BABY VERMA MODULES FOR RATIONAL CHEREDNIK ALGEBRAS

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Abstract. In this paper, we introduce *baby Verma modules* for symplectic reflection algebras of complex reflection groups at parameter $t = 0$ (the so–called rational Cherednik algebras at parameter $t = 0$) and present their most basic properties. We use baby Verma modules to answer several problems posed by Etingof and Ginzburg, [8], and give an elementary proof of a theorem of Finkelberg and Ginzburg, [9].

1. Introduction

1.1. Symplectic reflection algebras algebras of complex reflection groups (the so–called rational Cherednik algebras) arise in many different mathematical disciplines: integrable systems, Lie theory, representation theory, differential operators, symplectic geometry. Early on a dichotomy (depending on a single parameter t) appears in the behaviour of these algebras: when $t \neq 0$ the algebras are rather non–commutative and relations with differential operators come to the fore; when $t = 0$ the algebras have very large centres and connections between the geometry of the centre and representation theory are of interest. This paper concentrates on the second case. We introduce a family of finite dimensional modules called *baby Verma modules* and present their most basic properties. By analogy with the representation theory of reductive Lie algebras in positive characteristic, we believe these modules are fundamental to the understanding of the representation theory and associated geometry of the rational Cherednik algebras at parameter $t = 0$.

1.2. Let W be a complex reflection group and $\mathfrak{h}$ its reflection representation over $\mathbb{C}$. The rational Cherednik algebras introduced by Etingof and Ginzburg, [8], are deformations of the skew–group ring $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] * W$ depending on parameters t and c. We denote them by $H_{t,c}$. It can be shown that when $t = 0$ there is an embedding of $\mathbb{C}[\mathfrak{h}^*]^W \otimes \mathbb{C}[\mathfrak{h}]^W$ into Z_c, the centre of $H_{0,c}$, and thus a map

$$\Upsilon : \mathrm{Spec}(Z_c) \longrightarrow \mathfrak{h}^*/W \times \mathfrak{h}/W.$$

The baby Verma modules are $H_{0,c}$–modules naturally associated to points in the (reduced) zero fibre, $\Upsilon^{-1}(0)$. Studying their properties allows us to deduce some properties of Z_c and of the map Υ. We prove the following results:

(1) There is a surjective map, Θ, from the isomorphism classes of irreducible representations of W over $\mathbb{C}$ to the elements in the fibre $\Upsilon^{-1}(0)$.
(2) For each point $M \in \Upsilon^{-1}(0)$, M is smooth in $\mathrm{Spec}(Z_c)$ if and only if $\Theta^{-1}(M)$ consists of a unique irreducible W–isomorphism class. In particular, the map Θ is a bijection if and only if the points of $\Upsilon^{-1}(0)$ are smooth in $\mathrm{Spec}(Z_c)$.

(3) If Θ is a bijection and S a simple W–module, then the component of the scheme theoretic fibre $\Upsilon^*(0)$ corresponding to $\Theta(S)$ has a $\mathbb{C}^*$–action whose associated Poincaré polynomial is explicitly described. The component has, in particular, dimension $\dim(S)^2$.

(4) The variety $\mathrm{Spec}(Z_c)$ is singular for W a finite Coxeter group of type D_{2n}, E, F, H and $I_2(m)$ $(n \geq 2, m \geq 5)$.

These results are related to problems of Etingof and Ginzburg, confirming [8, Conjecture 17.14], answering a stronger version of [8, Question 17.15] and partially answering [8, Question 17.1].

1.3. The final result above should be compared to recent results on the non–existence of a crepant resolution for the orbit space $(\mathfrak{h}^* \times \mathfrak{h})/W$. It is shown in [16] that the orbit space admits no crepant resolution if W is the Coxeter group of type G_2; folding arguments then imply no such resolution for W of type D and E. It is an interesting problem to make the relationship between resolutions and deformations of $(\mathfrak{h} \times \mathfrak{h}^*)/W$ precise.

1.4. Recall Calogero–Moser phase space defined by

$$\mathcal{CM}_n = \{(X,Y) \in \mathrm{Mat}_n(\mathbb{C}) \times \mathrm{Mat}_n(\mathbb{C}) : [X,Y] + Id = \text{rank 1 matrix}\}//PGL_n(\mathbb{C}).$$

In the particular case of $W = \mathfrak{S}_n$, the symmetric group on n letters, it is known that for $c \neq 0$ there is an isomorphism between $\mathrm{Spec}(Z_c)$ and $\mathcal{CM}_n$ under which the map Υ corresponds to sending the pair of matrices (X,Y) to their eigenvalues, [8, Theorem 11.16]. Since it is known that $\mathcal{CM}_n$ is smooth, our results mentioned in 1.2 associate to each point in $\Upsilon^{-1}(0)$ a partition of n, as in [22]. Let $\lambda \vdash n$. We prove:

- The Poincaré polynomial of the component of the scheme–theoretic fibre $\Upsilon^*(0)$ corresponding to λ is $K_\lambda(t)K_\lambda(t^{-1})$, where K_λ is the Kostka polynomial corresponding to λ.
- For the point $M_\lambda \in \Upsilon^{-1}(0)$ corresponding to λ, there is a relationship between the finite dimensional algebra $H_{0,c}/M_\lambda H_{0,c}$ and Springer theory for the nilpotent orbit in $\mathrm{Mat}_n(\mathbb{C})$ with Jordan normal form of type λ.

These results confirm a pair of conjectures of Etingof and Ginzburg, [8, Conjectures 17.12 and 17.13], and give an elementary proof of a recent theorem of Finkelberg and Ginzburg, [9]. As noted later, the extension of the first result to $\Gamma \wr \mathfrak{S}_n$, the wreath product of a finite subgroup of $SL_2(\mathbb{C})$ by the symmetric group, is straightforward.

1.5. The paper is organised as follows. Sections 2–3 introduce notation and present the embedding of $\mathbb{C}[\mathfrak{h}^*]^W \otimes \mathbb{C}[\mathfrak{h}]^W$ into Z_c. In Section 4 we define and study baby Verma modules. Section 5 sees the proofs of all but the last result stated in 1.2, whilst Section 6 deals with the special case of the symmetric group, $\mathfrak{S}_n$. In Section 7 we prove the final result of 1.2 and present a couple of basic open problems.

2. Notation

2.1. The following is due to Etingof and Ginzburg, [8]. Let W be a complex reflection group and $\mathfrak{h}$ its reflection representation over $\mathbb{C}$. Let $\mathcal{S}$ denote the set of complex reflections in W. Let ω be the canonical symplectic form on $V = \mathfrak{h} \oplus \mathfrak{h}^*$. For $s \in \mathcal{S}$, let ω_s be the skew–symmetric form which coincides with ω on $\mathrm{im}(\mathrm{id}_V - s)$ and has $\ker(\mathrm{id}_V - s)$ as the radical. Let $c : \mathcal{S} \longrightarrow \mathbb{C}$ be a W–invariant function sending s to c_s. The rational Cherednik algebra at parameter $t = 0$ (depending on c) is the quotient of the skew group algebra of the tensor algebra on V, $TV * \mathbb{C}W$, by the relations

$$[x, y] = \sum_{s\in\mathcal{S}} c_s \omega_s(x, y)s$$

for all $y \in \mathfrak{h}^*$ and $x \in \mathfrak{h}$. This algebra is denoted by H_c.

2.2. We will denote a set of representatives of the isomorphism classes of simple W–modules by Λ.

2.3. Let $R = \oplus R_i$ be a finite dimensional $\mathbb{Z}$-graded $\mathbb{C}$–algebra. We will let $R-\mathrm{mod}$ ($R-\mathrm{mod}_{\mathbb{Z}}$) denote the category of finite dimensional (graded) left R–modules. Given an object M in $R-\mathrm{mod}_{\mathbb{Z}}$, let $M[i]$ denote the ith shift of M, defined by $M[i]_j = M_{j-i}$. The *graded endomorphism ring* of M is defined as

$$\mathrm{End}(M) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{R-\mathrm{mod}_{\mathbb{Z}}}(M, M[i]).$$

Given N, a $\mathbb{Z}$–graded right R–module, the *graded dual* $N^{\circledast}$ is the object of $R - \mathrm{mod}_{\mathbb{Z}}$ defined by

$$(N^{\circledast})_i = \{f : N \longrightarrow \mathbb{C} : f(N_j) = 0 \text{ for all } j \neq -i\}.$$

The forgetful functor is denoted $F : R - \mathrm{mod}_{\mathbb{Z}} \longrightarrow R - \mathrm{mod}$.

3. The centre of H_c

3.1. Throughout $Z_c = Z(H_c)$ and $A = \mathbb{C}[\mathfrak{h}^*]^W \otimes \mathbb{C}[\mathfrak{h}]^W$. Thanks to [8, Proposition 4.15], $A \subset Z_c$ for any finite complex reflection group W and any parameter c. We will give an elementary proof of this inclusion.

3.2. Let $s \in \mathcal{S}$. There exists $\alpha_s \in \mathfrak{h}$ such that $\mathfrak{h} = \mathfrak{h}_s \oplus \mathbb{C}\alpha_s$, where $\mathfrak{h}_s = \{h \in \mathfrak{h} : s(h) = h\}$ and $s(\alpha_s) = e^{2\pi i n_s}\alpha_s$ for some rational number $0 < n_s < 1$. Similarly, we have $\alpha_s^\vee$, $-n_s$ and $\mathfrak{h}_s^*$. We choose not to normalise α_s and $\alpha_s^\vee$, but note that for all $h \in \mathfrak{h}_s$ it follows from

$$\alpha_s^\vee(h) = \alpha_s^\vee(s(h)) = e^{2\pi i n_s}\alpha_s^\vee(h)$$

that $\alpha_s^\vee(\mathfrak{h}_s) = 0$. Similarly, $\alpha_s(\mathfrak{h}_s^*) = 0$.

3.3. For any reflection $s \in W$, an explicit calculation shows, for $y \in \mathfrak{h}^*$ and $x \in \mathfrak{h}$

$$\omega_s(x, y) = \frac{\alpha_s(y)\alpha_s^\vee(x)}{\alpha_s^\vee(\alpha_s)}.$$

The principal commutation relation for H_c is

$$[x, y] = \sum_{s\in S} c_s \frac{\alpha_s(y)\alpha_s^\vee(x)}{\alpha_s^\vee(\alpha_s)} s. \tag{1}$$

3.4. Let s be a reflection in W and set $\lambda_s = 1 - e^{2\pi i n_s} \in \mathbb{C}$. For $h \in \mathfrak{h}$

$$s(h) = h - \lambda_s \frac{\alpha_s^\vee(h)}{\alpha_s^\vee(\alpha_s)}\alpha_s.$$

This action extends multiplicatively to $\mathbb{C}[\mathfrak{h}^*]$.

3.5. The functional $\alpha_s^\vee$ extends inductively to a $\mathbb{C}$–linear operator on $\mathbb{C}[\mathfrak{h}^*]$ as follows: $\alpha_s^\vee|_{\mathbb{C}} \equiv 0$ and for $p, p' \in \mathbb{C}[\mathfrak{h}^*]$,

$$\alpha_s^\vee(pp') = \alpha_s^\vee(p)p' + p\alpha_s^\vee(p') - \lambda_s \frac{\alpha_s^\vee(p)\alpha_s^\vee(p')}{\alpha_s^\vee(\alpha_s)}\alpha_s. \tag{2}$$

Define a graded operator on $\mathbb{C}[\mathfrak{h}^*]$ by

$$\tilde{s}(p) = p - \lambda_s \frac{\alpha_s^\vee(p)}{\alpha_s^\vee(\alpha_s)}\alpha_s. \tag{3}$$

An inductive calculation proves that

$$\text{the operators } s \text{ and } \tilde{s} \text{ agree on } \mathbb{C}[\mathfrak{h}^*]. \tag{4}$$

In particular,

$$p \in \mathbb{C}[\mathfrak{h}^*]^W \text{ if and only if } \alpha_s^\vee(p) = 0 \text{ for all reflections } s \in W. \tag{5}$$

3.6. Let $y \in \mathfrak{h}^*$ and $p \in \mathbb{C}[\mathfrak{h}^*]$. One proves inductively using (1), (2), (3) and (4) that

$$[p, y] = \sum_s c_s \frac{y(\alpha_s)\alpha_s^\vee(p)}{\alpha_s^\vee(\alpha_s)} s.$$

It follows from (5) that if $p \in \mathbb{C}[\mathfrak{h}^*]^W$ then $p \in Z_c$. Of course similar arguments apply to $\mathbb{C}[\mathfrak{h}]^W$, so we have proved

Proposition. *Let H_c be a Cherednik algebra. Then $A \subset Z_c$.*

We denote the corresponding morphism of varieties

$$\Upsilon : \mathrm{Spec}(Z_c) \longrightarrow \mathrm{Spec}(A) = \mathfrak{h}^*/W \times \mathfrak{h}/W. \tag{6}$$

4. Baby Verma modules

4.1. Recall by Proposition 3.6 that $A \subset Z_c$. Let A_+ denote the elements of A of with no scalar term. Define

$$\overline{H_c} = \frac{H_c}{A_+ H_c}.$$

This is a $\mathbb{Z}$–graded algebra where $\deg(x) = 1$, $\deg(y) = -1$ and $\deg(w) = 0$ for $x \in \mathfrak{h}$, $y \in \mathfrak{h}^*$ and $w \in W$. The PBW theorem, [8, Theorem 1.3], shows that $\overline{H_c}$ has a (vector space) triangular decomposition

$$\overline{H_c} \xrightarrow{\ \sim\ } \mathbb{C}[\mathfrak{h}^*]^{coW} \otimes \mathbb{C}[\mathfrak{h}]^{coW} \otimes \mathbb{C}W, \tag{7}$$

where "coW" stands for W–coinvariants, that is $\mathbb{C}[\mathfrak{h}^*]^{coW} = \mathbb{C}[\mathfrak{h}^*]/\mathbb{C}[\mathfrak{h}^*]_+^W \mathbb{C}[\mathfrak{h}^*]$. Thanks to Chevalley's Theorem, [6, Theorem 6.1.2], these coinvariant rings are graded versions of the regular representation of W, so in particular $\dim(\overline{H_c}) = |W|^3$.

4.2. Let $\overline{H_c^-} = \mathbb{C}[\mathfrak{h}]^{coW} * \mathbb{C}W$, a subalgebra of $\overline{H_c}$. The algebra map $\overline{H_c^-} \longrightarrow \mathbb{C}W$ sending an element $q * w$ to $q(0)w$, makes any W–module into an $\overline{H_c^-}$–module.

Given $S \in \Lambda$, a simple W–module, we define $M(S)$, a *baby Verma module*, by

$$M(S) = \overline{H_c} \otimes_{\overline{H_c^-}} S.$$

By construction $M(S)$ is an object of $\overline{H_c} - \mathrm{mod}_{\mathbb{Z}}$. Equation (7) shows that $\dim(M(S)) = |W| \dim(S)$. Furthermore for $T \in \Lambda$, set

$$f_T(t) = \sum_{i \in \mathbb{Z}} (\mathbb{C}[\mathfrak{h}^*]^{coW} : T[i]) t^i. \tag{8}$$

The polynomials $f_T(t)$ are called the *fake degrees* of W and have the property $f_T(1) = \dim T$. They have been calculated for all finite Coxeter groups. In the graded Grothendieck group of W, we have by construction

$$[M(S)] = \sum_{T \in \Lambda} f_T(t)[T \otimes S]. \tag{9}$$

4.3. The following proposition gathers results from [15, Section 3]. Recall $F : \overline{H_c} - \mathrm{mod}_{\mathbb{Z}} \longrightarrow \overline{H_c} - \mathrm{mod}$ is the forgetful functor.

Proposition. *Let $S, T \in \Lambda$.*

(1) *The baby Verma module $M(S)$ has a simple head, denoted $L(S)$.*
(2) *$M(S)$ is isomorphic to $M(T)[i]$ if and only if S and T are the same element of Λ and $i = 0$.*
(3) *$\{L(S)[i] : S \in \Lambda, i \in \mathbb{Z}\}$ gives a complete set of pairwise non–isomorphic simple graded $\overline{H_c}$–modules.*
(4) *$F(L(S))$ is a simple $\overline{H_c}$–module, and $\{F(L(S)) : S \in \Lambda\}$ is a complete set of pairwise non–isomorphic simple $\overline{H_c}$–modules.*
(5) *Let $P(S)$ be the projective cover of $L(S)$. Then $F(P(S))$ is the projective cover $F(L(S))$.*

4.4. **Lemma.** *Let $S, T \in \Lambda$ and i be a positive integer. Then $(M(S)[i] : L(T)) = 0$ and $(M(S) : L(T)) = \delta_{S,T}$.*

Proof. By construction the baby Verma module $M(T)$ is concentrated in non–negative degree and generated by its degree zero component, $1 \otimes T = M(T)_0$. Since $M(T)$ is a homomorphic image of $P(T)$, we can find $\tilde{T}$, a W–submodule of $P(T)_0$ isomorphic to T which maps onto $M(T)_0$. Since $P(T)$ has a simple head, it follows that $\tilde{T}$ generates $P(T)$.

If $L(T)$ is a composition factor of $M(S)[i]$, there exists a non–zero graded homomorphism from $P(T)$ to $M(S)[i]$, sending $P(T)_0$ to $M(S)_{-i}$. As $M(S)_{-i}$ is zero for positive values of i, this proves the first claim of the lemma. If i is zero, then $\tilde{T}$ maps to $M(S)_0$ and the second claim follows. □

4.5. An object V in $\overline{H_c} - \mathrm{mod}_{\mathbb{Z}}$ is said to have an *M-filtration* if it has a filtration

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq V_n = V$$

such that for $1 \leq j \leq n$, $V_j/V_{j-1} \cong M(S)[i]$ for S a simple W–module and $i \in \mathbb{Z}$. By [15, Corollary 4.3] the numbers

$$[V : M(S)[i]] = |\{j : V_j/V_{j-1} \cong M(S)[i]\}|$$

are independent of the filtration used.

4.6. Let $\overline{H_c^+} = \mathbb{C}[\mathfrak{h}^*]^{coW} * \mathbb{C}W$. Let $S \in \Lambda$ and consider S^* as a right W–module. We define

$$M^-(S) = (S^* \otimes_{\overline{H_c^+}} \overline{H_c})^{\circledast}.$$

It is shown in [15, Theorem 4.5] that for a simple W–module S, the projective cover $P(S)$ has an M-filtration such that for any $i \in \mathbb{Z}$ and simple W–module T

$$[P(S) : M(T)[i]] = (M^-(T)[i] : L(S)). \tag{10}$$

4.7. For this paragraph only suppose that W is a finite Coxeter group. Then (10) can be improved to the following Brauer–type reciprocity formula

$$[P(S) : M(T)[i]] = (M(T)[i] : L(S)). \tag{11}$$

This follows from an adaptation of [15, Theorem 5.1] with two special ingredients. Firstly, we require the antiautomorphism, ω, of H_c which sends $x \in \mathfrak{h}$, respectively $y \in \mathfrak{h}^*$, to the corresponding element $\tilde{x}$ of $\mathfrak{h}^*$, respectively $\tilde{y}$ of $\mathfrak{h}$, under the W–invariant form $(-,-)$ on $\mathfrak{h}$ and w to w^{-1} for $w \in W$. Secondly, we must observe that any simple W–module is self–dual (since the characters of W all take values in $\mathbb{R}$).

5. On the fibre $\Upsilon^*(0)$

5.1. Recall the morphism Υ given in (6). We will study the scheme theoretic fibres $\Upsilon^*(0)$ of the point $0 \in \mathfrak{h}^*/W \times \mathfrak{h}/W$. The closed points of these fibres, denoted $\Upsilon^{-1}(0)$, are precisely the maximal ideals of Z_c lying over A_+. Given such an ideal, M, the component of $\Upsilon^*(0)$ corresponding to M is defined to be $\mathrm{Spec}(\mathcal{O}_M)$, where

$$\mathcal{O}_M = (Z_c)_M/A_+(Z_c)_M.$$

The algebra $\mathcal{O}_M$ inherits a $\mathbb{Z}$–grading from $\overline{H_c}$, seen as follows. The $\mathbb{Z}$–grading on $\overline{H_c}$ corresponds to an algebraic action of $\mathbb{C}^*$ which preserves both Z_c and A and makes Υ a $\mathbb{C}^*$-equivariant morphism. Since 0 is the unique fixed point of $\mathfrak{h}^*/W \times \mathfrak{h}/W$, each fibre $\Upsilon^*(0)$ inherits a $\mathbb{C}^*$–action, in other words a $\mathbb{Z}$–grading.

5.2. Recall that a point $M \in \mathrm{Spec}(Z_c)$ is called *Azumaya* if and only if there is a unique simple H_c–module annihilated by M and its dimension equals $|W|$. By [8, Theorem 1.7] the Azumaya points are precisely the smooth points of $\mathrm{Spec}(Z_c)$ and any simple H_c–module lying over an Azumaya point is isomorphic, as a W–module, to the regular representation of W.

5.3. The algebra $\overline{H_c}$ of 4.1 splits into a direct sum of indecomposable algebras, the blocks, which by [4, Corollary 2.7] are in one–to–one correspondence with the elements of $\Upsilon^{-1}(0)$

$$\overline{H_c} = \bigoplus_{M \in \Upsilon^{-1}(0)} \mathcal{B}_M.$$

If M is an Azumaya point of $\operatorname{Spec}(Z_c)$ there is an isomorphism

$$\mathcal{B}_M \cong \operatorname{Mat}_{|W|}(\mathcal{O}_M), \tag{12}$$

by [4, Proposition 2.2].

5.4. For any simple W–module S, the baby Verma module $M(S)$ is a $\mathbb{Z}$–graded indecomposable $\overline{H_c}$–module by Proposition 4.3(1), and so a non–trivial module for a unique block. This gives a mapping

$$\Theta : \{\text{isomorphism classes of simple } W\text{–modules}\} \longrightarrow \{\text{elements of } \Upsilon^{-1}(0)\}. \tag{13}$$

This mapping is surjective, since each simple $\overline{H_c}$–module is a quotient of a baby Verma by Proposition 4.3(3).

5.5. Suppose that $M \in \Upsilon^{-1}(0)$ is smooth in $\operatorname{Spec}(Z_c)$. It follows from 5.2 and (12) that

$$\mathcal{B}_M \cong \operatorname{Mat}_{|W|}(\mathcal{O}_M).$$

In particular, $\mathcal{B}_M$ has a unique simple module, implying that $M = \Theta(S)$ for a unique element S of Λ. Moreover, we have an isomorphism

$$\mathcal{O}_M \xrightarrow{\ \sim\ } \operatorname{End}(P(S)).$$

The map is graded since it is given by multiplication by elements of $\mathcal{O}_M$. Thus we have a formula for $p_S(t) \in \mathbb{Z}[t, t^{-1}]$, the Poincaré polynomial of $\mathcal{O}_M$,

$$\begin{aligned} p_S(t) &= \sum_{i \in \mathbb{Z}} \dim \left(\operatorname{Hom}_{\overline{H_c}-\mathrm{mod}_{\mathbb{Z}}}(P(S), P(S)[i]) \right) t^i \\ &= \sum_{i \in \mathbb{Z}} (P(S)[i] : L(S)) t^i. \end{aligned}$$

Since the block $\mathcal{B}_M$ has only the simple modules $L(S)[i]$ ($i \in \mathbb{Z}$) it follows from (10) that

$$\begin{aligned} p_S(t) &= \sum_{i,j \in \mathbb{Z}} [P(S)[i] : M(S)[j]](M(S)[j] : L(S)) t^i \\ &= \sum_{i,j \in \mathbb{Z}} (M^-(S)[j-i] : L(S))(M(S)[j] : L(S)) t^{i-j} t^j \\ &= \left(\sum_{i \in \mathbb{Z}} (M^-(S)[i] : L(S)) t^{-i} \right) \left(\sum_{i \in \mathbb{Z}} (M(S)[i] : L(S)) t^i \right). \end{aligned} \tag{14}$$

5.6. Recall the fake polynomials, $f_S(t)$, defined in (8). Let b_S be the lowest power of t appearing in $f_S(t)$.

Theorem. *Suppose $M \in \Upsilon^{-1}(0)$ is smooth in* $\operatorname{Spec}(Z_c)$. *Then there exists a unique simple module $S \in \Lambda$ such that $\Theta(S) = M$. Furthermore $p_S(t)$, the Poincaré polynomial of $\mathcal{O}_M$, is given by*

$$p_S(t) = t^{b_{S^*} - b_S} f_S(t) f_{S^*}(t^{-1}).$$

Proof. Following 5.5, it remains to prove the formula for the Poincaré polynomial $p_S(t)$. By 5.2 $L(S)$ is isomorphic to the regular representation of W and $L(S)$ is the only possible composition factor of $P(S)$. We begin by determining the unique integer l_S for which $(L(S)_{l_S} : \mathbf{1}) \neq 0$, where $\mathbf{1}$ denotes the trivial W–module. Let j be the smallest integer such that $(M(S)_j : \mathbf{1}) \neq 0$. Since $L(S)$ is a homomorphic image of $M(S)$, $j \leq l_S$. If $j < l_S$ then necessarily $(M(S)[l_S - j] : L(S)) \neq 0$, contradicting Lemma 4.4. Hence $j = l_S$.

If T is a simple W–module, $\mathbf{1}$ is a summand of $T \otimes S$ if and only if $T \cong S^*$. Hence, by (9),

$$\sum_{k\in\mathbb{Z}} (M(S)_k : \mathbf{1}) t^k = f_{S^*}(t).$$

Thus l_S equals b_{S^*}, the lowest power of t appearing in $f_{S^*}(t)$. Moreover, for each copy of $\mathbf{1}$ in $M(S)_k$ there occurs a corresponding copy of $L(S)[k - b_{S^*}]$ in $M(S)$, yielding

$$\sum_{i\in\mathbb{Z}} (M(S)[i] : L(S)) t^i = \sum_{i\in\mathbb{Z}} (M(S) : L(S)[-i])(t^{-1})^i = t^{b_{S^*}} f_{S^*}(t^{-1}).$$

A similar argument shows that

$$\sum_{i\in\mathbb{Z}} (M^-(S)[i] : L(S)) t^{-i} = t^{-b_S} f_S(t).$$

The result follows from the above and (14). $\square$

5.7. We remark that the statement of Theorem 5.6 simplifies if W is a finite Coxeter group. As observed in 4.7 every simple W–module is self–dual, so the Poincaré polynomial above becomes

$$p_S(t) = f_S(t) f_S(t^{-1}).$$

5.8. On specialising t to 1 and recalling that $f_S(1) = \dim S$, we obtain the following corollary, which generalises [8, Proposition 4.16], solves [8, Problem 17.15] and incidentally confirms [8, Conjecture 17.14].

Corollary. *Suppose $\Upsilon^{-1}(0)$ consists of smooth points in* $\operatorname{Spec}(Z_c)$. *Then $\Theta : \Lambda \longrightarrow \Upsilon^{-1}(0)$ is a bijection such that the scheme theoretic multiplicity of $\Upsilon^{-1}(0)$ at the point $\Theta(S)$ is* $\dim(S)^2$.

6. The symmetric group case

6.1. Throughout this section W will be the symmetric group $\mathfrak{S}_n$ and H_c will be the Cherednik algebra of W with *non–zero* parameter c. Since $Z_c \cong Z_{\lambda c}$ for any $\lambda \neq 0$ and $\operatorname{Spec}(Z_c)$ is smooth for a generic choice of c, [8, Corollary 16.2], we see that $\operatorname{Spec}(Z_c)$ is smooth for any non–zero value of c.

6.2. We recall notation and several basic facts on the representation theory of $\mathfrak{S}_n$ over $\mathbb{C}$. The simple $\mathfrak{S}_n$–modules are indexed by partitions of n, which we will write as $\lambda = (\lambda_1, \lambda_2, \ldots)$. For such a partition $\lambda \vdash n$, we denote the simple module by S_λ. In particular $S_{(n)} = \mathbf{1}$ and $S_{(1^n)} = \mathbf{sign}$, the trivial and sign representation of $\mathfrak{S}_n$ respectively. Given λ, a partition of n, we let $\lambda' \vdash n$ denote the transpose of λ. We have $S_{\lambda'} \cong S'_\lambda$, where, for any $\mathfrak{S}_n$–module V, we set $V' = V \otimes \mathbf{sign}$. The Young subgroup $\mathfrak{S}_\lambda \leq \mathfrak{S}_n$ is the row stabiliser of the Young tableau of shape λ, with the tableau numbered in the order that one reads a book. As an abstract group it is isomorphic to the product $\prod_i \mathfrak{S}_{\lambda_i}$.

6.3. We recall two polynomials which arise in the theory of symmetric functions. The (two variable) *Kostka–Macdonald coefficients* $K_{\lambda\mu}(q,t)$ are the transition functions between the Macdonald polynomials and Schur functions, see [17, VI.8]. The *Kostka polynomials* are defined as $K_{\lambda\mu}(0,t)$. In particular, we have

$$K_\lambda(t) \equiv t^{-n(\lambda')} K_{\lambda(1^n)}(0,t) = (1-t)\cdots(1-t^n)\prod_{u\in\lambda}(1-t^{h_\lambda(u)})^{-1} \in \mathbb{Z}[t,t^{-1}],$$

where $n(\lambda') = \sum_{i\geq 1}(i-1)\lambda'_i$ and $h_\lambda(u)$ is the hook–length of u in λ, meaning the number of boxes directly underneath or to the right of the box u (and including u) in the Young diagram of λ.

6.4. The first part of the following theorem recovers [9, Theorem 1.2].

Theorem. *Let λ be a partition of n.*
(1) *The Poincaré polynomial of $\mathcal{O}_{\Theta(S_\lambda)}$ is given by*

$$p_{S_\lambda}(t) = K_\lambda(t)K_\lambda(t^{-1}).$$

(2) *The image of the simple H_c–module $L(S_\lambda)$ in the graded Grothendieck group of W is*

$$[L(S_\lambda)] = \sum_{\mu\vdash n} K_{\mu\lambda}(t,t)[S_\mu].$$

Proof. Let μ, ρ be partitions of n. We let $H_\mu(t) = \prod_{u\in\mu}(1-t^{h_\mu(u)})$ denote the hook length polynomial. We will write χ^λ for the character of $\mathfrak{S}_n$ corresponding to S_λ, and χ^λ_ρ for the evaluation of χ^λ on a conjugacy class with cycle type ρ.
(1) By [21, Theorem 3.2]

$$f_{S_\lambda}(t) = (1-t)\cdots(1-t^n)s_\lambda(1,t,t^2,\ldots), \tag{15}$$

where s_λ is the Schur function for partition λ. The result follows from 5.7 and [17, Example I.3.2] which shows that

$$s_\lambda(1,t,t^2,\ldots) = t^{n(\lambda)}H_\lambda(t)^{-1}. \tag{16}$$

(2) Let $p_{\lambda,\mu}(t) = \sum_i [L(S_\lambda)_i : S_\mu]t^i$. In the Grothendieck group of graded W–modules we have, by the proof of Theorem 5.6,

$$\begin{aligned}
[M(S_\lambda)] &= \sum_\mu \left(\sum_i [M(S_\lambda)_i : S_\mu]t^i \right) [S_\mu] \\
&= \sum_\mu \left(\sum_{i,j} [M(S_\lambda)[j] : L(S_\lambda)][L(S_\lambda)_{i+j} : S_\mu]t^i \right) [S_\mu] \\
&= \sum_\mu t^{-b_\lambda} f_{S_\lambda}(t) p_{\lambda,\mu}(t)[S_\mu].
\end{aligned}$$

By Lemma 4.4, the transition matrix between the baby Verma modules and the simple $\overline{H_c}$–modules is invertible in $\mathbb{C}[[t]]$, so it is enough to show that

$$[M(S_\lambda)] = \sum_\mu t^{-b_\lambda} f_{S_\lambda}(t) K_{\mu\lambda}(t,t)[S_\mu]. \tag{17}$$

We will prove (17) by showing that the characters of both sides are equal.

By (9) the character of $[M(S_\lambda)]$ is $\sum_\mu f_{S_\mu}(t)\chi^\mu\chi^\lambda$, so we need to show that for all partitions ρ

$$\sum_\mu f_{S_\mu}(t)\chi^\mu_\rho\chi^\lambda_\rho = \sum_\mu t^{-b_\lambda} f_{S_\lambda}(t) K_{\mu\lambda}(t,t)\chi^\mu_\rho.$$

By [17, p.355]

$$\sum_\mu K_{\mu\lambda}(t,t)\chi^\mu_\rho = \frac{H_\lambda(t)}{\prod_{i\geq 1}(1-t^{\rho_i})}\chi^\lambda_\rho, \tag{18}$$

and by (15) and (16)

$$f_{S_\mu}(t) = t^{n(\mu)} \prod_{j=1}^n (1-t^j) H_\mu(t)^{-1}.$$

Note in particular this shows that $b_\mu = n(\mu)$ for all partitions μ. Therefore we have to prove that

$$\sum_\mu f_{S_\mu}(t)\chi^\mu_\rho\chi^\lambda_\rho = \frac{\prod_{j=1}^n(1-t^j)}{\prod_{i\geq 1}(1-t^{\rho_i})}\chi^\lambda_\rho$$

The particular case of (18) with $\lambda = (n)$ reduces the above equation to

$$\sum_\mu f_{S_\mu}(t)\chi^\mu_\rho\chi^\lambda_\rho = \sum_\mu K_{\mu,(n)}(t,t)\chi^\mu_\rho\chi^\lambda_\rho. \tag{19}$$

By [17, p.82 and p.362] $K_{\mu,(n)}(t,t) = f_{S_\mu}(t)$, proving that indeed (19) holds. □

6.5. In [9, Theorem 1.5] it is shown that an analogue of Theorem 6.4(1) holds for the wreath product $W = (\mathbb{Z}/N\mathbb{Z}) \wr \mathfrak{S}_n$. This result can be proved as above, using the fake degrees as described in [21, (5.5)].

6.6. **Springer theory.** Before proving Theorem 6.7 we need to recall several results from Springer theory. Conjugacy classes of nilpotent matrices in $\mathrm{Mat}_n(\mathbb{C})$ are classified by partitions of n thanks to the Jordan normal form: for each partition $\lambda \vdash n$, fix a representative of the corresponding nilpotent conjugacy class, e_λ. Let $\mathcal{B}$ be the flag variety of the Lie algebra $\mathrm{Mat}_n(\mathbb{C})$, consisting of all Borel subalgebras of $\mathrm{Mat}_n(\mathbb{C})$, and let $\mathcal{B}_\lambda \subseteq \mathcal{B}$ denote the Borel subalgebras containing e_λ. (Note that $\mathcal{B} = \mathcal{B}_{(1^n)}$.) Let $d_\lambda = 2\dim_{\mathbb{C}} \mathcal{B}_\lambda$. Let $H^*(\mathcal{B}_\lambda, \mathbb{C})$ denote the singular cohomology of $\mathcal{B}_\lambda$ with complex coefficients. The following theorem presents the results we will use.

Theorem. *For $\lambda \vdash n$ and $i \in \mathbb{N}$, there is an action of $\mathfrak{S}_n$ on $H^i(\mathcal{B}_\lambda, \mathbb{C})$ enjoying the following properties.*

1. *There is a $\mathfrak{S}_n$–equivariant isomorphism of algebras $\mathbb{C}[\mathfrak{h}^*]^{co\mathfrak{S}_n} \cong H^*(\mathcal{B}, \mathbb{C})$ doubling degree.*
2. *The simple module S_λ is not a component of $H^i(\mathcal{B}, \mathbb{C})$ for $i < d_\lambda$.*
3. *As an $\mathfrak{S}_n$–module, $H^{d_\lambda}(\mathcal{B}_\lambda, \mathbb{C}) \cong S_\lambda$, whilst for odd i or $i > d_\lambda$ $H^i(\mathcal{B}_\lambda, \mathbb{C}) = 0$.*
4. *As an $\mathfrak{S}_n$–module, $H^*(\mathcal{B}_\lambda, \mathbb{C}) \cong \mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} \mathbf{1}$.*
5. *The inclusion $\mathcal{B}_\lambda \subseteq \mathcal{B}$ induces a surjective $\mathfrak{S}_n$–equivariant graded algebra homomorphism*
$$\pi_\lambda : H^*(\mathcal{B}, \mathbb{C}) \longrightarrow H^*(\mathcal{B}_\lambda, \mathbb{C}).$$
6. *Under the $\mathbb{C}[\mathfrak{h}^*]^{coW} * \mathfrak{S}_n$–module structure induced by π_λ, $H^*(\mathcal{B}_\lambda, \mathbb{C})$ has socle S_λ.*

Proof. For (1)–(5) see [19],[20],[14], [7], [3] and [6, Sections 6.5 and 8.9].

(6) Define a partial order on monomials in $\mathbb{C}[x_1, \dots, x_n]$ by declaring $x_1^{i_1} \dots x_n^{i_n} \le x_1^{j_1} \dots x_n^{j_n}$ if and only if $i_k \le j_k$ for all $k = 1, \dots, n$. By [11, Proposition 4.2], the algebra $\mathbb{C}[\mathfrak{h}^*]^{coW} / \ker \pi_\lambda$ has a basis $\mathfrak{B}(\lambda)$ consisting of (the images of) monomials of $\mathbb{C}[\mathfrak{h}^*] = \mathbb{C}[x_1, \dots, x_n]$ with the following property: the elements of $\mathfrak{B}(\lambda)$ are the monomials which are smaller in the above partial order than the monomials in $\mathfrak{B}(\lambda)$ of degree d_λ. We see therefore that the socle of $H^*(\mathcal{B}_\lambda, \mathbb{C})$ as a $\mathbb{C}[\mathfrak{h}^*]^{coW}$–module is concentrated in degree d_λ. The result follows from (3). □

In this situation, the *Springer correspondence* is simply the association of the nilpotent class parametrised by λ to the simple $\mathfrak{S}_n$–module $S_\lambda \cong H^{d_\lambda}(\mathcal{B}_\lambda, \mathbb{C})$.

6.7. Let $M_\lambda = \Theta(S_\lambda) \in \Upsilon^{-1}(0)$. By 5.2 we have an isomorphism
$$\psi : \frac{H_c}{M_\lambda H_c} \longrightarrow \mathrm{End}_{\mathbb{C}}(L(S_\lambda)) \tag{20}$$
defined by the action of H_c on $L(S_\lambda)$. Let $\mathbf{e}, \mathbf{e}_- \in \mathbb{C}\mathfrak{S}_n \cong L(S_\lambda)$ denote the trivial and sign idempotents respectively. Since $\mathbb{C}\mathfrak{S}_n \subset H_c$ we can fix ϵ_λ (respectively ϵ'_λ), a base vector in the one–dimensional space $\mathbf{e}(H_c/M_\lambda H_c)\mathbf{e}_-$ (respectively $\mathbf{e}_-(H_c/M_\lambda H_c)\mathbf{e}$). The subalgebras $\mathbb{C}[\mathfrak{h}^*], \mathbb{C}[\mathfrak{h}] \subset H_c$ generate two subspaces $\mathbb{C}[\mathfrak{h}^*] \cdot \epsilon_\lambda, \epsilon_\lambda \cdot \mathbb{C}[\mathfrak{h}] \subset H_c/M_\lambda H_c$ which are invariant under the left, respectively right $\mathfrak{S}_n$-action on $H_c/M_\lambda H_c$ by multiplication.

The following theorem confirms [8, Conjecture 17.12].

Theorem. *In the above notation there are $\mathfrak{S}_n$–module isomorphisms*
$$\mathbb{C}[\mathfrak{h}^*] \cdot \epsilon_\lambda \cong \mathrm{Ind}_{\mathfrak{S}_{\lambda'}}^{\mathfrak{S}_n} \mathbf{1} \qquad , \qquad \epsilon_\lambda \cdot \mathbb{C}[\mathfrak{h}] \cong \mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} \mathbf{sign}.$$

In particular, these two $\mathfrak{S}_n$–modules have a single non–zero irreducible component in common: it is isomorphic to $S_{\lambda'}$.

Proof. Throughout the proof let $R = \mathbb{C}[\mathfrak{h}^*]^{co\mathfrak{S}_n}$. Let $N = n(n-1)/2 = d_{(1^n)}$. By Theorem 6.6 $R_j = 0$ if $j > N$ and R_N is the sign representation. Let $0 \neq s \in R_N$. Recall that given V, a $\mathfrak{S}_n$–module, we set $V' = V \otimes \mathbf{sign}$.

Under the isomorphism ψ of (20), the elements $\mathbf{e}$ and $\mathbf{e}_-$ correspond to projection onto the trivial and sign components of $L(S_\lambda)$ respectively. Thus ϵ_λ is the unique (up to scalars) transformation which sends the sign component of $L(S_\lambda)$ to the trivial component of $L(S_\lambda)$ and is zero on all other isotypic components. Similarly ϵ'_λ sends the trivial component of $L(S_\lambda)$ to the sign component of $L(S_\lambda)$ and is zero on all other isotypic components. As a left $\mathfrak{S}_n$–module therefore, $\mathbb{C}[\mathfrak{h}^*]\cdot\epsilon_\lambda$ and $\mathbb{C}[\mathfrak{h}^*]\cdot\epsilon'_\lambda$ correspond respectively to the subspaces $X = R\cdot t_\lambda \subseteq L(S_\lambda)$ and $Y = R\cdot s_\lambda$, where t_λ is a base vector for the trivial representation in $L(S_\lambda)$ and s_λ for the sign representation.

By [8, Theorem 1.7(ii)], H_c and Z_c are Morita equivalent, so in particular their injective dimensions both equal $\dim \operatorname{Spec}(Z_c) = 2(n-1)$. Since $\overline{H}_c$ is obtained from H_c by factoring out a regular sequence in Z_c of length $2(n-1)$ it follows that the injective dimension of $\overline{H}_c$ is 0, [5, Theorem 4.2]. Since the composition factors of the projective cover $P(S_\lambda)$ are all isomorphic to $L(S_\lambda)$ it follows that the socle of $P(S_\lambda)$ is $L(S_\lambda)$. Thus $M(S_\lambda)$ also has socle $L(S_\lambda)$, since $P(S_\lambda)$ is filtered by copies of $M(S_\lambda)$. Therefore t_λ belongs to the tensor product $S_\lambda\otimes S_\lambda \subset L(S_\lambda) \subseteq M(S_\lambda)$ where the first tensorand is the highest degree component of R isomorphic to S_λ, whilst s_λ belongs to the tensor product $S_{\lambda'}\otimes S_\lambda \subset L(S_\lambda)\subseteq M(S_\lambda)$ where the first tensorand is the highest degree component of R isomorphic to $S_{\lambda'}$. To be explicit, let $\{v_i\}$ be a basis of S_λ and let $\{f_i\}$ and $\{g_i\}$ be the corresponding bases for the copies of S_λ and $S_{\lambda'}$ in the highest possible degree of R. We take

$$t_\lambda = \sum_i f_i \otimes v_i \qquad \text{and} \qquad s_\lambda = \sum_i g_i\otimes v_i.$$

Consider the maps

$$\tau : R \longrightarrow X = R\cdot t_\lambda \qquad \text{and} \qquad \tau' : R' \longrightarrow Y = R\cdot s_\lambda,$$

defined by $\tau(p) = p\cdot t_\lambda$ and $\tau'(p\otimes 1) = p\cdot s_\lambda$. Both τ and τ' are $\mathfrak{S}_n$–equivariant surjective homomorphisms, since $t_\lambda \in \mathbf{1}$ and $s_\lambda\in\mathbf{sign}$ respectively.

We prove first that τ factors through $\pi_{\lambda'} : H^*(\mathcal{B},\mathbb{C}) \longrightarrow H^*(\mathcal{B}_{\lambda'},\mathbb{C})$, and that τ' factors through $\pi_\lambda\otimes \mathrm{id} : H^*(\mathcal{B},\mathbb{C})' \longrightarrow H^*(\mathcal{B}_\lambda,\mathbb{C})'$. Recall the non–degenerate form

$$(-,-) : R\times R \longrightarrow \mathbb{C},$$

defined by setting (p,q) equal to the coefficient of pq in R_N with respect to s. The form pairs R_i with R_{N-i} and, since $(wr, ws) = \mathbf{sign}(w)(r,s)$, a representation V with V'. Let $p\in\ker\pi_{\lambda'}$ and observe that since $\pi_{\lambda'}(Rp) = 0$ the intersection of Rp with the $S_{\lambda'}$–component of $H^{d_{\lambda'}}(\mathcal{B},\mathbb{C})$ is zero, by Parts (3) and (5) of Theorem 6.6. Thus $(R, pf_i) = (Rp, f_i) = 0$ for all i . This confirms that τ factors through $\pi_{\lambda'}$. Similarly, one sees if $q\in\ker\pi_\lambda$ then $\pi_\lambda(Rq) = 0$, ensuring that the intersection of Rq with the S_λ–component of $H^{d_\lambda}(\mathcal{B},\mathbb{C})$ is zero. Thus $(R, qg_i) = (Rq, g_i) = 0$ and so τ' factors through $\pi_\lambda\otimes\mathrm{id}$.

We have induced $R * \mathfrak{S}_n$–epimorphisms

$$\sigma : H^*(\mathcal{B}_{\lambda'}, \mathbb{C}) \longrightarrow X \qquad \text{and} \qquad \sigma' : H^*(\mathcal{B}_{\lambda}, \mathbb{C})' \longrightarrow Y.$$

Thanks to the non–degeneracy of the form $(-,-)$, we can find $f_i' \in R$ such that $f_i' f_j = \delta_{ij} s \in R_N$. Then, for any i, $f_i' \cdot t_\lambda = s \otimes v_i \in X$. Taking the span over all i shows that $S_{\lambda'} \cong R_N \otimes S_\lambda \subseteq X$. It follows from Parts (2) and (6) of Theorem 6.6 that the socle of $H^*(\mathcal{B}_{\lambda'}, \mathbb{C})$ does not lie in the kernel of σ, and so σ is injective. By Part (4) of Theorem 6.6 we deduce that

$$\mathrm{Ind}_{\mathfrak{S}_{\lambda'}}^{\mathfrak{S}_n} \mathbf{1} \cong H^*(\mathcal{B}_{\lambda'}, \mathbb{C}) \cong X \cong \mathbb{C}[\mathfrak{h}^*] \cdot \epsilon_\lambda.$$

Arguing as in the above paragraph we see that

$$\mathrm{Ind}_{\mathfrak{S}_{\lambda}}^{\mathfrak{S}_n} \mathbf{sign} \cong H^*(\mathcal{B}_{\lambda}, \mathbb{C})' \cong Y \cong \mathbb{C}[\mathfrak{h}^*] \cdot \epsilon_\lambda'.$$

Let ω be the antiautomorphism of H_c discussed in 4.7. Since $\omega(M_\lambda)$ annihilates $L(S_\lambda^*) \cong L(S_\lambda)$ there is an induced antiautomorphism on $H_c/M_\lambda H_c$. Under this antiautomorphism $\mathbb{C}[\mathfrak{h}^*] \cdot \epsilon_\lambda'$ is sent to $\epsilon_\lambda \cdot \mathbb{C}[\mathfrak{h}]$. Since all $\mathfrak{S}_n$–modules are self–dual, we deduce that

$$\epsilon_\lambda \cdot \mathbb{C}[\mathfrak{h}] \cong \mathrm{Ind}_{\mathfrak{S}_{\lambda}}^{\mathfrak{S}_n} \mathbf{sign}.$$

The final sentence of the theorem is a restatement of Young's construction of the irreducible representations of $\mathfrak{S}_n$, [13, Corollary 4.16]. □

6.8. We end this section by associating Theorem 6.7 to similar results found in the study of *principal nilpotent pairs.*

6.9. By [8, Theorem 11.16], there is an isomorphism of varieties

$$\mathrm{Spec}\, Z_c \longrightarrow \mathcal{CM}_n,$$

where $\mathcal{CM}_n$ denotes the Calogero–Moser space $\{(X,Y) \in \mathrm{Mat}_n(\mathbb{C}) \times \mathrm{Mat}_n(\mathbb{C}) : [X,Y] + \mathrm{Id} = \text{rank 1 matrix}\}//PGL_n(\mathbb{C})$. Under this isomorphism, the morphism $\Upsilon : \mathrm{Spec}\, Z_c \longrightarrow \mathfrak{h}^*/W \times \mathfrak{h}/W$ corresponds to the morphism sending the matrices (X,Y) to the pair of n–tuples consisting of their eigenvalues. An explicit description of the bijection between the partitions of n and the elements of $\mathcal{CM}_n$ in $\Upsilon^{-1}(0)$ is given in [22, Section 6].

6.10. Let $\mathrm{Hilb^n}(\mathbb{C}^2)$ be the Hilbert scheme of n points in the plane, whose points are codimension n ideals in $\mathbb{C}[X,Y]$, see [18] for details. The Chow morphism $\mathrm{Hilb^n}(\mathbb{C}^2) \longrightarrow (\mathfrak{h}^* \oplus \mathfrak{h})/W = \mathrm{Sym^n}(\mathbb{C}^2)$ sends such an ideal to its support (counted with multiplicity). There is a $\mathbb{C}^* \times \mathbb{C}^*$–action on $\mathrm{Hilb^n}(\mathbb{C}^2)$ induced from the embedding $\mathbb{C}^* \times \mathbb{C}^* \subseteq \mathbb{C}^2$. The $\mathbb{C}^* \times \mathbb{C}^*$–fixed points on $\mathrm{Hilb^n}(\mathbb{C}^2)$ are in bijection with the partitions of n: on the ideal corresponding to the fixed point of type λ, multiplication by X (respectively Y) yields a nilpotent transformation with Jordan form of type λ (respectively λ'). It follows that these fixed points correspond to conjugacy classes of principal nilpotent pairs in $\mathfrak{sl}_n(\mathbb{C})$, [13, Section 5].

6.11. Nakajima's quiver varieties relate Calogero–Moser space and the Hilbert scheme. There is a geometric family $(\mathcal{CM}_n)_z$ over $\mathbb{C}$ whose generic fibre $(\mathcal{CM}_n)_z$ for $z \neq 0$ is $\mathcal{CM}_n$ and whose degenerate point at $z = 0$ is $\mathrm{Hilb^n}(\mathbb{C}^2)$, [18, Section 3.2]. This family admits a $\mathbb{C}^* \times \mathbb{C}^*$–action, which agrees with the above $\mathbb{C}^* \times \mathbb{C}^*$–action on $\mathrm{Hilb^n}(\mathbb{C}^2)$ when $z = 0$. For $z \neq 0$, the diagonal $\mathbb{C}^*$–action dilates z and the anti–diagonal action corresponds to the $\mathbb{C}^*$–action on Z_c considered in 4.1. Fixing a partition $\lambda \vdash n$, the anti–diagonal $\mathbb{C}^*$–fixed points corresponding to λ give a section of the family $(\mathcal{CM})_z$ over $\mathbb{C} \setminus \{0\}$. The closure of this section yields the $\mathbb{C}^* \times \mathbb{C}^*$–fixed point of $\mathrm{Hilb^n}(\mathbb{C}^2)$ corresponding to λ and hence a representative (X_λ, Y_λ) in the conjugacy class of principal nilpotent pairs of type λ of $\mathfrak{sl}_n(\mathbb{C})$. The Springer correspondence associates to X_λ (respectively Y_λ) the simple module S_λ (respectively $S_{\lambda'}$), [13, Corollary 4.16]. This confirms (an amended form) of [8, Conjecture 17.13].

7. Remarks

7.1. A basic problem in the representation theory of H_c is:

Problem 1: Find the graded W–character of $L(S)$ for all $S \in \Lambda$.

This simply means finding the composition multiplicities in $[L(S)] = \sum_{i\in\mathbb{Z}, T\in\Lambda}(L(S) : T[i])[T]t^i$. By Lemma 4.4 this is equivalent to the problem:

Problem 1′: For all $S \in \Lambda$ find the polynomials $m_{S,T}(t) = \sum_{i\in\mathbb{Z}, T\in\Lambda}(M(S) : L(T)[i])t^i$.

Obviously, to solve Problem 1′ we can work in the blocks of the algebra $\overline{H_c}$. Thus an important subproblem is to describe which simple modules lie in a given block:

Problem 2: Determine the partition of simple W–modules yielding the blocks of $\overline{H_c}$.

In the notation of (13) this corresponds to determining the fibres of Θ.

In Section 4 we have solved Problems 1′ and 2 for simple W–modules, S, such that $\Theta(S)$ is a smooth point of $\mathrm{Spec}(Z_c)$ and in Section 5 we explictly solved Problem 1 under the same hypothesis.

7.2. Problem 2 is closely related to the determination of which complex reflection groups have $\mathrm{Spec}(Z_c)$ smooth for generic values of c. Thanks to Section 4, a point of $M \in \Upsilon^{-1}(0)$ is singular in $\mathrm{Spec}(Z_c)$ if $\Theta^{-1}(M)$ is not a singleton. The following lemma, due to Brown, proves the converse, showing that an arbitrary point $M' \in \mathrm{Spec}(Z_c)$ is Azumaya if and only if there is a unique simple H_c-module annihilated by M'. One might hope that if the points of $\Upsilon^{-1}(0)$ are all smooth then $\mathrm{Spec}(Z_c)$ is smooth.

Lemma (K.A.Brown)**.** *Let $\mathfrak{m}$ be a maximal ideal of H_c. Suppose there exists a unique maximal ideal M of H_c with $M \cap Z_c = \mathfrak{m}$. Then $\mathfrak{m}$ is an Azumaya point of* $\mathrm{Spec}\, Z_c$.

Proof. Let $H = H_c$ and $Z = Z_c$. Since M is the unique maximal ideal of H lying over $\mathfrak{m}$ we can localise at M in H to obtain H_M. Moreover $H \subseteq H_M$ since H is prime. Let $\mathbf{e}$ be the trivial idempotent of $\mathbb{C}W$, considered as an element of H. Since H_M is a local ring, by [10, Theorem 1] there exists a positive integer t such that $H_M\mathbf{e} = P^{\oplus t}$, where P is the unique indecomposable projective H_M–module. But, by [8, Theorem 3.1] $\mathbf{e}H\mathbf{e} \cong Z$, so that $\mathbf{e}H_M\mathbf{e} \cong Z_{\mathfrak{m}}$. Thus

$\mathrm{End}_{H_M}(H_M\mathbf{e}) \cong Z_\mathfrak{m}$, showing that $t = 1$. Since $H_M \cong P^{\oplus n}$ for some n we have

$$H_M \cong \mathrm{End}_{H_M}(H_M) \cong \mathrm{Mat}_n(Z_\mathfrak{m}).$$

Therefore Z_M and H_M are Morita equivalent, and since H_M has finite global dimension so too does $Z_\mathfrak{m}$. It follows that $\mathfrak{m}$ is a smooth point of $\mathrm{Spec}\, Z_c$. The lemma follows from 5.2. □

7.3. The analysis of Section 4 does provide a list of new examples for which $\mathrm{Spec}(Z_c)$ is not smooth at generic values of c. The following results generalises the dihedral case studied in [8, Section 16].

Proposition. *Suppose W is a finite Coxeter group of type D_{2n} ($n \geq 2$), E, F, H or $I_2(m)$ ($m \geq 5$). Then Z_c, the centre of H_c, is singular for all values of c.*

Proof. Suppose there exists a value c such that Z_c is smooth. Thanks to Corollary 6.4(1) the blocks of $\overline{H_c}$ are in bijective correspondence with the simple W–modules and by 5.2 each simple H_c–module, $L(S)$, is isomorphic to the regular representation of W. The proof of Theorem 5.6 shows that $(M(S) : L(S))$ equals the coefficient of t^{b_S} in the fake degree $f_{S^*}(t)$. Since Lemma 4.4 shows that $(M(S) : L(S)) = 1$ this coefficient is necessarily 1 for all fake degrees. But there are fake degrees for D_{2n}, E_7 and E_8 which have coefficient 2, [12, Remark 5.6.7], [2].

Let j_S be the degree of $f_{S^*}(t)$, so that the trivial module appears in $M(S)_{j_S}$, by (9). Since $L(S)$ is the only composition factor of $M(S)$ we thus find a bound on the dimension of $L(S)$ given by

$$\dim L(S) \leq \dim(S) \left(\dim \mathbb{C}[\mathfrak{h}^*]^{coW}_{<b_S} + \dim \mathbb{C}[\mathfrak{h}^*]^{coW}_{>j_S} + \min\{\dim \mathbb{C}[\mathfrak{h}^*]^{coW}_{b_S}, \dim \mathbb{C}[\mathfrak{h}^*]^{coW}_{j_S}\}\right). \quad (21)$$

Indeed, since $\mathbf{1}$ appears only in $L(S)_{b_S}$, this inequality expresses that all components of $L(S)$ of degree less than b_S (respectively greater than j_S) must occur in $\mathbb{C}[\mathfrak{h}^*]^{coW}_{<b_S}$ (respectively $\mathbb{C}[\mathfrak{h}^*]^{coW}_{>j_S}$), and the components of $L(S)$ of degree b_S belong to $\mathbb{C}[\mathfrak{h}^*]^{coW}_{b_S}$ or to $\mathbb{C}[\mathfrak{h}^*]^{coW}_{j_S}$. Using [2] and [1] we can calculate the right hand side of (21) and hence show that there exists a simple W–module S such that $\dim L(S) < |W|$ in types E_6, F_4, H_3, H_4 and $I_2(m)$ ($m \geq 5$).

For W the Coxeter group of type E_6 (respectively F_4, H_4, $I_2(m)$ for $m \geq 5$) we take S to be the unique 10–dimensional simple module (respectively the unique 12–dimensional simple module, the representation labelled χ_7 in [1], the reflection representation); the right hand side of (21) equals 49960 (respectively 1020, 8808, 8) which is less than the order of W.

For W the Coxeter group of type H_3 let S be the reflection representation, which has fake degree $f_S(t) = t + t^5 + t^9$. In this case the right hand side of (21) equals 120, the order of the group W. Let $S' = S \otimes \epsilon$ be the tensor product of S with the sign representation. The fake degree of S' is $t^{(9+5+1)} f_s(t^{-1}) = t^{14} + t^{10} + t^6$. Thus there are two copies of the sign representation in $M(S)_{>9}$, one in degree 10, the other in degree 14. Hence we can subtract 1 from our estimate in (21), showing that $\dim L(S) < 120 = |W|$, as required. □

There are a number of other complex reflection groups whose fake degrees have leading coefficient greater than 1, so we can prove non–smoothness in greater generality. The techniques above do not work for D_{2n+1} however.

7.4. Type G_2 is the smallest example of a Weyl group for which Z_c is singular for generic values of c. In this case, calculation shows that $\Upsilon^{-1}(0)$ has five elements, with $D_{\pm}$, the pair of two dimensional simple modules, being the only simple W–modules sent to the same element under Θ. The polynomials of Problem $1'$ are

$$m_{D_+,D_+}(t) = m_{D_-,D_-}(t) = 1 + t^4, \quad m_{D_+,D_-}(t) = m_{D_-,D_+}(t) = t + t^3.$$

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