

GELFAND-KIRILLOV CONJECTURE FOR SYMPLECTIC REFLECTION ALGEBRAS

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ABSTRACT. We construct functorially a class of algebras using the formalism of double derivations. These algebras extend to higher dimensions Crawley-Boevey and Holland’s construction of deformed preprojective algebras and encompass symplectic reflection algebras associated to wreath products. We use this construction to show that the quotient field of a symplectic reflection algebra is “rational”, confirming a pair of conjectures of Etingof and Ginzburg.

1. INTRODUCTION

1.1. Let Γ be a finite subgroup of $SL(2, \mathbb{C})$. For a positive integer n set Γ_n to be the wreath product $\Gamma \wr \mathfrak{S}_n$; it acts naturally on the vector space $V = (\mathbb{C}^2)^n$. The orbit space V/Γ_n is a singular symplectic variety which admits a symplectic resolution thanks to [W, Section 4.4]. Moreover, it can be shown that the resolution contains a Zariski open set isomorphic to $T^*(\mathbb{C}^n)$ and so it follows immediately that V/Γ_n is “Poisson rational”. This means that there is an algebra isomorphism between the quotient field of functions on V/Γ_n , $\mathbb{C}(V)^{\Gamma_n}$, and the quotient field of functions on \mathbb{A}^{2n} , $\mathbb{C}(x_1, \dots, x_n, y_1, \dots, y_n)$, which intertwines the Poisson structure on V/Γ_n with the standard Poisson structure on $2n$ -dimensional affine space.

1.2. In this paper we provide a noncommutative analogue of the above observation. The (spherical subalgebras of) symplectic reflection algebras produce a flat family of deformations of the ring of functions of V/Γ_n . They are denoted by $H_{t,k,c}(\Gamma_n)$ where $(t, k, c) \in \mathbb{C} \times \mathbb{C}[\Gamma]^{\text{class}}$ denote the deformation parameters, and spherical subalgebras by $eH_{t,k,c}$. We prove the following theorem, confirming [EG, Conjectures 17.6 and 17.7].

Theorem. *Let D_t denote the quotient division ring of $\mathbb{C}\langle X_1, \dots, X_n, Y_1, \dots, Y_n : [X_i, X_j] = [Y_i, Y_j] = 0, [Y_i, X_j] = \delta_{ijt} \rangle$. Then $\text{Quot}(H_{t,k,c}) \cong D_t \rtimes \Gamma_n$ and $\text{Quot}(eH_{t,k,c}) \cong D_t^{\Gamma_n}$, where Quot stands for simple artinian quotient ring.*

It is quite straightforward to show that $D_t^{\Gamma_n} \cong D_t$ and so the above result can be considered as a noncommutative rationality result, in other words as a confirmation of an analogue to the Gelfand-Kirillov conjecture for Lie algebras.

1.3. As pointed out in [EG, Theorem 17.7*] the above result already held true for symplectic reflection algebras associated to complex reflection groups – the so-called

rational Cherednik algebras – thanks to the Dunkl embedding. Thus it makes sense for us to restrict to Γ_n , the other family of groups generated by symplectic reflections.

1.4. Our method of proof is inspired by Crawley-Boevey’s proof in the case $n = 1$, [CB]. Associated to any algebra we define a family of algebras $\Pi_{n,x,\nu}(A)$, presented by generators and relations, and based on the structure of double derivations of A . When $A = kQ$, the path algebra of a quiver, the algebra produces the one parameter deformation of the tensor product of deformed preprojective algebras studied by Gan and Ginzburg, [GG]. For affine Dynkin quivers [loc.cit.] shows that the McKay correspondence ensures these deformations are Morita equivalent to symplectic reflection algebras, whilst the functoriality of our approach allows the comparison of different quivers, following the ideas of Schofield and Crawley-Boevey used in [CB].

We remark that the algebras $\Pi_{n,x,\nu}(A)$ should also be related to Etingof’s one parameter deformations of rational Cherednik algebras associated to the action of \mathfrak{S}_n on the n -fold product of a curve, [E], and give a concise description of these algebras by generators and relations.

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2. THE DEFINITION

2.1. Throughout k denotes an algebraically closed field and $K = \bigoplus_{v \in V} ke_v$ is a commutative semisimple artinian ring where V is some labeling set and e_v are idempotents. All unadorned tensor products are taken over k . Throughout we let $\mathbf{e} = \sum_{v \in V} e_v \otimes e_v \in K \otimes K$. By a K -algebra A we will mean that there exists an embedding $\sigma : K \rightarrow A$ such that $\sigma(k \cdot 1)$ is central.

2.2. Double algebra. Let A be a K -algebra. Let n be a positive integer greater than or equal to 2. There is the *outer* A -bimodule structure on $A^{\otimes n}$ given by

$$b(a_1 \otimes \cdots \otimes a_n)c = ba_1 \otimes \cdots \otimes a_nc.$$

We will also often use the *inner* A -bimodule structure on $A^{\otimes 2}$ given by

$$b * (a_1 \otimes a_2) * c = a_1c \otimes ba_2.$$

We will use the standard notation $a' \otimes a''$ to indicate a general element of $A \otimes A$.

2.3. We let $\mathbb{D}_K(A)$ be the space of double K -derivations with respect to the outer bimodule structure on $A^{\otimes 2}$, that is

$$\mathbb{D}_K(A) = \text{Der}_K(A, A^{\otimes 2}),$$

where $d(ab) = ad(b) + d(a)b$. Then $\mathbb{D}_K(A)$ is an A -bimodule thanks to the surviving inner bimodule action on $A^{\otimes 2}$. If $\Omega_{A/K}^1$ denotes the space of K -differentials of A , then there is an A -bimodule isomorphism

$$\Psi : \mathbb{D}_K(A) \longrightarrow \text{Hom}_{A \otimes A}(\Omega_{A/K}^1, A^{\otimes 2})$$

whose inverse is given by sending f to $\psi^{-1}(f)(a) = f(a \otimes_K 1 - 1 \otimes_K a)$. When there is no chance of ambiguity, we will abbreviate $\mathbb{D}_K(A)$ by \mathbb{D} .

Given two elements $\alpha, \beta \in \mathbb{D}$ we follow [vdB, Section 3.2], defining

$$\{\{\alpha, \beta\}\}_l = \tau_{(23)} \circ ((\alpha \otimes 1)\beta - (1 \otimes \beta)\alpha) \in \mathbb{D} \otimes A,$$

and

$$\{\{\alpha, \beta\}\}_r = \tau_{(12)} \circ ((1 \otimes \alpha)\beta - (\beta \otimes 1)\alpha) \in A \otimes \mathbb{D}.$$

These equations allow us to define a bracket

$$\{\{\alpha, \beta\}\} = \{\{\alpha, \beta\}\}_l + \{\{\alpha, \beta\}\}_r$$

which takes values in $\mathbb{D} \otimes A \oplus A \otimes \mathbb{D}$.

Following [CBEG, 3.1] there is a canonical element $\Delta \in \mathbb{D}$ given by $\Delta(a) = \mathbf{a}e - \mathbf{e}a$ for all $a \in A$.

2.4. The algebra. Given $x \in K, \nu \in k$ and n a positive integer greater than 1, define $\Pi_{n,x,\nu}(A)$ to be the following algebra. We let $T(A, n) = T_A \mathbb{D} * \cdots * T_A \mathbb{D}$ denote the free product over k of n copies of the tensor algebra $T_A \mathbb{D}$. The symmetric group \mathfrak{S}_n permutes the different copies of $T_A \mathbb{D}$ and so acts by algebra automorphisms on $T(A, n)$. Then $\Pi_{n,x,\nu}(A)$ is the quotient of $T(A, n) \rtimes \mathfrak{S}_n$ by the following relations:

$$(\Delta - x)_i = \nu \sum_{j \neq i} \mathbf{e}_{ij}(ij) \tag{1}$$

$$[a_i, b_j] = 0 \text{ if } a, b \in A \text{ and } i \neq j \tag{2}$$

$$[\alpha_i, \beta_j] = \nu \{\{\beta, \alpha\}\}_{ij}(ij) \text{ if } \alpha, \beta \in \mathbb{D} \text{ and } i \neq j \tag{3}$$

$$[a_i, \alpha_j] = \nu \alpha(a)_{ij}(ij) \text{ if } a \in A, \alpha \in \mathbb{D} \text{ and } i \neq j. \tag{4}$$

For $z \in A$ or $z \in D$ we have used the notation

$$z_i = 1 * \cdots * 1 * z * 1 \cdots * 1$$

where z appears in the i th position, and similarly for z_{ij} if $z \in A \otimes A$ or $z \in \mathbb{D} \otimes A \oplus A \otimes \mathbb{D}$.

Lemma. *The defining relations (2)–(4) are generated by $\alpha, \beta \in \mathbb{D}$ and $a, b \in A$ coming from an A -bimodule generating set and a k -algebra generating set respectively.*

Proof. This is a simple calculation, but we give the details for completeness.

It is obvious for (2) so we begin with (3). By [vdB, (3.5)] we have $\{\{\beta, a\alpha\}\} = a\{\{\beta, \alpha\}\} + \beta(a)\alpha$. It follows that

$$\begin{aligned}
[(a\alpha)_i, \beta_j] &= a_i[\alpha_i, \beta_j] + [a_i, \beta_j]\alpha_i \\
&= \nu a_i\{\{\beta, \alpha\}\}_{ij}(i j) + \nu\beta(a)_{ij}(i j)\alpha_i \\
&= \nu\left(a_i\{\{\beta, \alpha\}\}_{ij} + \beta(a)_{ij}\alpha_j\right)(i j) \\
&= \nu\{\{\beta, a\alpha\}\}_{ij}(i j),
\end{aligned}$$

as required. An analogous calculation using [vdB, (3.6)] deals with $[(\alpha a)_i, \beta_j]$.

We turn to (4). Let $a = bc \in A$. Then

$$\begin{aligned}
[(bc)_i, \alpha_j] &= b_i[c_i, \alpha_j] + [b_i, \alpha_j]c_i \\
&= b_i\nu\alpha(c)_{ij}(i j) + \nu\alpha(b)_{ij}(i j)c_i \\
&= \nu(b_i\alpha(c)_{ij} + \alpha(b)_{ij}c_j)(i j) \\
&= \nu\alpha(bc)_{ij}(i j),
\end{aligned}$$

as required. Similarly if $\alpha = b\beta c$ then we find

$$\begin{aligned}
[a_i, (b\beta c)_j] &= b_j[a_i, \beta_j]c_j \\
&= \nu b_j\beta(a)_{ij}(i j)c_j \\
&= \nu b_j\beta(a)_{ij}c_i(i j),
\end{aligned}$$

where the first equality holds since a_i commutes with both b_j and c_j because $i \neq j$. Since the A -bimodule structure on \mathbb{D} comes from the inner structure on $A^{\otimes 2}$, we have $b_j\beta(a)_{ij}c_i = (b\beta c)(a)_{ij}$, as required. \square

It is a straightforward calculation (left to the reader) to check that the defining relations (1) and (2) imply the relations (3) and (4) for any $\alpha \in A \cdot \Delta \cdot A$.

2.5. The following result will be crucial later.

Lemma. *Up to isomorphism $\Pi_{n,x,\nu}(A)$ depends only on the image of x in $H_0(A) = A/[A, A]$.*

Proof. We show that we get an isomorphic algebra by replacing x by $x - [y, z]$ for any $y, z \in A$ such that $[y, z] \in \sigma(K)$. To do this we follow [CB, Lemma 1.2] and replace each derivation α by $\theta(\alpha) \stackrel{\text{def}}{=} \alpha + \alpha(y)''z\alpha(y)'$ where $\alpha(y) = \sum \alpha(y)' \otimes \alpha(y)''$. Since $\Delta(y)''z\Delta(y)' = \sum_{v \in V} e_v[z, y]e_v = -[y, z]$ this has the effect of replacing the x in relation (1) with $x - [y, z]$. Clearly (2) is unchanged. We now check that (3) and (4) still hold.

We have

$$\begin{aligned}
& [(\alpha + \alpha(y)''z\alpha(y)')_i, (\beta + \beta(y)''z\beta(y)')_j] - \nu(\{\beta, \alpha\} + \{\beta, \alpha\}(y)''z\{\beta, \alpha\}(y)')_{ij}(ij) \\
&= \nu(\alpha(y)''\beta(z)' \otimes \beta(z)''\alpha(y)' - \alpha(z)''\beta(y)' \otimes \beta(y)''\alpha(z)')_{ij}(ij). \\
&= \nu(\alpha(y)''(\Psi(\beta)(z \otimes_K 1 - 1 \otimes_K z))\alpha(y)' - \alpha(z)''(\Psi(\beta)(y \otimes_K 1 - 1 \otimes_K y))\alpha(z)')_{ij}(ij) \\
&= \nu(\Psi(\beta)(\alpha(y)''(z \otimes_K 1 - 1 \otimes_K z)\alpha(y)' - \alpha(z)''(y \otimes_K 1 - 1 \otimes_K y)\alpha(z)'))_{ij}(ij) \\
&= \nu(\Psi(\beta)(\alpha(yz)'' \otimes_K \alpha(yz)' - \alpha(zy)'' \otimes_K \alpha(zy)'))_{ij}(ij) \\
&= \nu(\Psi(\beta)(\tau_{(12)} \circ \alpha([y, z])))_{ij}(ij) \\
&= 0,
\end{aligned}$$

where the last equality follows since $[y, z] \in K$. More easily,

$$[a_i, (\alpha + \alpha(y)''z\alpha(y)')_j] - \nu\alpha(a)_{ij}(ij) = [a_i, \alpha_j] - \nu\alpha(a)_{ij}(ij) = 0,$$

since if $i \neq j$ we have $[a_i, (\alpha(y)''z\alpha(y)')_j] = 0$. \square

2.6. Filtration. There is a filtration on $\Pi_{n,x,\nu}(A)$ induced by the grading on $T(A, n) \times \mathfrak{S}_n$. In this grading elements of \mathfrak{S}_n and of A have degree 0 and elements of \mathbb{D} have degree 1. There is an associated surjective homomorphism

$$\Pi_{n,0,0}(A) \longrightarrow \text{gr}(\Pi_{n,x,\nu}(A)). \quad (5)$$

In general this is not an isomorphism (but see 3.4).

3. EXAMPLE

3.1. Path algebras. Let Q be a quiver with vertex set V and let kQ be its path algebra. Set $K = \bigoplus_{v \in V} ke_v \subseteq kQ$. Given $x \in k^V$ we write $x = \sum_{v \in V} x_v e_v \in K$. Let \overline{Q} be the quiver obtained from Q by adjoining an arrow $a^* : v \rightarrow u$ for each arrow $a : u \rightarrow v$ in Q . Let $\nu \in k$ and take n to be a positive integer greater than 1.

3.2. The Gan-Ginzburg algebra. In [GG, Definition 1.2.3] an algebra $A_{n,x,\nu}$ is defined as follows. Let E be the vector space over k whose basis is given by the set of edges $\{a, a^* : a \in Q\}$. Set $\mathbf{B} := K^{\otimes n}$ and for any $1 \leq \ell \leq n$, define the \mathbf{B} -bimodules

$$\mathbf{E}_\ell := K^{\otimes(\ell-1)} \otimes E \otimes K^{\otimes(n-\ell)} \quad \text{and} \quad \mathbf{E} := \bigoplus_{1 \leq \ell \leq n} \mathbf{E}_\ell.$$

Given two elements $\varepsilon \in \mathbf{E}_\ell$ and $\varepsilon' \in \mathbf{E}_m$ of the form

$$\varepsilon = e_{v_1} \otimes e_{v_2} \otimes \cdots \otimes a \otimes \cdots \otimes e_{h(b)} \otimes \cdots \otimes e_{v_n}, \quad (6)$$

$$\varepsilon' = e_{v_1} \otimes e_{v_2} \otimes \cdots \otimes e_{t(a)} \otimes \cdots \otimes b \otimes \cdots \otimes e_{v_n}, \quad (7)$$

where $\ell \neq m$, $a, b \in \overline{Q}$ and $v_1, \dots, v_n \in V$, define

$$\begin{aligned}
[\varepsilon, \varepsilon'] &:= (e_{v_1} \otimes \cdots \otimes a \otimes \cdots \otimes e_{h(b)} \otimes \cdots \otimes e_{v_n})(e_{v_1} \otimes \cdots \otimes e_{t(a)} \otimes \cdots \otimes b \otimes \cdots \otimes e_{v_n}) \\
&\quad - (e_{v_1} \otimes \cdots \otimes e_{h(a)} \otimes \cdots \otimes b \otimes \cdots \otimes e_{v_n})(e_{v_1} \otimes \cdots \otimes a \otimes \cdots \otimes e_{t(b)} \otimes \cdots \otimes e_{v_n}).
\end{aligned}$$

Define the algebra $A_{n,\lambda,\nu}$ to be the quotient of $T_{\mathbb{B}}E \rtimes \mathfrak{S}_n$ by the following relations.

(i) For any $v_1, \dots, v_n \in V$ and $1 \leq \ell \leq n$:

$$e_{v_1} \otimes \cdots \otimes (r_{v_\ell} - \lambda_{v_\ell} e_{v_\ell}) \otimes \cdots \otimes e_{v_n} = \nu \sum_{\{j \neq \ell \mid v_j = v_\ell\}} (e_{v_1} \otimes \cdots \otimes e_{v_\ell} \otimes \cdots \otimes e_{v_n})(j \ell).$$

(ii) For any $\varepsilon, \varepsilon'$ of the form (6)–(7):

$$[\varepsilon, \varepsilon'] = \begin{cases} \nu(e_{v_1} \otimes \cdots \otimes e_{h(a)} \otimes \cdots \otimes e_{t(a)} \otimes \cdots \otimes e_{v_n})(\ell m) & \text{if } a = b^* \text{ and } b \in Q, \\ -\nu(e_{v_1} \otimes \cdots \otimes e_{h(a)} \otimes \cdots \otimes e_{t(a)} \otimes \cdots \otimes e_{v_n})(\ell m) & \text{if } b = a^* \text{ and } a \in Q, \\ 0 & \text{otherwise.} \end{cases}$$

3.3. The comparison.

Theorem. *There is an algebra isomorphism $\Pi_{n,x,\nu}(kQ) \cong A_{n,x,\nu}$.*

Proof. By [CB, Proof of Theorem 3.1] we have

$$\mathbb{D} = \text{Der}_K(kQ, kQ \otimes kQ) \cong \bigoplus_{a:u \rightarrow v} kQ e_u \otimes e_v kQ.$$

Let $\partial_a \in \mathbb{D}$ be a label for the generator $e_u \otimes e_v$ corresponding to $a : u \rightarrow v$. We have $\partial_a(e_v) = 0$ for all $v \in V$ and

$$\partial_a(b) = \begin{cases} -e_{h(a)} \otimes e_{t(a)} & \text{if } a = b \\ 0 & \text{otherwise.} \end{cases}$$

We define a mapping $\Phi : T(kQ, n) \rtimes \mathfrak{S}_n \longrightarrow A_{n,x,\nu}$ which is the identity on $(kQ)^{*n} \rtimes \mathfrak{S}_n$ and sends $(\partial_a)_i$ to a_i^* . The homomorphism Φ is clearly surjective and so we need to check that the kernel is generated by the relations (1)–(4).

Following [CB, Proof of Theorem 3.1] we see that the relation (1) for $1 \leq i \leq n$ becomes

$$\left(\sum_{a \in Q} (a \partial_a - \partial_a a) - \sum_{v \in V} x_v e_v \right)_i = \nu \sum_{j \neq i} \mathbf{e}_{ij}(i j). \quad (8)$$

If we fix $v \in V$ and multiply this identity by $(e_v)_i$ on the left and the right we find

$$\left(\sum_{\{a \in Q: h(a)=i\}} a \partial_a - \sum_{\{a \in Q: t(a)=i\}} \partial_a a - x_v e_v \right)_i = \nu \sum_{j \neq i} (e_v \otimes e_v)_{ij}(i j)$$

which corresponds to 3.2(i) under Φ . By contrast, if $v \neq w$ and we multiply by $(e_v)_i$ on the left and $(e_w)_i$ on the right, then both sides of (8) become zero.

The relation (2) just corresponds to the third case in 3.2(ii).

Now let $\alpha = \partial_a$ and $\beta = \partial_b$. From (3) we find $[(\partial_a)_i, (\partial_b)_j] = 0$ since $(\partial_a \otimes 1) \partial_b - (1 \otimes \partial_b) \partial_a = 0$: this is immediate from the definition of ∂_a and ∂_b above. From (4) we find for $a \in Q$

$$[a_i, (\partial_b)_j] = \nu \partial_b(a)_{ij}(i j) = -\nu \delta_{ab} (e_{h(a)} \otimes e_{t(a)})_{ij}(i j),$$

whilst $[(e_v)_i, (\partial_b)_j] = 0$ for any $v \in V$. Thus, under Φ , these relations correspond to the rest of 3.2(ii). \square

3.4. As a consequence of the above theorem, we find that if $A = kQ$ with Q affine Dynkin, then (5) is an isomorphism

$$\Pi_{n,0,0}(kQ) \xrightarrow{\sim} \text{gr}(\Pi_{n,x,\nu}(kQ)),$$

by [GG, Theorem 2.2.1].

3.5. Symplectic reflection algebras. We refer to [EG] and to [GG] for the definition of the symplectic reflection algebras $H_{t,k,c}(\Gamma_n)$.

Corollary ([GG]). *Suppose Q is the affine Dynkin quiver corresponding to the finite subgroup $\Gamma \leq SL(2, \mathbb{C})$. Then $\Pi_{n,x,\nu}(kQ)$ is Morita equivalent to a symplectic reflection algebra associated to $\Gamma_n = \Gamma \wr \mathfrak{S}_n$.*

More precisely, write $\mathbb{C}\Gamma = \bigoplus_{v \in V} \text{Mat}_{\delta_v}(\mathbb{C})$. If we let f_v be the idempotent e_{11}^v and then set $f = \sum_{v \in V} f_v \in \mathbb{C}\Gamma$, we produce an idempotent $f^{\otimes n} \in \mathbb{C}[\Gamma^n]$. By [GG, Theorem 3.5.2] and Theorem 3.3, an isomorphism for appropriate parameters $f^{\otimes n} H_{t,k,c}(\Gamma_n) f^{\otimes n} \cong \Pi_{n,x,\nu}(kQ)$ which induces the Morita equivalence above. In particular, if Γ is trivial then $H_{t,k,c}(\Gamma_n)$ and $\Pi_{n,x,\nu}(kQ)$ are isomorphic.

4. FUNCTORIALITY

4.1. Definitions. Recall that $A \rightarrow B$ is a *ring epimorphism* if any two ring homomorphisms $B \rightarrow C$ agreeing on the image of A are equal. Furthermore, recall that $A \rightarrow B$ is *pseudoflat* if $\text{Tor}_1^A(B, B) = 0$. We say the K -algebra A is *quasi-free* if $\Omega_{A/K}^1$ is a projective A -bimodule and *bimodule finite* if $\Omega_{A/K}^1$ is a finitely generated A -bimodule.

4.2. Suppose that $\theta : A \rightarrow B$ is a pseudoflat ring epimorphism and a K -algebra homomorphism and that A is a quasi-free bimodule finite K -algebra. Then by [CB, Lemma 9.1] the restriction mapping induces a bijection

$$\text{Der}_K(B, M) \xrightarrow{\sim} \text{Der}_K(A, M) \tag{9}$$

for any B -bimodule M . Moreover by [CB, Proof of Theorem 9.3], this induces a B -bimodule isomorphism

$$\tilde{\theta} : B \otimes_A \mathbb{D}_K(A) \otimes_A B = B \otimes_A \text{Der}_K(A, A^{\otimes 2}) \otimes_A B \rightarrow \text{Der}_K(B, B^{\otimes 2}) = \mathbb{D}_K(B) \tag{10}$$

such that $\tilde{\theta}(1 \otimes \Delta_A \otimes 1) = \Delta_B$.

Proposition. *The pseudoflat epimorphism $\theta : A \rightarrow B$ induces a pseudoflat epimorphism $\Theta_{n,x,\nu} : \Pi_{n,x,\nu}(A) \rightarrow \Pi_{n,\theta(x),\nu}(B)$, for $x \in K, \nu \in k$ and $n \geq 2$.*

Proof. We follow the methods of [CB, Section 9]. The homomorphisms θ and $\tilde{\theta}$ above extend to a homomorphism $T_A \mathbb{D}_K(A) \rightarrow T_B \mathbb{D}_K(B)$. This allows out to construct a homomorphism

$$\hat{\Theta} : T(A, n) \rtimes \mathfrak{S}_n \rightarrow T(B, n) \rtimes \mathfrak{S}_n.$$

It is clear that the relations (1)–(4) for $\Pi_{n,x,\nu}(A)$ are sent to the corresponding relations for $\Pi_{n,\theta(x),\nu}(B)$ under $\hat{\Theta}$ so we produce the asserted mapping $\Theta_{n,x,\nu}$.

Now we claim that there is a pushout diagram

$$\begin{array}{ccc} A^{*n} & \longrightarrow & \Pi_{n,x,\nu}(A) \\ \theta^{*n} \downarrow & & \downarrow \Theta_{n,x,\nu} \\ B^{*n} & \longrightarrow & \Pi_{n,\theta(x),\nu}(B). \end{array} \quad (11)$$

This is simply the assertion that the B^{*n} -bimodule generated by $\hat{\Theta}$ applied to the relations (1)–(4) defining $\Pi_{n,x,\nu}(A)$ gives the corresponding space of relations for $\Pi_{n,\theta(x),\nu}(B)$.

We have $\hat{\Theta}((\Delta_A - x)_i) = (\tilde{\theta}(\Delta_A) - \theta(x))_i = (\Delta_B - \theta(x))_i$, so (1) is straightforward. For (2) we will show that the ideal I of $T(B, n) \rtimes \mathfrak{S}_n$ generated by the relations of the form $[\theta(a)_i, \theta(a')_j]$ for $i \neq j$ and $a, a' \in A$ equals the ideal generated by the elements $[b_i, b'_j]$ for $b, b' \in B$. Let \bar{T} denote the quotient of $T(B, n) \rtimes \mathfrak{S}_n$ by I . Fix $a \in A$, and $i \neq j$. We have a mapping $a_{ij} : B \rightarrow \bar{T}$ sending b to $[a_i, b_j]$. This is a K -derivation for the B -bimodule structure on \bar{T} induced by the B -structure on the j th copy of $T_B \mathbb{D}_K(B)$. Since the restriction of c_{ij} to A is zero we deduce from (9) that a_{ij} is zero and thus that $[a_i, b_j] = 0$ in \bar{T} for all $a \in A, b \in B$ and $i \neq j$. Using this fact and repeating the above argument for the derivation b'_{ij} defined by sending b' to $[b'_i, b_j] \in \bar{T}$ we find that $[b'_i, b_j] = 0$ in \bar{T} for all $b, b' \in B$ and $i \neq j$, as required.

Now we move on to the relations (3) and (4). By Lemma 2.4, it is enough to check that we can obtain the relations on a generating set for $\mathbb{D}_K(B)$. For any $1 \leq i \leq n$, (10) ensures that we can find these generators for $\mathbb{D}_K(B)$ in the image of $\tilde{\theta}$. Since (3) holds for B if (2) holds (which we can now assume by the previous paragraph) and (3) holds for A and (4) holds for B , we are reduced to checking that (4) for A carries across to (4) for B . We argue as above, letting $\bar{\bar{T}}$ be the quotient of $T(B, n) \rtimes \mathfrak{S}_n$ by the relations generated by (4) for A . Fix a pair $i \neq j$ and consider $\bar{\bar{T}}$ as a B -bimodule via the i th copy of $T_B \mathbb{D}_K(B)$. Fix $\alpha \in \mathbb{D}_K(A)$. The mapping $\delta_{ij} : B \rightarrow \bar{\bar{T}}$ given by $\delta_{ij}(b) = [b_i, \alpha_j] - \nu\alpha(b)_{ij}(i j)$ defines a derivation since

$$\begin{aligned} \delta_{ij}(bb') &= [(bb')_i, \alpha_j] - \nu\alpha(bb')_{ij}(i j) \\ &= b_i([b'_i, \alpha_j] - \nu\alpha(b')_{ij}(i j)) + ([b_i, \alpha_j]b'_i - \nu\alpha(b)_{ij}b'_j(i j)) \\ &= b_i([b'_i, \alpha_j] - \nu\alpha(b')_{ij}(i j)) + ([b_i, \alpha_j] - \nu\alpha(b)_{ij}(i j))b'_i. \end{aligned}$$

The restriction of δ_{ij} to A is zero by the definition of $\bar{\bar{T}}$. Thus we have that $\delta_{ij} = 0$ by (9). We thus deduce that (4) holds in $\bar{\bar{T}}$ for any element of B and any $\alpha \in \mathbb{D}_A$.

It follows that the diagram (11) is indeed a pushout diagram. Since θ is a pseudoflat epimorphism so is θ^{*n} , and it follows from [BD, Proposition 5.2] that Θ is too. \square

5. MORITA EQUIVALENCE

We would like to understand how the algebras $\Pi_{n,x,\nu}(A)$ and $\Pi_{n,x',\nu'}(B)$ compare when A and B are Morita equivalent. Unfortunately, we have not managed to do this

in general, so we only present the very specific case here where $A = \text{Mat}_m(B)$ and $B = k[X]$.

So throughout this section set $S = \text{Mat}_m(k)$, $B = k[X]$, $A = S \otimes B$ and $K = \bigoplus_{v \in V} ke_v$ some diagonal subalgebra of S .

5.1. We begin by giving $S \otimes S$ the k -basis $f_{ab}^{pq} = e_{pb} \otimes e_{aq}$ for $1 \leq a, b, p, q \leq m$ and let $\rho : S \otimes S \rightarrow S \otimes_K S$ be the canonical surjective map. We set $\mathbf{1} = \sum_{a,p} f_{aa}^{pp} \in (S \otimes S)^{adS}$ and $\mathbf{1}_K = \rho(\sum_{a,p} f_{aa}^{pp}) \in (S \otimes_K S)^{adS}$.

Lemma. *There is an $S \otimes B$ -bimodule isomorphism*

$$\mathbb{D}_K(S \otimes B) \xrightarrow{\sim} \mathbb{D}_K(S) \otimes B^e \oplus S \otimes \mathbb{D}_k(B).$$

Under this isomorphism $\Delta_{S \otimes B}$ corresponds to $\Delta_S \otimes (1 \otimes 1) + 1 \otimes \Delta_B$. Furthermore the element $1 \otimes \delta \in S \otimes \mathbb{D}_k(B)$ corresponds to the double derivation $\hat{\delta} \in \mathbb{D}_K(S \otimes B)$ which sends $s \otimes b$ to $s\mathbf{1} \otimes \frac{1}{m}\delta(b)$.

Proof. We have an S -bimodule splitting

$$S \otimes_K S = \Omega_{S/K}^1 \oplus S' \tag{12}$$

where $\Omega_{S/K}^1$ is spanned by $\{\rho(f_{ab}^{pq}) : a \neq b\} \cup \{\rho(f_{aa}^{pq} - f_{11}^{pq}) : a \neq 1\}$ and S' has a basis given by $\{\rho(\sum_{a=1}^n f_{aa}^{pq}) : 1 \leq p, q \leq m\}$, an S -bimodule with generator $\mathbf{1}_K$ which is isomorphic to S . The splitting (12) induces $\Omega_{S \otimes B/K}^1 = \Omega_{S/K}^1 \otimes B^e \oplus S' \otimes \Omega_{B/k}^1$ which leads to

$$\begin{aligned} \mathbb{D}_K(S \otimes B) &\cong \text{Hom}_{(S \otimes B)^e}(\Omega_{S \otimes B/K}^1, (S \otimes B)^e) \\ &= \text{Hom}_{(S \otimes B)^e}(\Omega_{S/K}^1 \otimes B^e \oplus S' \otimes \Omega_{B/k}^1, (S \otimes B)^e) \\ &\cong \mathbb{D}_K(S) \otimes B^e \oplus \text{Hom}_{S^e}(S', S^e) \otimes \mathbb{D}_k(B) \\ &\cong \mathbb{D}_K(S) \otimes B^e \oplus S \otimes \mathbb{D}_k(B). \end{aligned}$$

Here we have used the S -bimodule isomorphism $S \rightarrow \text{Hom}_{S^e}(S', S^e)$ that sends 1 to the injection $S' \xrightarrow{\iota} S^e$ where ι sends $s' \otimes_K s''$ to $s'es''$; its inverse sends a mapping ψ to $\frac{1}{n}\psi(\mathbf{1})''\psi(\mathbf{1})'$.

Now take $\delta \in S \otimes \mathbb{D}_k(B)$. Observe that $(s \otimes 1_B) \otimes_K (1_S \otimes 1_B) - (1_S \otimes 1_B) \otimes_K (s \otimes 1_B) \in \Omega_{S \otimes B/K}^1$ corresponds to $(s \otimes_K 1_S - 1_S \otimes_K s) \otimes (1_B \otimes 1_B) \in \Omega_{S/K}^1 \otimes B^e$ so that $\hat{\delta}(s \otimes 1_B) = 0$. On the other hand, $(1_S \otimes b) \otimes_K (1_S \otimes 1_B) - (1_S \otimes 1_B) \otimes_K (1_S \otimes b) \in \Omega_{S \otimes B/K}^1$ corresponds to

$$(1_S \otimes_K 1_S - \frac{1}{m}\mathbf{1}_K) \otimes (b \otimes 1 - 1 \otimes b) + \frac{1}{m}(\mathbf{1}_K \otimes (b \otimes 1 - 1 \otimes b)) \in \Omega_{S/K}^1 \otimes B^e \oplus S' \otimes \Omega_{B/k}^1.$$

Thus $\hat{\delta}(1 \otimes b) = \frac{1}{m}\iota(\mathbf{1}_K) \otimes \delta(b) = \mathbf{1} \otimes \frac{1}{m}\delta(b)$.

The calculation of $\Delta_{S \otimes B}$ is similarly elementary. □

Note that since S is semisimple, all derivations related to S are inner, and so $\Delta_S \otimes (1 \otimes 1)$ generates the $(S \otimes B)$ -bimodule $\mathbb{D}_K(S) \otimes B^e$.

5.2. We now make our first comparison.

Lemma. *Let $x \in k$. Then there is a functor*

$$\mathbb{T} : \Pi_{n,x,\nu}(B) - \text{mod} \longrightarrow \Pi_{n,1 \otimes x, m\nu}(\text{Mat}_m(B)), \quad \mathbb{T}(V) = (k^m)^{\otimes n} \otimes V.$$

Proof. To define an action of $\Pi_{n,1 \otimes x, m\nu}(\text{Mat}_m(B))$ on $\mathbb{T}(V)$ we need to specify how \mathfrak{S}_n and each copy of $T_{S \otimes B} \mathbb{D}_K(S \otimes B)$ in $T(S \otimes B, n)$ acts. Now \mathfrak{S}_n acts by permutations on $(k^m)^{\otimes n}$ and it acts on V by hypothesis, so we let \mathfrak{S}_n act diagonally on $\mathbb{T}(V)$. We let each copy of B and $\mathbb{D}_k(B)$ act on V as they did in the original action. We let S_i act on the i th copy of k^m in $(k^m)^{\otimes n}$ by the natural matrix action and we let $(\Delta_{S \otimes B})_i$ act by the formula

$$(\Delta_{S \otimes B})_i \mapsto m\nu \sum_{j \neq i} \mathbf{e}_{ij}(i j) + x. \quad (13)$$

Since $(\Delta_{S \otimes B})_i = (\Delta_S \otimes (1 \otimes 1))_i + (1 \otimes \Delta_B)_i$ the above formula tells us that

$$(\Delta_S \otimes (1 \otimes 1))_i \mapsto m\nu \sum_{j \neq i} \mathbf{e}_{ij}(i j) - 1|_{(k^m)^{\otimes n}} \otimes \nu \sum_{j \neq i} (i j)|_V.$$

This then induces an action of the i th copy of $\mathbb{D}_K(S) \otimes B^e$, since by the comment following Lemma 5.1 $(\Delta_S \otimes (1 \otimes 1))_i$ is a generator for this space.

Since we have extended the action from $\Delta_S \otimes (1 \otimes 1)$ to all of $\mathbb{D}_K(S) \otimes B^e$ we need to show that this is actually a well-defined action. There are two checks to make. First, we must make sure that if $z \in (S \otimes B)^e$ annihilates $\Delta_{S \otimes B}$ then $z \cdot (m\nu \sum_{j \neq i} \mathbf{e}_{ij}(i j) + x)$ acts as zero on $\mathbb{T}(V)$. Second, we must check that if $\delta \in ((S \otimes B)^e \cdot \Delta_{S \otimes B}) \cap S \otimes \mathbb{D}_k(B)$ then the action induced from (13) agrees with the natural action of $S \otimes \mathbb{D}_k(B)$.

Let $z \in \text{Ann}_{(S \otimes B)^e}(\Delta_{S \otimes B})$. In particular we must have that $(z \cdot \Delta_{S \otimes B})(1 \otimes b) = \sum z' * (\mathbf{e} \otimes \Delta_B(b)) * z'' = 0$ for all $b \in B$. But since $B = k[X]$, we have that $\Omega_{B/k}^1$ is a free B^e -module and hence torsion-free. This forces $z \in \text{Ann}_{S^e}(\mathbf{e}) \otimes B^e$. Now $\text{Ann}_{S^e}(\mathbf{e}) \otimes B^e$ clearly annihilates $\mathbf{e}_{ij}(i j) = \mathbf{e}_{ji}(i j)$ for any $i \neq j$. Moreover $0 = \text{mult} \circ \tau_{(12)}(z * \mathbf{e}) = \text{mult}(z)$ so that $z \cdot x = x \text{mult}(z) = 0$. This completes the first check.

We have $\delta := z \cdot \Delta_{S \otimes B} \in S \otimes \mathbb{D}_k(B)$ if and only if $z \cdot (\Delta_S \otimes (1 \otimes 1)) = 0$. So $z \in \text{Ann}_{S^e}(\Delta_S) \otimes B^e$. Note that

$$\begin{aligned} (\mathbf{1} \cdot \Delta_S)(s) &= \sum_{a,p} \sum_v (s e_v e_{ap} \otimes e_{pa} e_v - e_v e_{ap} \otimes e_{pa} e_v s) \\ &= \sum_{a,p} (s e_{ap} \otimes e_{pa} - e_{ap} \otimes e_{pa} s) = 0, \end{aligned}$$

where the last equality holds because $\mathbf{1} \in (S \otimes S)^{adS}$. Thus $S^e \cdot \mathbf{1} \subseteq \text{Ann}_{S^e}(\Delta_S)$. On the other hand Δ_S generates $\mathbb{D}_K(S) = \text{Hom}_{S^e}(\Omega_{S/K}^1, S \otimes S) \cong \text{Hom}_{S^e}(S \otimes_K S, S \otimes S) / \text{Hom}_{S^e}(S', S \otimes S)$, so that $\dim_k \text{Ann}_{S^e}(\Delta_S) = \dim_k S' = \dim_k S^e \cdot \mathbf{1}$. Thus $S^e \cdot \mathbf{1} = \text{Ann}_{S^e}(\Delta_S)$. As a consequence $S \cdot \mathbf{1} \otimes B^e \cdot \Delta_B = ((S \otimes B)^e \cdot \Delta_{S \otimes B}) \cap S \otimes \mathbb{D}_k(B)$. So, without loss of generality, we may assume $\delta = \mathbf{1} \cdot (\Delta_{S \otimes B}) = \mathbf{1} \cdot 1 \otimes \Delta_B = m \otimes \Delta_B$. Now,

by definition,

$$(\mathbf{1} \cdot \Delta_{S \otimes B})_i \mapsto m\nu \sum_{a,p} \sum_{j \neq i} (e_{pa} e_{ap})_{ij}(i j) + mx = m(\nu \sum_{a,p} \sum_{j \neq i} (e_{pa} \otimes e_{ap})_{ij}(i j) + x).$$

We claim that $\sum_{a,p} (e_{pa} \otimes e_{ap})_{ij}(i j)$ acts as the identity on $(k^m \otimes k^m)_{ij}$. Indeed if v_k , $1 \leq k \leq m$, denotes the standard basis of k^m then

$$\begin{aligned} \sum_{a,p} (e_{pa} \otimes e_{ap})_{ij}(i j) \cdot (v_k \otimes v_\ell) &= \sum_{a,p} (e_{pa} \otimes e_{ap}) \cdot (v_\ell \otimes v_k) \\ &= \sum_{a,p} (e_{pa} \cdot v_\ell \otimes e_{ap} \cdot v_k) = v_k \otimes v_\ell. \end{aligned}$$

Thus

$$(\mathbf{1} \cdot \Delta_{S \otimes B})_i \mapsto m(1|_{(k^m)^{\otimes n}} \otimes \nu \sum_{j \neq i} (i j)|_V + x)$$

which agrees with the definition of the action of $(m \otimes \Delta_B)_i$. This completes the second check.

So far we have defined an action of $T(S \otimes B, n) \rtimes \mathfrak{S}_n$ on $\mathbb{T}(V)$. We now must check that the relations (1)-(4) are satisfied. This is immediate for (1). It is also clear that (2) holds since S_i and S_j act on distinct copies of k^m in $(k^m)^{\otimes n}$ and B_i acts only on V .

Similarly, the actions of S_i and $\mathbb{D}_k(B)_j$ commute by construction. Moreover, as we saw above, if $\delta \in S \otimes \mathbb{D}_k(B)$ and $s \in S$ then $\hat{\delta}(s \otimes 1) = 0$, so (4) holds in this special case. If we take $\hat{\alpha} \in \mathbb{D}_K(S \otimes B)$ corresponding to $\alpha \in 1 \otimes \mathbb{D}_k(B)$ then we find

$$[(s \otimes b)_i, \hat{\alpha}_j] = [(s \otimes b), (1 \otimes \alpha)_j] = s_i \otimes [b_i, \alpha_j] = \nu s_i \otimes \alpha(b)_{ij}(i j)|_V,$$

whilst

$$m\nu \hat{\alpha}(s \otimes b)_{ij}(i j) = \nu(s\mathbf{1} \otimes \alpha(b))_{ij}(i j) = \nu s_i \otimes \alpha(b)_{ij}(i j)|_V$$

where the first equality follows from Lemma 5.1 and the second equality from the calculation a couple of paragraphs above. Thus (4) holds for any $\alpha \in \mathbb{D}_k(B)$. This is enough to confirm (4) in general because we know that (4) holds now for $\Delta_{S \otimes B}$ (see the comment following Lemma 2.4) and for Δ_B and hence for $\Delta_S \otimes (1 \otimes 1) = \Delta_{S \otimes B} - 1 \otimes \Delta_B$, as required.

Finally we tackle (3). We argue again as above. Let $\hat{\alpha}, \hat{\beta} \in \mathbb{D}_K(S \otimes B)$ correspond to $\alpha, \beta \in \mathbb{D}_k(B)$. Then we have

$$[\hat{\alpha}_i, \hat{\beta}_j] = 1 \otimes [\alpha_i, \beta_j] = 1 \otimes \nu \{\{\beta, \alpha\}\}_{ij}(i j)|_V.$$

Calculation gives

$$\{\{\hat{\beta}, \hat{\alpha}\}\}_r = \frac{1}{m} \sum_{a,p} (e_{ap} \otimes \widehat{\{\{\beta, \alpha\}\}}'_r) \otimes e_{pa} \widehat{\{\{\beta, \alpha\}\}}''_r \in (S \otimes B) \otimes \mathbb{D}_K(S \otimes B),$$

and similarly for $\{\{\hat{\beta}, \hat{\alpha}\}\}_l$. It now follows as above that $m\nu \{\{\hat{\beta}, \hat{\alpha}\}\}_{ij}(i j) = 1 \otimes \nu \{\{\beta, \alpha\}\}_{ij}(i j)|_V$ and so

$$[\hat{\alpha}_i, \hat{\beta}_j] = m\nu \{\{\hat{\beta}, \hat{\alpha}\}\}_{ij}(i j).$$

We are at the point where we know that (3) holds for elements of $\mathbb{D}_k(B)$ and also for elements of $\Delta_{S \otimes B}$. It follows that

$$\begin{aligned}
[(1 \otimes \Delta_B)_i, (\Delta_S \otimes (1 \otimes 1))_j] &= [(1 \otimes \Delta_B)_i, (\Delta_{S \otimes B} - 1 \otimes \Delta_B)_j] \\
&= [(1 \otimes \Delta_B)_i, (\Delta_{S \otimes B})_j] - [(1 \otimes \Delta_B)_i, (1 \otimes \Delta_B)_j] \\
&= m\nu(\{\{\Delta_{S \otimes B}, 1 \otimes \Delta_B\} - \{1 \otimes \Delta_B, 1 \otimes \Delta_B\}\}_{ij}(ij)) \\
&= m\nu\{\{\Delta_S \otimes (1 \otimes 1), 1 \otimes \Delta_B\}\}_{ij}(ij),
\end{aligned}$$

which implies that it holds for $\alpha = 1 \otimes \Delta_B$ with any $\beta \in \mathbb{D}_K(S \otimes B)$. But this means that it also holds for $\alpha = \Delta_{S \otimes B} - 1 \otimes \Delta_B = \Delta_S \otimes (1 \otimes 1)$. Thus we are finished. \square

5.3. We can now see that things have not degenerated too far.

Lemma. *Let $\epsilon = e_{11} \in S$. Then there is a surjective k -algebra homomorphism*

$$\zeta : \epsilon^{\otimes n}(\Pi_{n,1 \otimes x, m\nu}(S \otimes B))\epsilon^{\otimes n} \longrightarrow \Pi_{n,x,\nu}(B) \quad (14)$$

In particular, $\epsilon^{\otimes n}(\Pi_{n,1 \otimes x, m\nu}(S \otimes B))\epsilon^{\otimes n}$ has Gelfand-Kirillov dimension at least $2n$.

Proof. If we apply $\epsilon^{\otimes n}$ to the functor \mathbb{T} then we send V to V and hence we endow every $\Pi_{n,x,\nu}(B)$ -module with the structure of a $\epsilon^{\otimes n}(\Pi_{n,1 \otimes x, m\nu}(S \otimes B))\epsilon^{\otimes n}$ -module. In particular, taking $V = \Pi_{n,x,\nu}(B)$ we see that $\epsilon^{\otimes n}(\Pi_{n,1 \otimes x, m\nu}(S \otimes B))\epsilon^{\otimes n}$ acts on $\Pi_{n,x,\nu}(B)$ in a way which, by construction, commutes with the right regular action on $\Pi_{n,x,\nu}(B)$. This produces a k -algebra homomorphism

$$\zeta : \epsilon^{\otimes n}(\Pi_{n,1 \otimes x, n\nu}(S \otimes B))\epsilon^{\otimes n} \longrightarrow \Pi_{n,x,\nu}(B).$$

This is surjective because the action of $\epsilon^{\otimes n}B$, $\epsilon^{\otimes n}\mathbb{D}_k(B)$ and $\epsilon^{\otimes n}\mathfrak{S}_n$ generate the action of $\Pi_{n,x,\nu}(B)$ on any module V , and in particular the left regular action on $\Pi_{n,x,\nu}(B)$.

The final sentence of the lemma follows since $\Pi_{n,x,\nu}(B)$ has Gelfand-Kirillov dimension $2n$ by Corollary 3.5 and [EG, Theorem 1.3]. \square

5.4. Continue with the idempotent $\epsilon = e_{11}$ and assume that $\epsilon \in K$. Set $\bar{\epsilon} = 1 - \epsilon$.

Proposition. *Let $y \in K$ and set $x = \text{Tr}(y) \in k$. Then there is an isomorphism $\epsilon^{\otimes n}(\Pi_{n,y,\nu}(S \otimes B))\epsilon^{\otimes n} \cong \Pi_{n,x,\nu}(B)$.*

Proof. By hypothesis we may write $y = x + \sum_i [x'_i, x''_i]$, so Lemma 2.5 shows that we may assume $x = y$ without loss of generality. We will show that there is a surjective k -algebra homomorphism

$$\psi : \Pi_{n,x,\nu}(B) \longrightarrow \epsilon^{\otimes n}(\Pi_{n,y,\nu}(S \otimes B))\epsilon^{\otimes n}.$$

As we noted above 3.5 and [EG, Theorem 1.3] show that $\Pi_{n,x,\nu}(B)$ is a prime algebra with Gelfand-Kirillov dimension $2n$. Thus if I is a non-zero ideal of $\Pi_{n,x,\nu}(B)$ then $\Pi_{n,x,\nu}(B)/I$ has Gelfand-Kirillov dimension strictly less than $2n$. Taking $I = \ker \psi$, we see that ψ is injective by Lemma 5.3, completing the proposition.

Set $A = S \otimes B$ and $\Delta = \Delta_{S \otimes B}$, and let \bar{T} denote the factor of $T(A, n) \rtimes \mathfrak{S}_n$ by the relations $[a_i, a'_j] = 0$ for all $a, a' \in A$ and $i \neq j$. Let $R_{1,i}$ (respectively $R_{2,i}, R_{3,i}$) be the k -space in \bar{T} generated by the relation $(\bar{\epsilon}\Delta\bar{\epsilon})_i = \nu \sum_{j \neq i} (\bar{\epsilon}\mathbf{e}\bar{\epsilon})_{ij}(i j)$ (respectively $(\epsilon\Delta\bar{\epsilon})_i = \nu \sum_{j \neq i} (\epsilon\mathbf{e}\bar{\epsilon})_{ij}(i j)$, $(\bar{\epsilon}\Delta\epsilon)_i = \nu \sum_{j \neq i} (\bar{\epsilon}\mathbf{e}\epsilon)_{ij}(i j)$), and let I_1 (respectively I_2, I_3) be the ideal of \bar{T} generated by all the $R_{1,i}$ (respectively $R_{2,i}, R_{3,i}$). Similarly, let $S_{1,i}$ (respectively $S_{2,i}, S_{3,i}$) be the A -subbimodule of the i th copy of $\mathbb{D}_K(S \otimes B)$ generated by $(\bar{\epsilon}\Delta\bar{\epsilon})_i$ (respectively $(\epsilon\Delta\bar{\epsilon})_i, (\bar{\epsilon}\Delta\epsilon)_i$) and let $J_{1,i}$ ($J_{2,i}, J_{3,i}$) be the ideal of $\epsilon^{\otimes n} \bar{T} \epsilon^{\otimes n}$ generated by the image of $\epsilon S_{1,i} \epsilon$ (respectively $\epsilon S_{2,i} \epsilon, \epsilon S_{3,i} \epsilon$). Let F be the ideal of \bar{T} generated by the relations (3) and (4). For $k = 1, 2$ or 3 we claim that $\epsilon^{\otimes n} I_k \epsilon^{\otimes n} \subseteq \sum_{i=1}^n J_{k,i} + \epsilon^{\otimes n} F \epsilon^{\otimes n}$. First note that since $1^{\otimes n} \in A^{\otimes n} \epsilon^{\otimes n} A^{\otimes n}$ we have

$$\epsilon^{\otimes n} I_k \epsilon^{\otimes n} = \epsilon^{\otimes n} \bar{T} \epsilon^{\otimes n} \left((\epsilon A)^{\otimes n} \left(\sum_{k=1}^n R_{k,i} \right) (A \epsilon)^{\otimes n} \right) \epsilon^{\otimes n} \bar{T} \epsilon^{\otimes n}.$$

This means that it is enough to check that $(\epsilon A)^{\otimes n} R_{k,i} (A \epsilon)^{\otimes n} \subseteq (\epsilon A \epsilon)^{\otimes n} \epsilon S_{k,i} \epsilon (\epsilon A \epsilon)^{\otimes n} + \epsilon^{\otimes n} F \epsilon^{\otimes n}$ for each i and k . First note that for $j \neq i$ we have

$$\begin{aligned} & (\epsilon a)_j [(\bar{\epsilon}\Delta\bar{\epsilon})_i - \nu(\bar{\epsilon}\mathbf{e}\bar{\epsilon})_{ij}(i j)] (a' \epsilon)_j \\ &= \bar{\epsilon}_i (\epsilon a)_j (\Delta\bar{\epsilon})_i (a' \epsilon)_j - \nu \bar{\epsilon}_i (\epsilon a)_j \mathbf{e}_{ij} \bar{\epsilon}_j (i j) (a' \epsilon)_j \\ &= (\bar{\epsilon}\Delta\bar{\epsilon})_i (\epsilon a)_j (a' \epsilon)_j + \nu \bar{\epsilon}_i \Delta (\epsilon a)_{ji} (i j) \bar{\epsilon}_i (a' \epsilon)_j - \nu \bar{\epsilon}_i (\epsilon a)_j \mathbf{e}_{ij} \bar{\epsilon}_j (i j) (a' \epsilon)_j + \bar{\epsilon}_i \epsilon_j F \bar{\epsilon}_i \epsilon_j \\ &= (\bar{\epsilon}\Delta\bar{\epsilon})_i (\epsilon a a' \epsilon)_j + \nu \bar{\epsilon}_i (\epsilon a \mathbf{e} - \mathbf{e} \epsilon a)_{ji} (i j) \bar{\epsilon}_i (a' \epsilon)_j - \nu \bar{\epsilon}_i (\epsilon a)_j \mathbf{e}_{ij} (i j) \bar{\epsilon}_i (a' \epsilon)_j + \bar{\epsilon}_i \epsilon_j F \bar{\epsilon}_i \epsilon_j \\ &= (\bar{\epsilon}\Delta\bar{\epsilon})_i (\epsilon a a' \epsilon)_j - \nu \bar{\epsilon}_i (\mathbf{e} \epsilon a)_{ji} (i j) \bar{\epsilon}_i (a' \epsilon)_j + \bar{\epsilon}_i \epsilon_j F \bar{\epsilon}_i \epsilon_j \\ &= (\bar{\epsilon}\Delta\bar{\epsilon})_i (\epsilon a a' \epsilon)_j + \bar{\epsilon}_i \epsilon_j F \bar{\epsilon}_i \epsilon_j. \end{aligned}$$

For the second last equality we have used that $\mathbf{e}_{ij} = \mathbf{e}_{ji}$ and for the last one that

$$\bar{\epsilon}_i (\mathbf{e} \epsilon)_{ji} = \sum_{v \in V} (e_v \otimes \bar{\epsilon} e_v \epsilon)_{ji} = 0$$

since $\bar{\epsilon} e_v \epsilon = 0$ for all $v \in V$. The above calculation shows that $[(\bar{\epsilon}\Delta\bar{\epsilon})_i, (\epsilon a \epsilon)_j] \in \bar{\epsilon}_i \epsilon_j F \bar{\epsilon}_i \epsilon_j$, so it follows that

$$\begin{aligned} & (\epsilon a)_1 \cdots (\epsilon b)_i \cdots (\epsilon c)_n \left[(\bar{\epsilon}\Delta\bar{\epsilon})_i - \nu \sum_{j \neq i} (\bar{\epsilon}\mathbf{e}\bar{\epsilon})_{ij}(i j) \right] (c' \epsilon)_n \cdots (b' \epsilon)_i \cdots (a' \epsilon)_1 \\ &= (\epsilon a a' \epsilon)_1 \cdots (\epsilon b \bar{\epsilon} \Delta \bar{\epsilon} b' \epsilon)_i \cdots (\epsilon c c' \epsilon)_n + \epsilon^{\otimes n} F \epsilon^{\otimes n}, \end{aligned}$$

proving the claim for $k = 1$.

Similarly we have

$$\begin{aligned} & (\epsilon a)_j [(\epsilon\Delta\bar{\epsilon})_i - \nu(\epsilon\mathbf{e}\bar{\epsilon})_{ij}(i j)] (a' \epsilon)_j \\ &= \epsilon_i (\epsilon a)_j (\Delta\bar{\epsilon})_i (a' \epsilon)_j - \nu (\epsilon a)_j \epsilon_i \mathbf{e}_{ij} \bar{\epsilon}_j (i j) (a' \epsilon)_j \\ &= \epsilon_j (\epsilon\Delta\bar{\epsilon})_i (\epsilon a)_j (a' \epsilon)_j + \epsilon_j \epsilon_i \nu (\epsilon a \mathbf{e} - \mathbf{e} \epsilon a)_{ji} (i j) \bar{\epsilon}_i (a' \epsilon)_j - \nu (\epsilon a)_j \epsilon_i \mathbf{e}_{ij} (i j) \bar{\epsilon}_i (a' \epsilon)_j + \epsilon_i \epsilon_j F \epsilon_i \epsilon_j \\ &= \epsilon_j (\epsilon\Delta\bar{\epsilon})_i (\epsilon a a' \epsilon)_j - \epsilon_j \epsilon_i \nu (\mathbf{e} \epsilon a)_{ji} \bar{\epsilon}_j (i j) (a' \epsilon)_j + \epsilon_i \epsilon_j F \bar{\epsilon}_i \epsilon_j \\ &= \epsilon_j (\epsilon\Delta\bar{\epsilon})_i (\epsilon a a' \epsilon)_j + \epsilon_i \epsilon_j F \bar{\epsilon}_i \epsilon_j \end{aligned}$$

where, for the last equality, we use

$$\epsilon_j(\mathbf{e}ea)_{ji}\bar{\epsilon}_j = \sum_{v \in V} (\epsilon e_v \bar{\epsilon} \otimes e_v ea)_{ji} = 0.$$

Thus again we find that

$$\begin{aligned} & (\epsilon a)_1 \cdots (\epsilon b)_i \cdots (\epsilon c)_n \left[(\epsilon \Delta \bar{\epsilon})_i - \nu \sum_{j \neq i} (\epsilon \mathbf{e} \bar{\epsilon})_{ij}(i j) \right] (c' \epsilon)_n \cdots (b' \epsilon)_i \cdots (a' \epsilon)_1 \\ &= \epsilon^{\otimes n} (\epsilon b \epsilon \Delta \bar{\epsilon} b' \epsilon)_i (\epsilon a a' \epsilon)_1 \cdots (\epsilon c c' \epsilon)_n + \epsilon^{\otimes n} F \epsilon^{\otimes n} \end{aligned}$$

which proves the case for $k = 2$. The $k = 3$ case is entirely analogous.

We have now proved that there is an isomorphism

$$\epsilon^{\otimes n} \left(\frac{\bar{T}}{I_1 + I_2 + I_3 + F} \right) \epsilon^{\otimes n} \xrightarrow{\sim} \frac{\epsilon^{\otimes n} \bar{T} \epsilon^{\otimes n}}{\sum_i (J_{1,i} + J_{2,i} + J_{3,i}) + \epsilon^{\otimes n} F \epsilon^{\otimes n}}.$$

Thanks to [CB, Lemma 5.1] the mapping ψ , sending α to $\epsilon \alpha \epsilon$, induces a surjection from $\mathbb{D}_K(A)$ to $\mathbb{D}_{\epsilon K \epsilon}(\epsilon A \epsilon) \cong \mathbb{D}_k(B)$ such that if J is the A -subbimodule of $\mathbb{D}_K(A)$ generated by $\epsilon \Delta \bar{\epsilon}$, $\bar{\epsilon} \Delta \epsilon$ and $\bar{\epsilon} \Delta \bar{\epsilon}$ then $\epsilon(\mathbb{D}_K(A)/J)\epsilon \cong \mathbb{D}_k(B)$. Thus ψ^{*n} induces an isomorphism

$$\frac{\epsilon^{\otimes n} \bar{T} \epsilon^{\otimes n}}{\sum_i (J_{1,i} + J_{2,i} + J_{3,i}) + \epsilon^{\otimes n} F \epsilon^{\otimes n}} \xrightarrow{\sim} \frac{\overline{T(\epsilon A \epsilon, n) \rtimes \mathfrak{S}_n}}{L},$$

where L is the ideal generated by $(\epsilon \Delta \epsilon - x)_i = \nu \sum_{j \neq i} (\epsilon \mathbf{e} \epsilon)_{ij}(i j)$ and $\psi^{*n}(\epsilon^{\otimes n} F \epsilon^{\otimes n})$.

It is clear that $\psi^{*n}(\epsilon^{\otimes n} F \epsilon^{\otimes n})$ contains the elements

$$\epsilon_1 \cdots \left([(\epsilon b a b' \epsilon)_i, (\epsilon c \beta c' \epsilon)_j] - \nu \{ \{ (\epsilon c \beta c' \epsilon), (\epsilon b a b' \epsilon) \} \}_{ij}(i j) \right) \cdots \epsilon_n,$$

and

$$\epsilon_1 \cdots \left([(\epsilon b z b' \epsilon)_i, (\epsilon c \alpha c' \epsilon)_j] - \nu (\epsilon c \alpha c' \epsilon \epsilon) (\epsilon b z b' \epsilon)_{ij}(i j) \right) \cdots \epsilon_n.$$

Thus $\overline{T(\epsilon A \epsilon, n) \rtimes \mathfrak{S}_n} / L$ is a quotient of $\Pi_{n,x,\nu}(\epsilon A \epsilon) \cong \Pi_{n,x,\nu}(B)$, as required. \square

5.5. There is one particular case of the above proposition that will interest us in the next section. Let N be a positive integer and let $1 = i_1 < i_2 < \cdots < i_t = N$. We take K to be the subalgebra of $\text{Mat}_N(k[X])$ consisting of diagonal matrices of the form

$$K = \{ \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_N) : \lambda_{i_{j-1}+1} = \lambda_{i_{j-1}+2} = \cdots = \lambda_{i_j} \text{ for } 1 \leq j \leq t \}, \quad (15)$$

where we set $i_0 = 0$

Corollary. *Let K be as above and take $y \in K$. There is a k -algebra isomorphism*

$$\Pi_{n,y,\nu}(\text{Mat}_N(k[X])) \cong \text{Mat}_{N^n}(\Pi_{n,x,\nu}(k[X]))$$

where $x = \sum_i y_{ii}$.

Proof. The matrix units $e_{ij} \in \text{Mat}_N(k)$ induce elements $e_{i_1 j_1} \otimes \cdots \otimes e_{i_n j_n}$ in $\Pi_{n,y,\nu}(\text{Mat}_N(k[X]))$ which can be considered as matrix units for $\text{Mat}_{N^n}(k)$. Thus $\Pi_{n,y,\nu}(\text{Mat}_N(k[X])) \cong \text{Mat}_{N^n}(R)$ where $R = \epsilon^{\otimes n} \Pi_{n,y,\nu}(\text{Mat}_N(k[X])) \epsilon^{\otimes n}$. Now applying Proposition 5.4 gives $R \cong \Pi_{n,x,\nu}(k[X])$, as required. \square

6. THE GELFAND-KIRILLOV CONJECTURE

6.1. Let Q be an affine Dynkin quiver and let Γ be the corresponding finite subgroup of $SL_2(\mathbb{C})$. Let δ be the fundamental root of Q so that by the McKay correspondence δ_v records the dimension of the irreducible Γ -representation associated to $v \in V$. Set $N = \sum_v \delta_v$. Let $K = k^V$. We will label the vertices V by the integers $1, \dots, t$ and assume that 1 corresponds to the trivial representation.

By [CB, Lemma 11.1] there is a pseudo-flat ring epimorphism $\theta : kQ \longrightarrow \text{Mat}_N(k[X])$. This is a K -algebra homomorphism where $K \subset \text{Mat}_N(k[X])$ consists of the diagonal matrices of the form (15) where $i_j = \delta_1 + \dots + \delta_j$. Thus, given $x \in K$ and $y = \sum_i x_i \delta_i$, we can combine Proposition 4.2 and Corollary 5.5 to produce a pseudo-flat k -algebra ring epimorphism

$$\Theta_{n,x,\nu} : \Pi_{n,x,\nu}(kQ) \longrightarrow \text{Mat}_{N^n}(\Pi_{n,y,\nu}(k[X])).$$

6.2. Before proving our main result, we require the following useful fact.

Lemma. *Let R be a k -algebra with a non-negative finite dimensional filtration such that $\text{gr } R$ is a commutative affine domain. Suppose that M is a non-zero left R -module such that $\text{gr } M$ is finitely generated and torsion-free of rank t . Then M has uniform dimension t .*

Proof. Let u be the uniform dimension of M . Since R is a prime Goldie ring and M is a torsion-free R -module, by [GW, Proposition 7.18] there exist non-zero left ideals I_j of R ($1 \leq j \leq u$) such that M has an essential submodule M' isomorphic to $\bigoplus_{j=1}^u I_j$.

Following [MR, Section 8.6] we let $e(-)$ denote the Bernstein multiplicity of a finitely generated R -module. Since R is a noetherian domain, I_j is an essential left ideal for any j , so we see that the $\text{GKdim}(R/I_j) < \text{GKdim}(R)$ by [MR, Corollary 8.3.6(i)]. By [MR, Corollary 6.20 (4.8(iii))] we have $e(M') = ue(R)$. On the other hand if $x_1 + M', \dots, x_n + M'$ are generators of M/M' then there is a surjection

$$R/J_1 \oplus \dots \oplus R/J_n \longrightarrow M/M'$$

where $J_k = \{r \in R : rx_k \in M'\}$. Each J_k is non-zero and so an essential left ideal of R since R is a domain. Thus $\text{GKdim}(M/M') < \text{GKdim}(R) = \text{GKdim}(M)$. Again by [MR, Corollary 6.20 (4.8(iii))] we have $e(M) = e(M')$. It follows that $u = e(M)/e(R)$.

However by definition we have $e(\text{gr } M) = e(M)$ and $e(\text{gr } R) = e(R)$ so that $u = e(\text{gr } M)/e(\text{gr } R)$. But by definition the Goldie rank and the torsion-free rank of $\text{gr } M$ agree, so we deduce that $t = e(\text{gr } M)/e(\text{gr } R)$ as required. \square

6.3. We have an immediate corollary. We state it only for the groups Γ_n , but the first statement holds with the same proof for any symplectic reflection algebra.

Corollary. *The Goldie rank of $H_{t,k,c}(\Gamma_n)$ equals $|\Gamma_n|$ and the Goldie rank of $\Pi_{n,x,\nu}(kQ)$ equals $n!N^n$.*

Proof. By [EG, Theorem 1.5(iv)] we have that

$$H_{t,k,c} \xrightarrow{\sim} \text{End}_{eH_{t,c}e}(eH_{t,k,c})^{op}. \quad (16)$$

Moreover there is a finite dimensional non-negative filtration on $H_{t,k,c}$ which induces filtrations on $eH_{t,k,c}e$ and $eH_{t,k,c}$ such that $\text{gr } eH_{t,k,c}e \cong \mathbb{C}[V]^{\Gamma_n}$ and $\text{gr } eH_{t,k,c} \cong \mathbb{C}[V]$. Thus the hypotheses of Lemma 6.2 are satisfied and we deduce that $eH_{t,k,c}$ has uniform dimension $|\Gamma_n|$, the torsion-free rank of $\mathbb{C}[V]$ over $\mathbb{C}[V]^{\Gamma_n}$. Tensoring (16) by the quotient division algebra of $eH_{t,k,c}e$ then shows that the Goldie rank of $H_{t,k,c}$ equals $|\Gamma_n|$, the uniform dimension of $eH_{t,k,c}$.

We apply the same argument to $\Pi_{n,x,\nu}(kQ)$. Let $f \in \mathbb{C}\Gamma_n$ be the idempotent introduced in 3.5. Since $ef = e$ we have $e_0\Pi_{n,x,\nu}(kQ)e_0 \cong eH_{t,k,c}e$ where $e_0 \in \Pi_{n,x,\nu}(kQ)$ is the image of ef . The isomorphism (16) induces an isomorphism

$$\Pi_{n,x,\nu}(kQ) = fH_{t,k,c}f \xrightarrow{\sim} \text{End}_{eH_{t,k,c}e}(eH_{t,k,c}f) = \text{End}_{eH_{t,k,c}e}(e_0\Pi_{n,x,\nu}(kQ)).$$

Now $\text{gr } e_0\Pi_{n,x,\nu}(kQ) = e_0\Pi_{n,0,0}(kQ) \cong (e_0\Pi_{1,0,0}(kQ))^{\otimes n}$. But $e\Pi_{1,0,0}(kQ)$ has torsion-free rank N over $e\Pi_{1,0,0}(kQ)e$ and so $(e_0\Pi_{1,0,0}(kQ))^{\otimes n}$ has torsion-free rank N^n over $(e_0\Pi_{1,0,0}(kQ)e_0)^{\otimes n}$. The result follows since $eH_{t,c}e = ((e_0\Pi_{1,0,0}(kQ)e_0)^{\otimes n})^{S_n}$ and the torsion-free rank of $(e_0\Pi_{1,0,0}(kQ)e_0)^{\otimes n}$ over $eH_{t,c}e$ is thus $n!$. \square

6.4. Let $C_\nu(n)$ be the algebra $\mathbb{C}\langle X_1, \dots, X_n, Y_1, \dots, Y_n : [X_i, X_j] = [Y_i, Y_j] = 0, [Y_i, X_j] = \delta_{ij}\nu \rangle$ and let $D_\nu(n)$ be its quotient division algebra.

Theorem. *The map $\Theta_{n,x,\nu}$ induces an isomorphism from the simple artinian quotient ring of $\Pi_{n,x,\nu}(kQ)$ to $\text{Mat}_r(D_\nu(n))$ where $r = n!N^n$.*

Proof. We observe first that $\Theta_{n,x,\nu}$ is injective. This is true because we have a commutative diagram

$$\begin{array}{ccc} \Pi_{n,0,0}(kQ) & \xrightarrow{\Theta_{n,0,0}} & \text{Mat}_{N^n}(\Pi_{n,0,0}(k[X])) \\ \downarrow & & \downarrow \\ \text{gr } \Pi_{n,x,\nu}(kQ) & \xrightarrow{\text{gr } \Theta_{n,x,\nu}} & \text{Mat}_{N^n}(\text{gr } \Pi_{n,y,\nu}(k[X])). \end{array}$$

Since both kQ and $k[X]$ are the path algebras of affine Dynkin quivers, the vertical arrows are isomorphisms by 3.4. Moreover if we identify $\Pi_{n,0,0}(kQ) \cong \Pi^0(Q)^{\otimes n} \rtimes \mathfrak{S}_n$ and $\text{Mat}_{N^n}(\Pi_{n,0,0}(k[X])) \cong \text{Mat}_N(k[X, Y])^{\otimes n} \rtimes \mathfrak{S}_n$, then $\Theta_{n,0,0}$ is induced from $\Theta_{1,0,0}^{\otimes n}$. By [CB, Lemma 11.3] $\Theta_{1,0,0}$ is injective. Thus we deduce that $\text{gr } \Theta_{n,x,\nu}$ is injective, and hence that $\Theta_{n,x,\nu}$ is too.

By 3.5 there is an isomorphism between $\Pi_{n,x,\nu}(k[X])$ and the rational Cherednik algebra of type \mathfrak{S}_n , so that $\Pi_{n,x,\nu}(k[X])$ has quotient ring $\text{Mat}_{n!}(D_\nu(n))$ thanks to the Dunkl embedding, [EG]. By [MR, Proposition 3.1.16] the quotient ring of $\Pi_{n,x,\nu}(kQ)$ thus embeds into $\text{Mat}_{N^n}(\text{Mat}_{n!}(D_\nu(n))) \cong \text{Mat}_r(D_\nu(n))$. Since the Goldie rank of $\Pi_{n,x,\nu}(kQ)$ equals r and $\Theta_{n,x,\nu}$ is a ring epimorphism the result now follows word-for-word from the proof of [CB, Theorem 11.6]. \square

6.5. We now get the result we wanted.

Corollary. *The quotient division ring of $eH_{t,k,c}e$ is isomorphic to $D_t(n)^{\Gamma_n}$ and the simple artinian quotient ring of $H_{t,k,c}(\Gamma_n)$ is isomorphic to $D_t(n) \rtimes \Gamma_n$.*

Proof. Since $eH_{t,k,c}e = e_0\Pi_{n,x,\nu}(kQ)e_0$ is a domain, we have

$$\text{Quot}(eH_{t,k,c}e) = e_0\text{Quot}(\Pi_{n,x,\nu}(kQ))e_0 = e_0\text{Mat}_r(D_t(n))e_0.$$

Since $e_0 = e_{\mathfrak{S}_n}e_{11}^{\otimes n}$ (where $e_{\mathfrak{S}_n}$ denotes the trivial idempotent for \mathfrak{S}_n) we see that this is isomorphic to $D_t(n)^{\mathfrak{S}_n}$. Now $D_t(n) \cong D_t(1)^{\otimes n}$ and $D_t(1)^{\Gamma} \cong \mathcal{D}_t(1)$ by [CB, Theorem 0.9]. Thus $\text{Quot}(eH_{t,k,c}e) \cong D_t(n)^{\mathfrak{S}_n} \cong D_t(n)^{\Gamma_n}$, proving the first claim. The second claim follows since $H_{t,k,c}$ has Goldie rank $|\Gamma_n|$ by Corollary 6.3 and the fact that $D_t \rtimes \Gamma_n \cong \text{Mat}_{|\Gamma_n|}(D_t^{\Gamma_n})$. \square

6.6. We note one other amusing consequence of the work in this paper.

Proposition. *Let (t, k, c) be deformation parameters for Γ_n . There is a fully faithful embedding*

$$F : H_{t,k}(S_n) - \text{mod} \longrightarrow H_{t,k,c}(\Gamma_n) - \text{mod} .$$

Proof. Since $\Theta_{n,x,\nu}$ is a ring epimorphism the restriction along $\Theta_{n,x,\nu}$ induces a fully faithful embedding on module categories. \square

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