$\mathcal{O}_\epsilon[G]$ IS A FREE MODULE OVER $\mathcal{O}[G]$

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Abstract. We show that the quantised function algebra $\mathcal{O}_\epsilon[G]$ of a simply connected semisimple algebraic group G at a root of unity is a free module over the subring isomorphic to $\mathcal{O}[G]$.

Let G be a simply-connected semisimple algebraic group over $\mathbb{C}$. Let $\ell > 1$ be an odd integer, prime to 3 if G has a component of type G_2, and let $\epsilon \in \mathbb{C}$ be a primitive ℓ^{th} root of unity. The quantised function algebra of G at ϵ, denoted $\mathcal{O}_\epsilon[G]$, is a noetherian $\mathbb{C}$-algebra containing the ring of regular functions of G, denoted $\mathcal{O}[G]$, in its centre, [4]. Since a false proof of the following theorem, and a proof of the special case $G = SL(2)$, have both recently appeared in the literature (see the remarks below for details), it seems worthwhile to record the full result.

Theorem. *As a module over $\mathcal{O}[G]$, the algebra $\mathcal{O}_\epsilon[G]$ is free of rank $\ell^{\dim G}$.*

Proof. Thanks to [4, Theorem 7.2] $\mathcal{O}_\epsilon[G]$ is a projective $\mathcal{O}[G]$-module of rank $\ell^{\dim G}$. By a result of Marlin, [7, Corollaire 3], the Grothendieck group of projective modules over $\mathcal{O}[G]$ is trivial, in other words

$$K_0(\mathcal{O}[G]) \cong \mathbb{Z}.$$

In particular, if P is a finitely generated projective $\mathcal{O}[G]$-module then P is stably free. Hence if the rank of P is greater than the Krull dimension of $\mathcal{O}[G]$, then P is necessarily free, [2, Corollary IV.3.5]. Since $\ell > 1$ we have $\text{Kdim}\mathcal{O}[G] = \dim G < \ell^{dimG} = \text{rank}\mathcal{O}_\epsilon[G]$, so the theorem follows. □

It is incorrectly stated in [12, Lemma 8] that this result follows from [13, Theorem 2.2] in the more general setting of a Hopf algebra, U, finitely generated over a central sub-Hopf algebra, O. However, there exist numerous examples in the literature of Hopf algebras that are not free over central sub-Hopf algebras. The ones closest in spirit to the present work occur in [14], where the author shows that, when n is even, $U = \mathcal{O}[SL_n(\mathbb{C})]$ is not free over the subring $O = \mathcal{O}[PSL_n]$. For example, consider the case $n = 2$. Then, O is the fixed ring U^A, where $A = \mathbb{Z}/2\mathbb{Z} = \langle\sigma\rangle$ acts on the generators x_{ij} of U by $\sigma(x_{ij}) = -x_{ij}$. This is even a Hopf-Galois extension with central invariants for the Hopf algebra $H = \mathbb{C}A$, in the notation of [12, §1.1]. This requires that U is an H-comodule algebra (use the map $\rho : U \to U \otimes H$ defined by $x_{ij} \mapsto x_{ij} \otimes \sigma$) such

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that $U^H = \{x \in U : \rho(x) = x \otimes 1\} = O$, that the natural map $U \otimes_O U \to U \otimes H$ given by $x \otimes y \mapsto (x \otimes 1)\rho(y)$ is bijective, together with certain naturality conditions. These are easy to check in this case.

Remarks. 1. It is clear the proof of the theorem generalises a little. Assume k is an algebraically closed field and U is a noetherian prime Hopf k-algebra, finitely generated as a module over the central sub-Hopf algebra O. Then U is a projective O-module, [6, Theorem 1.7]. If $K_0(O) \cong \mathbb{Z}$ then freeness follows as above when $\dim O$ is less than the rank of U over O.

2. One way to check that $K_0(O) \cong \mathbb{Z}$ is as follows. The algebra O is the ring of regular functions of an irreducible affine algebraic group, say G, [5, Section I.3]. Let $R_u(G)$ be the unipotent radical of G and set $G_{\text{red}} = G/R_u(G)$, by definition a reductive group. Thanks to [10, Proposition 4.1], the projection $G \longrightarrow G_{\text{red}}$ induces an isomorphism in K-theory, $K_0(G) \cong K_0(G_{\text{red}})$. Now, if the commutator subgroup of G_{red}, a semisimple algebraic group, is simply-connected we have an isomorphism $K_0(G_{\text{red}}) \cong \mathbb{Z}$, [9, Corollary 1.7 and Corollary 4.7].

3. For other situations where a Hopf algebra is free over particular subalgebras, see for example, [11].

4. [3] proves the theorem for the case $G = SL(2)$; in this case an explicit free basis is provided.

5. We do not know whether $\mathcal{O}_\epsilon[G]$ is a cleft extension of $\mathcal{O}[G]$. It would be interesting to find general conditions implying this.

6. The theorem appears in [1] too, where it is used in studying the representation theory of quantised function algebras at roots of unity.

References

[1] K.A. Brown and I. Gordon, The ramification of centres: quantised function algebras at roots of unity, math.RT/9912042.

[2] H. Bass, *Algebraic K-theory*, Benjamin, New York, 1968.

[3] L. Dabrowski, C. Reina and A. Zampa, $A[Sl_q(2)]$ at roots of unity is a free module over $A[Sl(2)]$, math.QA/0004092.

[4] C. De Concini and V. Lyubashenko, Quantum function algebras at roots of 1, *Adv. Math.*, **108** (1994), 205–262.

[5] G.P. Hochschild, *Basic theory of algebraic groups and Lie algebras*, Graduate Texts in Mathematics 75, Springer-Verlag, Berlin, 1981.

[6] H.F. Kreimer and M. Takeuchi, Hopf algebras and Galois extensions of an algebra, *Indiana U. Math. J.*, **30** (1981), 675–692.

[7] R. Marlin, Anneaux de Grothendieck des variétés de drapeaux, *Bull. Soc. math. France*, **104** (1976), 337-348.

[8] J.C. McConnell and J.C. Robson, *Noncommutative Noetherian Rings*, J. Wiley and Sons, 1988.

[9] A.S. Merkur'ev, Comparison of the equivariant and the ordinary K-theory of algebraic varieties, *St. Petersburg Math. J.* **9** (1998), 815–850.

[10] D. Quillen, Higher algebraic K-theory. I, *Algebraic K-theory I: Higher K-theories*, Springer Lecture Notes in Mathematics 341, (1973), 85–147.

[11] D. E. Radford, Freeness (projectivity) criteria for Hopf algebras over Hopf subalgebras, *J. Pure Appl. Algebra*, **11** (1977/78), 15–28.

[12] D. Rumynin, Hopf-Galois extensions with central invariants and their geometric properties, *Algebra and Rep. Theory*, **1** (1998), 353–381.

[13] H-J. Schneider, Normal basis and transitivity of crossed products for Hopf algebras, *J. Algebra*, **152** (1992), 289–312.

[14] W. C. Waterhouse, The module structure of certain Hopf algebra extensions, *Comm. Algebra*, **10** (1982), 115-120.

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