SUBREGULAR REPRESENTATIONS OF $\mathfrak{sl}_n$ AND SIMPLE SINGULARITIES OF TYPE A_{n-1}. II

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ABSTRACT. The aim of this paper is to show that the structures on K-theory used to formulate Lusztig's conjecture for subregular nilpotent $\mathfrak{sl}_n$–representations are, in fact, natural in the McKay correspondence. The main result is a categorification of these structures. The no-cycle algebra plays the special role of a bridge between complex geometry and representation theory in positive characteristic.

1. INTRODUCTION

1.1. There are two approaches to studying a Kleinian singularity. On the one hand, it is a quotient singularity $\mathbb{C}^2/\Gamma$ for a finite subgroup Γ of $SL_2(\mathbb{C})$. On the other hand, it is the inverse image $\pi^{-1}(0)$ where π is the restriction of the adjoint quotient $\mathfrak{g} \to \mathfrak{g}/G$ of a simple Lie algebra $\mathfrak{g}$ to a subregular special transversal slice $e+Z(f)$ where e, h, f is a subregular $\mathfrak{sl}_2$-triple. As far as "classical" (resolutions, semiuniversal deformations) geometry is concerned, the two approaches are equivalent and one can move between these two approaches [23].

Certain recent advances, however, have taken place that are not so easily translated between these two approaches. In particular, the derived version of McKay correspondence [3, 13] requires the finite groups realization. On the other hand, Lusztig's conjecture about equivariant K-theory and reduced enveloping algebras [15, 16] requires the Lie theoretic interpretation. The goal of this paper is to show that in type A the McKay correspondence can be used to understand Lusztig's conjecture. The link is provided by the no–cycle algebra that the authors have discovered on the proof of Premet's conjecture [7].

1.2. The current paper is a logical continuation of [7] and the reader is mildly encouraged to look at this paper first. Let us briefly explain the relation of the present paper to other relevant work on Lusztig's conjecture. In the case of a subregular representations of type A, Jantzen proves Lusztig's conjecture [11]. Lusztig notices a relation to geometry: his canonical basis in K-theory consists of Gonzalez-Sprinberg-Verdier sheaves [17]. We go a little further and derive Lusztig's pairing and duality on K-theory from geometry.

There are two approaches to Lusztig's general conjecture (that is, for any nilpotent character). Bezrukavnikov, Mirkovic and the second author develop localization techniques for D-modules in characteristic p and find an isomorphism between the K-groups in Lusztig's conjecture [2]. Alternatively, Premet suggests a possible relationship between quantisations of special transversal slices and subregular representations of a simple Lie algebra [20]. We follow the second approach in this paper and show that a isomorphism between K-groups can deduced from the McKay correspondence, preserving all relevant structure.

1.3. One of the main wonders of Lusztig's conjecture is that it bridges complex algebraic geometry and representation theory in characteristic p. One way to achieve this may be to show that the geometry is independent of the characteristic, as done in [2]. Here we use the alternative approach show that the representation theory (looked at from the right angle) is actually independent of the characteristic. To achieve this goal we work with algebraic material

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over an algebraically closed field $\mathbb{L}$ of characteristic $p \geq 0$, where p does not divide n. The geometry is always over $\mathbb{C}$.

1.4. Let $\mathbf{H}$ be the minimal desingularisation over $\mathbb{C}$ of a Kleinian singularity of type A and let $\mathbf{H}_0$ be the corresponding exceptional divisor. Both $\mathbf{H}$ and $\mathbf{H_0}$ admit an action of a two dimensional torus, T. Lusztig has constructed an action of an affine Iwahori-Hecke algebra on both $K_T(\mathbf{H})$ and $K_T(\mathbf{H}_0)$, the T-equivariant K-groups of $\mathbf{H}$ and $\mathbf{H_0}$ respectively. Using Kapranov and Vasserot's interpretation of the McKay correspondence in terms of derived categories, [13], we show in Theorem 5.14 that the above action on $K_T(\mathbf{H}_0)$ admits a categorification in $D_T(\mathbf{H}_0)$, the bounded derived category of T-equivariant sheaves on $\mathbf{H}_0$. This essentially means that $D_T(\mathbf{H}_0)$ admits an action of the affine braid group which specialises to Lusztig's Iwahori-Hecke algebra action on $K_T(\mathbf{H}_0)$. We also describe a form and a duality on $D_T(\mathbf{H}_0)$ which specialise to those considered by Lusztig on $K_T(\mathbf{H}_0)$ in [15]. The reader should beware that we do not concern ourselves with the existence of a braid group action as described in [6]; in the literature, our action is sometimes referred to as a weak action.

1.5. Reduced enveloping algebras appear only at the end of this paper, quantisations of transversal slices and simple singularities not at all. In the preceding paper [7], we show that the appropriate quotients of these are Morita equivalent to the no-cycle algebra, $C_{\mathbb{L}}(n)$. There is a natural bigrading on $C_{\mathbb{L}}(n)$. In the characteristic zero case there is a triangulated functor from $D(C_{\mathbb{C}}(n)\text{-grmod})$, the bounded derived category of bigraded $C_{\mathbb{C}}(n)$-modules, to $D_T(\mathbf{H}_0)$. This functor induces an isomorphism on K-theory, sending simple $C_{\mathbb{C}}(n)$-modules to the signed basis of $K_T(\mathbf{H}_0)$ constructed by Lusztig in [18]. In the modular case, forgetting half of the bigrading on $C_{\mathbb{K}}(n)$, there is an isomorphism between the K-groups of graded $C_{\mathbb{K}}(n)$-modules and of $U_{\chi,\lambda}$-T_0-modules for regular λ. Under this isomorphism the operations of wall-crossing functors on U_χ-modules correspond to the generators of an Iwahori-Hecke algebra action, a characteristic p analogue of that in 1.4. The category of bigraded $C_{\mathbb{K}}(n)$-modules is a mixed category lying over the usual category of $U_{\chi,\lambda}$-T_0-modules.

1.6. The paper is organised as follows. In Section 2 we recall some preliminaries on categories and Kleinian singularities. We remind the reader about equivariant sheaves and Hilbert schemes in Section 3. In Section 4 we find a new interpretation of Lusztig's signed basis (in the subregular situation) and in Section 5 we prove Theorem 5.14, on categorification. Section 6 ties together representation theory in characteristic p with geometry over $\mathbb{C}$.

2. Preliminaries

2.1. Let $\mathcal{C}$ be an abelian $\mathbb{L}$-category and let G be a group acting on $\mathcal{C}$. So, for every $g \in G$, we have an exact *shift functor*, $[g]$, together with natural isomorphisms $[g] \circ [g'] \to [gg']$. Sometimes in the literature such actions are called "weak" as opposed to "strong" actions, which satisfy associativity constraints for natural isomorphisms. We do not use the term "weak" since we are not interested in associativity constraints.

We call $\mathcal{C}$ a G-category if an action of G on $\mathcal{C}$ is fixed. If $\mathcal{C}$ and $\mathcal{D}$ are both G-categories, we say that the functor $\Phi : \mathcal{C} \to \mathcal{D}$ is a G-functor if functors $\Phi \circ [g]$ and $[g] \circ \Phi$ are naturally isomorphic for every $g \in G$.

A G-functor $\Phi : \mathcal{C} \to \mathcal{D}$ is a G-*equivalence* if there exists a G-functor $\Psi : \mathcal{D} \to \mathcal{C}$ such that $\Psi\Phi \cong 1_{\mathcal{C}}$ and $\Phi\Psi \cong 1_{\mathcal{D}}$. We say $\mathcal{C}$ and $\mathcal{D}$ are equivalent G-categories. Note that an equivalence that is a G-functor need not be a G-equivalence [8, Section 5].

2.2. Let $R = \oplus_{g \in G} R_g$ be a G-graded (noetherian) algebra, that is $R_g R_{g'} \subseteq R_{gg'}$. A G-graded R-module is an R-module, M, together with a $\mathbb{L}$-space decomposition $M = \oplus_{g\in G} M_g$ satisfying $R_g \cdot M_{g'} \subseteq M_{gg'}$. The category of G-graded (finitely generated) R-modules, denoted R-Grmod (R-grmod) is an example of a G-category. By definition, we have $(M[g])_{g'} = M_{g'g^{-1}}$. for all $g, g' \in G$.

If G acts on an algebra R by automorphisms then R-mod is another example of a G-category. The group G acts by twisting.

2.3. Given $X, Y \in \mathcal{C}$ and $g \in G$, set $\mathrm{Hom}(X,Y)_g = \mathrm{Hom}_{\mathcal{C}}(X[g], Y)$. We define

$$\mathrm{Hom}(X,Y) = \bigoplus_{g\in G} \mathrm{Hom}(X,Y)_g,$$

a G-graded vector space. The identification $\mathrm{Hom}_{\mathcal{C}}(X[g], Y) \cong \mathrm{Hom}_{\mathcal{C}}(X[gg'], Y[g'])$ yields a composition law for $X, Y, Z \in \mathcal{C}$: $\mathrm{Hom}(Y,Z)_{g'} \times \mathrm{Hom}(X,Y)_g \to \mathrm{Hom}(X,Z)_{gg'}$. Then, for the examples appearing in 2.2, the space $\mathrm{End}(X) = \mathrm{Hom}(X,X)$ becomes a G-graded $\mathbb{L}$-algebra and $\mathrm{Hom}(X,Y)$ a G-graded $\mathrm{End}(X)$-module. The functor $\mathrm{Hom}(X,-)$ is a G-functor from $\mathcal{C}$ to $\mathrm{End}(X)$-Grmod.

2.4. *A triangulated G-category* is a triangulated category with exact shift functors $[g]$, $g \in G$, and natural isomorphisms $[g] \circ [g'] \to [gg']$. The bounded derived category $D(\mathcal{C})$ of a G-category is a triangulated G-category. Analogous to 2.1, we have the notions of triangulated G-functors and equivalences of triangulated G-categories.

2.5. Let $K(\mathcal{C})$ denote the Grothendieck group of $\mathcal{C}$. The Grothendieck group of $D(\mathcal{C})$, denoted $K'(\mathcal{C})$, is defined to be the free abelian group generated by the isomorphism classes of objects of $D(\mathcal{C})$ subject to the relations $[M] - [M'] - [M'']$ for every distinguished triangle

$$M' \longrightarrow M \longrightarrow M'' \longrightarrow M'[1] \rightsquigarrow .$$

Taking the Euler characteristic induces an isomorphism $\mathbf{i} : K'(\mathcal{C}) \to K(\mathcal{C})$.

Let $\mathcal{C}$ be a G-category. Since shift functors are exact, they induce an action of $\mathbb{Z}[G]$, the group algebra of G, on both $K(\mathcal{C})$ and $K'(\mathcal{C})$, which commutes with $\mathbf{i}$. From now on, we will identify $K(\mathcal{C})$ and $K'(\mathcal{C})$.

2.6. **Kleinian singularities.** Let us assume $\zeta \in \mathbb{L}$ is a primitive root of unity of degree n. Set

$$\Gamma = \left\{ g^i : g = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \right\},$$

a subgroup of $SL_2(\mathbb{L})$. The natural action of Γ on $\mathbb{L}^2$ induces an action on $\mathbb{L}[X,Y]$: $g \cdot X = \zeta X$, $g \cdot Y = \zeta^{-1} Y$. The invariants of $\mathbb{L}[X,Y]$ under this action are generated by X^n, XY and Y^n. Thus, the orbit space $\mathbb{L}^2/\Gamma$ has co-ordinate ring

$$\mathcal{O}(\mathbb{L}^2/\Gamma) = \mathbb{L}[X^n, XY, Y^n] \cong \frac{\mathbb{L}[A,B,H]}{(AB - H^n)}.$$

The variety $\mathbb{L}^2/\Gamma$ has an isolated singularity at 0, a *Kleinian singularity of type A_{n-1}*.

3. Equivariant K-theory and Hilbert schemes

3.1. **Equivariant sheaves.** Let G be an affine algebraic group acting rationally on a quasi-projective noetherian variety X. Let $\mathrm{Coh}_G(X)$ denote the abelian category of G-equivariant coherent sheaves on X, [5, Chapter 5]. Let $D_G(X)$ denote the *bounded* derived category of $\mathrm{Coh}_G(X)$.

Let $f : X \to Y$ be a G-equivariant map. If f is proper then the right derived functor of f_* gives a pushforward

$$\mathbf{R}f_* : D_G(X) \longrightarrow D_G(Y).$$

Similarly, if f has finite Tor dimension (for example if f is flat or if Y is smooth), then the left derived functor of f^* gives a pullback

$$\mathbf{L}f^* : D_G(Y) \longrightarrow D_G(X).$$

3.2. Let $K_G(X)$ denote the Grothendieck group of $\mathrm{Coh}_G(X)$, or equally, by 2.5, of $D_G(X)$. The construction of 3.1 yields pushforward (respectively pullback) homomorphisms between the Grothendieck groups of X and Y for a proper (respectively finite Tor dimension) G-equivariant map $f : X \to Y$.

Let $f : X \to \{pt\}$ be projection to a point. Since $\mathrm{Coh}_G(pt)$ can be identified with the category of finite dimensional G-modules, any G-module, say M, can be pulled back to a G-equivariant locally free sheaf on X, f^*M. Since the operation of tensoring by a locally free sheaf is exact we obtain a functor $D_G(X) \to D_G(X)$ associated to each G-module and therefore an action of $K_G(pt) = \mathrm{Rep}(G)$, the representation ring of G, on $K_G(X)$.

3.3. Suppose Z is a closed G-stable subvariety of X. Let $D_G(X,Z)$ denote the full subcategory of $D_G(X)$ whose objects are complexes with homology supported on Z. If $i : Z \to X$ is the inclusion then $\mathbf{R}i_* : D_G(Z) \to D_G(X,Z)$ is a functor inducing an isomorphism of the Grothendieck groups..

3.4. Let $\mathcal{B}$ be the flag variety of the Lie algebra $\mathfrak{sl}_n(\mathbb{L})$ [5]. As a set this consists of all Borel subalgebras of $\mathfrak{sl}_n(\mathbb{L})$, that is those subalgebras which are conjugate under $SL_n(\mathbb{L})$ to the upper triangular matrices. The cotangent bundle of $\mathcal{B}$ is naturally identified with the variety

$$\tilde{\mathcal{N}} = \{(x, \mathfrak{b}) : x(\mathfrak{b}) = 0\} \subset \mathfrak{sl}_n(\mathbb{L})^* \times \mathcal{B}$$

where the first projection π_1 becomes the moment map. The Springer fibre $\mathcal{B}_x$ is the subvariety $\pi_1^{-1}(x)$ of $\mathcal{B}$.

Let us consider $\chi \in \mathfrak{sl}_n(\mathbb{L})^*$ such that $\chi(E_{i+1,i}) = 1,\ i = 1, 2, \ldots n-2$ and $\chi(X) = 0$ if X is another elementary matrix or a diagonal matrix. There is a simple way to parametrise $\mathcal{B}_\chi$, [23, Section 6.3]. Given a basis u_i of $\mathbb{L}^n$, let $\mathcal{F}(u_1, \ldots, u_n)$ be a flag with the span of $u_1, \ \ldots\ , u_k$ as the k-dimensional space. Let v_i be an element of the standard basis of $\mathbb{L}^n$ so that $E_{i,j}v_j = v_i$. We introduce the flag

$$\mathcal{F}_{k,\alpha} = \mathcal{F}(v_1, v_2, \ \ldots\ , v_{k-1}, v_k + \alpha v_n, v_n, v_{k+1}, v_{k+2}, \ \ldots\ , v_{n-1})$$

for all $(k, \alpha) \in (\{1, \ldots, n-1\} \times \mathbb{L}) \cup \{(0,0)\}$. The irreducible components of $\mathcal{B}_\chi$ are projective lines Π_k, $1 \le k \le n-1$ where

$$\Pi_k = \{\mathcal{F}_{n-k,\alpha} \mid \alpha \in \mathbb{L}\} \cup \{\mathcal{F}_{n-k-1,0}\}.$$

For $2 \le k \le n-1$ the components Π_{k-1} and Π_k intersect transversally at a point $p_{k-1,k} = \mathcal{F}_{n-k,0}$. Components Π_i and Π_j with $|i-j| > 1$ do not intersect.

Consider the following one parameter subgroup of the diagonal matrices in $SL_n(\mathbb{L})$

$$T_0 = \{\nu(\tau) = \tau E_{1,1} + \tau E_{2,2} \cdots + \tau E_{n-1,n-1} + \tau^{1-n} E_{n,n} : \tau \in \mathbb{L}^*\}.$$

Notice that T_0 stabilises $\mathcal{B}_\chi$, since $\nu(\tau) \cdot \mathcal{F}_{i,\alpha} = \mathcal{F}_{i,\tau^{-n}\alpha}$.

3.5. Let us further assume that $\mathbb{L} = \mathbb{C}$. By the Jacobson-Morozov theorem there exists an $\mathfrak{sl}_2$-triple $e, h, f \in \mathfrak{sl}_n(\mathbb{C})$ such that $\mathrm{Tr}(ex) = \chi(x)$ for each $x \in \mathfrak{sl}_n(\mathbb{C})$. Let $\mathcal{N}$ be the variety of nilpotent elements in $\mathfrak{sl}_n(\mathbb{C})$. Let

$$V_\chi = \{\mu \in \mathfrak{sl}_n(\mathbb{C})^* \mid \forall x \in \mathfrak{sl}_n(\mathbb{C})\ \mu([x,f]) = \chi([x,f])\}.$$

By [23, Theorem 6.4 and Section 7.4] V_χ is a Kleinian singularity of type A_{n-1} and $\Lambda_\chi = \pi_1^{-1}(V_\chi)$ is its minimal desingularisation with the exceptional fibre $\mathcal{B}_\chi$.

3.6. **Equivariant line bundles on $\mathbb{P}^1$,** [18, Section 5.4]**.** Fix k such that $1 \le k \le n-1$ and suppose $T = \mathbb{C}^* \times \mathbb{C}^*$ acts on $\mathbb{P}^1$ by $(\lambda, \mu) \cdot (a : b) = ((\lambda\mu)^k a : (\lambda^{-1}\mu)^{n-k} b)$. The following lemma can be proved by explicit calculation on the two standard charts of $\mathbb{P}^1$.

Lemma. *Let T act on $\mathbb{P}^1$ as above. For every collection of integers i, i', j, j' satisfying $i' - j' = nm$ and $i - j = (2k-n)m$ for some integer m, there exists a T-equivariant line bundle on $\mathbb{P}^1$, unique up to isomorphism, where (λ, μ) acts as $\lambda^{j'}\mu^j$ on the fibre above $(1:0)$ and as $\lambda^{i'}\mu^i$ on the fibre above $(0:1)$. All such T-equivariant line bundles arise in this way.*

We will denote the above line bundle $\mathcal{O}_k^{j',j;i',i}$.

3.7. **Braid groups.** The *affine Braid group* of type $\tilde{A}_{n-1}$, denoted B_{ad}, is the group with generators $\tilde{T}_i$ for $0 \leq i \leq n-1$, satisfying the braid relations $\tilde{T}_i\tilde{T}_{i+1}\tilde{T}_i = \tilde{T}_{i+1}\tilde{T}_i\tilde{T}_{i+1}$ (we set $\tilde{T}_n = \tilde{T}_0$). There is a natural action of $<\sigma>= \mathbb{Z}/n\mathbb{Z}$ on B_{ad} given by $\sigma(\tilde{T}_i) = \tilde{T}_{i+1}$. The *extended affine Braid group* is

$$B := \mathbb{Z}/n\mathbb{Z} \ltimes B_{ad}.$$

Let $\iota : B \to B$ be the involution which sends $\tilde{T}_i$ to $\tilde{T}_i^{-1}$ and fixes σ.

3.8. **Hecke algebras.** Let $\mathcal{A} = \mathbb{Z}[v^{\pm 1}]$. Let $\mathcal{A}[B]$ be the group algebra of B over $\mathcal{A}$ and let $\mathcal{H}$, the *extended affine Hecke algebra*, be the quotient of $\mathcal{A}[B]$ by the ideal generated by $(\tilde{T}_i + v^{-1})(\tilde{T}_i - v)$ for $1 \leq i \leq n$. The subalgebra generated by T_i for $1 \leq i \leq n-1$ (respectively $1 \leq i \leq n$) is the *finite Hecke algebra* (respectively *affine Hecke algebra*) and denoted by $\mathcal{H}_{fin}$ (respectively $\mathcal{H}_{ad}$).

The involution ι on B induces a ring automorphism on $\mathcal{H}$, which we denote by $^-$, sending v to v^{-1}, $\tilde{T}_i$ to $\tilde{T}_i^{-1} = \tilde{T}_i + (v^{-1} - v)$ and fixes σ.

There is a second presentation of $\mathcal{H}$, [14], discovered by Bernstein. In this presentation $\mathcal{H}$ is the $\mathcal{A}$-algebra generated by $\tilde{T}_i$, for $1 \leq i \leq n-1$, and θ_x, for $x \in P$, the weight lattice of $SL_n(\mathbb{C})$. For the fundamental weight ϖ_{n-1} we have $\theta_{\varpi_{n-1}} = \sigma\tilde{T}_{n-2}\tilde{T}_{n-3}\ldots\tilde{T}_1\tilde{T}_n = \tilde{T}_{n-1}\tilde{T}_{n-2}\ldots\tilde{T}_2\tilde{T}_1\sigma$.

3.9. Since $T = \mathbb{C}^* \times \mathbb{C}^*$ we can identify $\mathrm{Rep}(T)$ with $\mathbb{Z}[v'^{\pm 1}, v^{\pm 1}]$. The torus T acts on $\mathcal{B}_\chi$ and Λ_χ, [23, 7.5]. The T-action on Π_k agrees with the k-th action considered in 3.6, [18, 5.1]. There is an action of $\mathcal{H}$ on $K_T(\mathcal{B}_\chi)$, [15, Section 10]. We will denote this action by $\bullet$ always. The action of $\mathcal{A}$ arises from the action of $\mathrm{Rep}(T)$, whilst the action of $\tilde{T}_i$ for $1 \leq i \leq n-1$ is explained in [18, Sections 2,3]. For $x \in P$ the action of θ_x is given by tensoring by certain line bundles on $\mathcal{B}$, [18, Section 2].

3.10. There is an involution,

$$\tilde{\beta} : K_T(\mathcal{B}_\chi) \longrightarrow K_T(\mathcal{B}_\chi),$$

which is $\mathbb{Z}[v'^{\pm 1}]$-linear and twists the action of $\mathcal{H}$ by the involution $^-$, [15, Proposition 12.10].

Let † denote the $\mathcal{A}$-algebra involution of $\mathcal{A}[v'^{\pm 1}]$ which sends v' to v'^{-1} and let δ denote the $\mathcal{A}$-algebra anti-involution of $\mathcal{H}$ which fixes $\tilde{T}_i$ and sends σ to σ^{-1}. By [15, 12.16] there is a pairing

$$(\ |\ ) : K_T(\mathcal{B}_\chi) \times K_T(\mathcal{B}_\chi) \to \mathbb{Z}[v'^{\pm 1}, v^{\pm 1}],$$

which, for $F, F' \in K_T(\mathcal{B}_\chi)$, $p \in \mathbb{Z}[v'^{\pm 1}, v^{\pm 1}]$ and $\tilde{T} \in \mathcal{H}$, satisfies

(1) $(p.F|F') = (F|p^\dagger.F') = p(F|F')$;
(2) $(\tilde{T}\bullet F|F') = (F|\delta(\tilde{T})\bullet F')$;
(3) $(F|F') = (F'|F)^\dagger$.

As in [16, 5.11] we define

$$\mathbf{B}^{\pm}_{\mathcal{B}_\chi} = \{F \in K_T(\mathcal{B}_\chi) : \tilde{\beta}(F) = F,\ (F|F) \in 1 + \mathbb{Z}[v'^{\pm 1}, v^{-1}]\}.$$

Thanks to [18, Section 5] this set is a signed basis for the free $\mathcal{A}$-module $K_T(\mathcal{B}_\chi)$.

Let $\mathbf{p}_{k-1,k}$ denote the element in $K_T(\mathcal{B}_\chi)$ representing the skyscraper sheaf at $p_{k-1,k}$ with trivial T-action. Abusing notation we will let $\mathcal{O}_k^{j',j;i',i}$ denote the coherent sheaf on $\mathcal{B}_\chi$ obtained by extension by zero of the T-equivariant line bundle $\mathcal{O}_k^{j',j;i',i}$ on $\Pi_k \cong \mathbb{P}^1$. For $1 \leq k \leq n-1$ set $\mathbf{O}_k = \mathcal{O}_k^{0,-n+k;-n,-k}$. We define

$$\mathbf{O}_n = \mathbf{p}_{0,1} - \sum_{k=1}^{n-1} v^{n-k}\mathbf{O}_k = \mathbf{p}_{n-1,n} - \sum_{k=1}^{n-1} v'^n v^k \mathbf{O}_k.$$

Then, by [18, Proposition 5.25], we have

$$\mathbf{B}^{\pm}_{\mathcal{B}_\chi} = \{\pm v'^s \mathbf{O}_k : 1 \leq k \leq n,\ s \in \mathbb{Z}\}.$$

3.11. There is an explicit description of the action of $\mathcal{H}_{fin}$ on $K_T(\mathcal{B}_\chi)$ given in [18, 5.11]. For $1 \leq i, k \leq n-1$ we have

$$\tilde{T}_i \bullet \mathbf{O}_k = \begin{cases} v\mathbf{O}_k & \text{if } i = k \\ -v^{-1}\mathbf{O}_k - \mathbf{O}_{k\pm 1} & \text{if } i = k \pm 1 \\ -v^{-1}\mathbf{O}_k & \text{otherwise,} \end{cases}$$

whilst $\tilde{T}_i(\mathbf{p}_{0,1}) = -v^{-1}\mathbf{p}_{0,1} + \delta_{1,i}(-v'^n + v^n)\mathbf{O}_1$. By [18, Lemma 5.24], for $i \geq j$, the pairing is given by

$$(1) \qquad (\mathbf{O}_i, \mathbf{O}_j) = \begin{cases} 1 + v^{-2} & \text{if } i = j \\ -v'^n v^{-1} & \text{if } i = n \text{ and } j = 1 \\ v^{-1} & \text{if } j = i - 1 \\ 0 & \text{otherwise.} \end{cases}$$

3.12. **The equivariant Hilbert scheme.** Let Γ be the finite cyclic subgroup of $SL_2(\mathbb{C})$ of order n, defined as in 2.6. Let $\mathbf{H}$ be the Γ-equivariant Hilbert scheme of $\mathbb{C}^2$, [19, Chapter 4]. By definition the points of $\mathbf{H}$ are Γ-equivariant ideals of $\mathbb{C}[X,Y]$, I, such that $\mathbb{C}[X,Y]/I$ is isomorphic to the regular representation of Γ. The action of T on $\mathbb{C}[X,Y]$ induced by $(\lambda,\mu)\cdot X = \lambda\mu X$ and $(\lambda,\mu)\cdot Y = \lambda^{-1}\mu Y$, yields a T-action on $\mathbf{H}$. There is a morphism

$$\pi : \mathbf{H} \longrightarrow \mathbb{C}^2/\Gamma,$$

which, by [19, Theorem 4.1], is the minimal resolution of singularities of $\mathbb{C}^2/\Gamma$. Let $\mathbf{H}_0$ denote the zero fibre $\pi^{-1}(0)$. There is a T-equivariant isomorphism between $\mathbf{H}$ and Λ_χ which restricts to an isomorphism between $\mathbf{H}_0$ and $\mathcal{B}_\chi$.

3.13. **Derived equivalences.** Let $\mathcal{Z} = (\mathbf{H} \times_{\mathbb{C}^2/\Gamma} \mathbb{C}^2)_{\text{red}}$. By definition we have a commutative diagram

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{\phi} & \mathbb{C}^2 \\ \eta \downarrow & & \downarrow \sigma \\ \mathbf{H} & \xrightarrow{\pi} & \mathbb{C}^2/\Gamma, \end{array}$$

in which σ and η are finite, ϕ and π are proper and birational, and η is flat. Thanks to [13, Theorem 1.4] the functor

$$\overline{\Phi} = (\mathbf{R}\eta_* \circ \mathbf{L}\phi^*)^\Gamma : D_\Gamma(\mathbb{C}^2) \longrightarrow D(\mathbf{H})$$

is an equivalence of triangulated categories. Moreover $\overline{\Phi}$ restricts to an equivalence between $D_\Gamma(\mathbb{C}^2, 0)$ and $D(\mathbf{H}, \mathbf{H}_0)$, [3, 9.1].

Since ϕ and η are T-equivariant morphisms $\overline{\Phi}$ can be lifted to an equivalence of triangulated categories

$$\Phi = (\mathbf{R}\eta_* \circ \mathbf{L}\phi^*)^\Gamma : D_{\Gamma\times T}(\mathbb{C}^2) \longrightarrow D_T(\mathbf{H}).$$

3.14. Let X be any T-equivariant quasi-projective variety and let $p_X : X \to \{pt\}$ be projection onto a point. Let v' (respectively v) be the one dimensional T-module where $(\lambda,\mu).1 = \lambda$ (respectively $(\lambda,\mu).1 = \mu$). As in 3.2, pulling back yields a T-equivariant line bundle which we denote by v'_X (respectively v_X). Observe that tensoring by these line bundles realises an action of $\mathbb{Z}\times\mathbb{Z}$ on $\text{Coh}_T(X)$ and $D_T(X)$, making them into (triangulated) $\mathbb{Z}\times\mathbb{Z}$-categories. We will denote the shift functors by $v'^i_X v^j_X$ for $i, j \in \mathbb{Z}$. When there is no confusion as to which variety we mean we will suppress the subscript.

In particular the above discussion applies to $\mathbb{C}^2, \mathbf{H}, \mathbf{H}_0$ and $\mathcal{Z}$.

Lemma. *The equivalence Φ is a $\mathbb{Z}\times\mathbb{Z}$-equivalence.*

Proof. Let $\mathcal{F} \in D_{\Gamma\times T}(\mathbb{C}^2)$. We have the following natural isomorphisms

$$\begin{aligned}\mathbf{R}\eta_* \circ (\mathbf{L}\phi^*(- \otimes v_{\mathbb{C}^2})) &\cong \mathbf{R}\eta_* \circ (\mathbf{L}\phi^*(-) \otimes \phi^* v_{\mathbb{C}^2}) &&\cong \mathbf{R}\eta_* \circ (\mathbf{L}\phi^*(-) \otimes v_{\mathcal{Z}})\\ &\cong \mathbf{R}\eta_* \circ (\mathbf{L}\phi^*(-) \otimes \eta^* v_{\mathbf{H}}) &&\cong (\mathbf{R}\eta_* \circ \mathbf{L}\phi^*(-)) \otimes v_{\mathbf{H}},\end{aligned}$$

where the last isomorphism is the projection formula. The same equations hold for v', so Φ is a $\mathbb{Z}\times\mathbb{Z}$-functor. Using the same formalism, one checks the inverse is also a $\mathbb{Z}\times\mathbb{Z}$-functor. □

3.15. **Serre-Grothendieck duality.** Let $\omega_{\mathbb{C}^2}$ (respectively $\omega_{\mathbf{H}}$) be the canonical line bundle of $\mathbb{C}^2$ (respectively $\mathbf{H}$). As $\Gamma\times T$-equivariant bundles we have $\omega_{\mathbb{C}^2} \cong v^2_{\mathbb{C}^2}$. As T-equivariant bundles we have $\omega_{\mathbf{H}} \cong v^2_{\mathbf{H}}$ [15, Proposition 11.10]. We have contravariant isomorphisms of the derived categories $D_{\mathbb{C}^2}$ and $D_{\mathbf{H}}$ where

$$D_{\mathbb{C}^2} = \mathbf{R}\mathcal{H}om_{\mathbb{C}^2}(-, \omega_{\mathbb{C}^2}[2]), \qquad D_{\mathbf{H}} = \mathbf{R}\mathcal{H}om_{\mathbf{H}}(-, \omega_{\mathbf{H}}[2]).$$

Here $\mathcal{H}om$ denotes the sheaf of homomorphisms. Note that $D_{\mathbf{H}}$ sends $D_T(\mathbf{H}, \mathbf{H}_0)$ to itself.

3.16. **Skew group rings.** The algebra $R_{\mathbb{L}} = \mathbb{L}[X, Y]$ has a $\mathbb{Z}\times\mathbb{Z}$-grading with $\deg(X) = (1, 1)$ and $\deg(Y) = (-1, 1)$. Since Γ acts on $R_{\mathbb{L}}$ we can form the skew group ring $R_{\mathbb{L}} * \Gamma$. The $\mathbb{Z}\times\mathbb{Z}$-grading can be extended to $R_{\mathbb{L}} * \Gamma$ by giving the elements of Γ degree 0. We consider the category of finitely generated, bigraded $R_{\mathbb{L}} * \Gamma$-modules, denoted $R_{\mathbb{L}} * \Gamma$-grmod. As in 2.2 $R_{\mathbb{L}} * \Gamma$-grmod is a $\mathbb{Z}\times\mathbb{Z}$-category. As in 3.14, for $i, j \in \mathbb{Z}$ we will denote the associated shift functor by $v'^i v^j$. The following lemma is standard.

Lemma. *Taking global sections induces a $\mathbb{Z}\times\mathbb{Z}$-equivalence of categories*

$$\Upsilon : R_{\mathbb{C}} * \Gamma\text{-grmod} \longrightarrow Coh_{\Gamma\times T}(\mathbb{C}^2).$$

3.17. Let $\tau : R_{\mathbb{L}} * \Gamma \to R_{\mathbb{L}} * \Gamma$ be the anti-involution fixing $R_{\mathbb{L}}$ and sending $g \in \Gamma$ to g^{-1}. If M is a bigraded $R_{\mathbb{L}} * \Gamma$-module then τ ensures $\mathrm{Hom}_{R_{\mathbb{L}}}(M, R_{\mathbb{L}})$ is too. Serre-Grothendieck duality then induces the contravariant equivalence $D_{\mathbb{L}^2} = \mathbf{R}\mathrm{Hom}_{R_{\mathbb{L}}}(-, v^2 R_{\mathbb{L}}[2])$ on $D(R_{\mathbb{L}} * \Gamma\text{-grmod})$, the bounded derived category of $R_{\mathbb{L}} * \Gamma$-grmod.

Let $D_0(R_{\mathbb{L}} * \Gamma\text{-grmod})$ be the bounded derived category of finitely generated, bigraded $R_{\mathbb{L}} * \Gamma$-modules which are (X, Y)-primary, that is which are annihilated by some power of (X, Y). If M is a complex in $D_0(R_{\mathbb{L}} * \Gamma\text{-grmod})$ then $D_{\mathbb{L}^2}(M) \cong M^*$, the vector space dual of M, [25, (4.9)].

3.18. The results of this section can be summarised as follows.

Proposition. *There exists a $\mathbb{Z}\times\mathbb{Z}$-equivalence of triangulated categories*

$$\Psi = D_{\mathbf{H}} \circ \Phi \circ D_{\mathbb{C}^2} \circ \Upsilon : D(R_{\mathbb{C}} * \Gamma\text{-grmod}) \longrightarrow D_T(\mathbf{H}),$$

which restricts to a $\mathbb{Z}\times\mathbb{Z}$-equivalence between $D_0(R_{\mathbb{C}} * \Gamma\text{-grmod})$ and $D_T(\mathbf{H}_0)$.

4. The equivalence on K-theory

4.1. The simple Γ-modules are labelled by the elements of $\mathbb{Z}/n\mathbb{Z}$: the element g acts on the i-th simple module as scalar multiplication by ζ^i. Associated to the i-th simple Γ-module there are two bigraded $R_{\mathbb{L}} * \Gamma$-modules: the one dimensional module S_i which is annihilated by both X and Y, and the $R_{\mathbb{L}}$-projective cover of S_i, denoted R_i. As $\Gamma\times(\mathbb{L}^*)^2$-equivariant sheaves on $\mathbb{L}^2$ these correspond to the skyscraper sheaf supported at 0 and the trivial line bundle, with trivial $(\mathbb{L}^*)^2$-structure and Γ-structure given by the i-th simple Γ-module. There is a Koszul resolution relating the two types of module

$$0 \longrightarrow v^2 R_i \xrightarrow{(X,-Y)^t} v'^{-1} v R_{i-1} \oplus v' v R_{i+1} \xrightarrow{(Y,X)} R_i \longrightarrow S_i \longrightarrow 0. \tag{2}$$

By 3.15 we have $D(R_i) = v^2 R_{n-i}[2]$ and $D(S_i) = S_{n-i}$. Let $K_0(R_{\mathbb{L}} * \Gamma\text{-grmod})$ be the Grothendieck group of $D_0(R_{\mathbb{L}} * \Gamma\text{-grmod})$.

4.2. We switch from a field $\mathbb{L}$ to complex numbers for the rest of this section. The equivalence of Proposition 3.18 yields isomorphisms

$$\Sigma : K(R_{\mathbb{C}} * \Gamma\text{-grmod}) \longrightarrow K_T(\mathbf{H}) \quad \text{and} \quad \Sigma_0 : K_0(R_{\mathbb{C}} * \Gamma\text{-grmod}) \longrightarrow K_T(\mathbf{H}_0).$$

Moreover, since the equivalence preserves $\mathbb{Z} \times \mathbb{Z}$-action, both Σ and Σ_0 are $\mathbb{Z}[v'^{\pm 1}, v^{\pm 1}]$-module isomorphisms.

4.3. **Tautological bundles on H.** Since the projection $\eta : \mathcal{Z} \to \mathbf{H}$ is a flat and finite $\Gamma \times T$-equivariant morphism, the pushforward $\eta_* \mathcal{O}_{\mathcal{Z}}$ is naturally a $\Gamma \times T$-equivariant locally free sheaf, denoted $\mathcal{E}$. The fibre of $\mathcal{E}$ above the ideal $I \in \mathbf{H}$ is the vector space $R_{\mathbb{C}}/I$, the regular representation as a Γ-module. Now Γ-equivariance allows us to decompose $\mathcal{E}$ into a direct sum of T-equivariant bundles

$$\mathcal{E} = \bigoplus_{1 \leq i \leq n} \mathcal{E}_i.$$

By definition, the fibre of $\mathcal{E}_i$ above $I \in \mathbf{H}$ is the i-th isotopic component of $R_{\mathbb{C}}/I$. These are described in [18, 5.27].

Lemma. *The isomorphism Ψ sends R_i to $\mathcal{E}_i^{\vee}$, the dual of $\mathcal{E}_i$.*

Proof. Let $\pi_{\mathbb{C}^2} : \mathbf{H} \times \mathbb{C}^2 \to \mathbb{C}^2$ be the projection map. Since there is a natural isomorphism $\mathbf{L}\phi^* \cong \mathcal{O}_{\mathcal{Z}} \otimes^{\mathbf{L}} \pi_{\mathbb{C}^2}^*$, we have

$$\begin{aligned} \Psi(R_i) &\cong D_{\mathbf{H}}((\mathbf{R}\eta_* \mathbf{L}\phi^*(v_{\mathbb{C}^2}^2 R_{n-i}[2]))^{\Gamma}) \cong D_{\mathbf{H}}((\mathbf{R}\eta_*(\mathcal{O}_{\mathcal{Z}} \otimes^{\mathbf{L}} \pi_{\mathbb{C}^2}^*(v_{\mathbb{C}^2}^2 R_{n-i}[2]))^{\Gamma}) \\ &\cong D_{\mathbf{H}}((v_{\mathbf{H}}^2 \mathcal{E}[2] \otimes S_{n-i})^{\Gamma}) \cong D_{\mathbf{H}}(v_{\mathbf{H}}^2 \mathcal{E}_i[2]) \cong \mathcal{E}_i^{\vee}. \end{aligned}$$

□

4.4. Using the resolution (2) and Lemma 4.3 we find that under Ψ the module S_i is sent to the Koszul complex

$$v^2 \mathcal{E}_i^{\vee} \longrightarrow v'^{-1} v \mathcal{E}_{i-1}^{\vee} \oplus v' v \mathcal{E}_{i+1}^{\vee} \longrightarrow \mathcal{E}_i^{\vee}. \tag{3}$$

Proposition. *For $1 \leq i \leq n$ we have $\Sigma_0([S_i]) = v'^{n-i} \mathbf{O}_i$.*

Proof. In the non-equivariant setting the complex (3) was studied in [9]. It is shown in [9, Propositions 6.2] that its homology vanishes in degrees 1 and 2 and is $\mathcal{O}_{\Pi_i}(-1)$ in degree 0 if $1 \leq i \leq n-1$. Thus we have a quasi-isomorphism

$$\begin{array}{ccccc} v^2 \mathcal{E}_i^{\vee} & \longrightarrow & v'^{-1} v \mathcal{E}_{i-1}^{\vee} \oplus v' v \mathcal{E}_{i+1}^{\vee} & \xrightarrow{f} & \mathcal{E}_i^{\vee} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{E}_i^{\vee} / \operatorname{im} f. \end{array}$$

For $1 \leq i \leq n-1$ the bundle $\mathcal{E}_i^{\vee}$ restricted to Π_i is $v'^{n-i} \mathbf{O}_i$[18, 5.27]. If $i = n$ a similar argument using [9, 6.4], shows that $\Sigma_0(S_i) = D_{\mathbf{H}}(\mathcal{O}_{\mathbf{H}_0})$. It remains to prove that $[D_{\mathbf{H}}(\mathcal{O}_{\mathbf{H}_0})] = \mathbf{O}_n$.

For $1 \leq k \leq n-1$ let B_k be the subscheme of $\mathbf{H}_0$ consisting of the components $\Pi_1, \ldots, \Pi_k$. Associated to this subscheme we have the maps of coherent sheaves arising from the inclusions $p_{k-1,k} \in \Pi_k \subseteq B_k$ and $p_{k-1,k} \in B_{k-1} \subset B_k$

$$\mathcal{O}_{B_k} \longrightarrow \mathcal{O}_{B_{k-1}} \oplus \mathcal{O}_{\Pi_k} \longrightarrow p_{k-1,k}$$

This is an exact sequence; so we have $[\mathcal{O}_{B_k}] = [\mathcal{O}_{B_{k-1}}] + [\mathcal{O}_{\Pi_k}] - \mathbf{p}_{k-1,k}$. Induction yields

$$[\mathcal{O}_{\mathbf{H}_0}] = \sum_{k=1}^{n-1} [\mathcal{O}_{\Pi_k}] - \sum_{k=1}^{n-2} \mathbf{p}_{k,k+1}.$$

By [18, 5.4] we have $\mathbf{p}_{k,k+1} = [\mathcal{O}_{\Pi_{k+1}}] - [\mathcal{O}_{k+1}^{n,-n+2(k+1);0,0}]$ and $[\mathcal{O}_{\Pi_1}] = \mathbf{p}_{0,1} + [\mathcal{O}_1^{n,-n+2;0,0}]$, which yields

$$[\mathcal{O}_{\mathbf{H}_0}] = \mathbf{p}_{0,1} + \sum_{k=1}^{n-1} [\mathcal{O}_k^{n,-n+2k;0,0}] = \mathbf{p}_{0,1} + \sum_{k=1}^{n-1} v'^n v^k \mathbf{O}_k.$$

By [18, Lemma 5.16] we have $[D_{\mathbf{H}}(\mathbf{O}_k)] = -v'^n v^n [\mathbf{O}_k]$ and $[D_{\mathbf{H}}(p_{0,1})] = p_{0,1}$. The result follows. □

The proposition shows that the signed basis $\{\pm v'^s [S_i] : 1 \leq i \leq n, s \in \mathbb{Z}\}$ of the free $\mathcal{A}$-module $K_0(R_{\mathbb{C}} * \Gamma\text{-grmod})$ is sent by Σ_0 to $\mathbf{B}^{\pm}_{\mathcal{B}_\chi}$. A similar statement is true for Λ_χ, using Σ and Lemma 4.3.

5. Categorification

We switch back to a general field $\mathbb{L}$.

5.1. **Braid group action.** Given a pair of objects $M, N \in D_0(R_{\mathbb{L}} * \Gamma\text{-grmod})$, the functor $\mathbf{R}\mathrm{Hom}_{R_{\mathbb{L}} * \Gamma}(M, N)$ takes values in the derived category of bigraded, finite dimensional $\mathbb{L}$-vector spaces. We denote this category by $\mathcal{BV}_{\mathbb{L}}$. In particular, given the simple $R_{\mathbb{L}} * \Gamma$-modules S_i and S_j the resolution (2) shows

$$\mathbf{R}^n\mathrm{Hom}_{R_{\mathbb{L}}*\Gamma}(S_i, S_j) = \mathrm{Ext}^n_{R_{\mathbb{L}}*\Gamma}(S_i, S_j) = \begin{cases} \mathbb{L} & \text{if } n = 0 \text{ and } i = j \\ v'^{\pm 1} v^{-1} \mathbb{L} & \text{if } n = 1 \text{ and } i = j \pm 1 \\ v^{-2}\mathbb{L} & \text{if } n = 2 \text{ and } i = j \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Given any complex C belonging to $\mathcal{BV}_{\mathbb{L}}$ and $M \in D(R_{\mathbb{L}} * \Gamma\text{-grmod})$ we can form the tensor product $C \otimes D \in D(R_{\mathbb{L}} * \Gamma\text{-grmod})$ and the space of linear maps $lin(C, M) \in D(R_{\mathbb{L}} * \Gamma\text{-grmod})$, [22, 2a]. Given a complex of complexes $\cdots \to C^{-1} \to C^0 \to C^1 \to \cdots$ we write

$$\{\cdots \to C^{-1} \to C^0 \to C^1 \to \cdots\}$$

to denote the associated total complex.

For $0 \leq i \leq n-1$ let $\tau_i : D_0(R_{\mathbb{L}} * \Gamma\text{-grmod}) \to D_0(R_{\mathbb{L}} * \Gamma\text{-grmod})$ be the $\mathbb{Z} \times \mathbb{Z}$-functor defined on objects by

$$\tau_i(M) = \{\mathbf{R}\mathrm{Hom}_{R_{\mathbb{L}}*\Gamma}(S_i, M) \otimes S_i \xrightarrow{ev} M\},$$

where ev is the evaluation map and M is in degree 0, [22, Definition 2.5]. By construction, this is nothing but the mapping cone of ev, which turns out to be canonical for such complexes. The various τ_i's furnish an action of the affine braid group, B_{ad}, on $D_0(R_{\mathbb{L}} * \Gamma\text{-grmod})$, [22, Proposition 2.4, 2c and Example 3.9]. For our purposes it is better to work with an adjusted action for $0 \leq i \leq n-1$

$$\tilde{\tau}_i = D_{\mathbb{L}^2} \circ \tau_{n-i} \circ D_{\mathbb{L}^2} \circ v^{-1} \circ [1].$$

The $\mathbb{Z} \times \mathbb{Z}$-equivalences $\tilde{\tau}_i$ continue to satisfy the braid relations.

5.2. For $i \in \mathbb{Z}/n\mathbb{Z}$ let M_i^+ (respectively M_i^-) be the unique two dimensional bigraded $R_{\mathbb{L}} * \Gamma$-module with head isomorphic to S_i and socle isomorphic to $v'vS_{i+1}$ (respectively $v'^{-1}vS_{i-1}$), where multiplication by X (respectively Y) sends the head to the socle.

Lemma. *For $i, j \in \mathbb{Z}/n\mathbb{Z}$ we have*

$$\tilde{\tau}_i(S_j) = \begin{cases} vS_j & \textit{if } i = j \\ v^{-1}M_j^{\pm}[1] & \textit{if } i = j \pm 1 \\ v^{-1}S_j[1] & \textit{otherwise.} \end{cases}$$

Proof. We first calculate τ_i. If $i = j$ then, using (2), we see that $\tau_j(S_j)$ equals the complex

$$v^2 R_j \xrightarrow{(X,-Y)^t} v'^{-1}vR_{j-1} \oplus v'vR_{j+1} \xrightarrow{\begin{pmatrix} Y & 0 \\ X & 0 \end{pmatrix}} R_j \oplus R_j \xrightarrow{\begin{pmatrix} \epsilon_j & 0 & 0 \\ \epsilon_j & X & -Y \end{pmatrix}^t}$$

$$\to S_i \oplus v'^{-1}v^{-1}R_{j-1} \oplus v'v^{-1}R_{j+1} \xrightarrow{(0,Y,X)} v^{-2}R_j.$$

It is straightforward to check that this complex only has homology at its end term, equal to $v^{-2}S_j$. Therefore $\tau_j(S_j) = v^{-2}S_j[1]$. If $i = j+1$ (the case $i = j-1$ is analogous) we find $\tau_{j+1}(S_j)$ is the complex

$$v'vR_{j+1} \xrightarrow{(-X,Y)^t} R_j \oplus v'^2 R_{j+2} \xrightarrow{\begin{pmatrix} \epsilon_j & 0 \\ -Y & -X \end{pmatrix}} S_j \oplus v'v^{-1}R_{j+1}.$$

Again, it is straightforward to check that this complex only has homology at its end term. A basis for this homology group is $\{\overline{(1,0)}, \overline{(0,1)}\} \subset S_j \oplus v'v^{-1}R_{j+1}$, showing that this is the unique bigraded $R * \Gamma$-module with head $v'v^{-1}S_{j+1}$ and socle S_j (multiplication by Y sends the head $\overline{(0,1)}$ to the socle $\overline{(1,0)}$). Call this N_j. If $i \neq j, j \pm 1$, then $\tau_i(S_j) = S_j$ since the relevant Ext-group vanishes, (4).

The description of $\tilde{\tau}_i$ follows immediately, noting that $D_{\mathbb{L}^2}(N_{n-j}) \cong M_j^-$ thanks to 3.17. □

5.3. Let ς be the automorphism of $R_{\mathbb{L}} * \Gamma$ which fixes X and Y and sends g to ζg. Following 2.2, this induces an $\mathbb{Z} \times \mathbb{Z}$-equivariant action of $\mathbb{Z}/n\mathbb{Z}$ on the derived category $D(R_{\mathbb{L}} * \Gamma\text{-grmod})$. This satisfies $\varsigma \circ \tilde{\tau}_i \circ \varsigma^{-1} \cong \tilde{\tau}_{i+1}$. Therefore we have an action of the extended affine braid group B on $D(R_{\mathbb{L}} * \Gamma\text{-grmod})$ via $[\tilde{T}_i] = \tilde{\tau}_i$, $[\sigma] = \varsigma$.

5.4. The action of B on $D_0(R_{\mathbb{L}} * \Gamma\text{-grmod})$ induces an action of $\mathcal{A}[B]$ on $K_0(R_{\mathbb{L}} * \Gamma\text{-grmod})$ which, by Lemma 5.2, satisfies

$$\tilde{T}_i([S_j]) = \begin{cases} v[S_j] & \text{if } i = j \\ -v^{-1}[S_j] - v'^{\pm 1}[S_{j\pm 1}] & \text{if } i = j \pm 1 \\ -v^{-1}[S_j] & \text{otherwise.} \end{cases}$$

In particular, since $(\tilde{T}_i + v^{-1})(\tilde{T}_i - v)$ acts as zero, this factors through an action of $\mathcal{H}$. The following proposition follows.

Proposition. *The natural isomorphism* $\Psi : K_0(R_{\mathbb{L}} * \Gamma\text{-grmod}) \to K_0(R_{\mathbb{C}} * \Gamma\text{-grmod})$ *is* $\mathcal{H}$-*equivariant.*

5.5. The action on $K_T(\mathbf{H}_0)$ referred to in the proposition comes from 3.9. The proposition will be proved in the following six subsections, where we also identify the full $\mathcal{H}$-action.

Proposition. *The isomorphism* $\Sigma_0 : K_0(R_{\mathbb{C}} * \Gamma\text{-grmod}) \to K_T(\mathbf{H}_0)$ *is* $\mathcal{H}_{ad}$*-equivariant.*

5.6. Let us translate the action from $K_0(R_{\mathbb{C}} * \Gamma\text{-grmod})$ over to $K_T(\mathbf{H}_0)$. Using Proposition 4.4 it is straightforward to check we have

$$\tilde{T}_i(\mathbf{O}_j) = \begin{cases} v\mathbf{O}_j & \text{if } i = j \\ -v^{-1}\mathbf{O}_1 - v'^{-n}\mathbf{O}_n & \text{if } i = n, j = 1 \\ -v^{-1}\mathbf{O}_n - v'^{n}\mathbf{O}_1 & \text{if } i = 1, j = n \\ -v^{-1}\mathbf{O}_j - \mathbf{O}_{j\pm 1} & \text{if } i = j \pm 1 \text{ is not as above} \\ -v^{-1}\mathbf{O}_j & \text{otherwise.} \end{cases}$$

By 3.11 the action of $\mathcal{H}_{fin}$ here agrees with the $\bullet$-action on $\mathbf{O}_j$ for $1 \leq j \leq n-1$.

5.7. We first check that the actions of $\mathcal{H}_{fin}$ also agree on $\mathbf{O}_n$. Thanks to 3.11 we have

$$\tilde{T}_1\bullet\mathbf{O}_n = -v^{-1}\mathbf{p}_{0,1} + (-v'^n + v^n)\mathbf{O}_1 - v^n\mathbf{O}_1 - v^{n-2}(-v^{-1}\mathbf{O}_2 - \mathbf{O}_1) + \sum_{k=3}^{n-1} v^{n-k-1}\mathbf{O}_k$$

$$= -v^{-1}\mathbf{p}_{0,1} + (-v'^n + v^{n-2})\mathbf{O}_1 + \sum_{k=2}^{n-1} v^{n-k-1}\mathbf{O}_k = -v^{-1}\mathbf{O}_n - v'^n\mathbf{O}_1.$$

Similarly, for $2 \le i \le n-2$, we have $\tilde{T}_i\bullet\mathbf{O}_n = -v^{-1}\mathbf{O}_n$, whilst $\tilde{T}_{n-1}\bullet\mathbf{O}_n = -v^{-1}\mathbf{O}_n - \mathbf{O}_{n-1}$. This shows that Σ_0 is $\mathcal{H}_{fin}$-equivariant.

5.8. We now calculate the action of $\theta_{\varpi_{n-1}} = \sigma\tilde{T}_{n-2}\tilde{T}_{n-3}\dots\tilde{T}_1\tilde{T}_n$ in both cases. Tensoring by the line bundle $L_{n-1,n}$ on $\mathcal{B}$, described in [18, 5.1], corresponds to $\theta_{\varpi_{n-1}}\bullet$. The T-equivariant structure of $L_{n-1,n}$ is given in [18, 5.2] for $1 \le k \le n$ as follows

$$L_{n-1,n}|_{p_{k-1,k}} = \begin{cases} v'^{n-1} & \text{if } k = n \\ v'^{-1}v^{2-n} & \text{if } k < n. \end{cases}$$

In particular, if $k < n$ then the restriction of $L_{n-1,n}$ to Π_k is the trivial line bundle with equivariant structure $v'^{-1}v^{2-n}$. We deduce for $1 \le k \le n-2$

$$\theta_{\varpi_{n-1}}\bullet\mathbf{O}_k = L_{n-1,n} \otimes \mathbf{O}_k = v'^{-1}v^{2-n}\mathbf{O}_k.$$

We have

$$\begin{aligned}
\theta_{\varpi_{n-1}}\bullet\mathbf{O}_{n-1} &= L_{n-1,n} \otimes \mathbf{O}_{n-1} = [O_{n-1}^{-1,2-n;n-1,0}] \otimes [O_{n-1}^{0,-1;-n,1-n}] \\
&= [O_{n-1}^{-1,1-n;-1,1-n}] = v'^{-1}v^{1-n}[O_{n-1}^{0,0;0,0}] \\
&= v'^{-1}v^{1-n}(\mathbf{p}_{n-1,n} + [O_{n-1}^{0,0;-n,2-n}]) = v'^{-1}v^{1-n}(\mathbf{p}_{n-1,n} + v\mathbf{O}_{n-1}) \\
&= v'^{-1}v^{1-n}(\mathbf{p}_{n-1,n} - \sum_{k=1}^{n-1} v'^n v^k\mathbf{O}_k) + \sum_{k=1}^{n-2} v'^{n-1}v^{k+1-n}\mathbf{O}_k + (v'^{n-1} + v'^{-1}v^{2-n})\mathbf{O}_{n-1} \\
&= v'^{n-1}v^{1-n}(\sum_{k=1}^{n-1} v^i\mathbf{O}_i) + v'^{-1}v^{2-n}\mathbf{O}_{n-1} + v'^{-1}v^{1-n}\mathbf{O}_n,
\end{aligned}$$

and similarly

$$\begin{aligned}
\theta_{\varpi_{n-1}}\bullet\mathbf{O}_n &= L_{n-1,n} \otimes (\mathbf{p}_{0,1} - \sum_{k=1}^{n-1} v^{n-k}\mathbf{O}_k) = v'^{-1}v^{2-n}\mathbf{p}_{0,1} - \sum_{k=1}^{n-2} v'^{-1}v^{2-k}\mathbf{O}_k - \\
&- v(v'^{n-1}v^{1-n}(\sum_{k=1}^{n-1} v^i\mathbf{O}_i) + v'^{-1}v^{2-n}\mathbf{O}_{n-1} + v'^{-1}v^{1-n}\mathbf{O}_n) \\
&= -\sum_{k=1}^{n-2} v'^{n-1}v^{2+k-n}\mathbf{O}_k - v'^{n-1}v\mathbf{O}_{n-1} = -v'^{n-1}v^{2-n}(\sum_{k=1}^{n-2} v^k\mathbf{O}_k).
\end{aligned}$$

5.9. We calculate $\tilde{T}_{n-2}\tilde{T}_{n-3}\dots\tilde{T}_1\tilde{T}_0(\mathbf{O}_i)$ for $1 \le i \le n$. Note that $\tilde{T}_i\tilde{T}_{i-1}(\mathbf{O}_i) = v^{-1}\mathbf{O}_i$ for $2 \le i \le n-1$, whilst $\tilde{T}_1\tilde{T}_n(\mathbf{O}_1) = v'^{-n}v^{-1}\mathbf{O}_n$. It follows that, for $2 \le i \le n-1$,

$$\tilde{T}_{n-2}\tilde{T}_{n-3}\dots\tilde{T}_1\tilde{T}_0(\mathbf{O}_i) = (-1)^{n-1}v^{2-n}\mathbf{O}_{i-1}, \tag{5}$$

whilst

$$\tilde{T}_{n-2}\tilde{T}_{n-3}\dots\tilde{T}_1\tilde{T}_0(\mathbf{O}_1) = (-1)^{n-1}v'^{-n}v^{2-n}\mathbf{O}_n. \tag{6}$$

We have $\tilde{T}_j\dots\tilde{T}_1\tilde{T}_0(\mathbf{O}_{n-1}) = (-1)^{j+1}(v^{-j-1}\mathbf{O}_{n-1} + v^{-j}\mathbf{O}_n + v'^n v^{-j}\sum_{k=1}^{j} v^k\mathbf{O}_k)$ for $1 \le j \le n-3$: this can be proved by induction. We deduce that

$$\tilde{T}_{n-2}\tilde{T}_{n-3}\dots\tilde{T}_1\tilde{T}_0(\mathbf{O}_{n-1}) = \tilde{T}_{n-2}((-1)^{n-2}(v^{2-n}\mathbf{O}_{n-1} + v^{3-n}\mathbf{O}_n + v'^n v^{3-n}\sum_{k=1}^{n-3} v^k\mathbf{O}_k))$$

$$= (-1)^{n-1}(v^{1-n}\mathbf{O}_{n-1} + v^{2-n}\mathbf{O}_n + v'^n v^{2-n}\sum_{k=1}^{n-2} v^k\mathbf{O}_k + v^{2-n}\mathbf{O}_{n-2}). \tag{7}$$

Arguing by induction, we see $\tilde{T}_j \dots \tilde{T}_1\tilde{T}_0(\mathbf{O}_n) = (-1)^j(v^{1-j}\mathbf{O}_n + v'^n v^{-j+1}\sum_{k=1}^{j} v^k\mathbf{O}_k)$ for $1 \leq j \leq n-2$. In particular

$$\tilde{T}_{n-2}\tilde{T}_{n-3}\dots\tilde{T}_1\tilde{T}_0(\mathbf{O}_n) = (-1)^{n-2}(v^{3-n}\mathbf{O}_n + v'^n v^{3-n}\sum_{k=1}^{n-2} v^k\mathbf{O}_k). \tag{8}$$

5.10. To get $\theta_{\varpi_{n-1}}$ we must twist by the automorphism σ. This twist sends S_i to S_{i+1} so, by Theorem 4.4, sends $\mathbf{O}_i$ to $v'^{-1}\mathbf{O}_{i+1}$ for $1 \leq i \leq n-1$ and $\mathbf{O}_n$ to $v'^{n-1}\mathbf{O}_1$. Combining this with (5), (6), (7) and (8) yields

$$\theta_{\varpi_{n-1}}(\mathbf{O}_i) = \begin{cases} (-1)^{n-1}v'^{-1}v^{2-n}\mathbf{O}_i & \text{if } 1 \leq i \leq n-2 \\ (-1)^{n-1}(v'^{n-1}v^{1-n}(\sum_{k=1}^{n-1} v^i\mathbf{O}_i) + v'^{-1}v^{2-n}\mathbf{O}_{n-1} + v'^{-1}v^{1-n}\mathbf{O}_n) & \text{if } i = n-1 \\ (-1)^{n-2}v'^{n-1}v^{2-n}(\sum_{k=1}^{n-2} v^k\mathbf{O}_k) & \text{if } i = n. \end{cases}$$

5.11. Comparing 5.8 and 5.10 we see the actions of $\theta_{\varpi_{n-1}}$ differ by scalar multiplication by $(-1)^{n-1}$. Since $\theta_{\varpi_{n-1}} = \tilde{T}_{n-1}\tilde{T}_{n-2}\dots\tilde{T}_1\sigma$ and the actions of $\mathcal{H}_{fin}$ agree we deduce that the actions of σ must differ by scalar multiplication by $(-1)^{n-1}$. Since $T_0 = \sigma T_{n-1}\sigma^{-1}$ we deduce that the actions agree on T_n, and hence on $\mathcal{H}_{ad}$, proving Proposition 5.5.

There is an involution of $\mathcal{H}$ which fixes $\mathcal{H}_{ad}$ and sends σ to $(-1)^{n-1}\sigma$. The calculations show that the two module structures on $K_T(\mathbf{H}_0)$ are twists of each other under this involution.

5.12. **Duality.** Let ν be the automorphism of $R_{\mathbb{L}} * \Gamma$ which swaps X and Y and sends g to g^{-1}. As in 5.3 we can twist an object $M \in R_{\mathbb{L}} * \Gamma$-grmod by ν where we set

$$(M^\nu)_{i,j} = M_{-i,j}$$

for $i, j \in \mathbb{Z}$. Twisting by ν commutes with $D_{\mathbb{L}^2}$ by 3.17. There is a contravariant self-equivalence

$$\tilde{\beta} : D_0(R_{\mathbb{L}} * \Gamma\text{-grmod}) \longrightarrow D_0(R_{\mathbb{L}} * \Gamma\text{-grmod})$$

which sends M to $D_{\mathbb{L}^2}(M)^\nu$. Note that $\tilde{\beta}(v'^a v^b M) = v'^a v^{-b}\tilde{\beta}(M)$ for $a, b \in \mathbb{Z}$ and $M \in D_0(R_{\mathbb{L}} * \Gamma\text{-grmod})$. Moreover, $\tilde{\beta}(S_i) = S_i$ for $1 \leq i \leq n$.

Recall the involution ι on B, defined in 3.7.

Lemma. *For all $b \in B$ and $M \in D_0(R_{\mathbb{L}} * \Gamma\text{-grmod})$ we have $\tilde{\beta}(b(M)) \cong \iota(b)(\tilde{\beta}(M))$.*

Proof. Given any object $C \in \mathcal{BV}_{\mathbb{L}}$ and $M \in D(R_{\mathbb{L}} * \Gamma\text{-grmod})$ evaluation gives a natural isomorphism $lin(C, D_{\mathbb{L}^2}(M)) \longrightarrow D_{\mathbb{L}^2}(C \otimes M)$ in $D(R_{\mathbb{L}} * \Gamma\text{-grmod})$. Hence, by [22, Section 2],

$$\begin{aligned} T_i^{-1}(M) &= \{ M \xrightarrow{ev'} lin(\mathbf{R}\mathrm{Hom}_{R_{\mathbb{L}}*\Gamma}(M, D_{\mathbb{L}^2}S_{n-i}), D_{\mathbb{L}^2}S_{n-i}) \} \\ &\cong \{M \longrightarrow D_{\mathbb{L}^2}(\mathbf{R}\mathrm{Hom}_{R_{\mathbb{L}}*\Gamma}(M, D_{\mathbb{L}^2}S_{n-i}) \otimes S_{n-i})\} \\ &\cong \{M \longrightarrow D_{\mathbb{L}^2}(\mathbf{R}\mathrm{Hom}_{R_{\mathbb{L}}*\Gamma}(S_{n-i}, D_{\mathbb{L}^2}(M)) \otimes S_{n-i})\} \\ &\cong D_{\mathbb{L}^2}(\{ (\mathbf{R}\mathrm{Hom}_{R_{\mathbb{L}}*\Gamma}(S_{n-i}, D_{\mathbb{L}^2}(M)) \otimes S_{n-i}) \xrightarrow{ev} D_{\mathbb{L}^2}(M) \}) \\ &= D_{\mathbb{L}^2}T_{n-i}(D_{\mathbb{L}^2}(M)). \end{aligned}$$

By construction ${}^\nu \circ T_i = T_{n-i} \circ {}^\nu$ so we deduce that $\tilde{\beta} \circ T_i = T_i^{-1} \circ \tilde{\beta}$. The same statement with $\tilde{T}_i$ follows since

$$\tilde{T}_i \circ \tilde{\beta} = D_{\mathbb{L}^2} \circ T_{n-i} \circ D_{\mathbb{L}^2} \circ v^{-1} \circ [1] \circ \tilde{\beta} = \tilde{\beta} \circ D_{\mathbb{L}^2} \circ T_{n-i}^{-1} \circ D_{\mathbb{L}^2} \circ v \circ [-1] = \tilde{\beta} \circ \tilde{T}_i^{-1}.$$

Finally, since $\nu\sigma = \sigma^{-1}\nu$ and $(M^*)^\sigma = (M^{\sigma^{-1}})^*$ we find $(D_{\mathbb{L}^2}(M^\sigma))^\nu = (D_{\mathbb{L}^2}(M))^{\sigma^{-1}\nu} = (D_{\mathbb{L}^2}(M)^\nu)^\sigma$, as required. □

5.13. **A pairing.** There is a pairing

$$(\, , \,) : D(R_{\mathbb{L}} * \Gamma\text{-grmod}) \times D(R_{\mathbb{L}} * \Gamma\text{-grmod}) \longrightarrow \mathcal{BV}_{\mathbb{L}},$$

given by $(M, N) = \mathbf{R}\mathrm{Hom}_{R*\Gamma}(\tilde{\beta}(N), M)$.

Recall from 3.10 the involution † on $\mathbb{Z}[v'^{\pm 1}, v^{\pm 1}]$. This induces an involution on $\mathcal{BV}_{\mathbb{L}}$ which we denote also by †.

Lemma. *The pairing satisfies the following properties:*
1) $(v'^a v^b M, N) = (M, v'^{-a} v^b N) = v'^a v^b (M, N)$*;*
2) $(T_i M, N) = (M, T_i N)$ *and* $(\sigma M, N) = (M, \sigma^{-1} N)$*;*
3) $(M, N) = (N, M)^\dagger$*.*

Proof. Part 1 follows immediately from the variance properties of $\mathbf{R}\mathrm{Hom}$ and $\tilde{\beta}$. By Lemma 5.12 we have

$$\mathbf{R}\mathrm{Hom}_{R_{\mathbb{L}}*\Gamma}(\tilde{\beta}(N), T_i(M)) = \mathbf{R}\mathrm{Hom}_{R_{\mathbb{L}}*\Gamma}(T_i^{-1}(\tilde{\beta}(N)), M) = \mathbf{R}\mathrm{Hom}_{R_{\mathbb{L}}*\Gamma}(\tilde{\beta}(T_i(N)), M),$$

and

$$\mathbf{R}\mathrm{Hom}_{R_{\mathbb{L}}*\Gamma}(\tilde{\beta}(N), M^\sigma) = \mathbf{R}\mathrm{Hom}_{R_{\mathbb{L}}*\Gamma}(\tilde{\beta}(N)^{\sigma^{-1}}, M) = \mathbf{R}\mathrm{Hom}_{R_{\mathbb{L}}*\Gamma}(\tilde{\beta}(N^\sigma), M),$$

proving part 2. Finally, we have

$$\mathbf{R}\mathrm{Hom}_{R_{\mathbb{L}}*\Gamma}(\tilde{\beta}(N), M) \cong \mathbf{R}\mathrm{Hom}_{R_{\mathbb{L}}*\Gamma}(D_{\mathbb{L}^2}(M), D_{\mathbb{L}^2}(\tilde{\beta}(N))) \cong \mathbf{R}\mathrm{Hom}_{R_{\mathbb{L}}*\Gamma}(D_{\mathbb{L}^2}(M), N^\nu).$$

In $R_{\mathbb{L}} * \Gamma$-grmod the identity provides an identification $\mathrm{Hom}_{R_{\mathbb{L}}*\Gamma}(A, B) = \mathrm{Hom}_{R_{\mathbb{L}}*\Gamma}(A^\nu, B^\nu)^\dagger$, which, together with above equation, proves part 3. □

5.14. The following theorem shows that the structures defined in this chapter give a categorification of the action of $\mathcal{H}_{ad}$ on $K_T(\mathcal{B}_\chi)$ discussed in 3.9. It is clear we can extend this to a categorification of the $\mathcal{H}$-action. For more details about categorification, see [1].

Theorem. *The duality* $\tilde{\beta}$ *and pairing* $(\, , \,)$ *on* $D_0(R_{\mathbb{L}} * \Gamma\text{-grmod})$ *induce a duality and a pairing on* $K_0(R_{\mathbb{L}} * \Gamma\text{-grmod})$*, which, under the* $\mathcal{H}_{ad}$*-module isomorphism* $\Sigma_0 \circ \Psi$*, correspond to the duality and pairing defined in 3.10.*

Proof. By construction $\tilde{\beta}(v'^a v^b [S_i]) = v'^a v^{-b} [S_i]$ for $a, b \in \mathbb{Z}$ and $1 \le i \le n$. The claim about duality then follows from the sentence following the proof of Proposition 4.4.

The calculation of Ext-groups in (4) shows that

$$(S_i, S_j) = \begin{cases} 1 + v^{-2}[-2] & \text{if } j = i \\ v'^{\pm 1} v^{-1}[-1] & \text{if } j = i \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

Calculating the image of this in $K_T(\mathbf{H}_0)$ under $\Sigma_0 \circ \Psi$ yields (1). Thus the pairings agree since Claims 1,2 and 3 in Lemma 5.13 correspond to 3.10.1,3.10.2 and 3.10.3 □

6. An application

Throughout this section we consider an algebraically closed field $\mathbb{K}$ of characteristic p, $p > n$. More detailed explanations of enveloping algebras and their representations can be found in the sister paper [7].

6.1. The *no–cycle algebra* is

$$C_{\mathbb{L}}(n) = \frac{\mathbb{L}[X, Y]}{(X^n, XY, Y^n)} * \Gamma.$$

Since (X^n, XY, Y^n) is a homogeneous ideal for the bigrading on $\mathbb{L}[X, Y]$, 3.16, there is a natural bigraded structure on $C_{\mathbb{L}}(n)$. Forgetting the second $\mathbb{Z}$-grading yields $C_{\mathbb{L}}(n)$-grmod, the category of $\mathbb{Z}$–graded $C_{\mathbb{L}}(n)$–modules. (The definition here of the no–cycle algebra as a “skew coinvariant algebra” differs from the original definition in [7, Section 3]. As remarked in [7, 3.2], however, the algebras are bigraded isomorphic.)

6.2. Let λ be a regular weight and χ as defined in 3.4. Let $B_{\chi,\lambda}$ be the corresponding block and $U_{\chi,\lambda}$ the full central quotient of the reduced enveloping algebra $U_\chi(\mathfrak{sl}_n(\mathbb{K}))$, [7, 4.6 and 4.16]. By [7, Proposition 4.17], the category of $U_{\chi,\lambda}$-T_0-modules is $\mathbb{Z}$-equivalent to $C_{\mathbb{K}}(n)$-grmod. Under this equivalence, the simple $U_{\chi,\lambda}$-T_0-module L_i, [7, 4.12], corresponds to S_{n-i}. The duality D on $U_{\chi,\lambda}$-T_0-mod corresponds to $\tilde{\beta}_{\mathbb{K}}$.

6.3. Let $\gamma_i = T^{\lambda}_{\mu_i} \circ T^{\mu_i}_{\lambda} : B_{\chi,\lambda}\text{-}T_0\text{-mod} \longrightarrow B_{\chi,\lambda}\text{-}T_0\text{-mod}$ be the ith *wall-crossing functor*. By the adjunction property, there is a natural transformation $\epsilon_i : \gamma_i \to id$, which, as in [21, Section 3], induces a triangulated $\mathbb{Z}$-functor $\tilde{T}_i$ on the bounded derived category of $B_{\chi,\lambda}$-T_0-mod.

The functor $\tilde{T}_i$ does not necessarily restrict to $D(U_{\chi,\lambda}\text{-}T_0\text{-mod})$. However, there is an isomorphism between $K(B_{\chi,\lambda}\text{-}T_0\text{-mod})$ and $K(U_{\chi,\lambda}\text{-}T_0\text{-mod})$. By transfer of structure, the $\mathbb{Z}[v'^{\pm 1}]$-linear operators $\tilde{T}_i$ act on $K(U_{\chi,\lambda}\text{-}T_0\text{-mod})$.

Theorem. *Let $\mathcal{B}_\chi$ be the Springer fibre of χ over $\mathbb{C}$. There is an $\mathcal{H}_{ad}$-equivariant isomorphism between $K(U_{\chi,\lambda}\text{-}T_0\text{-mod})$ and $K_{T_0}(\mathcal{B}_\chi)$ preserving duality.*

Proof. It follows from 3.3 that there is an action of $\mathcal{H}$ on $K(C_{\mathbb{K}}(n)\text{-grmod})$, giving operators $\tilde{T}_i$ on $K(C_{\mathbb{K}}(n)\text{-grmod})$. Following [11, H.9], the isomorphism $\theta : K(U_{\chi,\lambda}\text{-}T_0\text{-mod}) \to K(C_{\mathbb{K}}(n)\text{-grmod})$ commutes with the operators $\tilde{T}_i$ for $0 \le i \le n-1$. The theorem follows from Theorem 5.14 and the comments above. □

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