

POISSON ORDERS, SYMPLECTIC REFLECTION ALGEBRAS AND REPRESENTATION THEORY

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ABSTRACT. We introduce a new class of algebras called Poisson orders. This class includes the symplectic reflection algebras of Etingof and Ginzburg, many quantum groups at roots of unity, and enveloping algebras of restricted Lie algebras in positive characteristic. Quite generally, we study this class of algebras from the point of view of Poisson geometry, exhibiting connections between their representation theory and some well-known geometric constructions. As an application, we employ our results in the study of symplectic reflection algebras, completing work of Etingof and Ginzburg on when these algebras are finite over their centres, and providing a framework for the study of their representation theory in the latter case.

1. INTRODUCTION

1.1. It is a truth universally acknowledged, that a geometric viewpoint benefits representation theory. Evidence is found in the success of the orbit method and the theory of characteristic varieties, whose tool is symplectic geometry, and the theory of algebras which are finite as modules over their centres, in which algebraic geometry is pervasive.

In this paper we introduce a new class of algebras, *Poisson orders*, which exhibit key features from both of the above exemplars. Poisson orders include as special cases the $t = 0$ case of symplectic reflection algebras (whose definition we recall in (7.1)), enveloping algebras of restricted Lie algebras in positive characteristic, and quantised enveloping algebras and function algebras at roots of unity. We prove several basic theorems which relate the representation theory and algebraic structure of Poisson orders to symplectic geometry and, as an extended example, study symplectic reflection algebras.

1.2. A Poisson order is an affine \mathbb{C} -algebra A , finitely generated as a module over a central subalgebra Z_0 , together with the datum of a \mathbb{C} -linear map $D : Z_0 \rightarrow \text{Der}_{\mathbb{C}}(A)$, which satisfies the Leibniz identity and makes Z_0 a Poisson algebra by restriction. The details of the definition are given in (2.1). An elementary but important example is the skew group algebra $A = \mathcal{O}(V) * \Gamma$, where $\mathcal{O}(V)$ is the algebra of polynomial functions on a symplectic \mathbb{C} -vector space V and Γ is a finite subgroup of $\text{Sp}(V)$. Then A is a Poisson order with $Z_0 = \mathcal{O}(V)^\Gamma$, the centre of A . The Poisson structure is induced from the symplectic form on V .

Another particularly rich source of examples is quantisation, recalled in (2.2). This yields in particular the examples of quantum groups and Lie algebras mentioned in (1.1), but it is also the mechanism whereby symplectic reflection algebras in the $t = 0$ case are given the structure of Poisson orders, (7.3).

1.3. Let $Z_0 \subseteq A$ be a Poisson order and set $\mathcal{Z} = \text{Maxspec} Z_0$, a Poisson variety. The Poisson structure induces several stratifications on \mathcal{Z} , as follows.

- The *rank stratification*, where \mathcal{Z} is stratified by the rank of the Poisson bracket at a point.
- The stratification by *symplectic cores*, refining the rank stratification. We expect this stratification is the finest possible algebraic stratification in which the Hamiltonian vector fields, $D(z)$ for $z \in Z_0$, are tangent.
- The stratification (when A is a \mathbb{C} -algebra) by *symplectic leaves*, a further refinement, which potentially leaves the algebraic category. This is a differential geometric notion, which produces the well-known foliation by symplectic leaves when \mathcal{Z} is smooth.

We introduce and discuss the relations between these stratifications in Section 3. When the stratification by symplectic cores and by symplectic leaves agree, we call the Poisson bracket on Z_0 *algebraic*. In favourable circumstances, for instance if there is only a finite number of symplectic leaves, these three stratifications essentially agree. In general, however, we know of no procedure to deduce when a bracket is algebraic.

1.4. We show in this paper that the symplectic cores have a profound influence on the representation theory of A . Indeed, the surjective map

$$\mathcal{Irr}(A) \longrightarrow \mathcal{Z},$$

sending an irreducible A -module, S , to its Z_0 -character $\text{Ann}_{Z_0}(S)$, *has constant fibres over the symplectic cores of \mathcal{Z}* . In fact more is true. Given $x \in \mathcal{Z}$, let A_x be the finite dimensional algebra obtained by taking the quotient of A by the ideal $\mathfrak{m}_x A$. We show in Theorem 4.2 that *if $x, y \in \mathcal{Z}$ and x and y belong to the same core, the algebras A_x and A_y are isomorphic*. Moreover, as we explain in Section 5, this isomorphism can be globalised, yielding a sheaf of *Azumaya* algebras over each core.

Results of this sort were first proved in the case of quantised function algebras at a root of unity in [4] and [5]. The more general result quoted above can also be fruitfully applied to symplectic reflection algebras. As we show in Section 7, *these algebras are sufficiently well-behaved to ensure that the stratifications in (1.3) agree*. In Section 7 we also take the opportunity to complete work begun in [8] *by determining when a symplectic reflection algebra is a finite module over its centre, and showing that the centre is \mathbb{C} in all other cases*. These results show that it is a fundamental problem to determine the symplectic leaves of the centres of symplectic algebras. We give a complete answer to this in the “degenerate” case of the skew group algebra $\mathcal{O}(V) * \Gamma$, so answering a question of [1], and we provide some preliminary information in the general case, in (7.4)-(7.8).

1.5. The paper is organised as follows. In Section 2 we introduce the basic definition of a Poisson order and its Poisson ideals. We follow this in Section 3 by studying the various stratifications of \mathcal{Z} , pausing to discuss relations to the *Dixmier-Moeglin equivalence*. In Sections 4 and 5 we prove our results relating the representation theory of A with the symplectic cores of \mathcal{Z} . Section 6 briefly recalls the (motivating) case of quantised function algebras at roots of unity. Finally, in Section 7 we discuss symplectic reflection algebras.

1.6. Although we work over a field of characteristic zero, which in later sections is assumed to be \mathbb{C} , most of the algebraic results can be generalised to arbitrary algebraically closed fields. However, there are serious obstacles to be overcome in the attempt to generalise the geometric picture to other (especially positive characteristic) fields. Given that the enveloping algebra of a restricted Lie algebra comprises part of our motivation, this is an important issue requiring attention.

2. POISSON ORDERS

2.1. Definition and notation. Let k be a field of characteristic 0 and let A be an affine k -algebra, finitely generated as a module over a central subalgebra Z_0 . By the Artin-Tate Lemma [15, 13.9.10], Z_0 is an affine k -algebra, whose (Krull) dimension we shall fix as d throughout. Suppose that there is a linear map

$$D : Z_0 \longrightarrow \text{Der}_k(A) : z \mapsto D_z,$$

satisfying

- (1) Z_0 is stable under $D(Z_0)$;
- (2) the resulting bracket $\{-, -\} : Z_0 \times Z_0 \longrightarrow Z_0$, defined by $\{z, z'\} = D_z(z')$, imposes a structure of Poisson k -algebra on Z_0 - that is, $D_{zz'} = zD_{z'} + z'D_z$ as derivations of Z_0 , and $(Z_0, \{-, -\})$ is a k -Lie algebra.

Then we shall say that A is a *Poisson Z_0 -order*. We can also express (2) as saying that $\mathcal{Z} := \text{Maxspec}(Z_0)$ is a *Poisson variety* with respect to the bracket defined in (2). The *algebra of Casimirs* of a Poisson algebra Z_0 is $\text{Cas}(Z_0) := \{z \in Z_0 : \{z, Z_0\} = 0\}$.

Throughout this section we'll assume that A is a Poisson Z_0 -order with associated linear map D .

2.2. Quantisation. Here is one important mechanism giving rise to a Poisson order. Let \hat{A} be a k -algebra, \hat{Z} a subalgebra, and $t \in \hat{Z}$ a central non-zero divisor. Assume that $Z_0 = \hat{Z}/t\hat{Z}$ is an affine central subalgebra of $A = \hat{A}/t\hat{A}$, and that A is a finitely generated Z_0 -module. Let $\pi : \hat{A} \longrightarrow A$ be the quotient map. Fix a k -basis $\{z_i : i \in \mathcal{I}\}$ of Z_0 , and lift these elements of Z_0 to elements \hat{z}_i of \hat{Z} .

Given $i \in \mathcal{I}$ there is a derivation of A , denoted D_{z_i} , defined by

$$D_{z_i}(a) = \pi([\hat{z}_i, \hat{a}]/t),$$

where $\hat{a} \in \hat{A}$ is a preimage under π of a . Now define $D : Z_0 \longrightarrow \text{Der}(A)$ by extending the above definition k -linearly. This construction satisfies the hypotheses of (2.1), [12]. Observe that alternative choices (for example of liftings of $\{z_i\}$) would yield the same outcomes, to within inner derivations of A .

2.3. Filtered and graded algebras. An important variant of the above is the following. Let A be an \mathbb{N} -filtered algebra whose i^{th} -filtered piece is denoted $F^i A$. Let Z be a subalgebra of A , and give it the induced filtration. Denote the associated graded rings of Z and A by $\text{gr}Z$ and $\text{gr}A$ respectively. Suppose that $\text{gr}Z$ is an affine central subalgebra of $\text{gr}A$, such that $\text{gr}A$ is a finitely generated $\text{gr}Z$ -module. Let $\sigma_i : F^i A \longrightarrow \text{gr}A$ be the i^{th} -principal symbol map,

sending an element of $F^i A \setminus F^{i-1} A$ to its leading term. Given a homogeneous element of $\text{gr} Z$, say $\sigma_m(z)$, there is a well-defined derivation of $\text{gr} A$, denoted $D_{\sigma_m(z)}$, given, for a homogeneous element $\sigma_n(a)$ of $\text{gr}(A)$, by

$$D_{\sigma_m(z)}(\sigma_n(a)) = \sigma_{m+n-1}([z, a]). \quad (1)$$

Extending this linearly yields a mapping $D : \text{gr} Z \rightarrow \text{Der}(\text{gr} A)$, satisfying the hypotheses of (2.1). Naturally, we'll call a Poisson bracket on a commutative graded algebra H *homogeneous* if the bracket of any two homogeneous elements of H is also homogeneous; in this case, if whenever $h \in H_i$ and $g \in H_j$ we have $\{h, g\} \in H_{i+j+d}$, we shall say that the bracket has *degree* d . Thus the bracket defined in (1) is homogeneous of degree -1 .

To see that this definition is really a special case of (2.2), form the Rees algebras $\hat{A} = \bigoplus_i F^i A t^i \subseteq A[t]$ and $\hat{Z} = \bigoplus_i F^i Z t^i \subseteq Z[t]$, where t is a central non-zero divisor. It can easily be checked that we recover the derivations in (1) from the construction in (2.2).

2.4. Poisson ideals and subsets. A two-sided ideal I of the Poisson Z_0 -order A (respectively J of Z_0) is called *Poisson* if it is stable under $D(Z_0)$. Thanks to [7, 3.3.2] if I (respectively J) is Poisson then so too are both \sqrt{I} (respectively \sqrt{J}) and the minimal prime ideals of A (respectively Z_0) over I (respectively J). We shall denote the space of prime Poisson ideals of Z_0 , with the topology induced from the Zariski topology on $\text{spec}(Z_0)$, by $\mathcal{P} - \text{spec}(Z_0)$. Clearly, if I is a Poisson ideal of Z_0 then there is an induced structure of Poisson algebra on Z_0/I . For a semiprime ideal I of Z_0 (respectively of A) we write $\mathcal{V}(I)$ for the closed subset of \mathcal{Z} (respectively of $\text{Maxspec}(A)$) defined by I . A closed subset $\mathcal{V}(I)$ of \mathcal{Z} is *Poisson closed* if its defining ideal is Poisson.

Extension ($e : J \rightarrow JA$) and contraction ($c : I \rightarrow I \cap Z_0$) are mappings between the ideals of Z_0 and A , which map Poisson ideals to Poisson ideals. It is an easy exercise using Going Up [15, 10.2.10(ii)] to show that

$$c \circ e \text{ is the identity on semiprime ideals of } Z_0.$$

Clearly, therefore, if I is a semiprime Poisson ideal of Z_0 then there is an induced structure of Poisson Z_0/I -order on A/IA , and if J is a Poisson ideal of A then there is an induced structure of Poisson $Z_0/J \cap Z_0$ -order on A/J .

3. STRATIFICATIONS

3.1. The rank stratification. Throughout Section 3 we consider an affine commutative Poisson k -algebra Z_0 . Let $\{z_1, \dots, z_m\}$ be a generating set for Z_0 and define the $m \times m$ -skew symmetric matrix $\mathcal{M} = (\{z_i, z_j\}) \in M_m(Z_0)$. The *rank* of the Poisson structure at $\mathfrak{m} \in \mathcal{Z}$, denoted $\text{rk}(\mathfrak{m})$, is defined to be the rank of the matrix $\mathcal{M}(\mathfrak{m}) = (\{z_i, z_j\} + \mathfrak{m}) \in M_m(k)$. It is independent of the choice of generators, [18, 2.6]. For each non-negative integer j we define

$$\mathcal{Z}_j^o(Z_0) := \{\mathfrak{m} \in \mathcal{Z} : \text{rk}(\mathfrak{m}) = j\}$$

and

$$\mathcal{Z}_j(Z_0) := \{\mathfrak{m} \in \mathcal{Z} : \text{rk}(\mathfrak{m}) \leq j\}.$$

We'll simply write \mathcal{Z}_j^o and \mathcal{Z}_j when the algebra involved is clear from the context.

Lemma. *Retain the above notation.*

(1) \mathcal{Z}_j is a closed subset of \mathcal{Z} , with

$$\mathcal{Z}_0 \subseteq \mathcal{Z}_1 \subseteq \dots \subseteq \mathcal{Z}_d = \mathcal{Z}.$$

(2) \mathcal{Z}_j is a Poisson subset of \mathcal{Z} , at least when $k = \mathbb{C}$.

(3) $\mathcal{Z}_j = \cup_{i \leq j} \mathcal{Z}_i^o = \mathcal{Z}_{j_0}$, where $j_0 := \max\{i : i \leq j, \mathcal{Z}_i^o \neq \emptyset\}$.

(4) The sets \mathcal{Z}_i^o are locally closed: if \mathcal{Z}_i^o is non-empty then $\overline{\mathcal{Z}_i^o}$ is equal to the union of some of the irreducible components of \mathcal{Z}_i .

(5) If $\mathcal{Z}_i^o \neq \emptyset$ then $\dim \mathcal{A} \geq i$ for each irreducible component \mathcal{A} of \mathcal{Z}_i^o .

(6) Suppose that $k = \mathbb{C}$. If the irreducible component \mathcal{A} of \mathcal{Z}_i^o has $\dim \mathcal{A} = i$ then \mathcal{A} is contained in the smooth locus of its closure $\overline{\mathcal{A}}$.

Proof. (1) It's clear that \mathcal{Z}_j is closed, since \mathcal{Z}_j is defined by the vanishing of all minors of \mathcal{M} of order $j + 1$. That $\mathcal{Z}_i \subseteq \mathcal{Z}_{i+1}$ is clear from the definition; and that $\mathcal{Z}_d = \mathcal{Z}$ is a consequence of (5), which we prove below.

(2) Assume that $k = \mathbb{C}$. To show that \mathcal{Z}_j is Poisson we argue by noetherian induction, and so assume that the result is true for all proper Poisson factors of Z_0 . Let J be the defining ideal of \mathcal{Z}_j , and let \mathcal{Z}_j have irreducible components $\mathcal{A}_1 = \mathcal{V}(P_1), \dots, \mathcal{A}_t = \mathcal{V}(P_t)$. Thus J is Poisson if and only if each P_i is Poisson, by (2.4). We thus aim to show that $P := P_1$ is Poisson. By (2.4) again, we can assume that Z_0 is a domain. Let I be the defining ideal of the singular locus of \mathcal{Z} , (so $I = Z_0$ if Z_0 is smooth). Let $0 \neq y \in I$. Then $Z_0[y^{-1}]$ is a Poisson algebra and if $P[y^{-1}]$ is a proper ideal of $Z_0[y^{-1}]$ then it defines a component of $\mathcal{Z}_j(Z_0[y^{-1}])$. Since $Z_0[y^{-1}]$ is smooth, $P[y^{-1}]$ is Poisson by [17, Corollary 2.3].

Define $\hat{P} := \bigcap_{0 \neq y \in I} P[y^{-1}] \cap Z_0$, so that \hat{P} is a Poisson ideal of Z_0 containing P , with $\hat{P}I \subseteq P$. Since P is prime either (i) $\hat{P} = P$ and so P is Poisson, or (ii) $I \subseteq P$. Now I is a Poisson ideal by [17, Corollary 2.4], and I is non-zero by definition. So if (ii) holds we can pass to Z_0/I and invoke our induction hypothesis. This completes the proof in either case.

(3), (4) If \mathcal{Z}_i^o is non-empty then it is precisely the subset of \mathcal{Z}_i on which some minor of \mathcal{M} of order i is not identically zero.

(5) Suppose that \mathcal{Z}_i^o is non-empty, and let \mathcal{A} be one of its irreducible components. Since \mathcal{A} is open in its closure \mathcal{T} , \mathcal{A} contains a point \mathfrak{m} which is smooth in \mathcal{T} . Let J be the defining ideal of \mathcal{T} , a Poisson ideal by (2) and (2.4). Note that \mathcal{M} defines a bilinear form of rank i on $\mathfrak{m}/\mathfrak{m}^2$, and since J is Poisson, $J + \mathfrak{m}^2/\mathfrak{m}^2$ is contained in the radical of this form. Therefore,

$$i \leq \dim(\mathfrak{m}/J + \mathfrak{m}^2) = \dim(\mathcal{T}) = \dim(\mathcal{A}).$$

(6) Assume now that $\dim \mathcal{A} = i$, and suppose that \mathcal{A} contains a singular point of $\overline{\mathcal{A}}$. Then the intersection of \mathcal{A} with the singular locus of $\overline{\mathcal{A}}$ is a non-empty open subset of the singular locus, and so contains a smooth point \mathfrak{n} of that singular locus. Let K be the ideal of Z_0 defining the singular locus of $\overline{\mathcal{A}}$, so K is a Poisson ideal by (2) and [17, Corollary 2.4]. Hence, the image of K in $\mathfrak{n}/\mathfrak{n}^2$ is in the radical of the form induced on $\mathfrak{n}/\mathfrak{n}^2$. Thus

$$i = \dim \overline{\mathcal{A}} > \text{Krull dim}(Z_0/K) = \dim_{\mathbb{C}}(\mathfrak{n}/\mathfrak{n}^2 + K) \geq i,$$

a contradiction. □

In the light of the lemma we define the *rank stratification* of \mathcal{Z} to be the disjoint union of locally closed subsets

$$\mathcal{Z} = \bigsqcup_j \mathcal{Z}_j^o. \quad (2)$$

3.2. Dixmier-Moeglin equivalence for Poisson ideals. Continue with the notation and assumptions already introduced for Z_0 . Given an ideal I of Z_0 we define the *Poisson core* of I to be the biggest Poisson ideal of Z_0 contained in I . This exists since the sum of two Poisson ideals is Poisson. We denote the Poisson core of I by $\mathcal{P}_{Z_0}(I)$, or just by $\mathcal{P}(I)$ when the algebra is clear from the context. If I is prime then by (2.4) so also is $\mathcal{P}(I)$. A prime ideal \mathfrak{p} of Z_0 is *Poisson primitive* if $\mathfrak{p} = \mathcal{P}(\mathfrak{m})$ for a maximal ideal \mathfrak{m} of Z_0 . Since every prime ideal of Z_0 is an intersection of maximal ideals,

$$\text{every prime Poisson ideal of } Z_0 \text{ is an intersection of Poisson primitive ideals.} \quad (3)$$

By analogy with well-known results of Dixmier and Moeglin for certain noncommutative affine algebras such as enveloping algebras of finite dimensional Lie algebras [15, Chapter 9] we are led to consider the following hypotheses which can be imposed on a prime Poisson ideal \mathfrak{p} of Z_0 :

- (A) \mathfrak{p} is Poisson primitive;
- (B) \mathfrak{p} is locally closed in $\mathcal{P} - \text{spec}(Z_0)$;
- (C) $\text{Cas}(Z_0/\mathfrak{p})$ is an algebraic extension of k ;
- (D) $\text{Cas}(\text{Fract}(Z_0/\mathfrak{p}))$ is an algebraic extension of k .

The following lemma is due to Oh [16, 1.7(i), 1.10].

Lemma. *Let $\mathfrak{p} \in \mathcal{P} - \text{spec}(Z_0)$, and fix hypotheses and notation as above. Then $(B) \implies (A) \implies (D) \implies (C)$.*

Remark: In general (C) does not imply (A) or (B). For consider $Z_0 = \mathbb{C}[x, y, z]$, with $\{x, z\} = x$, $\{y, z\} = y$, $\{x, y\} = 0$. If F is a Casimir, then $\partial F/\partial z = x\partial F/\partial x + y\partial F/\partial y = 0$, so it follows easily that F is a scalar. But one also calculates that $\mathcal{P} - \text{spec}(Z_0)$ consists of $\{0\}$, $\langle x - ay \rangle$, for $a \in \mathbb{C}$, and $\langle y \rangle$ and $\langle x, y \rangle$. All of these are thus Poisson primitive except for $\{0\}$, which also fails to be locally closed in $\mathcal{P} - \text{spec}(Z_0)$.

Note, however, that in the above example $\text{Cas}(\text{Fract}(Z_0)) = \mathbb{C}(xy^{-1})$. This motivates the following question, to which we will return in (3.4) and (4.2).

Question: Are properties (A), (B) and (D) always equivalent for an affine Poisson \mathbb{C} -algebra Z_0 ?

3.3. The stratification by symplectic cores. We again focus on Z_0 in this paragraph, and continue with the hypotheses as above. We define a relation \sim on \mathcal{Z} by:

$$\mathfrak{m} \sim \mathfrak{n} \iff \mathcal{P}(\mathfrak{m}) = \mathcal{P}(\mathfrak{n}).$$

Clearly \sim is an equivalence relation; we denote the equivalence class of \mathfrak{m} by $\mathcal{C}(\mathfrak{m})$, so that

$$\mathcal{Z} = \bigsqcup \mathcal{C}(\mathfrak{m}). \quad (4)$$

The set $\mathcal{C}(\mathfrak{m})$ is called the *symplectic core* (of \mathfrak{m}).

Lemma. *Fix notation as above.*

(1) *The subsets $\mathcal{C}(\mathfrak{m})$ of \mathcal{Z} are all locally closed, with $\overline{\mathcal{C}(\mathfrak{m})} = \mathcal{V}(\mathcal{P}(\mathfrak{m}))$ for all $\mathfrak{m} \in \mathcal{Z}$, if and only if (A) \implies (B) of Lemma 3.2 holds for Z_0 .*

(2) *Each symplectic core is smooth in its closure.*

(2) *At least when $k = \mathbb{C}$, the stratification (4) is a refinement of (2).*

Proof. (1) Assume that $\mathcal{C}(\mathfrak{m})$ is locally closed with $\overline{\mathcal{C}(\mathfrak{m})} = \mathcal{V}(\mathcal{P}(\mathfrak{m}))$, for all $\mathfrak{m} \in \mathcal{Z}$, and let $\mathfrak{n} \in \mathcal{Z}$. Thus

$$\mathcal{C}(\mathfrak{n}) = \mathcal{V}(\mathcal{P}(\mathfrak{n})) \setminus \mathcal{V}(K)$$

for an ideal K of Z_0 with $\mathcal{P}(\mathfrak{n}) \subsetneq K$. But then K is the intersection of all the prime Poisson ideals of Z_0 which strictly contain $\mathcal{P}(\mathfrak{n})$, so that K is Poisson and (3.2)(B) holds for $\mathcal{P}(\mathfrak{n})$, as required.

Conversely, assume that (3.2)(A) \implies (B) holds for Z_0 , and let $\mathfrak{n} \in \mathcal{Z}$. Let K be the intersection of Z_0 and the prime Poisson ideals of Z_0 which strictly contain $\mathcal{P}(\mathfrak{n})$, so, by hypothesis, $\mathcal{P}(\mathfrak{n}) \subsetneq K$. Clearly,

$$\mathcal{C}(\mathfrak{n}) = \mathcal{V}(\mathcal{P}(\mathfrak{n})) \setminus \mathcal{V}(K),$$

so that $\mathcal{C}(\mathfrak{n})$ is locally closed, and the proof is complete.

(2) Let J be the defining ideal of the singular locus of $\overline{\mathcal{C}(\mathfrak{m})}$. Since J is a Poisson ideal properly containing $\mathcal{P}(\mathfrak{m})$, it follows from the definition of a Poisson core that \mathfrak{m} does not contain J . Hence the point corresponding to \mathfrak{m} belongs to the smooth locus of $\overline{\mathcal{C}(\mathfrak{m})}$, as required.

(3) If $\mathfrak{n} \in \mathcal{C}(\mathfrak{m})$ is a point at which the rank of $\{-, -\}$ is minimal in $\mathcal{C}(\mathfrak{m})$, say $\text{rk}(\mathfrak{n}) = r$, then by Lemma 3.1.2, writing J_r for the defining ideal of \mathcal{Z}_r , $J_r \subseteq \mathcal{P}(\mathfrak{n}) = \mathcal{P}(\mathfrak{m})$, so that $\text{rk}(\mathfrak{a}) \leq r$ for all $\mathfrak{a} \in \mathcal{C}(\mathfrak{m})$. Then minimality of r implies that the rank is constant across $\mathcal{C}(\mathfrak{m})$. \square

The example of the trivial Poisson bracket on $Z_0 = \mathbb{C}[X]$ shows that in general (4) is finer than (2).

3.4. The Dixmier-Moeglin equivalence revisited. Here is one case in which Question 3.2 has a positive answer.

Lemma. *Suppose that Z_0 has only finitely many Poisson primitive ideals. Then $\mathcal{P} - \text{spec}(Z_0)$ is a finite set and (3.2)(A), (B) and (D) are all true.*

Proof. With the stated hypothesis, $\mathcal{P} - \text{spec}(Z_0)$ is a finite set by (3). Let $P \in \mathcal{P} - \text{spec}(Z_0)$. Then $P = \bigcap \{\mathfrak{m} : \mathfrak{m} \in \mathcal{Z}, P \subseteq \mathfrak{m}\}$, so P is the intersection of the Poisson cores of the maximal ideals which contain it. But by hypothesis this intersection is finite, and so since P is prime we have $P = \mathcal{P}(\mathfrak{m})$ for some maximal ideal \mathfrak{m} . The equivalence and validity of (3.2)(A), (B) and (D) now follows from Lemma 3.2. \square

Remark: In unpublished work, Goodearl obtains a considerable improvement of the above lemma – he shows that Question 3.2 has a positive answer provided Z_0 admits a torus $H = (k^\times)^r$ of k -algebra Poisson automorphisms acting rationally, such that $\mathcal{P} - \text{spec}(Z_0)$ has only finitely many H -orbits of Poisson primitive ideals.

3.5. The stratification by symplectic leaves. For the rest of the paper the underlying base field k will be \mathbb{C} .

We assume initially that Z_0 is smooth. By considering the complex topology, \mathcal{Z} is a complex analytic manifold. Let \hat{Z}_0 be the ring of complex analytic functions on \mathcal{Z} . Then \hat{Z}_0 is a Poisson algebra whose bracket we also denote by $\{-, -\}$. Indeed, the existence of a Poisson bracket is equivalent to the existence of a closed algebraic 2-form on \mathcal{Z} ; considering the given 2-form as complex analytic yields the Poisson bracket. Recall that the *symplectic leaf* $\mathcal{L}(\mathfrak{m})$ containing the point \mathfrak{m} of \mathcal{Z} is the maximal connected complex analytic manifold in \mathcal{Z} such that $\mathfrak{m} \in \mathcal{L}(\mathfrak{m})$ and $\{-, -\}$ is nondegenerate at every point of $\mathcal{L}(\mathfrak{m})$. By [19, Proposition 1.3] symplectic leaves exist and afford a stratification of \mathcal{Z} . By construction the points of the symplectic leaf $\mathcal{L}(\mathfrak{m})$ are exactly those which can be reached by travelling along the integral flows of Hamiltonian vector fields $\{H, -\}$ for $H \in \hat{Z}_0$.

We now extend this definition to an arbitrary commutative Poisson \mathbb{C} -algebra Z_0 . So, drop the hypothesis that Z_0 is smooth, and define inductively an ascending chain of ideals of Z_0 , as follows: set $I_0 = \sqrt{\{0\}}$ and let I_{t+1} be the ideal of Z_0 such that I_{t+1}/I_t defines the singular locus of Z_0/I_t . Fix m such that $I_m = Z_0$. By [15, Proposition 15.2.14(i)] or [17, Corollary 2.4] each I_t is a semiprime Poisson ideal of Z_0 . Set $Z_t = Z_0/I_t$ and $A_t = A/I_t A$. In view of (2.4) there is an inclusion $Z_t \subseteq A_t$ on which D induces a map such that the assumptions of (2.1) are satisfied.

Let $\mathcal{Z}_t = \text{Maxspec}(Z_t)$ and let $(\mathcal{Z}_t)_{\text{sm}}$ be the smooth locus of \mathcal{Z}_t . As $(\mathcal{Z}_t)_{\text{sm}}$ is a complex analytic Poisson manifold it has, as above, a foliation by symplectic leaves: for each $t = 0, \dots, m$, there is an index set \mathcal{I}_t such that

$$(\mathcal{Z}_t)_{\text{sm}} = \bigsqcup_{i \in \mathcal{I}_t} \mathcal{S}_{t,i}.$$

Since \mathcal{Z} is the disjoint union of the subsets $(\mathcal{Z}_t)_{\text{sm}}$, there is a stratification of \mathcal{Z} ,

$$\mathcal{Z} = \bigsqcup_{0 \leq t \leq m, i \in \mathcal{I}_t} \mathcal{S}_{t,i}. \quad (5)$$

We write $\mathcal{L}(\mathfrak{m})$ for the leaf containing $\mathfrak{m} \in \mathcal{Z}$, and $\overline{\mathcal{S}}_{t,i}$ for the Zariski closure of $\mathcal{S}_{t,i}$. Let $K_{t,i}$ be the defining ideal of $\overline{\mathcal{S}}_{t,i}$.

Lemma. *For all t , $0 \leq t \leq m$, and all $i \in \mathcal{I}_t$, $K_{t,i}$ is prime and Poisson. Indeed $K_{t,i}$ is the Poisson core $\mathcal{P}(\mathfrak{m})$ of every $\mathfrak{m} \in \mathcal{S}_{t,i}$.*

Proof. Let \mathfrak{m} be in $\mathcal{S}_{t,i}$. We show first that

$$\mathcal{P}(\mathfrak{m}) \subseteq K_{t,i}. \quad (6)$$

So, write $\overline{\mathcal{Z}} = \text{maxspec}(Z_0/\mathcal{P}(\mathfrak{m}))$, and use $\bar{}$ to denote images in $\hat{Z}_0/\mathcal{P}(\mathfrak{m})\hat{Z}_0$ below. Let $B(\epsilon)$ be the complex analytic ball with radius ϵ . Let $H \in \hat{Z}_0$, and let $\sigma_{\mathfrak{m}}(z) : B(\epsilon) \rightarrow \mathcal{Z}$ and $\overline{\sigma}_{\mathfrak{m}}(z) : B(\epsilon) \rightarrow \overline{\mathcal{Z}}$ be the integral curves to $\{H, -\}$ and $\{\overline{H}, -\}$ respectively, with $\sigma_{\mathfrak{m}}(0) = \mathfrak{m}$ and $\overline{\sigma}_{\mathfrak{m}}(0) = \overline{\mathfrak{m}}$. Thinking of $\overline{\mathcal{Z}}$ as a subset of \mathcal{Z} , it makes sense to claim, as we do, that

$$\overline{\sigma}_{\mathfrak{m}} = \sigma_{\mathfrak{m}} \quad (7)$$

in a neighbourhood of 0. To prove (7), let $f \in \hat{Z}_0$. By definition of an integral curve,

$$\frac{d}{dz}(f \circ \sigma_{\mathfrak{m}}) = \{H, f\} \circ \sigma_{\mathfrak{m}}, \quad (8)$$

and

$$\frac{d}{dz}(\bar{f} \circ \bar{\sigma}_{\mathfrak{m}}) = \{\bar{H}, \bar{f}\} \circ \bar{\sigma}_{\mathfrak{m}}. \quad (9)$$

But the left hand side of (9) is $\frac{d}{dz}(f \circ \bar{\sigma}_{\mathfrak{m}})$, while, because $\mathcal{P}(\mathfrak{m})\hat{Z}_0$ is a Poisson ideal, the right hand side is $\{H, f\} \circ \bar{\sigma}_{\mathfrak{m}}$. Hence, comparing this with (8), we deduce from the uniqueness of flows that $\sigma_{\mathfrak{m}} = \bar{\sigma}_{\mathfrak{m}}$ in a neighbourhood of 0. Since symplectic leaves are by definition obtained by travelling along integral curves to Hamiltonians, the leaf through \mathfrak{m} is contained in $\mathcal{V}(\mathcal{P}(\mathfrak{m}))$ and so (6) follows.

Now we complete the proof by showing that $K_{t,i}$ is a Poisson ideal. In doing so we may assume without loss that \mathcal{Z} is smooth, since, as in the definition of symplectic leaves above, we pass to a smooth variety containing \mathfrak{m} when determining $\mathcal{S}_{t,i}$. So let $f \in K_{t,i}$ and let $H \in Z_0$. Let $\sigma_{\mathfrak{m}}(z) : B(\epsilon) \rightarrow \mathcal{Z}$ be an integral curve to $\{H, -\}$ with $\sigma_{\mathfrak{m}}(0) = \mathfrak{m}$. Then, by definition of an integral curve, (8) holds. On a complex analytic neighbourhood of \mathfrak{m} , $f \circ \sigma_{\mathfrak{m}} = 0$ since $\sigma_{\mathfrak{m}}$ has image in $\mathcal{S}_{t,i}$. Hence

$$\frac{d}{dz}(f \circ \sigma_{\mathfrak{m}}) = 0 = \{H, f\} \circ \sigma_{\mathfrak{m}}(0) = \{H, f\}(\mathfrak{m}).$$

That is, $\{Z_0, K_{t,i}\} \subseteq \mathfrak{m}$. Repeating this argument with \mathfrak{m} replaced by each of the members of $\mathcal{L}(\mathfrak{m})$ in turn we conclude that $\{Z_0, K_{t,i}\} \subseteq K_{t,i}$ as required. \square

3.6. Of particular interest are the cases where, for all t and for all $i \in \mathcal{I}_t$,

$\mathcal{S}_{t,i}$ is a locally closed subvariety of \mathcal{Z} .

When this is so we shall say that the Poisson bracket $\{-, -\}$ is *algebraic*.

The following proposition relates the three stratifications discussed.

Proposition. *Let $\mathfrak{m} \in \mathcal{Z}$ with $\text{rk}(\mathfrak{m}) = j$.*

(1) *Then*

$$\mathcal{L}(\mathfrak{m}) \subseteq \mathcal{C}(\mathfrak{m}) \subseteq \mathcal{Z}_j^o.$$

(2) *Suppose that the Poisson bracket is algebraic. Then the first inclusion in (1) is an equality, so that the symplectic leaf of each maximal ideal is then determined by its core.*

Moreover (A) and (B) of (3.2) are equivalent. $\mathcal{L}(\mathfrak{m})$ is contained in the smooth locus of $\overline{\mathcal{L}(\mathfrak{m})}$, and

$$\dim \mathcal{L}(\mathfrak{m}) = \dim \overline{\mathcal{L}(\mathfrak{m})} = \text{Krull dim}(Z_0/\mathcal{P}(\mathfrak{m})) = j.$$

Proof. (1) The first inclusion is a restatement of part of the above lemma, and the second is Lemma 3.3.2.

(2) Suppose that the Poisson bracket is algebraic. Then, by the lemma, the defining ideal of the Zariski closure of the leaf $\mathcal{L}(\mathfrak{m})$ is the core $\mathcal{P}(\mathfrak{m})$ of \mathfrak{m} . Since this statement applies also to the leaves in the boundary of $\overline{\mathcal{L}(\mathfrak{m})}$, we deduce that in this case $\mathcal{L}(\mathfrak{m}) = \mathcal{C}(\mathfrak{m})$. The equivalence of (3.2)(A) and (B) now follows from Lemmas 3.2 and 3.3(1). That $\dim \overline{\mathcal{L}(\mathfrak{m})} = \text{Krull dim}(Z_0/\mathcal{P}(\mathfrak{m}))$ is now clear because $\mathcal{P}(\mathfrak{m})$ is the defining ideal of $\overline{\mathcal{L}(\mathfrak{m})}$, by Lemma 3.5.

Since $\mathcal{L}(\mathfrak{m})$ is open in $\overline{\mathcal{L}(\mathfrak{m})}$, $\mathcal{L}(\mathfrak{m})$ contains a point, say \mathfrak{q} , which is smooth in $\overline{\mathcal{L}(\mathfrak{m})}$. By definition of $\mathcal{L}(\mathfrak{m})$ the Poisson bracket is nondegenerate on $\mathfrak{q}/\mathfrak{q}^2 + \mathcal{P}(\mathfrak{m})$, so that

$$j = \text{rk}(\mathfrak{q}) = \dim_{\mathbb{C}}(\mathfrak{q}/\mathfrak{q}^2 + \mathcal{P}(\mathfrak{m})) = \text{Krull dim}(Z_0/\mathcal{P}(\mathfrak{m})),$$

the last equality holding thanks to smoothness of \mathfrak{q} in $\overline{\mathcal{L}(\mathfrak{m})}$. Finally, to see that $\mathcal{L}(\mathfrak{m})$ is contained in the smooth locus of $\overline{\mathcal{L}(\mathfrak{m})}$, let \mathfrak{n} be any point of $\mathcal{L}(\mathfrak{m})$. Nondegeneracy of the Poisson bracket on $\mathcal{L}(\mathfrak{m}) \subseteq \mathcal{Z}_j^0$ forces

$$\dim_{\mathbb{C}}(\mathfrak{n}/\mathfrak{n}^2 + \mathcal{P}(\mathfrak{m})) = j,$$

and since this integer is $\text{Krull dim}(Z_0/\mathcal{P}(\mathfrak{m}))$, smoothness follows. \square

Remarks: (1) The following example shows that symplectic leaves need not be algebraic, [18, Example 2.37]. Let $Z = \mathbb{C}[x, y, z]$ with Poisson bracket given by $\{x, y\} = 0, \{x, z\} = \alpha x$ and $\{y, z\} = y$ for some $\alpha \in \mathbb{R}$. A calculation shows that the symplectic leaves fall into two families: the single points $(0, 0, c)$ for $c \in \mathbb{C}$ and the sets cut out by the equation $xy^\alpha = C$, where $C \in \mathbb{C}$. If $\alpha = p/q \in \mathbb{Q}$ then the leaves are algebraic, being described by the equation $x^q y^p = C$. However, if $\alpha \notin \mathbb{Q}$ then the leaves are not algebraic varieties. In the second case the symplectic cores are locally closed: they are the points $(0, 0, c)$, for $c \in \mathbb{C}$, and $(\mathbb{C}^*)^2 \times \mathbb{C}$.

(2) The inclusion of $\mathcal{L}(\mathfrak{m})$ in the smooth locus of $\overline{\mathcal{L}(\mathfrak{m})}$ is in general strict: consider the example in Remark 3.2. Let $\mathfrak{m} = \langle x - \alpha, y - \beta, z - \gamma \rangle$ with $\alpha\beta \neq 0$. Then $\mathcal{P}(\mathfrak{m}) = \langle x - \beta^{-1}\alpha y \rangle$, so that $\overline{\mathcal{L}(\mathfrak{m})} = \mathcal{V}(\langle x - \beta^{-1}\alpha y \rangle)$, which is smooth; whereas $\mathcal{L}(\mathfrak{m}) = \mathcal{V}(\langle x - \beta^{-1}\alpha y \rangle) \setminus \langle x, y \rangle$.

3.7. The case of finitely many leaves. In later applications to symplectic reflection algebras the set of symplectic leaves will be finite. We now explore the ramifications of this hypothesis.

Proposition. *Continue with the notation and hypotheses of (3.5). Suppose that the stratification (5) of \mathcal{Z} into symplectic leaves is finite.*

- (1) *The Poisson bracket is algebraic.*
- (2) *Let $\mathfrak{m} \in \mathcal{Z}$ with $\text{rk}(\mathfrak{m}) = j$. Then the following subsets of \mathcal{Z} coincide:*
 - (a) *the symplectic leaf $\mathcal{L}(\mathfrak{m})$ containing \mathfrak{m} ;*
 - (b) *$\mathcal{C}(\mathfrak{m})$;*
 - (c) *the irreducible component of \mathcal{Z}_j^0 containing \mathfrak{m} ;*
 - (d) *the smooth locus of the irreducible component of \mathcal{Z}_j containing \mathfrak{m} .*

Proof. (1) We argue by noetherian induction. So we can assume that \mathcal{Z} is irreducible by (2.4). Since the singular locus of \mathcal{Z} is a union of leaves by the construction in (3.5), these leaves are locally closed by our induction assumption, and it remains to consider the leaves

$$\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_t \tag{10}$$

whose disjoint union equals the smooth locus \mathcal{S} of \mathcal{Z} . By Lemma 3.5, we can apply the induction hypothesis to any leaf \mathcal{L}_i whose closure is strictly contained in \mathcal{Z} , to see that such leaves are algebraic. Removing any such leaves from the list (10), it follows that $\cup_{i=1}^t \mathcal{L}_i$ is a non-empty open set of smooth points of \mathcal{Z} , with $\overline{\mathcal{L}_i} = \mathcal{Z}$ for all i .

Choose $\mathfrak{m} \in \cup_{i=1}^t \mathcal{L}_i$ for which $\text{rk}(\mathfrak{m})$ is maximal, say $\text{rk}(\mathfrak{m}) = j$ and $\mathfrak{m} \in \mathcal{L}_1$. Since $\overline{\mathcal{L}_1} = \mathcal{Z}$, the closed set \mathcal{Z}_{j-1} is proper in \mathcal{Z} . So, for all $i = 1, \dots, t$, $\overline{\mathcal{L}_i} = \mathcal{Z} \not\subseteq \mathcal{Z}_{j-1}$, implying that $\mathcal{L}_i \not\subseteq \mathcal{Z}_{j-1}$.

It follows from this and the maximality of $\text{rk}(\mathfrak{m})$ that the rank is constant and equal to j across $\cup_{i=1}^t \mathcal{L}_i$. Therefore $\cup_{i=1}^t \mathcal{L}_i$ is a j -dimensional irreducible smooth open subset of \mathcal{Z} on which the rank is constant and equal to j . Hence, $\cup_{i=1}^t \mathcal{L}_i$ is a leaf. That is, $t = 1$ and \mathcal{L}_1 is locally closed, as required.

(2) First, (a) = (b) and (b) is contained in (c) by the first part of the proposition and Proposition 3.6. Now let $I(\mathcal{A})$ be the defining ideal of the irreducible component \mathcal{A} of \mathcal{Z}_j^o containing \mathfrak{m} . We claim that

$$\text{Krull dim}(Z_0/I(\mathcal{A})) = j. \tag{11}$$

For certainly $\text{Krull dim}(Z_0/I(\mathcal{A})) \geq j$ by Lemma 3.1.5. Suppose that the inequality is strict. Then \mathcal{A} is a union of symplectic leaves by Proposition 3.6.1, and these leaves all have closures of dimension j by the first part of the proposition and Proposition 3.6.2. Hence infinitely many such leaves are needed to cover \mathcal{A} , contradicting the finiteness hypothesis. So (11) is proved.

Now \mathcal{A} is a union of j -dimensional algebraic leaves, but (11) and the irreducibility of \mathcal{A} means that the union consists of a single leaf, proving (b) = (c). Finally, Lemma 3.1.6 shows that (c) is contained in (d). To prove equality we adapt an idea from [8, Lemma 3.5(ii)]. Let \mathcal{S} be the smooth locus of $\overline{\mathcal{A}}$, and suppose for a contradiction that $\mathcal{A} \subsetneq \mathcal{S}$. Now \mathcal{S} is an (algebraic) Poisson manifold, for which the subvariety \mathcal{U} of points of rank less than j is non-empty and has codimension at least 2 in \mathcal{S} , since there are by hypothesis only a finite number of leaves and each one has even dimension. On the other hand \mathcal{U} has codimension 1 in \mathcal{S} since it is defined by the vanishing of the divisor associated with the Poisson form on \mathcal{S} . This contradiction shows that (c) = (d). \square

3.8. G-equivariant stratifications. Let G be an algebraic group. Suppose that G acts rationally by algebra automorphisms on A , that this action preserves Z_0 and that D is G -equivariant, in the sense that $D_{gz}(a) = gD_z(g^{-1}a)$ for all $g \in G, z \in Z_0$ and $a \in A$. Then G consists of Poisson automorphisms of Z_0 .

One can carry through a G -equivariant version of most of paragraphs (3.2) – (3.7), defining, for example, prime G -Poisson ideals, G -Poisson primitive ideals, G -symplectic cores, G -symplectic leaves (meaning G -orbits of symplectic leaves) and so on, and proving G -equivariant versions of most of the results of those paragraphs. In discussing the algebraicity of G -symplectic leaves, without loss of generality we may assume that G is connected, if necessary replacing G by its identity component G^o . We label the G -orbits of symplectic leaves \mathcal{G}_i , so that we have a stratification

$$\mathcal{Z} = \bigsqcup \mathcal{G}_i. \tag{12}$$

Proposition. *Assume that G is connected. If the Poisson bracket on Z_0 is algebraic, then each stratum \mathcal{G}_i is locally closed and irreducible.*

Proof. Consider the action morphism $\alpha : G \times \mathcal{Z} \rightarrow \mathcal{Z}$. The variety \mathcal{G}_i is irreducible since it is the image under α of the irreducible variety $G \times \mathcal{L}$ for some leaf \mathcal{L} . Moreover, \mathcal{L} is locally closed by hypothesis, and with it so is $G \times \mathcal{L}$. By Chevalley’s theorem, [11, Exercise II.3.19], the image under a morphism of a locally closed subvariety is constructible, so that \mathcal{G}_i is constructible,

meaning (thanks to [11, Exercise II.3.18]) that there is a finite union

$$\mathcal{G}_i = \bigcup_s \mathcal{U}_s \cap \mathcal{C}_s,$$

where \mathcal{U}_s is open in \mathcal{Z} and \mathcal{C}_s is closed in \mathcal{Z} for each s . Without loss of generality we may assume that $\mathcal{U}_s \cap \mathcal{C}_s \neq \emptyset$ and that $\mathcal{C}_s \subseteq \overline{\mathcal{G}}_i$ for all s . As \mathcal{G}_i is irreducible, there exists s' such that $\mathcal{C}_{s'} = \overline{\mathcal{G}}_i$. Hence $\mathcal{U}_{s'} \cap \mathcal{C}_{s'}$ is open and dense in $\overline{\mathcal{G}}_i$ and consequently \mathcal{G}_i itself is dense in $\overline{\mathcal{G}}_i$.

Thus the smooth locus of $\overline{\mathcal{G}}_i$ meets \mathcal{G}_i and, by applying the G -action if necessary, we see that this smooth locus actually meets the symplectic leaf \mathcal{L} . Since the defining ideals of $\overline{\mathcal{L}}$ and hence of $\overline{\mathcal{G}}_i$ are Poisson closed by Lemma 3.5, we can integrate the derivations arising from D to one-parameter families of complex analytic automorphisms of $(\overline{\mathcal{G}}_i)_{\text{sm}}$, acting transitively on \mathcal{L} . Combining this again with the action of G we deduce that each point of \mathcal{G}_i is smooth in $\overline{\mathcal{G}}_i$. Finally, $\mathcal{U}_{s'} \cap \mathcal{C}_{s'}$ is open in $\overline{\mathcal{G}}_i$, so that at least one point of \mathcal{G}_i has an open neighbourhood in $\overline{\mathcal{G}}_i$ which is contained in \mathcal{G}_i . Applying G and the automorphisms introduced above it follows that every point of \mathcal{G}_i has such a neighbourhood, so that \mathcal{G}_i is open in $\overline{\mathcal{G}}_i$. \square

4. THE BUNDLE OF FINITE DIMENSIONAL ALGEBRAS

In this section we continue to assume that A is a Poisson Z_0 -order, and we'll freely use the notation introduced in the earlier sections. The underlying field k will always be \mathbb{C} .

4.1. Let M be a Z_0 -module. We say that M is a *Poisson module* if there exists a bilinear form $\{-, -\}_M : Z_0 \times M \rightarrow M$ satisfying $\{z, z'm\}_M = \{z, z'\}m + z'\{z, m\}_M$ for all $z, z' \in Z_0$ and $m \in M$. The following is proved in [17, Lemma 2.1]

Lemma. *Let M be a finitely generated Z_0 -Poisson module. Then the annihilator of M is a Poisson ideal of Z_0 , and for any n the ideal defining the locus where the rank of M as a Z_0 -module is greater than n (in the sense of the minimal number of generators) is Poisson closed.*

Proof. Let $z'M = 0$ for some $z' \in Z_0$. Then for all $z \in Z_0$ and $m \in M$

$$0 = \{z, z'm\}_M = \{z, z'\}m + z'\{z, m\}_M = \{z, z'\}m,$$

proving that the annihilator of M is a Poisson ideal. The second claim follows by applying the first to the exterior powers of M . \square

4.2. The results we shall prove in this Section were proved by De Concini, Lyubashenko and Procesi [5, Corollary 11.8], [4, Corollary 9.2], in the special case of a Poisson Z_0 -order A with base field \mathbb{C} , such that Z_0 has finite global dimension and A is a free Z_0 -module, and an algebraic Poisson bracket. In the Hopf algebra setting of quantum groups these additional hypotheses are valid, but they no longer hold for symplectic reflection algebras and we are thus obliged to prove the stronger results given here.¹ Much of our proof is an adaptation of the earlier ones, but since the latter were somewhat brief we have included full details here as an aid to the reader.

¹We are grateful to several participants in the Oberwolfach meeting in Noncommutative Geometry of 14-19 April 2002, and in particular Alastair King, for helpful comments in connection with Theorem (4.2).

Theorem. *Let A be a Poisson Z_0 -order which is an affine \mathbb{C} -algebra. For each point $x \in \mathcal{Z}$, define the finite dimensional algebra*

$$A_x := \frac{A}{\mathfrak{m}_x A}.$$

If x, y belong to the symplectic core \mathcal{C} , then A_x and A_y are isomorphic \mathbb{C} -algebras.

Proof. Let I be the defining ideal of $\bar{\mathcal{C}}$, a prime Poisson ideal of Z_0 by Lemma 3.3. Both A_x and A_y are quotients of A/IA , an algebra satisfying the hypotheses of (2.1). In particular A/IA is a Poisson Z_0/I -module with bracket induced from D_z for $z \in Z_0$. By Lemma 4.1, \mathcal{D} , the subvariety of points of $\bar{\mathcal{C}}$ for which the rank of A/IA as a module over Z_0/I is non-minimal, is Poisson closed and, by definition, proper in $\bar{\mathcal{C}}$. Set $\mathcal{E} = (\bar{\mathcal{C}} \setminus \mathcal{D})_{sm}$, a locally closed variety with closure $\bar{\mathcal{C}}$. By construction the rank of A/IA as a module over Z_0/I is constant on \mathcal{E} . It follows from [11, Exercise II.5.8(c)] that A/IA is free (of finite rank) over \mathcal{E} .

The sets \mathcal{C} and \mathcal{D} are disjoint. Indeed, if $x \in \mathcal{C} \cap \mathcal{D}$ then, since \mathcal{C} is the Poisson core of x , $\bar{\mathcal{C}} \subseteq \mathcal{D}$, contradicting properness. It follows from Lemma 3.3(2) that $\mathcal{C} \subseteq \mathcal{E}$.

We will now argue as in [4, Section 9]. Thanks to the above, we may assume without loss of generality that $\mathcal{E} = \mathcal{Z}$ is smooth and A is a free Z_0 -module. Recall that \hat{Z}_0 denotes the ring of complex analytic functions on \mathcal{Z} . Define $\hat{A} = A \otimes_{Z_0} \hat{Z}_0$. Note that, for any value of x , the natural map

$$A_x = \frac{A}{\mathfrak{m}_x A} \longrightarrow \frac{\hat{A}}{\hat{\mathfrak{m}}_x \hat{A}} = \hat{A}_x$$

is an algebra isomorphism. Moreover, given any $H \in Z_0$, then the derivation D_H extends uniquely to a derivation, \hat{D}_H , on \hat{A} , thanks to the extension of the Poisson bracket on Z_0 to \hat{Z}_0 .

Consider \hat{A} as a vector bundle \mathcal{V} over \mathcal{Z} with fibres \hat{A}_x . Since this bundle is trivial, we have $\mathcal{V} \cong \mathcal{Z} \times V$ where V is a vector space isomorphic to A_x . Pick a basis for A over Z_0 , say $\{a_1, a_2, \dots, a_n\}$ with $a_1 = 1$. The above isomorphism identifies $\sum \lambda_i a_i \in \hat{A}_x$ with $(x, (\lambda_1, \dots, \lambda_n))$.

To the derivation \hat{D}_H we can associate a vector field, ξ_H , on \mathcal{V} . Since $T\mathcal{V} = T(\mathcal{Z} \times V)$, if $y = (x, (\lambda_1, \dots, \lambda_n))$ we have $T_y \mathcal{V} = T_x \mathcal{Z} \times \mathbb{C}^n$, and we write the vector field as

$$\xi_{H,y} = \left(\xi_{H,x}, \left(- \sum_{i=1}^n p_H^{i,j}(x) \right)_j \right),$$

where $\xi_{H,x}(f) = \{H, f\}(x)$ is the Hamiltonian vector field on \mathcal{Z} evaluated at x and $\{H, a_i\} = \sum_{j=1}^n p_H^{i,j} a_j$ with $p_H^{i,j} \in \hat{Z}_0$.

Let $\phi : B(\epsilon) \times \mathcal{V} \longrightarrow \mathcal{V}$ be a local flow around y lifting ξ_H . Note that ϕ_z sends fibres to fibres since the Hamiltonian vector field of ξ_H is independent of $\lambda_1, \dots, \lambda_n$. It follows from the definition of $\xi_{H,y}$ that ϕ_z is linear on the fibres also.

Let $\varrho : B(\epsilon) \times \mathcal{Z} \longrightarrow \mathcal{Z}$ be the local flow on \mathcal{Z} around x , induced by the Hamiltonian ξ_H . Splitting ϕ into components, we have a linear isomorphism

$$\psi_z : \hat{A}_{\varrho(0)} \longrightarrow \hat{A}_{\varrho(z)},$$

which is given explicitly in [4, 9.1]. We claim that ψ_z is an isomorphism of algebras. Up to first order, ψ_z is described as

$$\text{id} - \hat{D}_H : \hat{A}_{\varrho(z)} \longrightarrow \hat{A}_{\varrho(z+dz)}.$$

To prove this is an algebra isomorphism, let multiplication in \hat{A}_x be denoted by μ_x and $c_{ij}^k \in \hat{Z}_0$ be the structure constants of \hat{A} with respect to the chosen basis $\{a_1, \dots, a_n\}$. We have

$$\begin{aligned}
& \mu_{\varrho(z+dz)}((\text{id} - \hat{D}_{H,\varrho(z)}dz)(a_i) \otimes (\text{id} - \hat{D}_{H,\varrho(z)}dz)(a_j)) = \\
& = \mu_{\varrho(z+dz)}((a_i - \{H, a_i\}dz) \otimes (a_j - \{H, a_j\}dz)) \\
& = \sum_{k=1}^n c_{ij}^k(\varrho(z+dz))a_k - \mu_{\varrho(z)}(\{H, a_i\} \otimes a_j + a_i \otimes \{H, a_j\})dz \\
& = \left(\sum_{k=1}^n c_{ij}^k(\varrho(z))a_k + \sum_{k=1}^n \{H, c_{ij}^k\}(\varrho(z))dz \right) - \{H, a_i a_j\}dz \\
& = \sum_{k=1}^n c_{ij}^k(\varrho(z))a_k + \left(\sum_{k=1}^n \{H, c_{ij}^k\}(\varrho(z)) - \sum_{k=1}^n (\{H, c_{ij}^k\}(\varrho(z)) + c_{ij}^k(\varrho(z))\{H, a_k\}) \right) dz \\
& = (\text{id} - \hat{D}_H)\mu_{\varrho(z)}(a_i \otimes a_j).
\end{aligned}$$

Let $d_{ij}^k \in \hat{Z}_0$ be the structure constants obtained by transport of structure along ψ_z . Both c_{ij}^k and d_{ij}^k provide analytic maps $B(\epsilon) \rightarrow \text{Alg}_{\mathbb{C}}(n)$, where $\text{Alg}_{\mathbb{C}}(n)$ denotes the variety of n -dimensional algebras over \mathbb{C} . The above calculation shows that the derivatives of these maps agree everywhere. Therefore the maps are equal, proving the claim that ψ_z is an algebra isomorphism.

We have now shown that any Hamiltonian flow generated by an algebraic function lifts to an isomorphism of \mathbb{C} -algebras within the leaf \mathcal{L}_x . Since the set of algebraic functions is dense in the set of analytic functions it follows that we can trace out a dense subset of \mathcal{L}_x by algebraically generated Hamiltonian flows. Call this set $\mathcal{L}_x^{\text{alg}}$. By Lemma 3.5 we have

$$\overline{\mathcal{L}_x^{\text{alg}}} \cap \mathcal{E} = \overline{\mathcal{L}_x} \cap \mathcal{E} = \mathcal{E}.$$

There is an action of $GL_n(\mathbb{C})$ on $\text{Alg}_{\mathbb{C}}(n)$ by base change, whose orbits are the isomorphism classes of n -dimensional algebras over \mathbb{C} . Let $\Upsilon : \mathcal{E} \rightarrow \text{Alg}_{\mathbb{C}}(n)$ be the morphism obtained by sending a point $x \in \mathcal{E}$ to A_x . By the above paragraph $\Upsilon(\mathcal{L}_x^{\text{alg}})$ is contained in a unique $GL_n(\mathbb{C})$ -orbit, say \mathcal{O}_x . As $\mathcal{L}_x^{\text{alg}}$ is dense in \mathcal{E} , the image of Υ lies in $\overline{\mathcal{O}_x}$. Repeating the above argument for $y \in \mathcal{C} \subseteq \mathcal{E}$, we deduce that $\overline{\mathcal{O}_x} = \overline{\mathcal{O}_y}$. Therefore $\mathcal{O}_x = \mathcal{O}_y$, showing that $A_x \cong A_y$ as required. \square

Remark: In situations where Question 3.2 has a positive answer, Lemma 3.3(1) shows that the symplectic cores are locally closed in \mathcal{Z} . In this situation, the symplectic reflection algebra example in Section 7.6 (taking $\Gamma = \mathbb{Z}_2$) shows that the theorem cannot be improved: different symplectic cores can yield non-isomorphic algebras.

4.3. G -equivariant isomorphisms. Let G be a connected algebraic group, acting on A as in (3.8). We have a stratification by G -symplectic cores,

$$\mathcal{Z} = \bigsqcup \mathcal{GC}_i.$$

Clearly $\mathfrak{p}_i = I(\overline{\mathcal{GC}_i})$ is a Poisson closed prime ideal of Z_0 . Hence, by (2.4), both $\mathfrak{p}_i A$ and the minimal primes lying over it are Poisson ideals of A .

Proposition. *Retain the hypotheses of (3.8) and the notation just introduced. Let P be a prime ideal of A minimal over $\mathfrak{p}_i A$. For all $\mathfrak{m}_x, \mathfrak{m}_y \in \mathcal{GC}_i$ there are algebra isomorphisms*

$$A/\mathfrak{m}_x A \cong A/\mathfrak{m}_y A \quad \text{and} \quad \frac{A}{P + \mathfrak{m}_x A} \cong \frac{A}{P + \mathfrak{m}_y A}.$$

Proof. Since G is connected and preserves $\mathfrak{p}A$, $g(P) = P$ for all $g \in G$. Thus, by suitable application of elements of G , we may assume that \mathfrak{m}_x and \mathfrak{m}_y are in the same symplectic core. It is now clear that the algebra isomorphism from A_x to A_y afforded by Proposition 4.2 maps P to itself. The result follows. \square

5. AZUMAYA SHEAVES

5.1. In this section we examine in more detail the conclusion derived in Proposition 4.3. Thus we impose the following hypotheses throughout Section 5:

- (1) A is a noetherian \mathbb{C} -algebra, finitely generated over a central subalgebra Z_0 ;
- (2) Let $\mathcal{Z} = \text{Maxspec} Z_0$. There is a stratification

$$\mathcal{Z} = \bigsqcup \mathcal{Z}_i,$$

where each \mathcal{Z}_i is an irreducible locally closed subvariety of \mathcal{Z} such that $\overline{\mathcal{Z}}_i$, the closure of \mathcal{Z}_i , is a union of some of the \mathcal{Z}_j ;

- (3) Let $\mathfrak{p}_i = I(\overline{\mathcal{Z}}_i)$ be the prime ideal of Z_0 defining $\overline{\mathcal{Z}}_i$, and let $P_{i,j}$ be the minimal primes of A lying over $\mathfrak{p}_i A$. For all $x, y \in \mathcal{Z}_i$ and all j there is an algebra isomorphism

$$\frac{A}{P_{i,j} + \mathfrak{m}_x A} \cong \frac{A}{P_{i,j} + \mathfrak{m}_y A}.$$

5.2. Recall that \mathcal{A} is a sheaf of algebras over a variety \mathcal{V} if \mathcal{A} is a quasi-coherent sheaf over \mathcal{V} such that for each open set $\mathcal{U} \subseteq \mathcal{V}$, the sections $\mathcal{A}(\mathcal{U})$ yield an $\mathcal{O}_{\mathcal{V}}(\mathcal{U})$ -algebra. Moreover, if there exists an open affine covering of \mathcal{V} , say $\{\mathcal{V}_i\}$, such that $\mathcal{A}(\mathcal{V}_i)$ is an Azumaya algebra finitely generated as a module over $\mathcal{O}_{\mathcal{V}}(\mathcal{V}_i)$ for each i , then we shall call \mathcal{A} a *sheaf of Azumaya algebras* over \mathcal{V} . The proof of the following routine lemma is left to the reader.

Lemma. *Suppose \mathcal{A} is a sheaf of Azumaya algebras over a noetherian variety \mathcal{V} . Then, for any open irreducible affine subvariety \mathcal{U} , the algebra $\mathcal{A}(\mathcal{U})$ is Azumaya.*

We come now to the main result of this section.

Proposition. *Retain the assumptions of (5.1). Let $\mathcal{A}_{i,j}$ be the sheaf of algebras over $\overline{\mathcal{Z}}_i$ corresponding to $A/P_{i,j}$. Then the restriction of $\mathcal{A}_{i,j}$ to \mathcal{Z}_i is a sheaf of Azumaya algebras.*

Proof. By (5.1)(3), $P_{i,j} \cap Z_0 = \mathfrak{p}_i$, so we can consider the triple

$$Z_0/\mathfrak{p}_i \subseteq Z(A/P_{i,j}) \subseteq A/P_{i,j}.$$

By (5.1)(1) the extension $Z_0/\mathfrak{p}_i \subseteq Z(A/P_{i,j})$ is generically separable by [2, Lemma 2.1] and the prime ring $A/P_{i,j}$ is generically Azumaya by [15, 13.7.4 and 13.7.14]. Hence there exists a non-empty open set $\mathcal{U} \subseteq \overline{\mathcal{Z}}_i$ and integers d and s , such that for all $x \in \mathcal{U}$ we have

$$\frac{A}{P_{i,j} + \mathfrak{m}_x A} \cong \bigoplus_{i=1}^s \text{Mat}_d(\mathbb{C}). \tag{13}$$

By (5.1)(2) \mathcal{Z}_i and \mathcal{U} have non-empty intersection, and so $\mathcal{Z}_i \subseteq \mathcal{U}$ by (5.1)(3).

Let $f \in Z_0/\mathfrak{p}_i$ be any non-zero function vanishing on $\overline{\mathcal{Z}}_i \setminus \mathcal{Z}_i$. Then $(\overline{\mathcal{Z}}_i)_f \subseteq \mathcal{Z}_i$ and so, by (13) and the Artin-Procesi theorem, [15, 13.7.14], $A/P_{i,j}[f^{-1}]$ is an Azumaya algebra. The proposition follows by covering \mathcal{Z}_i by such distinguished open sets. \square

6. QUANTISED FUNCTION ALGEBRAS

6.1. Let G be a simply-connected, semisimple algebraic group over \mathbb{C} and let T be a maximal torus contained in a Borel subgroup B of G . Let B^- be the Borel subgroup of G opposite B and let W be the Weyl group of G with respect to T .

Let $A = \mathcal{O}_\epsilon[G]$ be the quantised function algebra of G at a primitive ℓ th root of unity ϵ , where ℓ is odd, and prime to 3 if G contains a factor of type G_2 . There is a central subalgebra Z_0 of $\mathcal{O}_\epsilon[G]$ isomorphic to $\mathcal{O}[G]$, with $\mathcal{O}_\epsilon[G]$ a finitely generated Z_0 -module. Using the construction in (2.2), it is shown in [4] that the pair A and Z_0 satisfy the conditions of (2.1). Moreover, the symplectic leaves are algebraic, [13, Appendix A].

6.2. There is an action of T on A by winding automorphisms, preserving Z_0 and acting as Poisson automorphisms of the latter, [4, Propositions 9.3 and 8.7(b)]. Under this action the orbits of the symplectic leaves are identified in [13, Appendix A] with the double Bruhat cells of G ,

$$G = \bigsqcup_{w_1, w_2 \in W} X_{w_1, w_2},$$

where $X_{w_1, w_2} = B\dot{w}_1B \cap B^-\dot{w}_2B^-$, with \dot{w}_i denoting an inverse image of $w_i \in W$ in $N_G(T)$.

6.3. For arbitrary $g \in G$ the algebras A_g of (4.2) are rather incompletely understood – for the current state of knowledge, see [3]. On the other hand, for any $w_1, w_2 \in W$ the sheaf of Azumaya algebras lying over X_{w_1, w_2} of Proposition 5.2 is in this case explicitly described in [6].

7. SYMPLECTIC REFLECTION ALGEBRAS

7.1. **Definition and fundamental properties.** Given the data of a $2n$ -dimensional complex symplectic vector space V and a finite subgroup Γ of $\mathrm{Sp}(V)$, Etingof and Ginzburg [8] construct a family of so-called *symplectic reflection algebras*, as follows. First, define an element $s \in \Gamma$ to be a *symplectic reflection* if (in its action on V) $\mathrm{rank}(1-s) = 2$. The set S of symplectic reflections in Γ is closed under conjugation. Take a complex number t and a Γ -invariant function $\mathbf{c} : S \rightarrow \mathbb{C} : s \mapsto c_s$. Note that for $s \in \Gamma$ there is a ω -orthogonal decomposition $V = \mathrm{Im}(1-s) \oplus \mathrm{Ker}(1-s)$. For $s \in S$, write ω_s for the skew-symmetric form on V which has $\mathrm{Ker}(1-s)$ as its radical, and coincides with ω on $\mathrm{Im}(1-s)$. Now define $A_{t, \mathbf{c}}$ to be the \mathbb{C} -algebra with generators V and Γ , and relations those for Γ , together with

$$\gamma x \gamma^{-1} = \gamma(x), \text{ and } xy - yx = t\omega(x, y)1_\Gamma + \sum_{s \in S} c_s \omega_s(x, y)s,$$

for $x, y \in V$ and $\gamma \in \Gamma$. Thus, for a given triple (V, ω, Γ) these algebras form a family parametrised by a complex number t together with the points of an affine space of dimension r (where r is the number of conjugacy classes of symplectic reflections in Γ). It's clear from the relations above that

$$A_{t, \mathbf{c}} \cong A_{\mu t, \mu \mathbf{c}} \tag{14}$$

for $\mu \in \mathbb{C}^*$, so that there is a space $\mathbb{P}^r(\mathbb{C})$ of symplectic reflection algebras arising from a given (V, ω, Γ) . This space includes the familiar special cases $A_{0,0}$ and $A_{1,0}$, the skew group algebras of Γ over the algebra $\mathcal{O}(V)$ and the Weyl algebra $A_n(\mathbb{C})$.

Clearly, $A_{t,\mathbf{c}}$ is a filtered \mathbb{C} -algebra: namely, set

$$F_0 = \mathbb{C}\Gamma; \quad F_1 = \mathbb{C}\Gamma + \mathbb{C}tV; \quad \text{and} \quad F_i = (F_1)^i, \quad \text{for } i \geq 1. \quad (15)$$

We can form the associated graded ring $\text{gr}(A_{t,\mathbf{c}})$ of $A_{t,\mathbf{c}}$. There is an obvious epimorphism of algebras

$$\rho : \mathcal{O}(V) * \Gamma \rightarrow \text{gr}(A_{t,\mathbf{c}}).$$

Etingof and Ginzburg [8, Theorem 1.3] prove the beautiful result that

$$\rho \text{ is an isomorphism.} \quad (16)$$

This is *the PBW theorem for symplectic reflection algebras*. Notice that we can immediately conclude from (16) that $A_{t,\mathbf{c}}$ is a deformation of $\mathcal{O}(V) * \Gamma$, the family $A_{t,\mathbf{c}}$ over $\mathbb{P}^r(\mathbb{C})$ is flat, and (using standard filtered-graded techniques) deduce that $A_{t,\mathbf{c}}$ is a prime noetherian \mathbb{C} -algebra with good homological properties.

7.2. The centre of $A_{t,\mathbf{c}}$. Let $e = |\Gamma|^{-1} \sum_{\gamma \in \Gamma} \gamma \in \mathbb{C}\Gamma \subseteq A_{t,\mathbf{c}}$, so e is the familiar averaging idempotent. The filtration (15) induces by intersection a filtration of $eA_{t,\mathbf{c}}e$, whose associated graded ring is clearly $e\mathcal{O}(V) * \Gamma e$. It's easy to see that the latter algebra is isomorphic to the invariant ring $\mathcal{O}(V)^\Gamma$, which is the centre of $\mathcal{O}(V) * \Gamma$. In particular, $eA_{t,\mathbf{c}}e$ is a (not necessarily commutative) noetherian integral domain. Let $Z_{t,\mathbf{c}}$ be the centre of $A_{t,\mathbf{c}}$. One proves by a straightforward adaptation of the proof of [8, Theorem 3.1] that $Z_{t,\mathbf{c}}$ is isomorphic to the centre of the integral domain $eA_{t,\mathbf{c}}e$ via the map $z \mapsto ez$.

Proposition. *Retain the above notation.*

- (1) *If $t = 0$ then $Z_{0,\mathbf{c}} \cong eA_{0,\mathbf{c}}e$ and $A_{0,\mathbf{c}}$ is a finitely generated $Z_{0,\mathbf{c}}$ -module for all $\mathbf{c} \in \mathbb{A}^r$.*
- (2) *If $t \neq 0$ then $Z_{t,\mathbf{c}} = \mathbb{C}$ for all $\mathbf{c} \in \mathbb{A}^r$.*

Proof. (1) This is [8, Theorem 3.1].

(2) Assume that $t \neq 0$. We work in the algebra $eA_{t,\mathbf{c}}e$. As explained above, this algebra has a filtration $F^0 \subseteq F^1 \subseteq F^2 \subseteq \dots$, whose associated graded ring equals $eA_{0,0}e \cong \mathcal{O}(V)^\Gamma$. By [8, Claim 2.25], $eA_{t,\mathbf{c}}e$ is noncommutative: more precisely, given $u \in F^m$ and $v \in F^n$, the commutator $[u, v]$ is contained in F^{m+n-2} , and moreover there exist m, n, u and v such that $[u, v] \notin F^{m+n-3}$. Thus, following (1) in (2.3) with $m + n - 2$ replacing $m + n - 1$, we have a non-trivial Poisson bracket of degree -2 on $\mathcal{O}(V)^\Gamma$, say $B(-, -)$. Since $\Gamma \subseteq \text{Sp}(V)$, the Poisson bracket on $\mathcal{O}(V)$ defined by the symplectic form on V restricts to $\mathcal{O}(V)^\Gamma$ – that is, $\{\mathcal{O}(V)^\Gamma, \mathcal{O}(V)^\Gamma\} \subseteq \mathcal{O}(V)^\Gamma$. It's a consequence of Hartog's theorem that, up to scalar multiplication, this is the unique bracket of degree -2 , [8, Lemma 2.23]. We thus have

$$B(-, -) = \lambda\{-, -\}$$

for some $\lambda \neq 0$. The principal symbol of any element z in the centre of $eA_{t,\mathbf{c}}e$ belongs to $\text{Cas}(\mathcal{O}(V)^\Gamma)$, the algebra of Casimirs on $\mathcal{O}(V)^\Gamma$, since

$$\{\sigma_m(z), \sigma_n(u)\} = \lambda^{-1}B(\sigma_m(z), \sigma_n(u)) = \lambda^{-1}([z, u] + F^{m+n-3}) = 0 + F^{m+n-3}.$$

Thus it suffices to prove that $\text{Cas}(\mathcal{O}(V)^\Gamma) = \mathbb{C}$, since the scalars are the only elements of $eA_{t,\mathbf{c}}e$ for which the principal symbol lies in \mathbb{C} .

Since V is a symplectic vector space, $\text{Cas}(\mathcal{O}(V)) = \mathbb{C}$. Let $p \in \text{Cas}(\mathcal{O}(V)^\Gamma)$ and let $u \in \mathcal{O}(V)$. Since $\mathcal{O}(V)$ is integral over $\mathcal{O}(V)^\Gamma$ there exists a polynomial of minimal degree $\sum_{i=0}^n a_i X^i$ with $a_n = 1$ and $a_i \in \mathcal{O}(V)^\Gamma$ such that $\sum a_i u^i = 0$. Then

$$0 = \{p, \sum_{i=0}^n a_i u^i\} = \sum_{i=0}^n (\{p, a_i\} u^i + a_i \{p, u^i\}) = \left(\sum_{i=0}^n i a_i u^{i-1} \right) \{p, u\}.$$

By minimality $\sum i a_i u^{i-1} \neq 0$, so $\{p, u\} = 0$. Thus $p \in \text{Cas}(\mathcal{O}(V)) = \mathbb{C}$ as required. \square

7.3. Poisson structure. Henceforth we will concentrate on the case $t = 0$. The results in this paragraph can be found in [8]. We let $A_{\mathbf{c}}$, respectively $Z_{\mathbf{c}}$, denote the algebra $A_{0,\mathbf{c}}$, respectively $Z_{0,\mathbf{c}}$, for any $\mathbf{c} \in \mathbb{A}^r$. The family $Z_{\mathbf{c}}$ for $\mathbf{c} \in \mathbb{A}^r$, is flat, and, by (14), there is a \mathbb{C}^* -action by algebra automorphisms on the family lifting the natural action on \mathbb{A}^r .

As we've already explained, in the degenerate case $\mathbf{c} = \mathbf{0} \in \mathbb{A}^r$, the algebra $A_{\mathbf{0}} \cong \mathcal{O}(V) * \Gamma$ and $Z_{\mathbf{0}} \cong \mathcal{O}(V)^\Gamma$, the skew group ring and fixed point ring respectively. Thus, in view of Proposition 7.2 and the discussion in the first paragraph of (7.2) the variety $\text{Maxspec}(Z_{\mathbf{c}})$ is a deformation of the quotient variety V/Γ .

Fix $\mathbf{c} \in \mathbb{A}^r$. The algebra $A_{\mathbf{c}}$ can be lifted to a \mathbb{C} -algebra $\hat{A}_{\mathbf{c}}$ as described in (2.2) – to be precise, define $\hat{A}_{\mathbf{c}}$ to be the algebra $A_{t,\mathbf{c}}$, but with t an indeterminate rather than a complex number, so that $\hat{A}_{\mathbf{c}}$ is a $\mathbb{C}[t]$ -algebra with $\hat{A}_{\mathbf{c}}/t\hat{A}_{\mathbf{c}} \cong A_{\mathbf{c}}$. So by (2.2) $Z_{\mathbf{c}}$ admits a structure of Poisson algebra in such a way that the pair $Z_{\mathbf{c}} \subseteq A_{\mathbf{c}}$ satisfies the conditions of (2.1) – that is, $A_{\mathbf{c}}$ is a Poisson $Z_{\mathbf{c}}$ -order. Recall the filtration (15) on $A_{\mathbf{c}}$ with induced filtration on $Z_{\mathbf{c}}$, with associated graded rings $\mathcal{O}(V) * \Gamma$ and $\mathcal{O}(V)^\Gamma$ respectively.

There exists a $\mathbb{C}[t]$ -algebra $\hat{Z}_{\mathbf{c}}$ such that the flat family $Z_{\lambda\mathbf{c}}$ ($\lambda \in \mathbb{C}$) is realised by specialisation, that is $Z_{\lambda\mathbf{c}} \cong \hat{Z}_{\mathbf{c}} \otimes_{\mathbb{C}[t]} \mathbb{C}_\lambda$. There is, moreover, a Poisson structure on $\hat{Z}_{\mathbf{c}}$ which is compatible with specialisation and a \mathbb{C}^* -action on $\hat{Z}_{\mathbf{c}}$ (by \mathbb{C} -algebra automorphisms) lifting the action on the $Z_{\lambda\mathbf{c}}$'s.

7.4. Symplectic leaves of V/Γ . The restriction of the Poisson bracket on $\mathcal{O}(V)$ to $\mathcal{O}(V)^\Gamma$ agrees with the Poisson bracket on the Poisson $Z_{\mathbf{0}}$ -order described in the previous paragraph. We shall determine the symplectic leaves of V/Γ . In particular, we'll see that they are finite in number, so that Proposition 3.7 applies.

Given $v \in V$ let $\Gamma_v = \{\gamma \in \Gamma : \gamma v = v\}$, the stabiliser of v , and given $H \leq \Gamma$ let $V_H^o = \{v \in V : H = \Gamma_v\}$, and $V_H = \{v \in V : H \subseteq \Gamma_v\}$. Let $I(H) = \{x^h - x : x \in \mathcal{O}(V), h \in H\}$, an ideal of $\mathcal{O}(V)$, and set

$$J(H) = I(H) \cap \mathcal{O}(V)^\Gamma = \bigcap_{\gamma \in \Gamma} I(H^\gamma) \cap \mathcal{O}(V)^\Gamma,$$

an ideal of $\mathcal{O}(V)^\Gamma$. Clearly V_H is a closed subset of V with $I(V_H) = I(H)$, and V_H^o is open in V_H , being the complement of the closed subset of points with stabiliser strictly containing H . Letting H vary over subgroups of Γ thus gives a stratification of V by locally closed subsets,

$$V = \bigsqcup_{H \leq \Gamma} V_H^o.$$

Let $\pi : V \longrightarrow V/\Gamma$ be the orbit map, and for $H \leq \Gamma$ set $\mathcal{Z}_H^o = \pi(V_H^o)$, a locally closed subset of V/Γ which depends only on the conjugacy class of H in Γ . So there is a stratification of V/Γ by the locally closed sets \mathcal{Z}_H^o ,

$$V/\Gamma = \bigsqcup_{H \leq \Gamma} \mathcal{Z}_H^o, \quad (17)$$

and

$$\mathcal{Z}_H := \overline{\mathcal{Z}_H^o} = \pi(V_H),$$

with $J(H)$ being the defining ideal of \mathcal{Z}_H .

Proposition. *The symplectic leaves of V/Γ are precisely the sets \mathcal{Z}_H^o as H runs through the conjugacy classes of subgroups of Γ for which $V_H^o \neq \emptyset$. The various leaves \mathcal{Z}_H^o coincide with the smooth points of the irreducible components of the rank stratification of V/Γ .*

Proof. To show $J(H)$, and hence \mathcal{Z}_H , is Poisson closed take $x, x' \in \mathcal{O}(V)$, $y \in \mathcal{O}(V)^\Gamma$ and $h \in H$. Then, since the Poisson bracket is induced from the symplectic form on V ,

$$\begin{aligned} \{(x^h - x)x', y\} &= \{x^h - x, y\}x' + (x^h - x)\{x', y\} \\ &= (\{x, y\}^h - \{x, y\})x' + (x^h - x)\{x', y\}, \end{aligned}$$

proving that $I(H)$, and therefore $J(H)$, is stable under the Poisson action of $\mathcal{O}(V)^\Gamma$. It follows from this and Proposition 3.6.1 that the stratification of V/Γ by symplectic leaves is a refinement of (17).

Taking the union over subgroups conjugate to H yields

$$\pi^{-1}(\mathcal{Z}_H^o) = \bigsqcup_{\gamma \in \Gamma} V_{H\gamma}^o \quad (18)$$

and the restriction of π to this space is a covering map whose fibres have $[\Gamma : H]$ elements. Since V_H^o is an open subset of the vector space V_H , it follows that

\mathcal{Z}_H^o is smooth.

Moreover the restriction of the symplectic form to V_H is non-degenerate. Indeed, suppose $x \in V_H$ is in the radical. For all $y \in V$, since $\Gamma \leq \text{Sp}(V)$,

$$(x, y) = |H|^{-1} \sum_{h \in H} (h.x, h.y) = (x, |H|^{-1} \sum_{h \in H} h.y) = 0$$

ensuring $x = 0$ as required. As a result, under the covering (18), the form passes to the Poisson form on \mathcal{Z}_H^o , proving non-degeneracy of the restriction of the form to \mathcal{Z}_H^o . Thus each point of \mathcal{Z}_H^o has rank equal to $\dim(\mathcal{Z}_H^o)$ and so each subset \mathcal{Z}_H^o is a symplectic leaf of V/Γ .

In particular there are only finitely many leaves in V/Γ , so that the second sentence of the proposition follows from Proposition 3.7.2. \square

7.5. It follows from the above the description and Lemma 3.4 that the Poisson prime ideals of $\mathcal{O}(V)^\Gamma$ are precisely the ideals $J(H)$, where H runs through representatives of conjugacy classes of stabilisers of points in V . That this should be so was speculated in [1, Section 5].

7.6. Finite dimensional algebras. Given $v \in V$, recall that Γ_v is its stabiliser, and let $\mathfrak{m}_{\pi(v)}$ be the maximal ideal of $\mathcal{O}(V)^\Gamma$ corresponding to $\pi(v)$. Recall the finite dimensional algebra of (4.2),

$$(\mathcal{O}(V) * \Gamma)_{\pi(v)} = \frac{\mathcal{O}(V) * \Gamma}{\mathfrak{m}_{\pi(v)} \mathcal{O}(V) * \Gamma}.$$

Let $V = V_{\Gamma_v} \oplus V'$ be a Γ_v -equivariant vector space decomposition of V . It can be shown that Γ_v is generated by symplectic reflections on V' and that there is an algebra isomorphism

$$(\mathcal{O}(V) * \Gamma)_{\pi(v)} \cong \text{Mat}_{[\Gamma: \Gamma_v]}((\mathcal{O}(V') * \Gamma_v)_{\pi(0)}). \quad (19)$$

In particular, this explicitly demonstrates that the representation theory of $\mathcal{O}(V) * \Gamma$ is constant along the symplectic leaves, as follows from Theorem 4.2.

Remark. It can be shown quite generally that given an arbitrary vector space V and finite group $\Gamma \leq GL(V)$, an exact analogue of (19) holds.

7.7. Let W be a finite Weyl group. If \mathfrak{h} denotes the reflection representation of W , there is an induced action of W by symplectic reflections on $\mathfrak{h} \oplus \mathfrak{h}^*$. It was suggested in [1, Introduction] that the number of Poisson prime ideals of $\mathcal{O}(\mathfrak{h} \oplus \mathfrak{h}^*)^W$ with height $2k$ should equal a_k , the number of conjugacy classes of W having 1 as an eigenvalue with multiplicity k in the space \mathfrak{h} .

Proposition. *Retain the above notation. The number of Poisson prime ideals in $\mathcal{O}(\mathfrak{h} \oplus \mathfrak{h}^*)^W$ of height $2k$ equals the number of conjugacy classes of parabolic subgroups of W of rank k . In particular, this agrees with a_k if and only if all irreducible factors of W are of type A .*

Proof. Without loss of generality we may assume that W is irreducible, and identify \mathfrak{h}^* with \mathfrak{h} via the Killing form. Let H be the stabiliser in W of a point $(x, y) \in \mathfrak{h} \oplus \mathfrak{h}$. It is well-known that H is a parabolic subgroup. Indeed, by definition H is the intersection the parabolic subgroups W_x and W_y , [14, Theorem 1.12]. To prove the claim we may assume that x and y are linearly independent in \mathfrak{h} . We will show that H is the stabiliser of $\lambda x + \mu y \in \mathfrak{h}$ for generic values of $(\lambda : \mu) \in \mathbb{P}^1$. Let $w \in W$, and write $wx = \alpha x + \beta y + z$, where x, y and z are linearly independent. Then w stabilises $\lambda x + \mu y \in \mathfrak{h}$ for a generic value of $(\lambda : \mu) \in \mathbb{P}^1$ if and only if

$$wy = \frac{1}{\mu}(\lambda(1 - \alpha)x + (\mu - \lambda\beta)y - \lambda z).$$

By genericity we must have $\alpha = 1$, $\beta = 0$ and $z = 0$, proving that w stabilises x and hence y , as required.

Given a parabolic subgroup, H , the height of the corresponding ideal $J(H) \subseteq \mathcal{O}(V)^\Gamma$ is the codimension of the vector space

$$(\mathfrak{h} \oplus \mathfrak{h}^*)_H = \{(x, y) : H \leq W_x \text{ and } H \leq W_y\} = \mathfrak{h}_H \oplus \mathfrak{h}_H^*.$$

By [14, 1.15] $\dim \mathfrak{h}_H = \dim \mathfrak{h} - \text{rank}(H)$, so the first claim of the proposition follows.

Note that there is a well-defined injective mapping, say θ , from conjugacy classes of parabolic subgroups of W to conjugacy classes of W , sending a parabolic subgroup H to the product of its generating reflections, that is to a Coxeter element of H , [9, Proposition 3.1.15]. Let s be the rank function on W , that is the function which assigns to an element w , the codimension of $\{x \in \mathfrak{h} : w x = x\}$ in \mathfrak{h} . In other words, $s(w)$ is the number of eigenvalues of w on \mathfrak{h} , not equal to 1. By [14, Theorem 1.12(d)], $\dim \mathfrak{h}_H = \dim \mathfrak{h} - s(\theta(H))$. Therefore, there is an one-to-one

association from Poisson prime ideals of $\mathcal{O}(\mathfrak{h} \oplus \mathfrak{h}^*)^W$ to conjugacy classes of elements in W , sending primes of height $2k$ to elements of rank k .

The second claim of the proposition follows from [9, Theorem 3.2.12, Section 3.4 and Appendix B]: conjugacy classes of parabolic subgroups provide a first approximation to conjugacy classes of elements in W via their Coxeter elements – this approximation is exact if and only if W is of type A . \square

7.8. We now show that for any $\mathbf{c} \in \mathbb{A}^r$ an analogue of Proposition 7.4 holds for $\mathcal{Z}_{\mathbf{c}} = \text{Maxspec}(Z_{\mathbf{c}})$.

Theorem. *For any $\mathbf{c} \in \mathbb{A}^r$ the symplectic leaves of $\mathcal{Z}_{\mathbf{c}}$ are precisely the smooth points of the irreducible components of the rank stratification. In particular they are algebraic and finite in number.*

Proof. Let $\mathcal{X} \subseteq \mathcal{Z}_{\mathbf{c}}$ be a closed subvariety and define a closed subvariety of $\hat{\mathcal{Z}}_{\mathbf{c}}$ by $\hat{\mathcal{X}} = \overline{\mathbb{C}^* \mathcal{X}}$, the Zariski closure of $\mathbb{C}^* \mathcal{X}$. Taking the intersection of $\hat{\mathcal{X}}$ with \mathcal{Z}_0 gives a closed subvariety of \mathcal{Z}_0 which we denote $\text{gr}\mathcal{X}$. On the level of ideals, this construction sends a radical ideal I of $Z_{\mathbf{c}}$ to the ideal $\sqrt{\text{gr}I}$ of Z_0 . Therefore $\dim \mathcal{X} = \dim \text{gr}\mathcal{X}$ and if \mathcal{X} is Poisson closed, then so too is $\text{gr}\mathcal{X}$.

Let $r \in \mathbb{N}$ and suppose that some point $\mathfrak{m} \in \mathcal{Z}_{\mathbf{c}}$ has rank r . Let \mathcal{U} be the subvariety of $\mathcal{Z}_{\mathbf{c}}$ consisting of the points whose rank is no more than r , and let \mathcal{X} be an irreducible component of \mathcal{U} . By Lemma 3.1.5, $\dim \mathcal{X} \geq r$. Now suppose that $\mathfrak{n} \in \text{gr}\mathcal{X}$ has $\text{rank}(\mathfrak{n}) = s > r$. Since $\hat{\mathcal{X}}$ is Poisson closed, there is a non-empty open set of points of $\hat{\mathcal{X}}$ whose rank is at least s . But, thanks to the \mathbb{C}^* -action, $\hat{\mathcal{X}}$ is irreducible since \mathcal{X} is and has a non-empty open subset consisting of points of rank r since \mathcal{X} does. This contradiction shows that all points of $\text{gr}\mathcal{X}$ have rank at most r . Therefore, since $\text{gr}\mathcal{X}$ is Poisson closed $\dim \text{gr}\mathcal{X} \leq r$ by Proposition 7.4. Thus

$$r \geq \dim \text{gr}\mathcal{X} = \dim \mathcal{X} \geq r.$$

Thus the set \mathcal{X}^o of points of \mathcal{X} of rank r is non-empty, open, irreducible and of dimension r ; that is, it is a symplectic manifold and hence is a leaf. The other claims now follow immediately from Proposition 3.7. \square

7.9. **Remarks. 1.** Lemma 7.8 shows that for any $\mathbf{c} \in \mathbb{A}^r$ the pairs $Z_{\mathbf{c}} \subseteq A_{\mathbf{c}}$ satisfy the hypotheses of Sections 4 and 5. In general it is an interesting problem to describe the associated families of finite dimensional algebras and the Azumaya stratifications. Such a description for all \mathbf{c} should among other consequences give definitive information on the existence of a symplectic resolution of V/Γ .

2. In the case $t \neq 0$ where $A_{t,\mathbf{c}}$ does *not* satisfy a polynomial identity the most obvious open problem is to describe the (two-sided) ideals of $A_{t,\mathbf{c}}$. The mechanism of the proof of Proposition 7.2.2 allows us to induce the Poisson bracket on $\mathcal{O}(V)^\Gamma$ by exploiting the fact that $\mathcal{O}(V) * \Gamma$ is the associated graded algebra of $A_{t,\mathbf{c}}$. It therefore follows easily that if I is an ideal of $A_{t,\mathbf{c}}$ then $\text{gr}I$ is a Poisson ideal of $A_{0,0}$, so that some information about I can be read off from Proposition 7.4. But this is unsatisfactory - one would prefer to associate I with an ideal of $A_{0,\mathbf{c}}$, and hence to make use of Proposition 7.8. Whether this can be done, we leave as an open question.

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