another variable.

I would suggest the survey article by Crandall, Ishii, and Lions that appeared in the Bulletin of the American Mathematical Society in 1992 for those who want to read more about this area.

The second body of work that I want to discuss has to do with the Boltzmann equation and similar equations. During the last six or seven years Pierre-Louis Lions has played the central role in new developments in the theory of the Boltzmann equation and similar transport equations. These are important in kinetic theory and arise in a wide variety of physical applications. We will for simplicity stay within the context of the Boltzmann equation. In R^3 we have a collection of particles moving along and interacting through "collisions" among themselves. As we do not want to keep track of the positions and velocities of the particles individually, we abstract the situation by the density $f(x, v)$ of particles that are at position x with velocity v. Even if there is no interaction, the density $f(x, v)$ will change in time due to uniform motion of the particles. The time-dependent density $f(t, x, v)$ will

satisfy the equation

$$\frac{\partial f}{\partial t} + v.\nabla_x f = 0\,.$$

The collisions will change this equation to

$$\frac{\partial f}{\partial t} + v.\nabla_x f = Q(f,f)\,.$$

Here Q is a quadratic quantity that represents binary collisions and its precise form depends on the nature of the interaction. Generally it looks like

$$Q(f,f) = \int\int_{R^3\times s^2} dv_* d\omega B(v-v_*,\,\omega)\,\{f'f'_* - ff_*\}\,.$$

The notation here is standard: v and v_* are the incoming velocities and v' and v'_* are the outgoing velocities. B is the collision kernel. For given incoming velocities v and v_*, ω on the sphere S^2 parametrizes all the outgoing velocities compatible with the conservation of energy and momenta.

$$v' = v - (v-v_*,\,\omega)\omega, \quad v'_* = v + (v-v_*,\,\omega)\omega$$

and $f', f_*,\, f'_*$ are $f(t,x,v)$ with v replaced by the correspondingly changed v', v_*, and v'_*.

This problem of course has a long history. Smooth and unique solutions had been obtained for small time or globally for initial data close to equilibrium. Carleman had studied spatially homogeneous solutions. But a general global existence theorem had never been proved. The work of Lions (in collaboration with DiPerna) is a breakthrough for this and many other related transport problems of great physical interest.

Let me spend a few minutes giving you some idea of the method as developed by Lions and others (mostly his collaborators).

Although the nonlinearity looks somewhat benign it causes a serious problem in trying to establish any existence results. The collision term is quadratic and involves both positive and negative terms. To carry out any analysis one must control each piece separately. One gets certain a priori estimates from the conservation of mass and energy. The Boltzmann H-theorem gives an important additional control if one starts with an initial data with finite entropy. If we denote by Q^+ and Q^- the positive and negative terms in the collision term with considerable effort one is able to obtain only local L_1 bounds on $(1+f)^{-1}Q^{\pm}(f,f)$. The weak solutions are therefore formulated in terms of $\log(1+f)$. As there are no smoothness estimates in x one has to show that the velocity integrals contained in Q provide the compactification needed to make the weak limit behave properly.

This idea of "velocity averaging", which is central to these methods, is easy to state in a simple context. Suppose we have a function $g(x,v)$ in $R^N \times R^N$ and for some reasonable function $a(v)$ we have a local L_p estimate on $a(v).\nabla_x g(x,v)$. Then

for a good test function ψ the velocity integral

$$\int_{R^N} \psi(v) g(x,v)\, dv$$

is in a suitable Sobolev space. Another important step that is needed in dealing with the Vlasov equation is the ability to integrate vector fields with minimal regularity. In nonlinear problems you have to learn to live with the regularity that the problem gives you. The writeup by Lions in the Proceedings of the last ICM (Kyoto 1990) provides a survey with references.

The third and final topic that I would like to touch on is the contribution Lions has made to a class of variational problems. There are many nonlinear PDEs that are Euler equations for variational problems. The first step in solving such equations by the variational method is to show that the extremum is attained. This requires some coercivity or compactness. If the quantity to be minimized has an "energy"-like term involving derivatives, then one has control on local regularity along a minimizing sequence. This usually works if the domain is compact. If the domain is noncompact the situation is far from clear. Take for instance the problem of minimizing

$$\int_{R^N} |(\nabla f)(x)|^2 dx - \int\int V(x-y) f^2(x) f^2(y)\, dx\, dy$$

over functions f with L_2 norm λ (fixed positive number). Here $V(.)$ is a reasonable function decaying at ∞. Because of translation invariance, the minimizing sequence must be centered properly in order to have a chance of converging. The key idea is that, in some complicated but precise sense, if the minimizing sequence cannot be centered, then any member of the sequence can be thought of as two functions with supports very far away from each other. If we denote the infimum by $\sigma(\lambda)$, then along such sequences the infimum will be $\sigma(\lambda_1) + \sigma(\lambda_2)$ with $\lambda_1 + \lambda_2 = \lambda$ $0 < \lambda_1, \lambda_2 < \lambda$ rather than $\sigma(\lambda)$. If independently one can show that $\sigma(\lambda)$ is strictly subadditive, then one can prove the existence of a minimizer. This idea has been developed fully and applied successfully by Lions to many important and interesting problems.

See the papers in Annales de l'Institut Henri Poincaré, Analyse Non Linéaire 1984 by Lions for many examples where this point of view is successfully used.

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ON SOME RECENT METHODS FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Dedicated to the memory of Ron DiPerna (1947–1989)

1. Introduction

We wish to present here some aspects of a few general methods that have been introduced recently in order to solve nonlinear partial differential equations and related problems in nonlinear analysis.

As is well known, nonlinear partial differential equations have become a rather vast subject with a long history of deep and fruitful connections with many other areas of mathematics and various sciences like physics, mechanics, chemistry, engineering sciences, etc. And we shall not pretend to make any attempt at surveying all recent activities in that field. Also, we shall concentrate on rather theoretical issues leaving completely aside more applied issues such as mathematical modelling, numerical questions that go hand in hand in a fundamental way with the theories. For a discussion of the interaction between nonlinear analysis and modern applied mathematics, we refer the reader to the report by Majda [56] in the preceding Congress.

We shall mainly discuss here recent methods that have been developed recently for the analysis of the major mathematical models of gas dynamics (and compressible fluid mechanics), namely the Boltzmann equation and compressible Euler and Navier–Stokes equations (essentially in the so-called "isentropic regime"). These methods include velocity averaging, regularization by collisions that we shall apply to the solution of the Boltzmann equation (Section 2 below), and compactness via commutators and in particular compensated compactness, which we illustrate on isentropic compressible Euler and Navier–Stokes equations.

This selection of topics (equations and methods) is by no means an exhaustive treatment of all the exciting progresses that have taken place recently in nonlinear partial differential equations: many more important problems have been investigated — see for instance the various reports in this Congress related to Nonlinear

Partial Differential Equations — and other methods and theories have been developed. We briefly mention a few in Section 4. And even for the methods that we describe here, much more could be said in particular about applications to other relevant problems.

We only hope that our selection will serve as a good illustration of recent activities. It will also emphasize some current trends that go far beyond the material discussed here. The first one is the analysis of the qualitative behavior of solutions (regularity, compactness, classification of possible behaviors, etc.). The second one, related to the preceding one, concerns the structure of specific nonlinearities and its interplay with the behavior (or possible behaviors) of solutions. Finally, this requires theories and methods that are connected with many branches of mathematics and analysis in particular.

2. Boltzmann Equation

2.1. *Existence and compactness results*

The Boltzmann equation is given by

$$\frac{\partial f}{\partial t} + v.\nabla_x f = Q(f,f) \qquad (x,v) \in \mathbb{R}^{2N},\ t \geq 0 \tag{1}$$

where the unknown f is a nonnegative function on $\mathbb{R}^{2N} \times [0,\infty)$, $N \geq 2$, ∇_x denotes the gradient with respect to x, and we denote by $x \cdot y$ or (x,y) the scalar product in $\mathbb{R}^N$. The nonlinear operator Q can be written as

$$Q(f,f) = Q^+(f,f) - Q^-(f,f) \tag{2}$$

$$Q^+(f,f) = \int_{\mathbb{R}^N} dv_* \int_{S^{N-1}} d\omega\, B(v-v_*,\omega) f' f'_* \tag{3}$$

$$Q^-(f,f) = \int_{\mathbb{R}^N} dv_* \int_{S^{N-1}} d\omega\, B(v-v_*,\omega) f f_* = f L(f), \quad L(f) = f \underset{v}{*} A \tag{4}$$

where $f_* = f(x,v_*,t)$, $f' = f(x,v',t)$, $f'_* = f(x,v'_*,t)$, $A(z) = \int_{S^{N-1}} B(z,\omega)\, d\omega$, and $B = B(z,\omega)$ is a given nonnegative function of $|z|$ and $|(z,\omega)|$, is called the scattering cross-section or the collision kernel, which depends on the physical interactions of the gas particles (or molecules) and

$$v' = v - (v-v_*,\omega)\omega \quad , \quad v'_* = v_* + (v-v_*,\omega)\omega\,. \tag{5}$$

A typical example (the so-called hard spheres case) of B is given by; $B = |(z,\omega)|$. We always assume that $A \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $(1+|z|^2)^{-1} \cdot \int_{|\xi|<R} A(z-\xi) d\xi \to 0$ as $|z| \to \infty$, for all $R \in (0,\infty)$.

Of course, we wish to solve (1) given an initial condition that is the values of f at $t=0$

$$f|_{t=0} = f_0 \qquad \text{in} \quad \mathbb{R}^{2N}\,. \tag{6}$$

The initial value problem (1), (6) is a deceivingly simple-looking first-order partial differential equation with nonlinear (quadratic) nonlocal terms. It is a relevant model for the study of a rarefied gas and is currently used for flights in the upper layers of the atmosphere (Mach 20–24, altitude of 70–120 km). The statistical description of a gas in terms of the evolution of the density f of molecules was originally obtained by Boltzmann [6] (see also Maxwell [57], [58]). There is a long history of important mathematical contributions to the study of (1) by Hilbert [31], Carleman [8], [9] etc. Further details on the derivation of (1) and references to earlier mathematical contributions can be found in Grad [28], Cercignani [10], and DiPerna and Lions [18].

The major mathematical difficulty of (1), (6) is the lack of a priori estimates on solutions: only bounds on f in L^1 (with weights) and on $f \log f$ in L^1 are known! Nevertheless, the following result, taken from [18], [20], holds:

Theorem 2.1. *Let $f_0 \geq 0$ satisfy:*

$$\int_{\mathbb{R}^{2N}} f_0(1 + |x|^2 + |v|^2 + |\log\ f_0|)\, dx\, dv < \infty\,.$$

Then there exists a global weak solution of (1), (6)$f \in C([0,\infty);\ L^1(\mathbb{R}^{2N}_{x,v}))$ *satisfying*

$$\sup_{t\in[0,\infty)} \int_{\mathbb{R}^{2n}} f(t)(1 + |x - vt|^2 + |v|^2 + |\log\ f(t)|)\, dx\, dv < \infty$$

and the following entropy inequality for all $t \geq 0$

$$\int_{\mathbb{R}^{2N}} f(t)\ \log\ f(t)\, dx\, dv + \frac{1}{4}\int_0^t ds \int_{\mathbb{R}^N} dx\, D[f] \leq \int_{\mathbb{R}^{2N}} f_0\ \log\ f_0\, dx\, dv \qquad (7)$$

where $D[f] = \iint_{\mathbb{R}^{2N}} dv\, dv_* \int_{S^{N-1}} B\, d\omega (f'f'_* - ff_*) \log\ \frac{f'f'_*}{ff_*}$.

Remarks 2.1. (i) We do not want to give here the precise definition of global weak solution as it is a bit too technical. Let us mention that the notion introduced in [18], [20] is modified in Lions [48] (additional properties are imposed on f in [48]).

(ii) Further regularity properties of solutions are an outstanding open problem. It is only known that the regularity of solutions is not "created by the evolution" and has to come from the initial condition f_0. It is tempting to think, in view of the results shown in [48] (see sections 2.2, 2.3 below), that, at least in the model case when $B = \varphi\big(|z|, \frac{|(z,\omega)|}{|z|}\big)$ with $\varphi \in C_0^\infty((0,\infty)\times(0,1))$, f is smooth if f_0 is smooth. Related to the regularity issue is the uniqueness question: uniqueness of weak solutions is not known (it is shown in [48] that any weak solution is equal to a solution with improved bounds assuming that the latter exists!).

(iii) The assumption made upon B corresponds to the so-called angular cut-off.

(iv) Boundary conditions for Boltzmann's equation can be treated: see Hamdache [29] for an analogue of the above result in that case. Realistic boundary conditions require some new a priori estimates and are treated in Lions [46].

(v) Other kinetic models of physical and mathematical interest can be studied by the methods of proof of Theorem 2.1: see for instance DiPerna and Lions [19], Arkeryd and Cercignani [2], Esteban and Perthame [22], and Lions [48].

The strategy of proof for Theorem 2.1 is a classical one, which is almost always the one followed for the proofs of *global existence results*: one *approximates* the problem by a sequence of simpler problems having the same structure (and the same a priori bounds) for which one shows easily the existence of global solutions, and then one tries to *pass to the limit*. This strategy is also useful for the mathematical analysis of numerical methods because one can view numerical solutions as approximated solutions or solutions of approximated problems. This is why the main mathematical problem behind the proof of Theorem 2.2 is the analysis of the behavior of *sequences of solutions* (we could as well consider approximated solutions ...) and in particular of passage to the limit in the equation. This is a delicate question because the available a priori bounds only yield *weak convergences* that are not enough to *pass to the limit in nonlinear terms*. This theme will be recurrent in this report (as it was already in Majda's report [56]).

We thus consider a sequence of (weak or even smooth) nonnegative solutions f^n of (1) corresponding to initial conditions (6) with f_0 replaced by f_0^n and we assume

$$\sup_{n\geq 1} \int_{\mathbb{R}^{2N}} f_0^n (1 + |x|^2 + |v|^2 + |\log\ f_0^n|)\, dx\, dv < \infty \tag{8}$$

$$\sup_{n\geq 1}\ \sup_{t\geq 0} \int_{\mathbb{R}^{2N}} f^n(t)(1 + |x - vt|^2 + |v|^2 + |\log\ f^n|)\, dx\, dv < \infty \tag{9}$$

$$\sup_{n\geq 1} \int_0^\infty dt \int_{\mathbb{R}^N} dv\, D[f^n] < \infty\,. \tag{10}$$

Without loss of generality — extracting subsequences if necessary — we may assume that f_0^n, f^n converge weakly in $L^1(\mathbb{R}^{2N}),\ L^1(\mathbb{R}^{2N} \times (0,T))(\forall\ T \in (0,\infty))$ respectively to f_0, f.

Theorem 2.2. *We have for all* $\psi \in C_0^\infty(\mathbb{R}_v^N),\ T, R \in (0,\infty)$

$$\int_{\mathbb{R}^N} f^n \psi\, dv \underset{n}{\longrightarrow} \int_{\mathbb{R}^N} f\psi\, dv \qquad \textit{in} \quad L^1(\mathbb{R}_x^N \times (0,T))\,, \tag{11}$$

$$\left.\begin{array}{c} \displaystyle\int_{\mathbb{R}^N} Q^+(f^n, f^n)\psi\, dv \underset{n}{\longrightarrow} \int_{\mathbb{R}^N} Q^+(f,f)\psi\, dv\,, \\ \displaystyle\int_{\mathbb{R}^N} Q^-(f^n, f^n)\psi\, dv \underset{n}{\longrightarrow} \int_{\mathbb{R}^N} Q^-(f,f)\psi\, dv \\ \textit{in measure for } |x| < R,\ t \in (0,T)\,, \end{array}\right\} \tag{12}$$

and f is a global weak solution of (1), (6).

Theorem 2.3. (1) *We have for all* $R, T \in (0, \infty)$

$$Q^+(f^n, f^n) \underset{n}{\longrightarrow} Q^+(f, f) \textit{in measure for } |x| < R,\ |v| < R,\ t \in (0, T)\,. \tag{13}$$

(2) *If* f_o^n *converges in* $L^1(\mathbb{R}^{2N})$ *to* f_0*, then* f^n *converges to* f *in* $C([0,T];\ L^1(\mathbb{R}^{2N}_{x,v}))$ *for all* $T \in (0, \infty)$.

Remarks 2.2. (i) Theorem 2.2 is shown in [18] — a simplification of the proof of the passage to the limit (using (13)) is given in [48]. Theorem 2.3 is taken from [48].

(ii) The heart of the matter in Theorem 2.2 is (11), which is a consequence of the velocity averaging phenomenon detailed in Section 2.2 below. The proof of Theorem 2.3 relies upon the results of Section 2.3 below.

(iii) It is shown in Lions [48] that the conclusion in (2) of Theorem 2.3 implies that f_0^n converges to f_0 in $L^1(\mathbb{R}^{2N})$: in other words, no compactification and in particular no regularization is taking place for $t > 0$. As indicated in [47] (see also the recent result of Desvillettes [16]) this fact might be related to the angular cut-off assumption because grazing collisions seem to generate some compactification ("nonlinear hypoelliptic features" in the model studied in [47]).

2.2. *Velocity averaging*

A typical example of the so-called velocity averaging results is the following

Theorem 2.4. *Let* $m \geq 0$*, let* $\theta \in [0, 1)$*, and let* $f, g \in L^p(\mathbb{R}^N_x \times \mathbb{R}^N_v \times \mathbb{R}_t)$ *with* $1 < p \leq 2$*. We assume*

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = (-\Delta_{x,t} + 1)^{\theta/2}(-\Delta_v + 1)^{m/2} g \qquad \textit{in} \quad \mathcal{D}'(\mathbb{R}^{2N+1})\,. \tag{14}$$

Then, for all $\psi \in C_0^\infty(\mathbb{R}^N)$*,* $\int_{\mathbb{R}^N} f(x, v, t)\psi(v)\, dv$ *belongs to the (Besov) space* $B_2^{s,p}(\mathbb{R}^N \times \mathbb{R})$ *— and thus to* $H^{s',p}(\mathbb{R}^N)$ *for all* $0 < s' < s$ *— where* $s = (1 - \theta)\frac{p-1}{p}(1 + m)^{-1}$.

Remarks 2.3. (i) If $m = 0$, $\int_{\mathbb{R}^N} f\psi\, dv \in H^{s,p}$ with $s = \frac{p-1}{p}$. The above exponent s is optimal in general (this is shown in a work to appear by the author). Similar results are available if $2 < p < \infty$ or in more general settings: we refer the reader to DiPerna, Lions, and Meyer [21].

(ii) Such velocity averages are known in statistical physics (or mechanics) as macroscopic quantities. The above result shows that transport equations induce some improved partial regularity on velocity averages (by some kind of dispersive effect).

(iii) The first results in this direction were obtained in Golse, Perthame, and Sentis [27], Golse, Lions, Perthame, and Sentis [26] (where the case $m = 0$ is considered).

The case $m \geq 0,\ p = 2$, was treated in DiPerna and Lions [19] while the general case is due to DiPerna, Lions, and Meyer [21] — a slight improvement of the Besov space can be found in Bézard [5]. Two related strategies of proof are proposed in [21] that both rely on Fourier analysis, one uses some harmonic analysis, namely product Hardy spaces and interpolation theory, while the second one uses classical multipliers theory and careful Littlewood–Paley dyadic decompositions. However, the main idea is rather elementary and described below in extremely rough terms.

As indicated in the preceding remark, we give a caricatural (but accurate!) explanation of the phenomena illustrated by Theorem 2.4. If we Fourier transform (14) in (x,t), we see that we gain decay (=regularity) in (ξ,τ) — dual variables of (x,t) — provided $|\tau + v\cdot\xi| \geq \delta|v\cdot\xi|$ for some $\delta > 0$. On the other hand, the set of v on which we do not gain that regularity, namely $\{v \in \text{Supp }(\psi)/|\tau + v\cdot\xi| < \delta|(\tau,\xi)|\}$, has a measure of order δ, and hence contributes little to the integral $(\int_{\mathbb{R}^n} \hat{f}(\xi,v,\tau)\psi(v)\,dv)$. Balancing the two contributions, we obtain some (fractional) regularity.

Of course, such improved regularity yields local compactness (in (x,t)) of the velocity averages and leads (after some work) to (11).

2.3. *Gain terms and Radon transforms*

We set for $f, g \in C_0^\infty(\mathbb{R}^N_v)$

$$Q^+(f,g) = \int_{\mathbb{R}^N} dv_* \int_{S^{N-1}} d\omega\, B(v - v_*, \omega)\, f' g'_* \tag{15}$$

and we assume (to simplify the presentation) that B satisfies:
$B(z,\omega) = \varphi\big(|z|,\ \frac{|(z,\omega)|}{|z|}\big)$ (this is always the case in the context of (1)) and $\varphi \in C_0^\infty((0,\infty)\times(0,1))$. We denote by $\mathcal{M}(\mathbb{R}^N)$ the space of bounded measures on $\mathbb{R}^N$.

Theorem 2.5. *The operator Q^+ from $\mathcal{M}(\mathbb{R}^N)\times H^s(\mathbb{R}^N)$ and $H^s(\mathbb{R}^N)\times\mathcal{M}(\mathbb{R}^N)$ into $H^{s+\frac{N-1}{2}}(\mathbb{R}^N)$ is bounded for all $s\in\mathbb{R}$.*

Remark 2.4. This result is taken from Lions [48] using generalized Radon transforms; a variant of this proof making direct connection with the classical Radon transform has been recently given by Wennberg [72] (this proof, contrarily to the one in [48], does not extend to more general situations such as collision models for mixtures or relativistic models — this case is treated in Andréasson [1]).

The above gain of regularity ($\frac{N-1}{2}$ derivatives) can be shown by writing Q^+ or its adjoint as a "linear combination" of translates of some Radon-like transforms given by

$$R\psi(v) = \int_{S^{N-1}} B(v,\omega)\psi((v,\omega)\omega)\,d\omega \quad , \quad \forall\psi\in C_0^\infty(\mathbb{R}^N) \tag{16}$$

or

$$R\psi(v) = \int_{S^{N-1}} B(v,\omega)\psi(v-(v,\omega)\omega)\,d\omega \quad , \quad \forall\psi\in C_0^\infty(\mathbb{R}^N)\,. \tag{17}$$

In both cases, one integrates φ over the set $\{(v,\omega)\omega \,|\, \omega \in S^{N-1}\} = \{v-(v,\omega)\omega | \omega \in S^{N-1}\}$, which is the sphere centered at $\frac{v}{2}$ and of radius $\frac{|v|}{2}$. These operators are rather special Fourier integral operators often called generalized Radon transforms (see for instance Phong and Stein [62], Stein [66]). The crucial fact is that the set over which ψ is integrated "movies" with v — except that all these spheres go through 0, but this does not create difficulties because B vanishes if $(v,\omega)\omega = 0$ or if $v-(v,\omega)\omega = 0$. This is the main reason why one can prove that R is bounded from $H^s(\mathbb{R}^N)$ into $H^{s+\frac{N-1}{2}}(\mathbb{R}^N)$ for all $s \in \mathbb{R}(\frac{N-1}{2}$ comes from the stationary phase principle...).

3. Compressible Euler and Navier–Stokes Equations

The compressible Euler and Navier–Stokes equations are the basic models for the evolution of a compressible gas. In the case of aeronautical applications, the main difference between the domains of validity of the Boltzmann equation and the Euler–Navier–Stokes systems is the altitude of the aircraft. This indicates that there should be a transition from the Boltzmann model to those mentioned here. Mathematically, this corresponds to replacing B by $\frac{1}{\varepsilon}\, B$ in (1) and letting ε go to 0 (at least formally): as is well known, one recovers, taking velocity averages of the limit f(i.e. $\rho = \int_{\mathbb{R}^N} f\, dv$, $\rho u = \int_{\mathbb{R}^N} fv\, dv$, $\rho E = \int_{\mathbb{R}^N} f|v|^2\, dv$), the compressible Euler equations (with $\gamma = \frac{N+2}{N}$) — see Cercignani [10] for more details. This heuristic limit (and related limits) remains completely open from a mathematical viewpoint: partial results can be found in Nishida [61], and Ukaï and Asano [71], and recent progress based upon the material described in Section 2 above is due to Bardos, Golse, and Levermore [4]. Related problems are described in Varadhan's report in this Congress.

The compressible Euler and Navier–Stokes equations take the following form:

$$\frac{\partial p}{\partial t} + \operatorname{div}\,(\rho u) = 0 \qquad x \in \mathbb{R}^N\,,\, t \geq 0 \tag{18}$$

$$\frac{\partial}{\partial t}(\rho u) + \operatorname{div}\,(\rho u \otimes u) - \lambda \Delta u - (\lambda+\mu)\nabla \operatorname{div} u + \nabla p = 0 \quad x \in \mathbb{R}^N\,, t \geq\ 0 \tag{19}$$

and an equation for the pressure p (or equivalently for the total energy or the temperature) that we do not wish to write for reasons explained below. The unknowns ρ, u correspond respectively to the density of the gas ($\rho \geq 0$) and its velocity u (where $u(x,t) \in \mathbb{R}^N$). The constants λ, μ are the viscosity coefficients of the fluid: if $\lambda = \mu = 0$, the above system is called the compressible Euler equations, whereas if $\lambda > 0,\ 2\lambda + \mu > 0$, it is called the compressible Navier–Stokes equations. Despite the long history of these problems, the global existence of solutions "in the large" is still open for the full (i.e. with the temperature equation) systems except in the case of compressible Navier–Stokes equations when $N = 1$: in that case, general existence and uniqueness results can be found in Kazhikov and Shelukhin [37], Kazhikov [36], Serre [64], [63], and Hoff [34]. This is why we shall restrict

ourselves here to the so-called "isentropic" (or barotropic) case where one postulates that p is a function of ρ only, and in order to fix ideas we take

$$p = a\rho^\gamma \quad , \quad a > 0 \quad , \quad \gamma > 1\,. \tag{20}$$

This condition is a severe restriction from the mechanical viewpoint (in the Navier–Stokes case, it essentially means considering the adiabatic case and neglecting the viscous heating). Mathematically, it leads to an interesting model problem that is supposed to capture some of the difficulties of the exact systems.

Of course we complement (18)–(19) with initial conditions

$$\rho|_{t=0} = \rho_0 \quad , \quad \rho u|_{t=0} = m_0 \quad , \quad \text{in} \quad \mathbb{R}^N \tag{21}$$

where $\rho_0 \geq 0,\ m_0$ are given function on $\mathbb{R}^N$.

We study the case of compressible isentropic Euler equations in Section 3.1. The analogous problem for Navier–Stokes equations is considered in Section 3.3.

3.1. *1D isentropic gas dynamics*

We thus consider the following system

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \quad , \quad \frac{\partial(\rho u)}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 + a\rho^\gamma) = 0 \qquad x \in \mathbb{R}\,,\, t > 0 \tag{22}$$

where $\rho \geq 0$, and $a > 0,\ \gamma > 1$ are given constants. Without loss of generality (by a simple scaling) we can take $a = \frac{(\gamma-1)^2}{4\gamma}$ (to simplify some of the constants below).

As is well known for such systems of nonlinear hyperbolic (first-order) equations, singularities develop in finite time: even if $\rho_0 = \bar\rho + \rho_1 > 0$ on $\mathbb{R}$ with $\bar\rho \in \mathbb{R}$, $\bar\rho > 0,\ \rho_1, u_0 \in C_0^\infty(\mathbb{R})$, then u_x and ρ_x become infinite in finite time (see Lax [38], [39], [41], and Majda [54], [55] for more details). In addition, bounded solution of (22), (21) are not unique and additional requirements known as (Lax) entropy conditions on the solutions are needed (Lax [41], [40], see also the report by Dafermos in this Congress).

In the case of (22), these requirements take the following form (see Diperna [17], Chen [11], and Lions, Perthame, and Tadmor [53]):

$$\frac{\partial}{\partial t}[\varphi(\rho, \rho u)] + \frac{\partial}{\partial x}[\psi(\rho, \rho u)] \leq 0 \qquad \text{in} \quad \mathcal{D}'(\mathbb{R} \times (0, \infty)) \tag{23}$$

and φ, ψ are given by

$$\begin{cases} \varphi = \displaystyle\int_{\mathbb{R}} dv\, \omega(v)(\rho^{\gamma-1} - (v-u)^2)_+^\lambda\,, \\ \psi = \displaystyle\int_{\mathbb{R}} dv\, [\theta v + (1-\theta)u]\omega(v)(\rho^{\gamma-1} - (v-u)^2)_+^\lambda \end{cases} \tag{24}$$

where ω is an arbitrary convex function on $\mathbb{R}$ such that ω'' is bounded on $\mathbb{R}$, $\lambda = \frac{3-\gamma}{2(\gamma-1)}$, $\theta = \frac{\gamma-1}{2}$.

Theorem 3.1. *Let $\rho_0, m_0 \in L^\infty(\mathbb{R})$ satisfy: $\rho_0 \geq 0$, $|m_0| \leq C\rho_0$ a.e. in $\mathbb{R}$ for some $C \geq 0$. Then there exists $(\rho, u) \in L^\infty(\mathbb{R} \times (0,\infty))$ $(\rho \geq 0)$ solution of* (21) – (22) *satisfying* (23).

As explained in Section 2.2, the proof of the existence results depends very much upon the stability and compactness results shown below (in fact one approximates (22) by the vanishing viscosity method; i.e., adding $-\varepsilon\frac{\partial^2\rho}{\partial x^2}, -\varepsilon\frac{\partial^2(\rho u)}{\partial x^2}$ in the equations respectively satisfied by $\rho, \rho u$ where $\varepsilon > 0$, and one lets ε go to 0). We thus consider a sequence (ρ^n, u^n) of solutions of (22) satisfying (23) and we assume that (ρ^n, u^n) is bounded uniformly in n in $L^\infty(\mathbb{R}\times(0,\infty))$ ($\rho^n \geq 0$ a.e.). Without loss of generality, we may assume that (ρ^n, u^n) converges weakly in $L^\infty(\mathbb{R} \times (0,\infty))$–weak* to some $(\rho, u) \in L^\infty(\mathbb{R}\times(0,\infty))$ ($\rho \geq 0$ a.e.). The main mathematical difficulty is the lack of any a priori estimate (except for $\gamma = 3$, the so-called monoatomic case) that would ensure the pointwise compactness needed to pass to the limit in $\rho^n(u^n)^2$ or $(\rho^n)^\gamma$.

Theorem 3.2. *$\rho^n, \rho^n u^n$ converge in measure on $(-R, R) \times (0, T)$ (for all $0 < R, T < \infty$) to $\rho, \rho u$ respectively. And (ρ, u) is a solution of* (22) *satisfying* (23).

Remarks 3.1. (i) This result shows that the hyperbolic system (22) has compactifying properties because initially at $t = 0$ we did not require that ρ^n or $\rho^n u^n$ converge in measure.

(ii) Theorem 3.2 is essentially due to DiPerna [17] if $\gamma = \frac{2k+3}{2k+1}(k \geq 1)$, Chen [11] if $1 < \gamma \leq \frac{5}{3}$. It is shown in Lions, Perthame, and Tadmor [53] if $\gamma \geq 3$ and in Lions, Perthame, and Souganidis [51] if $1 < \gamma < 3$. The existence result (Theorem 3.1) for $1 < \gamma < \infty$ is taken from [51].

(iii) The proofs in [53], [51] use two main tools: the method introduced by Tartar [69] (and developed by DiPerna [17]) which combines the compensated-compactness theory of Tartar [68], [69], Murat [60] and the entropy inequalities (23), and the kinetic formulation of (22) introduced in [52], [53] where one adds a new "velocity" variable, and writes the unknowns $(\rho, \rho u)$ in terms of macroscopic quantities (velocity averages) associated with a density $f(x, v, t)$ that has a fixed "profile" in v (a "pseudo-maxwellian"). This formulation connects the Boltzmann theory as described in Section 2 and the study of compressible hydrodynamic (or gas dynamics) macroscopic models. More details on this new approach are to be found in Perthame's report in this Congress. In the next section, we present some aspects of the compensated-compactness theory.

3.2. *Compensated compactness and Hardy spaces*

One important point in the compensated-compactness theory developed by Tartar [68], [69] and F. Murat [60] is the systematic detection of nonlinear quantities that

enjoy "weak compactness" properties. A typical example known as the div-curl example — it is precisely the one used in the proof of Theorem 3.2 — is given by the following result taken from [60].

Theorem 3.3. *Let* (E^n, B^n) *converge weakly to* (E, B) *in* $L^p(\mathbb{R}^N)^N \times L^q(\mathbb{R}^N)^N$ *with* $1 < p < \infty$, $\frac{1}{q} + \frac{1}{p} = 1$, $N \geq 2$. *We assume that* curl E^n, div B^n *are relatively compact in* $W^{-1,p}(\mathbb{R}^N)$, $W^{-1,q}(\mathbb{R}^N)$ *respectively. Then,* $E^n \cdot B^n$ *converges weakly* (*in the sense of measures or in distributions sense*) *to* $E \cdot B$.

Remark 3.2. Let us sketch a proof. We write: $E^n = \nabla\pi^n + \tilde{E}^n$ where div $\tilde{E}^n = 0$, $\tilde{E}^n$ is compact in $L^p(\mathbb{R}^N)$ (Hodge–De Rham decompositions), $\pi^n \in L^p_{\text{loc}}(\mathbb{R}^N), \nabla\pi^n \in L^p(\mathbb{R}^N)$. Then, we only have to pass to the limit in $B^n \cdot \nabla\pi^n = \text{div } (\pi^n B^n) - \pi^n \text{div } B^n$. The first term passes to the limit because π^n is compact in $L^p_{\text{loc}}(\mathbb{R}^N)$ (Rellich–Kondrakov theorem) while the second term also does because div B^n is relatively compact in $W^{-1,q}(\mathbb{R}^N)$ and $\nabla\pi^n$ is bounded in $L^p(\mathbb{R}^N)$.

As shown in Coifman, Lions, Meyer, and Semmes [12], the above nonlinear phenomenon is intimately connected with some general results in harmonic analysis associated with the (multi-dimensional) Hardy spaces denoted here by $H_p(\mathbb{R}^N)$ $(0 < p \leq 1)$: see Stein and Weiss [67], Fefferman and Stein [23], and Coifman and Weiss [14] for more details on Hardy spaces.

In particular, the following result holds.

Theorem 3.4. *Let* $E \in L^p(\mathbb{R}^N)$ *satisfy* curl $E = 0$ *in* $\mathcal{D}'(\mathbb{R}^N)$, *let* $B \in L^q(\mathbb{R}^N)$ *satisfy* div $B = 0$ *in* $\mathcal{D}'(\mathbb{R}^N)$ *with* $1 < p, q < \infty$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} < 1 + \frac{1}{N}$. *Then* $E \cdot B \in H_r(\mathbb{R}^N)$.

Remarks 3.3. (i) This result is taken from [12] (and was inspired by a surprising observation due to Müller [59]).

(ii) The relations between the weak compactness result (Theorem 3.3) and the regularity result (Theorem 3.4) are made clear in [12] and follow from some general considerations on dilation and translation invariant multilinear forms that enjoy a crucial cancellation property ($\int_{\mathbb{R}^N} E \cdot B\, dx = 0$ in Theorem 3.4 above).

(iii) Theorem 3.4 is one of the tools used in the proof by Hélein [30] of the regularity of two-dimensional harmonic maps.

(iv) It is shown in [12] that any element of $H_1(\mathbb{R}^N)$ can be decomposed in a series $\sum_{n\geq 1} \lambda_n E_n \cdot B_n$ where $\|E_n\|_{L^2} = \|B_n\|_{L^2} = 1$, div B_n = curl $E_n = 0$, $\sum_{n\geq 1} |\lambda_n| < \infty$.

If we denote by R_k the Riesz transform $(= \partial_k(-\Delta)^{-1/2})$, then, under the conditions of Theorem 3.4, there exists $\hat{\pi} \in L^p(\mathbb{R}^N)$ such that $E = R\hat{\pi}$. And $E \cdot B = B \cdot R\hat{\pi} = B \cdot R\hat{\pi} + (R \cdot B)\hat{\pi}$ because $R \cdot B = (-\Delta)^{-1/2}\text{div } B = 0$. Then we can recover the case $r = 1$ in Theorem 3.4 using the H_1-BMO duality and the

result on commutators due to Coifman, Rochberg, and Weiss [13]: indeed, we then obtain $f(R_k g) + (R_k f)g \in H_1(\mathbb{R}^N)$ for each $k \geq 1$, $f \in L^p(\mathbb{R}^N)$, $g \in L^q(\mathbb{R}^N)$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

3.3. *Isentropic Navier–Stokes equations*

We now consider the system

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}\,(\rho u) = 0\,, \\ \dfrac{\partial \rho u}{\partial t} + \operatorname{div}\,(\rho u \otimes u) - \lambda \Delta u - (\lambda + \mu)\nabla \operatorname{div}\, u + a\nabla \rho^\gamma = 0\,, \\ \qquad\qquad x \in \mathbb{R}^N\,,\, t > 0\,, \end{cases} \tag{25}$$

where $a > 0$, $1 < \gamma < \infty, \lambda > 0$, $2\lambda + \mu > 0$, $\rho(x,t) \geq 0$ on $\mathbb{R}^N \times (0,\infty)$, with the initial conditions (21) that are required to satisfy

$$\begin{cases} \rho_0 \in L^1(\mathbb{R}^N) \cap L^\gamma(\mathbb{R}^N)\,, \quad \rho_0 \geq 0\,, \\ m_0 = \sqrt{\rho_0} v_0 \quad \text{a.e. with} \quad v_0 \in L^2(\mathbb{R}^N)\,. \end{cases} \tag{26}$$

Theorem 3.5. *We assume* (26) *and* $\gamma \geq \frac{3}{2}$ *if* $N = 2$, $\gamma \geq \frac{9}{5}$ *if* $N = 3$, $\gamma > \frac{N}{2}$ *if* $N \geq 4$. *Then there exists a solution* $(\rho, u) \in L^\infty(0,\infty; L^\gamma(\mathbb{R}^N)) \cap L^2(0,T; H^1(B_R))$ $(\forall\, R, T \in (0,\infty))$ *of* (25), (21) *satisfying in addition*: $\rho \in C([0,\infty); L^p(\mathbb{R}^N))$ *if* $1 \leq p < \gamma$, $\rho|u|^2 \in L^\infty(0,\infty; L^1(\Omega))$, $\rho \in L^q(\mathbb{R}^N \times (0,T))$ *for* $1 \leq q \leq \gamma + \frac{2\gamma}{N} - 1$ *if* $N \geq 2$.

$$\begin{cases} \displaystyle\int_\Omega \frac{1}{2}\rho(t)|u(t)|^2 + \frac{a}{\gamma - 1}\rho(t)^\gamma\, dx + \int_0^t ds \int_\Omega \lambda|\nabla u|^2 + (\lambda + \mu)(\operatorname{div}\, u)^2\, dx \\ \qquad\qquad \displaystyle\leq \int_\Omega \frac{1}{2}\frac{|m_0|^2}{\rho_0} + \frac{a}{\gamma - 1}\rho_0^\gamma\, dx \end{cases} \tag{27}$$

for almost all $t \geq 0$.

Remarks 3.4. (i) This result is taken from Lions [45] (see also [50]). If $N = 1$, more general results are available and we refer to Serre [63], Hoff [32], [33].

(ii) If $N \geq 2$, the uniqueness and further regularity of solutions are completely open as is the case of a general $\gamma > 1$. The case $\gamma = 1$ is also an interesting mathematical problem (see [49]).

(iii) Of course, the equations contained in (25) hold in the sense of distributions.

(iv) The preceding result is rather similar to the results obtained by Leray [42], [43], [44] on the global existence of weak solutions of three-dimensional incompressible Navier–Stokes equations satisfying an energy inequality like (27). Despite many important contributions (like the partial regularity results obtained by Caffarelli,

Kohn, and Nirenberg [7]), the uniqueness and regularity of solutions are still open questions.

As explained in the previous sections, the above existence result is based upon a convergence result for sequences of solutions ρ^n, u^n satisfying uniformly in n the properties mentioned in the above result. Hence, without loss of generality, we may assume that (ρ^n, u^n) converge weakly to (ρ, u) in $L^\gamma(\mathbb{R}^N \times (0,T)) \times L^2(0,T; H^1(B_R))$ $(\forall\, R, T \in (0,\infty))$. Then it is shown in [45], [49] that if ρ_0^n $(= \rho^n|_{t=0})$ converges in $L^1(\mathbb{R}^N)$, then ρ^n converges in $C([0,T]; L^p(\mathbb{R}^N)) \cap L^q(\mathbb{R}^N \times (0,T))$ for all $T \in (0,\infty)$, $1 \le p < \gamma$, $1 \le q < \gamma + \frac{2\gamma}{N} - 1$. And (ρ, u) is a solution of (25) with the properties listed in the preceding result. It is also shown in [45], [49] (see also Serre [65]) that the analogue of Theorem 3.2 for the system (25) *does not hold*: in other words, the compactification that took place for the hyperbolic system (22) *is lost* when we add viscous terms while we could expect (from a linear-linearized inspection) that the introduction of viscous terms regularizes the problem! These delicate and surprising phenomena depend in a subtle way on the nonlinearities of the systems we consider. Let us also mention that the proof of the above convergence result is rather delicate and uses in particular the structure of the convective derivatives $(\frac{\partial}{\partial t} + u \cdot \nabla_x)$ that lead with the analysis detailed in [45], [49] to terms like

$$\rho^n R_i R_j(\rho^n u_i^n u_j^n) - \rho^n u_i^n R_i R_j(\rho^n u_j^n)\,,$$

which are shown to converge weakly to $\rho R_i R_j(\rho u_i u_j) - \rho u_i R_i, R_j(\rho u_j)$ under the sole weak convergence stated above on ρ^n, u^n. This weak continuity follows from regularizing properties of the commutators $[u_i^n, R_i R_j]$. It is worth noting that the incompressible limit of such compressible models yields $\rho^n =$ cst (say 1), div $u^n = 0$, in which case the above term reduces to $R_i R_j(u_i^n u_j^n)$ and $R_j(u_i^n u_j^n) = (-\Delta)^{-1/2}$ $\{u^n \cdot \nabla u^n\}$ because div $u^n = 0$. Obviously, curl $(\nabla u^n) = 0$, div $u^n = 0$, and $u^n \cdot \nabla u^n$ is precisely a nonlinear expression for which the compensated-compactness theory applies (see Section 3.2).

4. Perspectives, Trends, Problems and Methods

Let us immediately emphasize that this brief section will select topics in a biased way that reflects the author's tastes.

First of all, we have mentioned above some of the progress made recently and many remaining open questions in gas dynamics and fluid mechanics. There is much more to say and in particular we have not touched here the incompressible models (Euler and Navier–Stokes equations) — see the reports by Beale, Chemin, Constantin, and Avellaneda in this Congress and Majda [56]. Even if many fundamental questions are left open, progress is being made (step by step).

We should also make clear that the topics covered here do not reflect fully the scope of nonlinear partial differential equations and in particular those arising from applications, the variety of mathematical problems and methods developed recently, and their relationships with other fields of mathematics. Let us briefly mention a

few more examples of themes covering several related areas that all have important scientific and technological implications: (i) propagation of fronts and interfaces, geometric equations, viscosity solutions, image processing (see the reports by Spruck, Souganidis, and Osher in this Congress), (ii) quantum chemistry, N-body problems, density-dependent and meanfield models, binding, thermodynamic limits, (iii) twinning and defects in solids and crystals, phase transitions, Young measures (see for instance Ball and James [3], James and Kinderlehrer [35], and the report by Sverak).

However, we wish to emphasize that the trends mentioned in the Introduction can also be found in the above themes.

Finally, it is important to develop at the same time the methods — some of which have been briefly presented in this paper — which are certainly interesting by themselves, and we would like to conclude with a few examples of such developments: (i) H-measures of Tartar [70], Gérard [24] (and the related Wigner measures by Lions and Paul [50], Gérard [25]), (ii) nonlinear partial differential equations in infinite dimensions (and in particular the viscosity solutions approach of Crandall and Lions [15]).

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PRÉSENTATION DE JEAN-CHRISTOPHE YOCCOZ

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1. Curriculum

Jean-Christophe Yoccoz est un pur produit, et du meilleur cru, du système français. Ancien élève de l'Ecole Normale Supérieure où il fut reçu 1er en 1975, reçu ^{er} à l'Ecole Polytechnique la même année, 1er ex-aequo à l'Agrégation de Mathématiques en 1977, il soutient sa Thèse de Doctorat d'Etat en 1985 et fut invité à enseigner le Cours Peccot au Collège de France en 1987. Aujourd'hui, âgé de 37 ans, il est Professeur à l'Université de Paris–Sud (Orsay), membre de l'I.U.F. (Institut Universitaire de France) et de l'URA "Topologie et Dynamique" du CNRS à Orsay.

I1 effectua son Service National en Coopération à l'IMPA à Rio de Janeiro, et cela devait le marquer profondément. I1 visite régulièrement le Brésil, mais aussi le Centre International de Trieste. D'ailleurs sa femme est brésilienne.

Yoccoz est un étudiant de Michel Herman, et c'est ainsi qu'il est devenu peut-être le meilleur spécialiste de la Théorie des Systèmes Dynamiques.

2. La Théorie des Systèmes Dynamiques

Cette théorie cherche à décrire *l'évolution à long terme* d'un système quand on en connait la loi d'évolution élémentaire. Le temps peut y être continu ou discret.

Dans le cas d'un *temps continu*, la loi d'évolution infinitésimale se traduit par une équation différentielle, qui est donnée par un *champ de vecteurs*, et le problème est de comprendre l'évolution à long terme des solutions. On obtient parfois des attracteurs étranges.

Un exemple typique est le problème de la *stabilité du Système Solaire*, qui a amené Poincaré à fonder la théorie au tournant du siècle.

Dans le cas d'un *temps discret*, l'évolution élémentaire est donnée par une application f, qui donne l'état du système au temps $n+1$ en fonction de l'état au temps n. I1 s'agit alors d'itérer f un grand nombre de fois.

Quand deux applications f et g décrivent le même phénomène dans des représentations différentes, elles sont *conjuguées* par l'application h qui traduit le

changement de représentation. Toute conjugaison peut être interprétée de cette façon. Deux applications conjuguées ont donc les mêmes propriétés dynamiques. Par suite la classification des applications à conjugaison près est un problème central dans la théorie.

3. Conjugaison C^∞ à la Rotation

L'exemple le plus simple est celui où l'espace des états est un cercle, et où l'application à itérer est indéfiniment différentiable ainsi que son inverse, autrement dit un *difféomorphisme* C^∞. Pour une telle application f, Poincaré a défini le *nombre de rotation* $\alpha = \mathrm{Rot}\ (f) \in \mathbf{T} = \mathbf{R}/\mathbf{Z}$. Les question est alors: quand-est-ce que f est C^∞-conjuguée à la rotation $\mathcal{R}_\alpha : t \mapsto t + \alpha$?

Si α en rationnel, disons $\alpha = \frac{p}{q}$, ceci exige que l'on ait $f^q = I$, ce qui ne se produit essentiellement jamais. Le cas intéressant est donc celui où α est irrationnel. I1 a été étudié par Denjoy — qui a montré que f est toujours topologiqement conjuguée à $\mathcal{R}_\alpha$ —, Birkhoff, Arnold, Herman et beaucoup d'autres, et bien sûr Yoccoz. Ils ont tous insisté sur l'importance des *propriétés arithmétiques* de α.

Pour α rationnel, il se produit des résonnances. Si α est irrationnel, il se produit presque toujours des compensations et on observe certaines régularités. Mais si α, tout en étant irrationnel, se trouve extrêmement proche de rationnels avec des dénominateurs modérément grands, il arrive qu'une résonnance s'amorce et qu'avant qu'elle soit amortie une autre prenne le relais, et on peut obtenir une situation très compliquée. Ce qui importe est donc la distance $\delta_q(\alpha)$ de α à l'ensemble des rationnels à d'enominateur borné par q, et la façon dont cette distance tend vers 0 quand q tend vers l'infini.

4. Conditions Diophantiennes

On dit que α est diophantien si $\delta_q(\alpha)$ est minoré par une expression de la forme $\frac{c}{q^\beta}$.

Pour un difféomorphisme C^∞ du cercle de nombre de rotation α, Herman a montré que f est nécessairement C^∞-conjuguée à la rotation $\mathcal{R}_\alpha$ si α est diophantien d'exposant 2. Ce résultat constituait une percée importante. En fait Herman avait démontré un théorème plus fort: le même résultat sous une hypothèse plus faible, satisfaite pour presque toute valeur de α.

Dans sa thèse, Yoccoz a amélioré le théorème de Herman: il a donné une démonstration plus simple et obtenu le résultat sous l'hypothèse que α est diophantien sans restriction d'exposant — hypothèse plus faible que celle de Herman.

Des contre-exemples de Herman montrent que ce résultat est optimal.

5. Les Cas R-analytique

On peut se poser la même question dans le cadre $\mathbf{R}$-analytique.

Yoccoz a démontré dans sa thèse qu'un difféomorphisme $\mathbf{R}$-analytique du cercle à nombre de rotation α diophantien est nécessairement $\mathbf{R}$-analytiquement conjugué à la rotation $\mathcal{R}_\alpha$. Récemment il a donné une description de l'ensemble exact des

nombres de rotation α ayant cette propriété. C'est un ensemble compliqué: alors que les ensembles qu'on définit de cette façon sont en général du type F_σ, celui-ci est seulement $F_{\sigma\delta}$.

Les fonctions C^∞ et $\mathbf{R}$-analytique ont une consistance différente. Quand on travaille dans le cadre $\mathbf{R}$-analytique, la première chose que l'on fait est d'étendre les applications aux valeurs complexes de la variable. Une application $f; \mathbf{T} \to \mathbf{T}$ s'étend ainsi à un voisinage annulaire Ω de $\mathbf{T}$ dans le cylindre $\mathbf{C}/\mathbf{Z}$, et si f est $\mathbf{R}$-analytiquement conjugué à $\mathcal{R}_\alpha$ il y a un anneau A dans Ω qui et invariant par f. Pour $z \in A$, la fermeture de l'orbite de A est une courbe $\mathbf{R}$-analytique, correspondant à un cercle parallèle à l'équateur $\mathbf{T}$ de $\mathbf{C}/\mathbf{Z}$. L'épaisseur minimale de A, son module, ce qui se produit au voisinage de sa frontière sont autant de propriétés sur lesquelles le raisonnement géométrique a prise.

6. Réciproque du Théorème de Bruno

La question de linéarisabilité locale des difféomorphismes holomorphes au voisinage d'un point fixe est étroitement liée à la précédente. C'est la suivante:

Une fonction $f : z \mapsto a_1 z + a_2 z^2 + a_3 z^3 + \cdots$, est-elle holomorphiquement conjuguée au voisinage de 0 à sa partie linéaire $z \mapsto a_1 z$?

Le résultat est facile si $|a_1| \neq 1$ (Schröder, Böttcher), le cas intéressant est celui où a_1 est de la forme $e^{2i\pi\alpha}$. I1 a été étudié par Fatou — qui a traité le cas où α est rationnel, Cremer — qui a donné des exemples de non-linéarisabilité, Siegel — qui a montré que f est linéarisable dès que α est Diophantien (et ce quelle que soit la queue $a_2 z^2 + \cdots$), Bruno — qui a amélioré le théorème de Siegel en démontrant le résultat sous l'hypothèse plus faible $\Sigma \frac{\operatorname{Log} q_{n+1}}{q_n} < \infty$ (où les $\frac{p_n}{q_n}$ sont les réduites du développement de α en fraction continue), et enfin Yoccoz qui a démontré la réciproque du théorème de Bruno.

Siegel et Bruno travaillaient en force, résolvant le problème formellement et majorant les coefficients de la conjugante. Yoccoz a une approache plus géométrique et plus fine. Il y a une construction que l'on appelle renormalisation, qui associe à une application f d'angle α une application f_1, ayant un angle α_1, dont le développment en fraction continue est le même que celui de α mais décalé avec perte du premier terme. Par une étude quantitative pousséee des propriétés de cette opération et de ses itérées, Yoccoz a obtenu une démonstration très éclairante du théorème de Bruno, et il a pu prouver la réciproque: que pour tout α ne satisfaisant pas à la condition de Bruno on peut chosir la queue de façon que f ait des points périodiques arbitrairement proches de 0, ce qui exclut la linéarisabilité. En fait la queue la plus simple ($f = a_1 z + z^2$) fait l'affaire.

Restait une question: La non-linéarisabilité est-elle toujours due à la présence de petits cycles? Les exemples construits par Cremer et Yoccoz pouvaient le laisser croire. La question a été résolue par la négative par Perez–Marco, un élève de Yoccoz qui a encore affiné sa méthode. Elle fait très curieusement intervenir la condition $\Sigma \frac{\operatorname{Log}\operatorname{Log} q_{n+1}}{q_n} < \infty$, plus faible que celle de Bruno.

7. Jeu de Cadres

La Géométrie et l'Analyse interviennent dans toute les parties de la Théorie des Systèmes Dynamiques. Mais elles ont une façon particulière d'interagir dans les Systèmes Dynamiques Complexes, grâce aux inégalités de Schwarz, Koebe, Groetzsch etc., inégalités puissantes qu'on peut appliquer sous des hypothèses purement topologiques. C'est une méthode que Yoccoz a énormément développée.

8. MLC: la Taille des Membres

L'essentiel de notre connaissance des propriétés dynamiques de la famille des polynômes quadratiques complexes est concentré dans le propriétés topologiques de son lieu de connexité M, connu sons le nom d'*ensemble de Mandelbrot*. Ses propriétés combinatoires sont maintenant bien comprises, et Thurston en a proposé un modèle synthétique. Mais pour savoir que M est effectivement homéomorphe à son modèle, il manque une information: que M est localement connexe.

L'ensemble M contient des copies de lui-même. Yoccoz a montré que M est localement connexe en tout point "non infiniment renormalisable", c'est à dire qui n'est pas contenu dans l'intersection d'une suite décroissante de copies de M. Pour démontrer la conjecture MLC complète, il reste aujourd'hui à montrer que l'intersection d'une suite décroissante de copies de M est réduite à un point.

Le premier cas est constitué par les points de la cardioïde: L'ensemble M est formé d'une grande cardioïde Γ, remplie, et de *membres* attachés aux points de Γ d'argument interne rationnel (les arguments internes définissent la paramétrisation naturelle de Γ).

Yoccoz a montré que le diamètre d'un membre $M_{p/q}$ attaché au point d'argument interne p/q est majoré par une expression de la forme $\frac{c}{q}$ où c est une constante. Ce résultat n'est surement pas optimal (d'après Hubbard ou pourrait espérer $\frac{\operatorname{Log} q}{q^2}$), mais il suffit à montrer que M est localement connexe aux points de Γ. Par la même méthode, il obtient la connexité locale en tout point qui est sur le bord d'une composante hyperbolique.

9. MLC: les Puzzles de Yoccoz

Pour montrer que M est localement connexe aux point c qui ne sont ni infiniment renormalisables ni sur le bord d'une composante hyperbolique, Yoccoz emploie la méthode dite des "*Puzzles de Yoccoz*". Selon de principe que je défends en Dynamique Complexe.

On laboure dans le plan dynamique.

On moissonne dans le plan des paramètres.

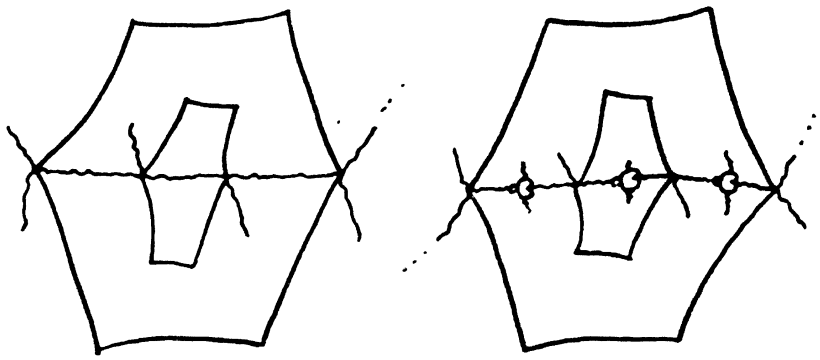
Il y a en effet des figures dans le plan dynamique qui se trouvent reproduites plus on moins fidèlement dans le plan des paramètres.

Le point de départ est un article de Branner–Hubbard, qui traite d'une certaine famille de polynômes cubiques (voir l'exposé de Lyubich à ce Congrès). Il leur faut montrer que certains ensembles dans le plan des paramètres, dont on s'attend à ce qu'ils soient réduits à un point, le sont effectivement. D'après l'inégalité de Grötzsch, il suffit d'enfermer un tel ensemble dans une suite d'anneaux emboités dont la somme des modules est infinie. C'est ce qu'ils font, mais d'abord dans le plan dynamique où les anneaux considérés sont des revêtements les uns des autres, de sorte qu'à une constante près les modules sont des inverses d'entiers et la divergence résulte d'une étude combinatoire très poussée.

Dans leur cas, le transfert dans le plan des paramètre est facile, car les anneaux considérés s'y retrouvent reproduits conformément.

La Conjecture MLC est aussi en un sens un énoncé de la forme "les points sont effectivement des points". Dans le plan des paramètres qui contient M, on peut définir des *pièces* limitées par des rayons externes et des arcs d'équipotentielles. Une telle pièce découpe dans M un ensemble connexe, et pour démontrer MLC en un point c il suffit démontrer que l'intersection des pièces qui sont des voisinages de c est réduite à c.

La situation est analogue à celle de Branner–Hubbard, et la démonstration dans le plan dynamique peut se faire suivant les mêmes lignes. L'essentiel de la difficulté réside dans le passage au plan des paramètres, et Yoccoz réalise là un tour de force d'Analyse. En effet, en dehors de M et de K, il y a une correspondance conforme entre le plan dynamique et le plan des pramètres, mais sur ces ensembles il n'y a plus de correspondance (dans M il y a des petites copies de M qui ne se retrouvent pas dans le plan dynamique), les anneaux n'ont pas même module et il faut faire de l'Analyse fine pour montrer que le rapport des modules est borné, et que la divergence est donc préservée.



Plan dynamique Plan des paramètres

Yoccoz n'a pas fait taper son manuscrit, mais on peut lire une démonstration dans les rédactions qu'en ont faites Milnor dans un preprint, et Hubbard (Three theorems of Yoccoz) dans le livre dédié à Milnor.

10. Conjugaison C^∞

Je me suis étendu longuement sur la Dynamique Complexe, parce que c'est ce que je comprends le mieux, mais les travaux de Yoccoz en Dynamique Réelle sont tout aussi importants. La plupart sont en collaboration.

Palis et Yoccoz ont obtenu un système complet d'invariants de conjugaison C^∞ pour les difféomorphismes de Morse–Smale.

- des invariants locaux qui décrivent les formes normales aux points attractifs ou répulsifs;
- des invariants globaux qui comparent les coordonnées adaptées à ces formes normales là où les bassins se recouvrent.

Le cas d'une dynamique Nord–Sud sur S^n en facile: le second invariant est le changement de cartes. Mais dans le cas général il y a des points-selle. Palis et Yoccoz montrent que ces points-selle ne produisent pas de nouveaux invariants en raison d'un théorème de singularités inessentielles: Si deux difféomorphismes de Morse–Smale sont C^∞-conjugués sur la réunion des bassins attractifs et répulsifs, la conjugaison s'étend de façon C^∞ à la variété toute entière.

11. Autres Travaux avec Palis

Yoccoz a écrit au moins trois autres articles avec Palis. Un sur les centralisateurs des difféomorphismes, où ils démontrent que, sous certaines conditions assez générales, en partant d'un difféomorphisme hyperbolique f on peut obtenir par une perturbation arbitrairement petite un difféomorphisme f, qui ne commute qu'avec lui-même et ses itérés, et tel que tout difféomorphisme f_2 suffisamment voisin de f_1 ait la même propriété.

Un autre article sur les bifurcations homoclines complète un résultat de Newhouse et établit une réciproque à un résultat de Palis–Takens: si la dimension de Hausdorff de l'ensemble hyperbolique créant cette bifurcation est plus grande que un, les applications structurellement stables ne sont pas prévalentes au voisinage, détruisant un vieux rêve de Thom.

A en croire Michel Serres, dans une telle collaboration, il y a toujours un renard qui va à la chasse et un sanglier qui creuse. Combien de fois Palis et Yoccoz ont-ils échangé les rôles?

12. Travaux avec Le Calvez et Raphael Douady

Avec Le Calvez, autre étudiant de Herman, Yoccoz a démontré qu'il n'y a pas d'homéomorphisme minimal de l'anneau $S^1 \times \mathbf{R}$. Autrement dit il n'y a pas

d'homéomorphisme de la sphère S^2 préservant les deux poles, et tel que tout autre point ait une orbite dense.

Les méthodes sont celles de la topologie en dimension 2. Le lemme central est que, au voisinage d'un point fixe qui n'est ni atttractif ni répulsif, et pour lequel il y a un voisinage ne contenant aucune orbite complète, l'application se comporte du point de vue de l'indice comme $z \mapsto e^{2\pi i p/q} z(1 - \overline{z}^{rq})$ pour certains entiers r et q.

Je veux aussi citer un article avec Raphael Douady. Pour un difféomorphisme f du cercle conjugué à $\mathcal{R}_\alpha$ par une application h de classe φ^1, la mesme μ_s de densité $(h')^{1-s}$ satisfait $f_*((f')^s \mu_s) = \mu_s$. Douady et Yoccoz montrent qu'il existe une mesure unique satisfaisant cette propriéte des que f, difféomorphisme de classe φ^2, a un nombre de rotation irrationnel, même si la conjugante est seulement un homéomorphisme.

13. Cocorico

Ce retour aux difféomorphismes du cercle termine notre brève visite guidée des travaux de Yoccoz.

Avec deux médailles Fields pour la France, même s'il s'agit d'une coincidence nous pouvons pavoiser. Mais dans une occasion pareille il convient de se rappeler un proverbe de nos jardiniers

Si la rose est belle, c'est que le fumier est gras.

Il est de la responsabilité de chaque communauté nationale, en Mathématiques, de veiller à ce que la qualité de l'enseignement en mathématiques, en particulier au niveau secondaire, soit préservée. Pour nous Français, au moment où des réductions d'horaires draconniennes menacent, cette tâche sera rude.

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RECENT DEVELOPMENTS IN DYNAMICS

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1. Introduction

1.1 Broadly speaking, the goal of the theory of dynamical systems is, as it should be, to understand *most* of the dynamics of *most* systems.

The dynamical systems that we will consider in this survey are smooth maps f from a smooth manifold M to itself; the time variable then runs amongst non-negative integers.

Frequently, we will also assume that the map f is a diffeomorphism, allowing the time variable to take all integer values. We could also consider smooth flows on M, with a real time variable: the ideas and concepts are pretty much the same in this case.

Given two dynamical systems $f : M \to M$ and $g : N \to N$, a morphism from f to g is a smooth map $h : M \to N$ such that $g \circ h = h \circ f$, in other words the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "h"'] & M \arrow[d, "h"] \\ N \arrow[r, "g"'] & N \end{tikzcd}$$

is commutative.

When h is a diffeomorphism, we will say that f and g are (smoothly) *conjugated*. When h is an embedding, f is a *subsystem* of g. When h is a submersion, f is an *extension* of g, and g is a *factor* of f.

The ultimate goal of the theory should be to classify dynamical systems up to conjugacy. This can be achieved for some classes of simple systems [PY1]; but even for (say) smooth diffeomorphisms of the two-dimensional torus, such a goal is totally unrealistic. Hence we have to settle to the more limited, but still formidable, task to understand most of the dynamics of most systems.

The word "most" in the last sentence may assume both a topological and metrical meaning. From a topological point of view, it means open and dense, or more frequently G_δ-dense (i.e. countable intersection of open and dense); from a metrical point of view, we would like to understand the trajectories of Lebesgue for almost every point of the system; when considering a smoothly parametrized family of maps or diffeomorphisms, we would also like to understand the dynamics for almost all values of the parameter.

1.2 The dynamical features that we are able to understand fall into two classes, hyperbolic dynamics and quasiperiodic dynamics; it may very well happen, especially in the conservative case, that a system exhibits both hyperbolic and quasiperiodic features.

I will not try to give a precise definition of what is hyperbolic or quasiperiodic: actually, we seek to extend these concepts, keeping a reasonable understanding of the dynamics, in order to account for as many systems as we can. The big question is then: Are these concepts sufficient to understand most systems?

1.3 The prototype of a quasiperiodic dynamical system is a translation T in a compact abelian group G; typically, G is the n-dimensional torus $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$, but the additive group $\mathbf{Z}_p$ of p-adic integers (or more generally any profinite abelian group) is also relevant.

Every translation commutes with T, hence is a symmetry of the dynamics of T: this makes the dynamics homogeneous, with a group of symmetries acting transitively. Another significant feature is that the family of iterates of T is equicontinuous; the topological entropy of T is zero.

Finally, the Haar measure on G is invariant under T, and the unitary operator $\varphi \mapsto \varphi \circ T$ of $L^2(G)$ induced by T has a discrete spectrum.

1.4 As prototypes of hyperbolic dynamical systems, we will consider two examples.

The first one is the Bernoulli shift σ on the profinite abelian group $\Sigma = \{0,1\}^{\mathbf{z}}$, defined by

$$\sigma((x_i)_{i\in\mathbf{z}}) = (y_i)_{i\in\mathbf{z}}\,, \quad y_i = x_{i+1}\,.$$

For the second one, we consider a matrix $A \in \mathrm{GL}(n,\mathbf{Z})$ which is hyperbolic, i.e. no eigenvalue has modulus one. Such a matrix induces an automorphism of $\mathbf{T}^n$, which is a typical example of Anosov diffeomorphism.

Let us consider some significant features of the dynamics (in both examples).

Perhaps the most important is the *shadowing property*: define an ε-pseudo orbit as a sequence $(z_i)\_i \in \mathbf{Z}$ such that $d(f(z_i), z_{i+1}) < \varepsilon$ for all i; then, for given $\delta > 0$, there exists $\varepsilon > 0$ such that every ε-pseudo orbit $(z_i)\_i \in \mathbf{Z}$ is "shadowed" by a true orbit $(w_i)\_i \in \mathbf{Z}$ in the sense that $d(w_i, z_i) < \delta$ for all i.

A counterpart of the shadowing property is the *expansivity property*: there exists $\varepsilon_o > 0$ such that

$$\operatorname*{Sup}_n d(f^n x, f^n y) \geq \varepsilon_o$$

for all distinct x, y: this makes the shadowing orbit unique (for δ small enough) and is in contrast with the equicontinuity of iterates of the quasiperiodic case.

In both examples, the topological entropy is strictly positive. As automorphisms of compact abelian groups, the two examples preserve the Haar measure; the corresponding unitary operators have a Lebesgue spectrum.

2. Quasiperiodic Dynamics

2.1 Before giving some specific results, let us begin with a broad overview.

There are three approaches to quasiperiodic dynamics that have been very fruitful.

The first one is the function-theoretical approach, dealing with the stability of diophantine quasiperiodic motions. This includes the so-called KAM-theory, and techniques where functional equations are solved via Newton's method (combined with smoothing operators) or implicit function theorems in Fréchet spaces (which are conceptual analogues of Newton's method). In several special but important contexts. Herman [H2] has been able to solve the functional equations via the Schauder–Tichonoff fixed point theorem.

Finally, Rüssmann [Ru2] has announced the proof of several KAM-theorems relying only on the standard fixed point theorem.

In the symplectic context, the variational approach has also been quite successful; there is a huge number of results related to the existence of periodic orbits. We will present briefly the pioneering work of Mather on quasiperiodic dynamics in this context.

The last approach to quasiperiodic phenomena is more geometric, and frequently coined as "renormalization". Roughly speaking, the combinatorics of the recurrence are unravelled in an infinite sequence of simple successive steps, each of them involving a change of scales both in time and space. Typically, for a circle diffeomorphism f, with irrational rotation number α having convergents $(p_n/q_n)\_n \geq 0$, two successive iterates f^{q_n}, $f^{q_{n+1}}$ give rise to a circle diffeomorphism f_n, which is the "nth-renormalization" of f (Herman, Yoccoz). Sullivan has developed this approach when the recurrence is combinatorially described as a translation in a profinite abelian group.

2.2 Let us consider a holomorphic germ $f(z) = \lambda z + O(z^2)$, $\lambda \in \mathbf{C}^*$, in one complex variable.

We are interested in the dynamics near the fixed point 0, when the eigenvalue λ has modulus 1 but is not a root of unity; we write $\lambda = e^{2\pi i\alpha}$, with irrational $\alpha \in (0, 1)$.

It is convenient to assume some normalization on f; we will consider the class S_α of germs as above that are defined and univalent in the unit disk $\mathbf{D}$.

The germ f is always formally linearizable: there exists a unique formal power series $h_f(z) = z + O(z^2)$ satisfying $h_f \circ R_\lambda = f \circ h_f$, where $R_\lambda : z \to \lambda z$ is the linear part of f.

Consider $V_f = \operatorname{int}\left(\bigcap_{n\geq 0} f^{-n}(\mathbf{D})\right)$; it is easy to see that $0 \in V_f$ if and only if h_f is convergent, and that in this case there exists $r_f > 0$ such that the restriction of h_f to $\{|z| < r_f\}$ is a conformal representation of the component U_f of 0 in V_f. Actually, when $\overline{U}_f \subset \mathbf{D}$, r_f is the radius of convergence of h_f.

Let us define

$$r(\alpha) = \inf_{S_\alpha} r_f\,,$$

and denote by $(p_n/q_n)\_n \geq 0$ the convergents of α.

Siegel [Si] proved in 1942 that $r(\alpha) > 0$ as soon as the diophantine condition $\operatorname{Log} q_{n+1} = O(\operatorname{Log} q_n)$ holds; he achieved this first breakthrough through small divisors problems by a direct estimation of the coefficients of h_f. Later, Brjuno [Br], through a refinement of Siegel's estimates, proved that if

$$\text{(B)} \qquad \Phi(\alpha) = \sum_{n\geq 0} q_n^{-1} \operatorname{Log} q_{n+1} < +\infty\,,$$

then $r(\alpha) > 0$ and even $\operatorname{Log} r(\alpha) > 2\Phi(\alpha) - c$ (for some $c > 0$ independent of α). See also [C].

Using a "renormalization" approach based on a geometric construction, I gave a new proof of the Siegel–Brjuno theorem and proved the converse ([Y5], [Y4]).

Theorem. (1) *If* $\Phi(\alpha) < +\infty$, *then*

$$|\operatorname{Log} r(\alpha) + \Phi(\alpha)| < c\,,$$

for some $c > 0$ *independent of* α.

(2) *If* $\Phi(\alpha) = +\infty$, *the quadratic polynomial* $P_\lambda(z) = \lambda z + z^2$ *is not linearizable*: *every neighborhood of* 0 *contains a periodic orbit*, *distinct from* 0.

Actually, one first constructs a nonlinearizable germ with this property, and then shows that the same holds for the quadratic polynomial, via Douady–Hubbard's theory of quadratic-like maps.

Significant progress has been achieved by Perez–Marco [PM2], [PM3] in the understandings of the dynamics in the nonlinearizable case. He first showed that for a germ $f \in S_\alpha$ that is not linearizable and has no periodic orbit in $\mathbf{D}$ (except for 0) to exist, it is necessary and sufficient that

$$\sum_{n\geq 0} q_n^{-1} \operatorname{Log}\ \operatorname{Log} q_{n+1} = +\infty\,.$$

He also defines "degenerate" Siegel disks as follows: assuming f to be univalent in a neighborhood of $\overline{\mathbf{D}}$, the connected component K_f of 0 in $\bigcap_{\mathbf{z}} f^{-n}(\overline{\mathbf{D}})$ is a full, compact, connected, invariant subset of $\overline{\mathbf{D}}$ that meets S^1. When α satisfies the diophantine condition (H) (see 2.3), one has just $K_f = \overline{U}_f$.

These invariant sets provide a rich connection with the theory of analytic circle diffeomorphisms; if $k : \mathbf{H}/\mathbf{Z} \to \mathbf{C} - K_f$ is a conformal representation, the map $g = k^{-1} f\ k$ is defined in some strip $\{0 < \operatorname{Im} z < \delta\}$ and extends by Schwarz's reflection principle to a circle diffeomorphism with the same rotation number as f.

2.3 Let us now consider analytic circle diffeomorphisms. For $\delta > 0$, define $B_\delta = \{z \in \mathbf{C}/\mathbf{Z}, |\operatorname{Im} z| < \delta\}$. For irrational $\alpha \in \mathbf{R}/\mathbf{Z}$, let $S_\alpha(\delta)$ be the set of orientation preserving analytic diffeomorphisms f of $\mathbf{R}/\mathbf{Z}$ with rotation number α that extend to a univalent map from B_δ to $\mathbf{C}/\mathbf{Z}$.

By Denjoy's theorem, f is conjugated to the translation $R_\alpha : z \mapsto z + \alpha$ on the circle by a homeomorphism h_f of $\mathbf{R}/\mathbf{Z}$ (uniquely defined if we require $h_f(0) = 0$). As for germs, h_f is analytic if and only if the circle $\mathbf{R}/\mathbf{Z}$ is contained in the interior of $\bigcap_{n \geq 0} f^{-n}(B_\delta)$. There are two kinds of results, depending on whether we assume or not that f is near the translation R_α; the breakthroughs (under more restrictive arithmetic conditions) are respectively due to Arnold (1960) and Herman (1976). We state the results in their final form before some comments.

Theorem 1. (Arnold [A], Rüssmann, Yoccoz [Y6]) *Assume that* $\Sigma q_n^{-1} \operatorname{Log} q_{n+1} < +\infty$. *There exists* $\varepsilon = \varepsilon(\alpha, \delta)$ *such that, if*

$$\|f - R_\alpha\|_{C^o(B_\delta)} < \varepsilon(\alpha, \delta)\,,$$

then h_f *is analytic. Moreover, the diophantine condition is optimal.*

Theorem 2. (Herman, Yoccoz) *Assume that the rotation number satisfies the diophantine condition* (H) *below. Then* h_f *is analytic. Moreover, the diophantine condition is optimal.*

The Arithmetic Condition (H)

Assume that $0 < \alpha < 1$ and define $\alpha_o = \alpha, \alpha_n = \left\{\alpha_{n-1}^{-1}\right\}$ for $n \geq 1$. For $m \geq n \geq 0$, define inductively $\Delta(m, n)$ as follows:

$$\Delta(n, n) = 0\,, \quad \forall n \geq 0$$

$$\Delta(m+1, n) = \begin{cases} \exp \Delta(m, n) & \text{if} \quad \Delta(m, n) \leq \operatorname{Log} \alpha_m^{-1} \\ \alpha_m^{-1}(\Delta(m, n) - \operatorname{Log} \alpha_m^{-1} + 1) & \text{if} \quad \Delta(m, n) \geq \operatorname{Log} \alpha_m^{-1} \end{cases}$$

Then α satisfies (H) if for every $n \geq 0$ we have $\Delta(m, n) \geq \operatorname{Log} \alpha_m^{-1}$ for $m \geq m_o = m_o(n)$.

The set of numbers α satisfying (H) is a $F_{\sigma\delta}$ set (a countable intersection of F_σ sets) but neither a F_σ or a G_δ set (this explains why the definition has to be complicated). Numbers α such that

$$\text{Log } q_{n+1} = 0((\text{Log } q_b)^c)\,, \quad \text{for some } c > 0$$

satisfy (H). On the other hand, condition (H) is strictly stronger than condition (B). Indeed, for numbers $\alpha = 1/(a_1 + 1/(a_2 + \cdots$ such that

$$a_i \leq a_{i+1} \leq \exp(a_i)$$

condition (B) is always fulfilled; on the other hand, defining $b_o = 0$, $b_n = \exp(b_{n-1})$, the number α satisfies (H) if and only if, for any $k \geq 0$, we have $a_{m+k} \leq b_m$ for m large enough; for instance, if $a_{i+1} \geq \exp(a_i^\theta)$, for some $\theta \in (0,1)$, α does not satisfy (H).

Conditions (B) and (H) are closely related: let $\mathcal{H}_o$ be the set of irrational α such that $\Delta(m,0) \geq \text{Log } \alpha_m^{-1}$ for large m; then α satisfies (H) if and only if its orbit under $\text{GL}(2,\mathbf{Z})$ is contained in $\mathcal{H}_o$, whereas it satisfies (B) if and only if its orbit meets $\mathcal{H}_o$.

Remarks. (1) The fact that the optimal arithmetic condition is not the same in the local and global conjugacy theorems is in strong contrast with the smooth (C^∞) case; then in both theorems, the optimal arithmetic condition is the standard one

$$\text{Log } q_{n+1} = O(\text{Log } q_n)$$

(Moser [Mo2], Herman [H1], Yoccoz [Y2]).

(2) Another important difference between the smooth and analytic cases is that the effect of good rational approximations is cumulative in the analytic case, but not in the smooth case. Another way to see this difference is to observe that the arithmetic condition in the smooth case is given by the linearized equation, whereas both conditions (H) and (B) do not appear naturally when looking at linear difference equations.

(3) All known proofs ([H1], [Y2], [KO1], [KO2], [KS]) of *global* conjugacy theorems (smooth or analytic) are based on a renormalization scheme that relies in an essential way on the relationship between the good rational approximations of the rotation number (given by the continued fraction).

2.4 When several frequencies are involved, KAM techniques are available, but they do not give as much geometric insight as we would like to have. One would like to have some geometric renormalization scheme as above, but the problem, of a purely arithmetical nature, is then to understand thoroughly the relationships between good rational approximations.

Here is a test case. Consider the following two theorems.

Theorem 1. (Arnold [A], Moser [Mo2]) *Let* $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{T}^n$ *satisfy the diophantine condition*: $\exists_\gamma > 0,\ \tau \geq 0$ *s.t.*

$$|\langle k, \alpha\rangle + k_o| \geq \gamma \|k\|^{-n-\tau}$$

for all $(k_o, k_1 \cdots k_n) \in \mathbf{Z}^{n+1} - \{0\}$.

There exists $\varepsilon = \varepsilon(\alpha)$ *and* $k = k(\tau)$ *such that if* f *is a smooth diffeomorphism of* $\mathbf{T}^n$ *satisfying*

$$\|f - R_\alpha\|_{C^k} < \varepsilon\,,$$

then there exists a (small) translation R_λ *and a smooth diffeomorphism* h *such that*

$$f = R_\lambda \circ h \circ R_\alpha \circ h^{-1}\,.$$

Theorem 2. (Moser [Mo3]) *Let* $(\alpha_1, \ldots, \alpha_n) \in \mathbf{T}^n$ *satisfy the diophantine condition*: $\exists_\gamma > 0,\ \tau \geq 0$ *s.t.*

$$|k_o\alpha_i - k_i| \geq \gamma \|k\|^{-\frac{1}{n}-\tau}\,, \quad 1 \leq i \leq n$$

for all $(k_o, \ldots, k_n) \in \mathbf{Z}^{n+1} - \{0\}$.

There exists $\varepsilon = \varepsilon(\alpha)$ *and* $k = k(\tau)$ *such that if* $f_1, \ldots, f_n$ *are smooth commuting diffeomorphisms of* $\mathbf{T}^1$ *satisfying*

$$\|f_i - R_{\alpha_i}\|_{C^k} < \varepsilon\,, \quad \rho(f_i) = \alpha_i$$

then there exists a smooth diffeomorphism h *such that*

$$f_i = h\, R_{\alpha_i} h^{-1}\,, \quad 1 \leq i \leq n\,.$$

Problem 1: Prove Theorem 2 without assuming that f_i is close to R_{α_i}.

Problem 2: Find the *optimal* arithmetical conditions in Theorem 1 and Theorem 2 in the *analytic* case.

The first problem should be easier: diophantine conditions in smooth small divisors problems tend to be more "stable" than in analytic ones.

2.5 Codimension 1 invariant tori

The fundamental result of Moser [Mo1] on the existence of invariant curves for near integrable area-preserving twist diffeomorphisms of the annulus was first generalized by Rüssmann as a "translated curve" theorem (removing the area-preserving hypothesis) [Ru1], [H3]. This has recently been further generalized to higher dimensions as follows (by Cheng–Sun [CS] and Herman [H6]).

Let L be a smooth orientation preserving diffeomorphism of $\mathbf{T}^n \times \mathbf{R}$ such that $L(\mathbf{T}^n \times \{0\}) = \mathbf{T}^n \times \{0\}$ and the restriction of L to $\mathbf{T}^n \times \{0\}$ is a translation; let also $\alpha \in \mathbf{T}^n$ satisfy the standard diophantine condition (see Theorem 1 in 2.4).

Then, if F is a smooth diffeomorphism of $\mathbf{T}^n \times \mathbf{R}$ close enough to L, there exists a translation R in $\mathbf{T}^n \times \mathbf{R}$ such that $R \circ F$ leaves invariant a codimension one torus T, going through the origin, C^∞-close to $\mathbf{T}^n \times \{0\}$, and $R \circ F/T$ is smoothly conjugated to the given translation R_α.

Herman has derived important consequences of this result.

The first is the failure of the quasi-ergodic hypothesis. The ergodic (resp. quasi-ergodic) hypothesis states that the generic Hamiltonian flow is ergodic (resp. has a dense orbit) on the generic (compact, connected) energy surface. The classical KAM-theorems provide for open sets of Hamiltonian flows a set of positive measure (on each energy surface) made of diophantine invariant tori; hence the ergodic hypothesis fails. Herman has discovered a rigidity property of the rotation number in the symplectic context that guarantees a similar phenomenon: there exist a nonempty open set of Hamiltonian flows and energy values for which the energy surface contains a Cantor set of *codimension one* diophantine invariant tori; the orbits "between" the tori are thus constrained to stay there.

Another important consequence is the failure of a conjecture of Pesin: Herman shows that on any manifold M (of dimension ≥ 3) there exists a nonempty open set of volume-preserving diffeomorphisms whose Lyapunov exponents are all 0 on a set of positive volume. In dimension 2, this follows from Moser's twist theorem.

2.6 In the symplectic context, Mather has been pioneering the study of quasiperiodic motions through a variational approach. In one degree of freedom, we now have (due to Aubry [AD], Mather [Ma1], Le Calvez [LeC1], ...) a fairly satisfactory theory of Aubry–Mather Cantor sets. In more degrees of freedom, Mather has obtained a yet partial generalization that seems quite promising in understanding somewhat Arnold diffusion ([Ma3], [Ma2]).

Let me explain this very roughly for discrete time (diffeomorphisms). Consider an integrable diffeomorphism L of $\mathbf{T}^n \times \mathbf{R}^n = T^*\mathbf{T}^n$:

$$L(\theta, r) = (\theta + \nabla\ell(r), r)\,,$$

with strictly convex ℓ superlinear at ∞.

To each invariant lagrangian torus $\{r = r_o\}$, we can associate the cohomology class $r_o \in H^1(\mathbf{T}^n, \mathbf{R})$ and the rotation number $\alpha = \nabla\ell(r_o)$ that belongs in a natural way to $H_1(\mathbf{T}^n, \mathbf{R})$.

Let now $F : (\theta, r) \mapsto (\Theta, R)$ be an exact symplectic diffeomorphism close to L. Writing

$$\Sigma R_i \, d\Theta_i - \Sigma r_i \, d\theta_i = dH\ (\theta, \Theta)$$

we obtain the generating function H of F, defined on $\mathbf{R}^n \times \mathbf{R}^n$ and satisfying

$$H(\theta + k, \Theta + k) = H(\theta, \Theta)\,, \quad k \in \mathbf{Z}^n\,.$$

Given an invariant measure μ with compact support, we transport it via $(\theta, r) \mapsto (\theta, \Theta)$ to the diagonal quotient $(\mathbf{R}^n \times \mathbf{R}^n)/\mathbf{Z}^n$ and consider, for $\omega \in \mathbf{R}^n = H^1(\mathbf{T}^n, \mathbf{R})$, the ω-action:

$$A_\omega(\mu) = \int [H(\theta, \Theta) - \langle \omega, \Theta - \theta \rangle] \, d\mu \, .$$

The invariant measure is *minimal* if it minimizes the ω-action (amongst invariant measures, or equivalently amongst all measures) for some cohomology class ω.

On the other hand, to any invariant measure, one can associate a rotation number

$$\alpha(\mu) = \int (\Theta - \theta) \, d\mu \in \mathbf{R}^n = H_1(\mathbf{T}^n, \mathbf{R}) \, .$$

Then μ is minimal if and only if it minimizes the action A_o amongst all invariant measures with the same rotation number.

For any ω, there exist ω-minimal measures; there also exist minimal measures with any given rotation number α. The correspondence between α and ω is realized by Legendre transform (in a nonsmooth, nonstrictly convex context).

The support of such a minimal measure is an invariant torus in the integrable case and shares in the general case some properties of Aubry–Master sets: in particular, it is the *graph* of the restriction to a closed subset of a *Lipschitz* map from $\mathbf{T}^n$ to $\mathbf{R}^n$.

The key point for further progresses is to understand the "shadowing" properties of these minimal measures. With one degree of freedom, Mather has proved (see also Le Calvez) that there are no obstructions except for the obvious ones: if $(\Lambda_n)\_n \in \mathbf{Z}$ is a sequence of Aubry–Mather sets, *not separated by an invariant curve*, there exists an orbit coming successively (in the prescribed order) close to each of the Λ_i. In more degrees of freedom, invariant tori do not separate and there is no obvious obstruction preventing an orbit to come successively close to the supports of any given sequence of minimal measures (for a generic diffeomorphsim). Mather has a partial result in this direction.

2.7 Renormalization theory for quadratic polynomials

The Aubry–Mather sets and minimal measures we have just discussed are important generalizations of the classical KAM quasiperiodic motions. Another nonstandard "generalization" is provided by the dynamics of infinitely renormalizable quadratic polynomials.

The key tool is the Douady–Hubbard theory of quadratic-like maps [DH2], i.e. ramified covering $f : U \to U'$ of degree 2, with U, U' simply connected and $U \subset\subset U'$. Such a map is quasiconformally conjugated to a quadratic polynomial, its filled-in Julia set is $K_f = \bigcap_{n \geq 0} f^{-n}(U')$.

An integer $n \geq 2$ is a renormalization period for the quadratic polynomial $P_c : z \mapsto z^2 + c$ if there exist open neighborhoods $U_n \subset\subset U'_n$ of 0 such that $P_c^n : U_n \to U'_n$

is quadratic-like with connected filled-in Julia set. The quadratic polynomial is infinitely renormalizable if the set $\mathcal{N} = \{n_1 < n_2 < \cdots\}$ of its renormalization periods is infinite; then n_k divides n_{k+1} (we write $n_{k+1} = p_{k+1}n_k,\ p_1 = n_1$); let $f_o = P_c$ and $f_k = P_c^{n_k}/U_{n_k}$. Then f_{k+1} is the restriction of f_k^{pk} to the smaller domain $U_{n_{k+1}}$; it is called the renormalization of f_k. (See Figure 1).

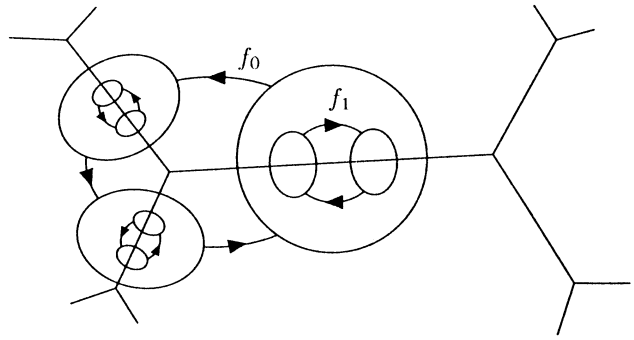

Figure 1

In the study of the dynamics of rational maps, a key point is to understand the geometry and the dynamics of the post-critical set:

$$P(f) = \{\overline{f^n(c), n \geq 1, c \text{ critical value}}\}\,.$$

In our case, $P(f_o)$ is contained in the fully invariant compact set

$$K_\infty = \bigcap_{k\geq 0} \bigcup_{0\leq j<n_k} f_o^j(K_{f_k})\,.$$

We have a natural continuous surjective map

$$K_\infty \xrightarrow{p} \mathbf{Z}_{\mathcal{N}} = \lim_{\leftarrow} \mathbf{Z}/n_k\mathbf{Z}$$

onto the profinite abelian group of $\mathcal{N}$-adic integers, the dynamics on $\mathbf{Z}_{\mathcal{N}}$ being translation by 1; the post-critical set is sent to the set of positive integers. The map p is known to be a homeomorphism for real c (Sullivan), but it is also known that it is not always injective.

Problem: Find a necessary and sufficient condition on the combinatorics for p to be a homeomorphism.

Very little is known in the general case (with complex c). On the other hand, a beautiful approach pioneered by Sullivan [Su1], [Su2] has been fruitful in important particular cases. The general strategy is the following; one first constructs, from an infinitely renormalizable quadratic-like map, a geometric object that is

a compact set laminated by Riemann surfaces (Riemann lamination). The dynamical properties of the initial quadratic-like map correspond to some properties of the complex geometry of this Riemann lamination. Such laminations have, as usual Riemann surfaces, a Teichmüller space; the renormalization operator (from f_k to f_{k+1}) corresponds to a map between such Teichmüller spaces and we are led to study the *dynamics* of this new map (at the "parameter" level). This map does not increase the Teichmüller distance, and the central problem is to understand to which extent it is contracting. There are partial results in this direction by Sullivan (for real c, with $(p_k)\_k \geq 1$ bounded) and McMullen (under a potentially more general geometric assumption) [McM].

3. Hyperbolic Dynamics

3.1 Before we discuss some recent developments, we recall some "classical" hyperbolic dynamics, as developed by Anosov, Sinaï, Smale, Palis, ... in the 1960s [Bo], [Sm], [Sh], [Y9].

The central concept is that of a *basic set* : if f is a smooth diffeomorphism of a manifold M, a basic set of f is a *compact*, *invariant* subset K of M that is *transitive* (f/K has a dense orbit), *locally maximal* (K is the maximal invariant set in an open neighborhood), and *hyperbolic* : the tanget bundle $E = TM/K$ admits an invariant splitting $E = E^s \oplus E^u$ in a stable subbundle E^s uniformly contracted by Tf and an unstable subbundle E^u uniformly contracted by Tf^{-1}.

The dynamics on a basic set are fairly well understood (and completely so when $\dim M = 2$); in particular, the existence of Markov partitions allows us to reduce the study of periodic orbits, invariant measures, ... to the same problems in symbolic dynamics, i.e. subshifts of a finite type on a finite alphabet.

The existence of a basic set K for a diffeomorphism is a semilocal property: it only involves the dynamics of f near K. One gets to more global properties (Anosov diffeomorphisms, Axiom A diffeomorphisms, ...) if one asks that some big invariant subset, carrying "most" of the dynamical properties of f, is hyperbolic.

For instance the *chain recurrent set* $C(f)$ of a smooth diffeomorphism of a compact manifold is the locus of points that are periodic for some arbitrarily small C^o-perturbation of f. Let us say that f is *uniformly hyperbolic* if $C(f)$ is hyperbolic.

It can be proven that $C(f)$ is then a finite union of disjoint basic sets. Uniformly hyperbolic diffeomorphisms form an open subset of $\mathrm{Diff}^\infty(M)$ (it is even open in the C^1-topology), and they are *stable* [R], [Ro]: two C^1-close uniformly hyperbolic diffeomorphisms are topologically conjugated on a neighborhood of their respective chain recurrent sets. Actually, a deep theorem of Mañé (extended by Palis) states that the converse is also true: a diffeomorphism that is C^1 stable in this sense is uniformly hyperbolic [M3].

It was hoped at some point that such globally hyperbolic diffeomorphisms could account, at least in the dissipative case, for most diffeomorphisms. This was shown to be too optimistic when Newhouse [N1], [N2] discovered in the 1970s that there

exist open sets of diffeomorphisms that exhibit generically infinitely many attractive periodic orbits (this is not compatible to any global uniform hyperbolic behaviour). Nevertheless, uniformly hyperbolic diffeomorphisms still constitute a good starting point from which one can bifurcate and study more complicated diffeomorphisms. Also, there are many important classes of diffeomorphisms that are not uniformly hyperbolic, but that admit many basic sets that together should carry a lot of information on the dynamics.

3.2 The conceptual apparatus to study weaker forms of hyperbolicity is based on Oseledets' theorem (1968) [O] and Pesin's theory (1976) [Pe1], [Pe2], [FHY]. Oseledets theorem, itself based on a subadditive ergodic theorem, asserts the existence of Lyapunov exponents of a diffeomorphisms on a (Borel) set of points that has full measure with respect to all invariant measures. From this starting point, Pesin then constructed the stable and unstable "foliations" associated to the nonzero exponents and proved the crucial fact that they are absolutely continuous.

How frequently are all (or some) of the Lyapunov exponents of a non-uniformly hyperbolic diffeomorphism different from 0? I have mentioned above Herman's theorem (see 2.5), which indicates that we cannot be too optimistic. On the other hand, there have been several breakthroughs showing that it tends to happen with positive measure in the parameter space.

3.3 The first crucial step in this direction is Jakobson's theorem (1981) [J]. He considers real quadratic polynomials $P_c(x) = x^2 + c$, for c in some subset $A_\varepsilon \subset [-2, -2+\varepsilon]$, whose relative measure tends to 1 as ε goes to zero. For such a parameter, let α be the negative fixed point of P_c and $I = (\alpha, -\alpha)$; he constructs a countable partition $I = \bigcup I_i$ (mod 0) into disjoint open intervals and a map $T : \bigcup I_i \to I$ that is *uniformly expanding* (with bounded distortion) and whose restriction to each I_i is an iterate $P_c^{k_i}$ realizing a diffeomorphism onto I (see Figure 2).

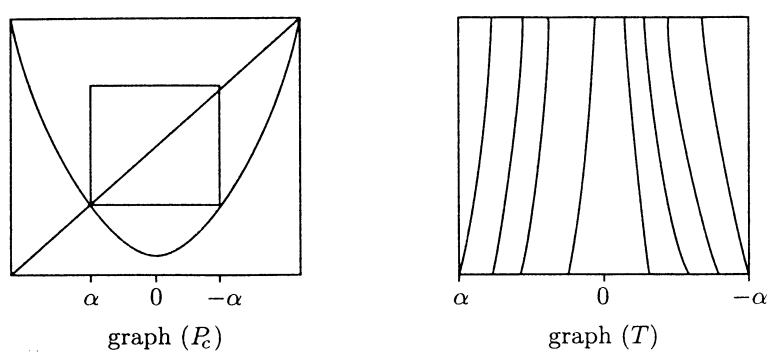

Figure 2

An important point is that although the k_i are not bounded, the measures of the I_i for which $k_i \geq k$ is exponentially small with k. From the existence of such a map T, it is easy to deduce that P_c has an ergodic invariant measure that is equivalent to Lebesgue measure on the (real) Julia set, and that the corresponding Lyapunov exponent is positive.

This kind of result has since been extended in several directions. Rees [Re] has proved that a similar statement holds for an holomorphic family of rational maps (see also [Be]). Jakobson and Swiatek have extended the set of values of c for which the map T is constructed (putting no restriction on the k_i's) [JS].

For *complex* quadratic polynomials whose all periodic orbits are repulsive and that are not infinitely renormalizable, I proved that the dynamics are still sufficiently expanding to guarantee that the Julia sets are locally connected, as a consequence of a (very weak) self-similarity property [Y10]. Lyubich [Ly1] then went on to prove that such a Julia set has measure 0. These results are related to previous work of Branner–Hubbard [BH] and McMullen [McM] on complex cubic polynomials.

3.4 Another very important breakthrough, going to higher dimensions, was achieved by Benedicks–Carleson (1989). They consider Hénon's family [He] of polynomial diffeomorphisms of the plane:

$$H_{a,b}(x,y) = (x^2 + a - y,\ bx)\,.$$

The parameter b is the constant value of the Jacobian; it is fixed and very small. The parameter a belongs to a subset $A_\varepsilon \subset [-2+\varepsilon,\ -2+2\varepsilon]$ of relative large measure, with $0 < |b| \ll \varepsilon \ll 1$. For such parameters, the rectangle $U = \{(x,y), |x| < 2 - \frac{3}{4}\varepsilon, |y| < 3b\}$ satisfies $H_{a,b}(U) \subset\subset U$, and one wants to describe the "attractor"

$$\Lambda_{a,b} = \bigcap_{n\geq 0} H^n_{a,b}(U)\,.$$

What emerges from Benedicks–Carleson's study [BC2], together with more recent work of Benedicks–Young [BY] and Jakobson–Newhouse [JN] is the following structure (see Figure 3): one can construct an open subrectangle $V \subset U$, a countable family of disjoint subrectangles $V_i \subset V$ such that $\bigcup_i V_i$ "essentially" covers $V \cap \Lambda$, and a map $T : \bigcup_i V_i \to V$, whose restriction to V_i is some iterate H^{k_i} of H, and that is *uniformly* hyperbolic; the k_i's are not bounded, but they take big values on very small sets.

From there, one constructs a nice Sinaï–Bowen–Ruelle invariant measure on Λ; it describes the asymptotics of a positive *Lebesgue* measure set of orbits in U, and the Lyapunov exponents with respect to that measure are nonzero. One also recovers many classical properties of uniformly hyperbolic attractors (it is

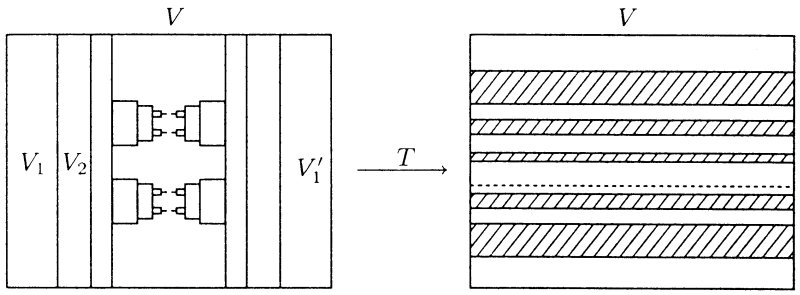

Figure 3

easy to see that Λ cannot be uniformly hyperbolic; in fact, there is a dense subset of Λ where the stable and unstable manifolds are tangent).

One should note that the admissible set A_ε of values of the parameter a has an empty interior; also, two distinct values of a give rise to attractors that admit the same qualitative description, but are definitely not conjugated. This is in strong contrast to the uniformly hyperbolic case.

The kind of phenomenon that we have tried to describe is not particular to the Hénon family. A first extension of these results, extremely important for applications (see below), was given by Mora–Viana [MV], who introduced the concept of "Hénon-like" families. More recently, Viana [V] proved similar results for some families in higher dimensions, for instance skew products:

$$T : \mathbf{T}^2 \times \mathbf{R}^2 \to \mathbf{T}^2 \times \mathbf{R}^2$$

$$T(\theta, (x, y)) = (A(\theta),\ H_{a(\theta),b}(x, y))\,,$$

where A is an Anosov diffeomorphisms of $\mathbf{T}^2$ (for instance linear hyperbolic) and a is a Morse function on $\mathbf{T}^2$ (subjected to some conditions). In this context, because of the uniform hyperbolicity on the base, it is no more necessary to exclude parameters.

3.5 What we would like to do in the next few years is to obtain a conceptual theory of "weakly hyperbolic basic sets" (including of course the striking examples considered above). For a smooth diffeomorphisms f of a manifold M, such a "weakly hyperbolic basic set" should again be a *compact*, *invariant*, *transitive*, *locally maximal* subset K of M satisfying moreover some kind (?) of *weak hyperbolicity* condition. Let me speculate, based on the examples above, on what could be some aspects of this theory.

(1) One would be able to cover "most" of K with a countable family of disjoint open sets V_i and define on $\bigcup V_i$ a *uniformly hyperbolic* map T whose restriction to V_i is some iterate f^{k_i} of f;

(2) One would thus obtain K as the limit (for the Hausdoff distance on compact sets) of an increasing sequence of (uniformly hyperbolic) basic sets

$$K_n = \bigcap_{m \in \mathbf{z}} f^{-m} \left(\overline{\bigcup_{k_i \leq n} V_i} \right) ;$$

(3) One should be able to construct some kind of Sinaï–Bowen–Ruelle invariant measure, whose "restriction" to most unstable manifolds would be absolutely continuous with respect to some Hausdorff measure on the unstable manifold. The Lyapunov exponents with respect to this measure would be nonzero;

(4) There would exist some "infinite Markov partition" (as the V_i above) allowing a description by symbolic dynamics (with an infinite alphabet).

4. Parameter Space

4.1 I want to discuss now "how many" dynamical systems we are able to understand.

Let us start with a "test case", the family of quadratic (real or complex) polynomials $P_c : x \mapsto x^2 + c$, where only one (real or complex) parameter c is involved.

If the critical point 0 escapes under iteration to infinity or converges to some attractive periodic orbit, the dynamics on J_c are *uniformly hyperbolic* (expanding) and stable; such parameters c form an open set U_{hyp}.

If there is a periodic orbit with eigenvalue λ of modulus 1, the parameter c is determined (algebraically) by the period and λ (up to a finite number of choices): the dynamics on the Julia set is of "weak-hyperbolic" type when λ is a root of unity, whereas quasiperiodic features are dominant when λ is not a root of unity (see 2.2).

We are left with the case where all periodic orbits are repulsive, but the critical point belongs to the Julia set J_c (preventing it to be uniformly hyperbolic): it is here natural to discuss separately the cases where P_c is infinitely renormalizable and it is not.

In the non-infinitely renormalizable case, the dynamics on the Julia set still exhibit some (very weak) form of hyperbolicity (see 3.3). As consequence, I proved that such parameters are *rigid*, i.e. they are determined by the combinatorics of the Julia set [Y10].

In the infinitely renormalizable case, the dynamics on the Julia set exhibit many quasiperiodic features (see 2.7). Swiatek [Sw] and Lyubic [Ly2] have proved that for *real* quadratic polynomials, these parameters are rigid (in the real sense). As a consequence, the open set $U_{\text{hyp}} \cap \mathbf{R}$ of (uniform) hyperbolicity is dense in $\mathbf{R}$. Actually, one would expect that even the following stronger statement should be true: for almost all $c \in \mathbf{R}$, either $c \in U_{\text{hyp}}$ or there exists (as in Jakobson's theorem) an invariant measure on $J_c \cap \mathbf{R}$, absolutely continuous with respect to Lebesgue measure, for which the Lyapunov exponent is nonzero.

On the other hand, although there is a partial result by Lyubich, it is not known whether infinitely renormalizable parameters are rigid in the complex sense. This

is the missing step in the Douady–Hubbard's conjecture that the Mandelbrot set is locally connected; actually, assuming this rigidity, we would have from Douady–Hubbard [DH1] and Thurston a complete *topological* description of the Mandelbrot set (whereas we have only a *combinatorial* one at the moment).

4.2 In higher dimensions, such as for instance for diffeomorphisms of surfaces, we are still very far from having a near complete understanding of the "parameter space".

Nevertheless, the results that we have discussed above and others in the same line have led to a change of point of view in looking at these problems.

The classical, uniformly hyperbolic, basic sets have a strong stability property known as hyperbolic continuation: a nearby diffeomorphism admits basic set close to the original one and the dynamics on the two basic sets are conjugated by a homeomorphisms close to the identity. The "parameter space", for instance the space of all smooth diffeomorphisms of the given compact manifold, was in the 1960s and 1970s mostly considered from a topological point of view; one was looking for dynamical features appearing on some open set, or some G_δ set (dense into some open set).

Although this point of view remains important, properties of "weakly hyperbolic basic set" such as Hénon-like attractors have given a strong impetus on the measure-theoretic point of view of the parameter space: typically, in generic parameter family of diffeomorphisms, on expects to meet these weakly hyperbolic features on a F_σ subset of the parameter space (because one needs to exclude parameters), but one that has positive Lebesgue measure.

4.3 I would like here to emphasize the importance of the Hénon family, or rather of the Hénon-like families introduced by Mora and Viana (see 3.4), for the study of surface diffeomorphisms.

Consider a smooth diffeomorphism f_o of a surface M that exhibits a homoclinic tangency: this means that f_o has a fixed saddle point p such that the stable manifold $W^s(p)$ and the unstable manifold $W^u(p)$ are tangent along a homoclinic orbit $(f^n(q))_{n\in\mathbb{Z}}$. This is a codimension one phenomenon.

Assume that the tangency is quadratic and consider a one-parameter family of diffeomorphisms $(f_t)_{-|t|<\varepsilon}$ unfolding generically the tangency.

One would like to understand the orbits under f_t, $|t|$ small, which remain in an appropriately small neighborhood of the orbit of q under f_o. To do this, the first step is to compute the return map R_t into some small neighborhood V of q; it is a disjoint union $V = \bigcup_{n\geq n_o} V_n$; R_t is equal to f_t^n on V_n and is a Hénon-like family of approximately constant Jacobian. The striking conclusion is that every dynamical feature exhibited by the Hénon family, or Hénon-like families, actually occurs in a generic one-parameter family near homoclinic bifurcations. For instance, if the fixed point p is dissipative, the Jacobian of these Hénon-like families will be very

small and we will get for a positive measure set of parameters Hénon-like attractors. Let me recall in this context an older result of Newhouse: there are arbitrarily close to 0 intervals of parameter values in which for generic t the diffeomorphisms f_t has *infinitely many* periodic orbits; still we suspect (but we don't know) that this only happens for a set of t of Lebesgue measure zero.

Homoclinic bifurcations are by no means the only codimension one bifurcations where Hénon-like families occur; another such example, studied by Diaz, Rocha, and Viana, is the critical saddle-node bifurcation [DRV1], [DRV2].

4.4 Palis has proposed a general program to study the dynamics of (non-uniformly hyperbolic) diffeomorphisms (of compact surfaces, to begin with). He suggests that one should look first to a *dense* subset in the space of non-uniformly hyperbolic diffeomorphisms for which we have at least some grasp of the dynamics; he conjectures actually that homoclinic tangencies could be such a subset. Then one should consider generic parametrized families through these "simple" diffeomorphisms, and study which dynamical features are "persistent" in the measure theoretical sense, i.e. they occur on sets of parameters of positive measure or even relative positive density near the initial diffeomorphism.

Starting with Newhouse–Palis–Takens [NPT] and Palis–Takens ([PT1], [PT2]), there has been a great deal of effort and results related to the study of homoclinic bifurcations, which give rise to an extremely rich number of complicated phenomena; still we are quite far from a satisfactory understanding of f. One very interesting feature of these results, when the fixed saddle point belongs to some (uniformly hyperbolic) basic set, is the subtle relationship between the geometry of the basic set (Hausdorff dimension, thickness, ...) and the dynamics near the bifurcating parameter [PY2].

4.5 Still there is a very central difficulty in carrying out Palis's program, a difficulty that occurs at very many places in dynamical systems, the so-called closing-lemma problem. Pugh's closing lemma [Pu] asserts that if p is a recurrent point for a smooth diffeomorphism f, then one can perturb f *in the C^1-topology* in order for p to become periodic. See also [M1].

We still have no idea whether it is possible to achieve the same goal by a C^2 (or even $C^{1+\varepsilon}$) perturbation. In particular, we still don't know whether the C^2 diffeomorphisms of $\mathbf{T}^2$ that have a periodic orbit form a C^2-dense subset of $\mathrm{Diff}^2(\mathbf{T}^2)$! Guttiercz [G] has constructed an example (on the noncompact surface $\mathbf{T}^2 - \{0\}$ that indicates that the localized perturbations used by Pugh in the C^1-case cannot be sufficient in the C^2-case. Also Herman [H4], [H5] has constructed a Hamiltonian flow on a compact sympletic manifold, for which no periodic orbit can be created by smooth perturbations of the Hamiltonian (because of KAM-theory and a sympletic rigidity of the rotation number).

In a similar vein, recent results of Herman suggest that it would be very interesting to know the answer to the following question: Let $\mathcal{C}$ be the set of

smooth diffeomorphisms (of a compact manifold M) that are of finite order on a nonempty open set (depending on the diffeomorphism); what is the closure of $\mathcal{C}$ in the C^∞-topology?

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THE WORK OF EFIM ZELMANOV

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0. Introduction

Efim Zelmanov has received a Fields Medal for the solution of the restricted Burnside problem. This problem in group theory had long been known to be related to the theory of Lie algebras. In fact, to a large extent *it is* the problem in Lie algebras. A precise statement of it can be found in Section 2 below.

In proving the necessary properties of Lie algebras, Zelmanov built on the work of many others, though he went far beyond what had previously been done in this direction. For instance, he greatly simplified Kostrikin's results [K] which settled the case of prime exponent and then extended these methods to handle the prime power case.

However, while the case of exponent 2 is trivial, the case of exponent 2^k for arbitrary k is the most difficult case that needed to be addressed. The results from Lie algebras that work for exponent p^k with p an odd prime are not adequate for exponent 2^k. This indicated that a new approach was necessary here. Zelmanov was the first to realize that in the case of groups of exponent 2^k the theory of Jordan algebras is of great significance. Even though Vaughan–Lee later removed the need for Jordan algebras [V], it seems probable that this proof could not have been discovered without them, as the ideas used arise most naturally from Jordan algebras.

Zelmanov had earlier made fundamental contributions to Jordan algebras and was an expert in this area, thus he was uniquely qualified to attack the restricted Burnside problem.

Below the background from the theory of Jordan algebras and some of Zelmanov's contributions to this theory are first discussed (I am grateful to McCrimmon and Jacobson for much of this material). See [J1] and [J2] for the general theory of Jordan algebras. Then the Burnside problems are described and some of the things that were earlier known about them are listed. Section 4 contains some consequences of the restricted Burnside problem. Finally, some relevant results from Lie and Jordan algebras are mentioned (in a necessarily sketchy manner).

Zelmanov himself has written a set of expository notes on these topics [Z11]. It contains all the appropriate definitions and some of the material used in the proof of the restricted Burnside problem. It also includes material on several related questions, such as the Kurosh–Levitzky problem. Of course, ultimately, the details are the heart of the matter, and for these the reader should consult [Z8] and [Z9], or [V].

This circle of ideas illustrates the unity of mathematics once again. Although many formal identities are used to settle the restricted Burnside problem, it seems unlikely that they could have been discovered without the conceptual framework provided by the seemingly unrelated and diverse fields of Lie and Jordan algebras.

1. Jordan Algebras

Jordan algebras were introduced in the 1930s by the physicist P. Jordan in an attempt to find an algebraic setting for quantum mechanics, essentially different from the standard setting of hermitian matrices. Hermitian matrices or operators are not closed under the associative product xy, but are closed under the symmetric products $xy + yx$, xyx, x^n. An empirical investigation indicated that the basic operation was the Jordan product

$$x \cdot y = \frac{1}{2}(xy + yx)\,,$$

and that all other properties flowed from the commutative law $x \cdot y = y \cdot x$ and the Jordan identity $(x^2 \cdot y) \cdot x = x^2 \cdot (y \cdot x)$. (For example, the Jordan triple product $\{xyz\} = \frac{1}{2}(xyz + zyx)$ can be expressed as $x \cdot (y \cdot z) + (x \cdot y) \cdot z - (x \cdot z) \cdot y$, though the tetrad

$$\{xyzw\} = \frac{1}{2}(xyzw + wzyx)$$

cannot be expressed in terms of the Jordan product.) Jordan took these as axioms for the variety of Jordan algebras. Algebras resulting from the Jordan product in an associative algebra were called special, so the physicists were seeking algebras that were exceptional (= nonspecial). In a fundamental paper [JNW] Jordan, von Neumann, and Wigner classified all finite-dimensional formally-real Jordan algebras. These are direct sums of five types of simple algebras: algebras determined by a quadratic form on a vector space (a special subalgebra of the Clifford algebra of the quadratic form) and four types of algebras of hermitian $(n \times n)$-matrices over the four composition algebras (the reals, complexes, quaternions, and octonions). The algebra of hermitian matrices over the octonions is Jordan only for $n \leq 3$, and is exceptional if $n = 3$, so there was only one exceptional simple algebra in their list (now known as the Albert algebra, of dimension 27). At the end of their paper Jordan, von Neumann, and Wigner expressed the hope that by dropping the assumption of finite dimensionality one might obtain exceptional simple algebras other than Albert algebras.

Algebraists developed a rich structure theory of Jordan algebras over fields of characteristic $\neq 2$. First, the analogue of Wedderburn's theory of finite-dimensional

associative algebras was obtained by Albert. Next this was extended by Jacobson to an analogue of the Wedderburn–Artin theory of semisimple rings with minimum condition on left or right ideals. In this, the role of the one-sided ideals was played by inner deals, defined as subspaces B such that $U_b x$ is in B for all x in the algebra A and all b in B where $U_a = 2L_a^2 - L_{a^2}$ and L_a is the left multiplication by a in the Jordan algebra A. If A is the Jordan subalgebra of an associative algebra the $U_b x = bxb$ in the associative product. Using the definition of semi-simplicity (called nondegeneracy) that A contains no $z \neq 0$ such that $U_z = 0$, Jacobson showed that every nondegenerate Jordan algebra with d.c.c. on inner ideals is the direct sum of simple algebras that are of classical type (analogues of those found in [JNW]: (Type I) Jordan algebras of nondegenerate quadratic forms; (Type II) algebras $H(A, *)$ of hermitian elements in a $*$-simple artinian associative algebra $A((n \times n)$-matrices over a division algebra with involution, or over a direct sum of a division algebra and its opposite under the exchange involution, or matrices over a split quaternion algebra with standard involution); (Type III) 27-dimensional exceptional Albert algebras; (Type IV) Jordan division algebras, defined by the condition that U_a is invertible for every $a \neq 0$.

Up to this point, the structure theory treated only algebras with finiteness conditions because the primary tool was the use of primitive idempotents to introduce coordinates. In 1975 Alfsen, Schultz, and Stömer obtained a Gelfand–Naimark theorem for Jordan C^*-algebras, and once again the basic dimensional structure theorem, but here again it was crucial that the hypotheses guaranteed a rich supply of idempotents.

In three papers [Z1], [Z2], [Z3], Zelmanov revolutionized the structure theory of Jordan algebras. These deal with prime Jordan algebras, where A is called prime if $U_B C = 0$ for ideals B and C in A implies that either B or $C = 0$. In [Z1] Zelmanov proved the remarkable result that a prime Jordan algebra without nil ideals (improved in [Z3] to prime and nondegenerate) is either i-special (a homomorphic image of a special Jordan algebra) or is a form of the 27-dimensional exceptional algebra. This applied in particular to simple algebras. The proof required the introduction of a host of novel concepts and techniques as well as sharpening of earlier methods, e.g. the coordinatization theorem of Jacobson and analogues of results on radicals due to Amitsur.

The paper [Z3] is devoted to the study of i-special Jordan algebras. Zelmanov showed that a prime nondegenerate i-special algebra is special, and he determined their structure as either of hermitian type or of Clifford type. Paper [Z2], which preceded [Z3] obtained these results for Jordan division algebras.

The principal tool in both papers is the study of the free associative algebra $\Phi\langle X\rangle$ on $X = \{x_1, x_2, \ldots\}$. This becomes a Jordan algebra $\Phi\langle X\rangle^+$ by replacing the given associative multiplication ab by $a \cdot b = \frac{1}{2}(ab + ba)$. The subalgebra $SJ\langle X\rangle$ of $\Phi\langle X\rangle^+$ generated by X is called the free special Jordan algebra.

We also have the subalgebra $H\langle X\rangle$ of $\Phi\langle X\rangle^+$ of symmetric elements ($a^* = a$) under the involution in $\Phi\langle X\rangle$ fixing the elements of X. It was shown by Paul Cohn

in 1954 that $SJ\langle X\rangle \subsetneqq H\langle X\rangle$ and $H\langle X\rangle$ is the subalgebra of $\Phi\langle X\rangle^+$ generated by X and all the tetrads $x_ix_jx_kx_l$ with $i < j < k < l$. Zelmanov has obtained a completely unanticipated supplement to Cohn's theorem: the existence of elements f in $SJ\langle X\rangle$ such that if $I(f)$ denotes the ideal generated by f then

$$\{I(f), p, q, r\} \in SJ\langle X\rangle$$

for p, q, r in $SJ\langle X\rangle$. This is used to sort out the two types of i-special algebras: Clifford types characterized by the identity $f \equiv 0$ and hermitian types by the nonidentity $f \not\equiv 0$.

One of the consequences of Zelmanov's theory is that the only exceptional simple Jordan algebras, even including infinite-dimensional ones, are the forms of the 27-dimensional Albert algebras. This laid to rest the hope that had been raised by Jordan, von Neumann, and Wigner in [JNW]. Another consequence of Zelmanov's results is that the free Jordan algebra in three or more generators has zero divisors (elements a such that U_a is not injective). This is in sharp contrast to the theorem of Malcev and Neumann that any free associative algebra can be imbedded in a division algebra.

Motivated by applications to analysis and differential geometry, Koecher, Loos, and Myberg extended the structure theory of Jordan algebras to triple systems and Jordan pairs. Zelmanov applied his methods to obtain new results on these.

Lie methods were used in these papers based on the Tits–Koecher construction. The final work in this line of investigation was [Z4] in which Zelmanov applied the theory of Jordan triple systems to study graded Lie algebras with finite gradings in which the homogeneous parts could be infinite dimensional.

To encompass characteristic 2 (which is essential for applications to the restricted Burnside problem) it is necessary to deal with quadratic Jordan algebras [JM]. These were introduced by McCrimmon in [Mc] as the natural extension of Jordan algebras to algebras over any commutative ring. This amounted to replacing the product $a \cdot b = \frac{1}{2}(ab + ba)$ in an associative algebra by the product $U_a b$.

In the joint paper with McCrimmon [ZM], the results of [Z3] were extended to quadratic Jordan algebras.

2. Burnside Problems

We begin with some definitions and notation.

A group is *locally finite* if every finite subset generates a finite group. In 1902 Burnside [B1] studied torsion groups and asked when such groups are locally finite. The most general form of the question is the Generalized Burnside Problem (GBP).

(GBP) Is a torsion group necessarily locally finite?

Equivalently

(GBP)$'$ Is every finitely generated torsion group finite?

A group G has a *finite exponent* e if $x^e = 1$ for all x in G and e is the smallest natural number with this property. Clearly a group with a finite exponent is a torsion group. A more restricted version of GBP, which already occurs in Burnside's work, is the ordinary Burnside Problem (BP).

(BP) Is every group that has a finite exponent locally finite?

There is a universal object $B(r, e)$, (the Burnside group of exponent e on r generators), which is the quotient of the free group on r generators by the subgroup generated by all eth powers. BP is equivalent to

(BP)$'$ Is $B(r, e)$ finite for all natural numbers e and r?

Burnside proved that groups of exponent 2 (trivial) and exponent 3 are locally finite. In 1905 Burnside [B2] showed that a subgroup of $GL(n, \mathbf{C})$ of finite exponent is finite. Schur in 1911 [Sc] proved that a finitely generated torsion subgroup of $GL(n, \mathbf{C})$ has finite exponent, and hence a torsion subgroup of $GL(n, \mathbf{C})$ is locally finite. This was very important as it showed that answers to BP or GBP would necessarily involve groups not describable in terms of linear transformations over $\mathbf{C}$. Other methods were required. In handling groups of exponent 3 Burnside had used only the multiplication table of a group. However, his methods were totally inadequate to handle, for instance, groups of prime exponent greater than 3.

During the 1930s people began to study finite quotients of $B(r, e)$ and considered the following statement.

(RBP) $B(r, e)$ has only finitely many finite quotients.

This is equivalent to

(RBP)$'$ $B(r, e)$ has a unique maximal finite quotient $RB(r, e)$.

W. Magnus called the question of the truth or falsity of RBP *the restricted Burnside problem.* If such a unique maximal finite quotient $RB(r, e)$ exists for some e and r, then necessarily every finite group on r generators and exponent e is a homomorphic image of $RB(r, e)$. If $RB(r, e)$ exists for some e and all r we say that RBP is true for e.

3. Results

In 1964 Golod [G] constructed infinite groups for every prime p, which are generated by 2 elements and in which every element has order a power of p, thus giving a negative answer to GBP. A few years later in 1968 Adian and Novikov [AN] showed that $B(2, e)$ is infinite for e odd and $e > 4380$, thus giving a negative answer to BP. The bound has been improved since then as $B(r, e)$ is finite for $e = 2, 3, 4$, or 6, but in no other case with $r > 1$ is it known to be finite.

In a seminal paper Hall and Higman [HH] in 1956 proved a series of results concerning RBP. Let π be a set of primes. Consider the following two statements.

(1) There are only finitely many finite simple π-groups of any given exponent.

(2) The Schreier conjecture is true for π-groups, i.e. for any finite simple π-group G, $\mathrm{Aut}(G)/G$ is solvable.

A special case of one of their results is the following.

Theorem [HH]. *Suppose that statements* (1) *and* (2) *are true for the set* π. *Then if for every prime* p *in* π, *and natural numbers* m *and* r, $RB(r, p^m)$ *exists*; *then RBP is true for any exponent* e *that is a* π*-number.*

The classification of the finite simple groups shows that (1) and (2) are true for any set of primes π. Hence the truth of RBP will follow once it is proved that $RB(r, p^m)$ exists for all primes p and all natural numbers m and r.

In 1959 Kostrikin announced that $RB(r, p)$ exists for p a prime and any natural number r. Kostrikin's original argument had some difficulties. He published a corrected and updated version of his proof in his book [K], which contains numerous references to Zelmanov.

In 1989 Zelmanov announced that RBP is true for all exponents p^m with p any prime, and hence for all exponents by the remarks above. The proof appeared in 1990–91 in Russian. The English translation appeared in [Z8] and [Z9].

It should be mentioned that analogous questions have been raised for associative, Lie, and Jordan algebras. Golod's work was actually motivated by the associative algebra question and the counterexamples for groups arose as corollaries. The questions for Lie and Jordan algebras will be discussed below.

4. Some Consequences

This section contains some consequences of RBP. The ideas used in the proof, in addition to the actual result, have also been applied widely.

The next three results were proved by Zelmanov [Z10] as direct consequences of RBP.

Theorem 1. *Every periodic pro-p-group is locally finite.*

Corollary 2. *Every infinite compact* (*Hausdorff*) *group contains an infinite abelian subgroup.*

Theorem 3. *Every periodic compact* (*Hausdorff*) *group is locally finite.*

Theorem 3 was conjectured by Platonov [Ko].

Shalev showed that RBP implies the following.

Theorem 4 [Sh]. *A pro-p-group is p-adic analytic if and only if there exists a natural number n such that the wreath product* $Z_p \wr Z_{p^n}$ *is not a homomorphic image of any subgroup of* G.

The "only if" part of Theorem 4 is elementary, but the converse is equivalent to RBP. Since then, Zelmanov jointly with others, has made several further contributions to the study of pro-p-groups, see e.g. [ZS], [ZW].

5. Lie Algebras

Let G be a finite group of exponent p^k, p a prime. Let $G = G_0$ and $G_{i+1} = [G, G_i]$ for all i. Choose s with $G_s \neq \langle 1 \rangle$, $G_{s+1} = \langle 1 \rangle$. Then

$$G = G_0 > \cdots > G_{s+1} = \langle 1 \rangle$$

is the lower central series of G. Define

$$L(G) = \sum_{i=0}^{s} G_i/G_{i+1}$$

as abelian groups. Then $L(G)$ becomes a Lie ring with $[a_iG_i, a_jG_j] = [a_i, a_j]G_{i+j+1}$, and $L(G)$ has the same nilpotency class as G. Furthermore $L(G)/pL(G)$ is a Lie algebra over Z_p.

Let L be a Lie algebra.

L satisfies the *Engel identity* (E_n) if $\mathrm{ad}(x)^n = 0$ for all x in L.

An element x in L is *nilpotent* if $\mathrm{ad}(x)^n = 0$ for some n.

If G has exponent p then $L(G)$ is a Lie algebra over Z_p that satisfies (E_{p-1}). Kostrikin proved

Theorem 5 [K]. *If L is a Lie algebra over Z_p that satisfies (E_{p-1}) then L is locally nilpotent.*

Theorem 1 implies the existence of $RB(r, p)$ and so yields RBP for prime exponent. Observe that for prime exponent $e = p$, the case $p = 2$ is trivial, so that it may be assumed that $p > 2$. This is in sharp contrast to prime power exponents $e = p^k$, where $p = 2$ is the most complicated case.

An element a of L is a *sandwich* if $[[L, a], a] = 0$ and $[[[L, a], L], a] = 0$. L is a *sandwich algebra* if it is generated by finitely many sandwiches. This concept was introduced by Kostrikin and is of fundamental importance for the proof of RBP. A first critical result is

Theorem 6 [ZK]. *Every sandwich Lie algebra is locally nilpotent.*

Theorem 6 is essential for the proof of Theorem 5.

The main result in [Z8] is rather technical but it has the following consequence.

Theorem 7 [Z8]. *Every Lie ring satisfying an Engel condition is locally nilpotent.*

More importantly, it implies

Theorem 8. *$RB(r, p^k)$ exists for p an odd prime.*

Once again an essential part of the proof requires Theorem 2. Let L be a Lie algebra over an infinite field of characteristic p that satisfies an Engel condition. They way to apply Theorem 2 is to construct a polynomial $f(x_1, \ldots, x_t)$ that is not identically zero, such that every element in $f(L)$ is a sandwich in L. Actually such a polynomial is not constructed but its existence for $p > 2$ follows only after a very complicated series of arguments, which constitute the bulk of the paper [Z8]. This of course settles RBP for odd exponent. (It might be mentioned that the classification of finite simple groups is not required here, only that groups of odd order are solvable.)

6. The Case of Exponent 2^k

The outline of the proof of RBP for exponent 2^k is similar to that for exponent p^k with $p > 2$ described in the previous section. However, the construction of the function f is vastly more complicated. It is here that quadratic Jordan algebras play an essential role, most especially the results of [ZM]. The details are extremely technical and cannot be presented here. The reader should consult [Z9] for a complete proof.

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Leonardo da Vinci Lecture, Milan Italy, June 1995
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ON THE RESTRICTED BURNSIDE PROBLEM

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In 1902 W. Burnside formulated his famous problems on periodic groups.

A group G is said to be *periodic* if for an arbitrary element $g \in G$ there exists a natural number $n = n(g)$ depending on g such that $g = 1$. A group G is said to be *periodic of bounded exponent* if there exists $n \geq 1$ such that for an arbitrary element $g \in G$ there holds $g^n = 1$. The minimal n with this property is called the exponent of G.

The General Burnside Problem: Is it true that a finitely generated periodic group is finite?

The Burnside Problem: Is it true that a finitely generated group of bounded exponent is finite?

After many unsuccessful attempts to solve the problems in positive the following weaker version of The Burnside Problem was formulated.

The Restricted Burnside Problem: Is it true that there are only finitely many *finite* m generated groups of exponent n?

Let F_m be the free group of rank m. Let F_m^n be the subgroup of F_m generated by all n-th powers $g^n, g \in F_m$. The Burnside Problem asks whether the group $B(m,n) = F_m/F_m^n$ is finite. The Restricted Burnside Problem asks whether the group $B(m,n)$, even if infinite, still has only finitely many subgroups of finite index. If yes, then factoring out the intersection of all subgroups of finite index in $B(m,n)$ we'll get $B_0(m,n)$, the universal finite m generated group of exponent n having all other finite m generated groups of exponent n its homomorphic images.

In 1964 Golod and Shafarevich constructed counter examples to the General Burnside Problem. Since then new infinite finitely generated periodic groups were constructed by S. V. Alyoshin [2], R. I. Grigorchuk [13], N. Gupta and S. Sidki [16], V. I. Sushchansky [47].

In 1968 P. S. Novikov and S. I. Adian [40] were able to construct counter examples to The Burnside Problem for groups of odd exponent $n \geq 4381$ (later cut to $n \geq 665$ by S. I. Adian still under assumption that n is odd).

Only very recently S. Ivanov [20] proved that the group $B(2, 2^k)$, where k is sufficiently big is also infinite. Thus The Burnside Problem has negative solution for all sufficiently big exponents whether even or odd.

Tarsky Monsters (see A. Yu. Olshansky [41]) and other Monsters brought to life by geometric methods show how wild infinite periodic groups can be. In a sense, this is the strongest form of a negative solution of The Burnside Problem.

Speaking of what is known on the positive side, the General Burnside Problem has positive solution for linear groups (W. Burnside, I. Schur) and The Burnside Problem has positive solution for groups of exponent 3 (W. Burnside), exponent 4 (I. N. Sanov) and exponent 6 (M. Hall). For all other exponents the problem is either open or solved in negative.

At the same time there were two major reasons to believe that the Restricted Burnside Problem would have a positive solution. One of these reasons was the Reduction Theorem obtained by P. Hall and G. Higman [17]. Let $n = p_1^{k_1} \cdots p_r^{k_r}$, where p_i are distinct prime numbers, $k_i \geq 1$, and assume that (a) the Restricted Burnside Problem for groups of exponents $p_i^{k_i}$ has a positive solution, (b) there are finitely many finite simple groups of exponent n, (c) the group of outer automorphisms $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is solvable for any finite simple group of exponent n. Then the Restricted Burnside Problem for groups of exponent n also has positive solution.

The announced classification of finite simple groups (see [11]) implies the conditions (b) and (c). Remark that if the exponent n is odd or $n = p^\alpha q^\beta$, where p, q are prime numbers, then by the celebrated theorems of Feit and Thompson [8] and W. Burnside there are no finite simple groups of exponent n and the conditions (b), (c) are satisfied automatically.

Another reason was the close relation of the problem to Lie algebras. Suppose that $n = p^k$, where p is a prime number. A finite group of exponent p^k is nilpotent. Let G be an m generated finite group of exponent $n = p^k$. To solve the Restricted Burnside Problem is to find an upper bound for the order $|G|$ that depends only on m and n. It is easy to see, however, that it is sufficient to find an upper bound $f(m, n)$ for the class of nilpotency of G.

Consider the Zassenhaus filtration of G : $G = G_1 > G_2 > \cdots > G_s = (1)$. The subgroup G_k is generated by p-powers of left-normed group commutators $(a_1, \ldots, a_i)^{p^j}$, where

$$i \cdot p^j \geq k, (a_1, a_2) = a_1^{-1} a_2^{-1} a_1 a_2, (a_1, \ldots, a_i) = (\cdots (a_1, a_2), a_3), \ldots, a_i)$$

and $a_1, a_2, \ldots$ are arbitrary elements from G. Factors G_i/G_{i+1} are elementary abelian p-groups and thus can be viewed as vector spaces over the field $\mathbb{Z}/p\mathbb{Z}$. Consider the direct sum

$$\tilde{L}(G) = \bigoplus_{i \geq 1} G_i/G_{i+1} \,.$$

The brackets $[a_i G_{i+1}, b_j G_{j+1}] = (a_i, b_j) G_{i+j+1}$, where $a_i \in G_i, b_j \in G_j$, defines a structure of a Lie algebra on $\tilde{L}(G)$.

Suppose that the group G is generated by elements $g_1, \dots, g_m$ and let $L(G)$ be the subalgebra of $\tilde{L}(G)$ generated by cosets $g_1G_2, \dots, g_mG_2$. If $L(G)^c = (0)$, that is, if $[[\dots[a_1, a_2], a_3], \dots, a_c] = 0$ for arbitrary elements $a_1, a_2, \dots, a_c \in L(G)$, then an arbitrary group commutator in $g_1, \dots, g_m$ of length c is a product of elements $\rho_k^{p^{m_k}}$, where ρ_k is a commutator in $g_1, \dots, g_m$ of length i and $i \cdot p^{m_k} > c$. Let $\rho_1, \rho_2, \dots, \rho_r$ be all left-normed group commutators in $g_1, \dots, g_m$ of length $\leq c$, $r \leq m^c$. Then an arbitrary element $g \in G$ can be represented as $g = \rho_1^{k_1} \cdots \rho_r^{k_r}$, $0 \leq k_i < n$. Hence,

$$|G| \leq n^{m^c}.$$

Suppose that $n = p$ is a prime number. Then Zassenhous filtration of G coincides with the lower central series, $G_{i+1} = (G_i, G)$. W. Magnus [36] proved that in this case the Lie algebra $L(G)$ satisfies Engel's identity

$$[\dots[y, \underbrace{x], x], \dots, x]}_{p-1} = 0 \qquad (E_{p-1}).$$

Thus the problem for groups of prime exponent p has been reduced to the following problem in Lie algebras:

Is it true that a finitely generated Lie algebra over $\mathbb{Z}/p\mathbb{Z}$ that satisfies Engel's identity E_{p-1} is nilpotent?

If the answer is "yes" then the degree of nilpotency can depend only on p and on the number of generators. This problem has been successfully solved by A. I. Kostrikin [28, 29] who solved in this way the Restricted Burnside Problem for groups of prime exponent.

If G is a finite group of prime power exponent $p^k, k > 1$, then the Lie algebra $L(G)$ does not necessarily satisfy Engel's identity E_{p^k-1} ([15, 18]). But

(1) $L(G)$ satisfies the linearized Engel's identity E_{p^k-1}, that is, for arbitrary elements $a_1, a_{p^k-1} \in L(G)$ we have

$$\sum \operatorname{ad}(a_{\sigma(1)}) \cdots \operatorname{ad}(a_{\sigma(p^k-1)})) = 0,$$

where σ runs over the whole symmetric group S_{p^k-1} (G. Higman, [19]),

(2) for an arbitrary commutator ρ on the generators $g_iG_2, 1 \leq i \leq m$ of $L(G)$ we have

$$\operatorname{ad}(\rho)^{p^k} = 0.$$

(I. N. Sanov [43]).

Now let us turn to what was happening in associative and Lie nil algebras.

An associative algebra A is said to be *nil* if for an arbitrary element $a \in A$ there exists a natural number $n = n(a)$ such that $a^{n(a)} = 0$. The algebra A is said to be *nil of bounded degree* if all numbers $n(a), a \in A$ are uniformly bounded from above.

A. G. Kurosh [32] and J. Levitzky (see [3]) formulated two problems for nil algebras that were similar to Burnside's problems.

The General Kurosh–Levitzky Problem: Is every finitely generated nil algebra nilpotent?

The Kurosh–Levitzky Problem: Is every finitely generated nil algebra of bounded degree nilpotent?

In fact, it was a counter example to the General Kurosh–Levitzky Problem that was constructed by the method of E. S. Golod and I. R. Shafarevich. Then this counter example was used to construct the first counter example to the General Burnside Problem. We have already mentioned above that since 1964 there appeared many new examples of finitely generated infinite periodic groups. But so far the example of E. S. Golod and I. R. Shafarevich remains the only example of a nonnilpotent finitely generated nil ring.

For nil algebras of bounded degree the situation is quite different. Unlike its group theoretic counterpart The Kurosh–Levitzky Problem has only positive solutions in all important classes of algebras.

The most general class of algebras where the General Kurosh–Levitzky Problem has positive solution is the class of PI-algebra. Let $f(x_1, \dots, x_k)$ be a nonzero element of the free associative algebra on the free generators $x_1, \dots, x_k$. We say that an associative algebra A satisfies the polynomial identity $f = 0$ (and thus is a PI-algebra) if $f(a_1, \dots, a_k) = 0$ for arbitrary elements $a_1, \dots, a_k \in A$.

Theorem (I. Kaplansky [25]**).** *A finitely generated nil PI-algebra is nilpotent.*

In 1956 A. I. Shirshov suggested another purely combinatorial direct approach to Kurosh–Levitzky problems.

Theorem (A. I. Shirshov [46]**).** *Suppose that an associative algebra A is generated by elements $a_1, \dots, a_m$ and assume that* (1) *A satisfies a polynomial identity of degree n,* (2) *every product of a_i's of length $\leq n$ is a nilpotent element. Then the algebra A is nilpotent.*

It is very important that the nilpotency assumption is imposed here not on every element of A but only on words in generators (even on finitely many of them).

Now let us go back to Lie algebras.

It is natural to call an element $a \in L$ nilpotent of the generator $\mathrm{ad}(a)$ is nilpotent. With this definition both Kurosh–Levitzky problems become meaningful for Lie algebras. Moreover, by the results of G. Higman and I. N. Sanov (see above) the Lie algebra of a finite group G of exponent p^n satisfies the assumptions of The Kurosh–Levitzky Problem in the form of A. I. Shirshov (the role of words is played by commutators).

In [56, 57] we solved this problem for Lie algebras satisfying a linearized Engel's identity E_n.

Theorem 1. *Suppose that a Lie algebra L is generated by elements $a_1, \ldots, a_m$ and assume that there exist integers $n \geq 1, m \geq 1$ such that* (1) *L satisfies the linearized Engel's identity E_n,* (2) *for an arbitrary commutator ρ in a_i's we have* $\mathrm{ad}(\rho)^m = 0$. *Then L is nilpotent.*

Corollary. *A Lie ring that satisfies E_n is locally nilpotent.*

From Theorem 1 we derive:

Theorem 2. *The Restricted Burnside Problem has a positive solution for groups of exponent p^k.*

In view of the Reduction Theorem of P. Hall and G. Higman and the announced classification of finite simple groups this implies that the Restricted Burnside Problem has positive solution for groups of an arbitrary exponent.

Now we shall try to explain briefly the idea of the proof of Theorem 1.

Recall that an algebra is said to be locally nilpotent if every finitely generated subalgebra of it is nilpotent.

In [54] we showed that to prove Theorems 1, 2 it suffices to prove that a Lie algebra over an infinite field which satisfies an Engel's identity is locally nilpotent.

An element a of a Lie algebra L is called a *sandwich* if

$$[[L, a], a] = (0), [[[L, a], L], a] = (0)\,,$$

(see A. I. Kostrikin, [29]). In case of algebras of odd characteristics the second equality easily follows from the first one. However if char $= 2$ both conditions are necessary. We call a Lie algebra a *sandwich algebra* if it is generated by a finite collection of sandwiches. The following theorem is due to A. I. Kostrikin and the author.

Theorem about Sandwich Algebras [32]. *A sandwich Lie algebra is nilpotent.*

This theorem suggests the following plan of attack on Theorem 1 (which has been outlined in [54]).

Assume that there exists a nonzero Lie algebra $\mathcal{L}$ over an infinite field K which satisfies an Engel's identity but isn't locally nilpotent. Factoring out the locally nilpotent radical of $\mathcal{L}$ (see [27, 42]) we may assume that $\mathcal{L}$ does not contain any nonzero locally nilpotent ideals.

Suppose we managed to construct a Lie polynomial $f(x_1, \ldots, x_r)$ (an element of the free Lie algebra) such that f is not identically zero on $\mathcal{L}$ and for arbitrary

elements $a_1, \ldots, a_r \in \mathcal{L}$ the value $f(a_1, \ldots, a_r)$ is a sandwich of $\mathcal{L}$. The K-linear span of $f(\mathcal{L}) = \{f(a_1, \ldots, a_r) \mid a_1, \ldots, a_r \in \mathcal{L}\}$ is an ideal in $\mathcal{L}$. By the theorem about sandwich algebras the ideal $Kf(L)$ is locally nilpotent, which contradicts our assumption.

A year of effort that followed [54] did not bring us a desired sandwich-valued polynomial (its existence a posteriori followed from Theorem 1). Instead in November 1988 we constructed an even sandwich-valued superpolynomial f. This means that for a Lie superalgebra $L = L_0 + L_1$ satisfying a superization of E_n every value of f on L is a sandwich of L_0. It turned out to be a good substitute of sandwich-valued polynomials. The sketch of this rather complicated construction appeared in [55]. Unfortunately it worked only for characteristic $\neq 2, 3$.

In January of 1989 we constructed another "generalized" nonzero sandwich-valued polynomial (this time involving "divided powers" of ad-generators). The full linearization of a "generalized polynomial" is an ordinary polynomial and every value of such a linearization is a linear combination of sandwiches. This approach worked for an arbitrary characteristic ([56, 57]).

Some lengthy computations from the proof (which are really hard to read) may be explained better within the framework of Jordan Algebra Theory (see [21, 22]). We shall demonstrate the idea for the simpler case $p \neq 2, 3$.

The first (less computational) part of the construction of a sandwich-valued polynomial is a construction of a polynomial f such that $f(\mathcal{L}) \neq (0)$ and for an arbitrary element $a \in f(\mathcal{L})$ we have $\mathrm{ad}(a)^3 = 0$.

Choose arbitrary elements $a, b \in f(\mathcal{L})$ and consider the subspaces $\mathcal{L}^+ = \mathcal{L}\ \mathrm{ad}(a)^2$, $\mathcal{L}^- = \mathcal{L}\ \mathrm{ad}(b)^2$. Then for an arbitrary element $c \in \mathcal{L}^+$ the operation $x \circ_c y = [[x, c], y]$; $x, y \in \mathcal{L}^-$, defines the structure of a Jordan algebra on $\mathcal{L}^-$ (see [6, 24, 26]). The pair of subspaces $(\mathcal{L}^-, \mathcal{L}^+)$ is a so-called Jordan Pair (see [34, 39]).

For $p = 2$ or $p = 3$ we define $\mathcal{L}^-$ and $\mathcal{L}^+$ with the divided squares of adjoint operators and apply Kevin McCrimman's theory of Quadratic Jordan Algebras [22, 37, 38].

For odd p we managed to translate Jordan arguments into the language of elementary computations in [56]. Later M. Vaughan–Lee (see [48]) was able to do this even for $p = 2$. In both cases, however, there was a price to pay: lengthy computations.

The question about upper bounds for orders (or classes of nilpotency) of finite m generated groups of exponent n seems to be rather complicated. Exact orders of universal finite Burnside groups $B_0(m, n)$ are known for $n = 2, 3, 4, 5$ (see [48]). G. Higman found an upper bound for the class of nilpotency of $B_0(m, 5)$ that grows linearly in m. M. Vaughan–Lee found an upper bound for the class of $B_0(m, 7)$ which is polynomial in m.

The difficulty of the problem is related to the fact that the Burnside Problem is a Ramsey-type problem and, conversely, many Ramsey-type problems can be reformulated as Burnside-type problems. It has been an open problem for many years whether or not there exists a primitive recursive upper bound in van der

Waerden's theorem and in other major Ramsey-type theorems. This problem was solved by S. Shelah [12, 45], whose bound lies in the fifth class of the Grzegorchyk hierarchy.

Our original proof in [56] yielded an upper bound that is a variant of Ackermann's function and thus grows faster than any primitive recursive function.

The primitive recursive upper bound for $|B_0(m,n)|$ was found in [49]. Let $T(m,1) = m,\ T(m,n+1) = m^{T(m,n)}$.

Theorem 3. *Let G be a finite m generated groups of exponent n. Then $|G| \leq T(m, n^{n^n})$.*

S. I. Adian and N. N. Repin [1] found a lower bound for the class of $B_0(2,p),\ p$ is prime, which is exponential in p.

Remark, that A. Belov [5] proved that there exists a constant $\alpha > 0$ such that an arbitrary m generated associative ring satisfying the identity $x^n = 0$ is nilpotent of class $\leq m^{\alpha n}$.

The following generalization of Theorem 1 solves The General Kurosh–Levitsky Problem (in Shirshov's form) in the class of Lie PI-algebras.

Theorem 4. *There exists a function $h(m,n)$ with the following property. Suppose that Lie algebra L is generated by m elements $a_1, \ldots, a_m$ and* (1) *L satisfies a polynomial identity of degree n,* (2) *for an arbitrary commutator ρ in $a_1, \ldots, a_m$ of length $\leq h(m,n)$ the operator* $\mathrm{ad}(\rho)$ *is nilpotent. Then the algebra L is nilpotent.*

It is known that an infinite finitely generated p-group can be residually finite (such are the examples of E. S. Golod, R. I. Grigorchuk, N. Gupta-S. Sidki). However, Theorem 4 implies

Theorem 5. *A finitely generated residually finite p-group G whose Lie algebra $L(G)$ is PI, is finite.*

In particular, the Lie algebra $L(G)$ is PI if the pro-p completion $G_{\hat{p}}$ does not contain an abstract free subgroup of rank 2 (see [52]).

V. P. Platonov conjectured that periodic compact groups are locally finite. J. S. Wilson [51] proved that (under the assumption that there are finitely many simple sporadic groups) it suffices to prove the conjecture for pro-p groups. A periodic pro-p group G clearly does not contain a free abstract subgroup, thus the Lie algebra $L(G)$ is PI.

Theorem 6 [58]. *A periodic pro-p group is locally finite.*

A. Shalev [44] proved that the positive solution of The Restricted Burnside Problem implies (and in fact, is equivalent to) the following criterion of analyticity

of a pro-p group. A finite p-group P is said to be a section of a pro-p group G if there exist an open subgroup K of G and a closed normal subgroup H in K such that $K/H \cong P$. Let C_m denote the cyclic group of order m.

Theorem 7 (A. Shalev, [44]**).** *A finitely generated pro-p group G is not analytic if and only if for any n the wreath product C_p wr C_{p^n} is a section of G.*

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