CALOGERO-MOSER SPACE, REDUCED RATIONAL CHEREDNIK ALGEBRAS AND TWO-SIDED CELLS.

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Abstract. We conjecture that the "nilpotent points" of Calogero-Moser space for reflection groups are parametrised naturally by the two-sided cells of the group with unequal parameters. The nilpotent points correspond to blocks of restricted Cherednik algebras and we describe these blocks in the case $G = \mu_\ell \wr \mathfrak{S}_n$ and show that in type B our description produces an existing conjectural description of two-sided cells.

1. Introduction

1.1. Smooth points are all alike; every singular point is singular in its own way. Calogero-Moser space associated to the symmetric group has remarkable applications in a broad range of topics; in [3], Etingof and Ginzburg introduced a generalisation associated to any complex reflection group which has also found a variety of uses. The Calogero-Moser spaces associated to a complex reflection group, however, exhibit new behaviour: they are often singular. The nature of these singularities remains a mystery, but their existence has been used to solve the problem of the existence of crepant resolutions of symplectic quotient singularities. The generalised Calogero-Moser spaces are moduli spaces of representations of rational Cherednik algebras and so their geometry reflects the representation theory of these algebras: smooth points correspond to irreducible representations of maximal dimension; singular points to smaller, more interesting representations. In this note we conjecture a strong link between the representations corresponding to some particularly interesting "nilpotent points" of Calogero-Moser space and Kazhdan-Lusztig cell theory for Hecke algebras with unequal parameters. To justify the conjecture we give a combinatorial parametrisation of these points, thus answering a question of [7], and then relate this parametrisation to the conjectures of [1] on the cell theory for Weyl groups of type B.

1.2. Let W be a complex reflection group and $\mathfrak{h}$ its reflection representation over $\mathbb{C}$. Let $\mathcal{S}$ denote the set of complex reflections in W. Let ω be the canonical symplectic form on $V = \mathfrak{h} \oplus \mathfrak{h}^*$. For $s \in \mathcal{S}$, let ω_s be the skew-symmetric form that coincides with ω on $\mathrm{im}(\mathrm{id}_V - s)$ and has $\ker(\mathrm{id}_V - s)$ as its radical. Let $\mathbf{c} : \mathcal{S} \longrightarrow \mathbb{C}$ be a W-invariant function sending s to c_s. The *rational Cherednik algebra* at parameter $t = 0$ (depending on $\mathbf{c}$) is the quotient of the skew group algebra of the tensor algebra on V, $TV * W$, by the relations

$$[x, y] = \sum_{s \in \mathcal{S}} c_s \omega_s(x, y) s \tag{1}$$

for all $x, y \in V$. This algebra is denoted by $H_{\mathbf{c}}$.

Let $Z_{\mathbf{c}}$ denote the centre of $H_{\mathbf{c}}$ and set $A = \mathbb{C}[\mathfrak{h}^*]^W \otimes \mathbb{C}[\mathfrak{h}]^W$. Thanks to [3, Proposition 4.15] $A \subset Z_{\mathbf{c}}$ for any parameter $\mathbf{c}$ and $Z_{\mathbf{c}}$ is a free A-module of rank $|W|$. Let $X_{\mathbf{c}}$ denote the spectrum of $Z_{\mathbf{c}}$: this is called the *Calogero-Moser space* associated to W. Corresponding to the inclusion $A \subset Z_{\mathbf{c}}$ there is a finite surjective morphism $\Upsilon_{\mathbf{c}} : X_{\mathbf{c}} \longrightarrow \mathfrak{h}^*/W \times \mathfrak{h}/W$.

Let $\mathfrak{m}$ be the homogeneous maximal ideal of A. The *restricted rational Cherednik algebra* is $\overline{H}_{\mathbf{c}} = H_{\mathbf{c}}/\mathfrak{m}H_{\mathbf{c}}$. By [3, PBW theorem 1.3] it has dimension $|W|^3$ over $\mathbb{C}$. General theory asserts that the blocks of $\overline{H}_{\mathbf{c}}$ are labelled by the closed points of the scheme-theoretic fibre $\Upsilon^*(0)$. We call these points the *nilpotent points* of $X_{\mathbf{c}}$. By [7, 5.4] there is a surjective mapping

$$\Theta_{\mathbf{c}} : \mathsf{Irr}\, W \longrightarrow \{\text{closed points of } \Upsilon_{\mathbf{c}}^*(0)\} = \{\text{blocks of } \overline{H}_{\mathbf{c}}\},$$

constructed by associating to any $\boldsymbol{\lambda} \in \mathsf{Irr}\, W$ an indecomposable $\overline{H}_{\mathbf{c}}$-module, the baby Verma module $M_{\mathbf{c}}(\boldsymbol{\lambda})$. The fibres of $\Theta_{\mathbf{c}}$ partition $\mathsf{Irr}\, W$. We will call this the *$CM_{\mathbf{c}}$-partition*.

1.3. Let W be a Weyl group. Let $\mathbf{L} : W \to \mathbb{Q}$ be a weight function (in the sense of [1, Section 2]). Let $\mathcal{H}$ be the corresponding Iwahori-Hecke algebra at unequal parameters, an algebra over the group algebra of $\mathbb{Q}$, $A = \oplus_{q\in\mathbb{Q}} \mathbb{Z}v^q$, which has a basis T_w for $w \in W$, with multiplication given by the rule

$$T_sT_w = \begin{cases} T_{sw} & \text{if } l(sw) = l(w) + 1 \\ T_{sw} + (v^{\mathbf{L}(s)} - v^{-\mathbf{L}(s)})T_w & \text{if } l(sw) = l(w) - 1 \end{cases}$$

where $s \in \mathcal{S}$ and $w \in W$. There is an associated partition of W into two-sided cells, see [9, Chapter 8]. We call these the $KL_{\mathbf{L}}$-cells.

Conjecture. Let W be a Weyl group and let $\mathbf{L}$ be the weight function generated by $\mathbf{L}(s) = c_s$ for each $s \in \mathcal{S}$.

(1) There is a natural identification of the $CM_{\mathbf{c}}$-partition and the $KL_{\mathbf{L}}$-partition; this is induced by attaching a $KL_{\mathbf{L}}$-cell to an irreducible W-representation via the asymptotic algebra J, [9, 20.2].

(2) Let $\mathcal{F}$ be a $KL_{\mathbf{L}}$-cell of W and let $M_{\mathcal{F}}$ be the closed point of $\Upsilon_{\mathbf{c}}^*(0)$ corresponding to $\mathcal{F}$ by (1). Then $\dim_{\mathbb{C}}(\Upsilon_{\mathbf{c}}^*(0)_{M_{\mathcal{F}}}) = |\mathcal{F}|$.

The existence of the asymptotic algebra mentioned in (1) is still a conjecture, depending on Lusztig's conjectures P1-P15 in [9, Conjecture 14.2].

1.4. This conjecture generalises the known results about the blocks of $\overline{H}_{\mathbf{c}}$ and about the fibre $\Upsilon_{\mathbf{c}}^*(0)$.

- [7, Corollary 5.8] If $X_{\mathbf{c}}$ is smooth then $\Theta_{\mathbf{c}}$ is bijective, making the $CM_{\mathbf{c}}$-partition trivial. If $S \in \mathsf{Irr}\, W$ then $\dim_{\mathbb{C}}(\Upsilon_{\mathbf{c}}^*(0)_{M_S}) = \dim_{\mathbb{C}}(S)^2$.
- $\Theta_{\mathbf{c}}$ is not bijective when W is a finite Coxeter group of type D_{2n}, E, F, H or $I_2(m)$ $(m \geq 5)$, [7, Proposition 7.3]. In all of these cases there are non-trivial two-sided cells.

- Both the a-function and the c-function are constant across fibres of $\Theta_{\mathbf{c}}$, [6, Lemma 5.3 and Proposition 9.2]. This should be a property of two-sided cells.

An advantage of the Cherednik algeras is that the $CM_{\mathbf{c}}$-partition exists for any complex reflection group whereas, at the moment, a cell theory only exists for Coxeter groups.

1.5. In Theorem 2.5 we will give a combinatorial description of the $CM_{\mathbf{c}}$-partition when $W = G(\ell, 1, n) = \mu_\ell \wr \mathfrak{S}_n$, and then in Theorem 3.3 we will provide evidence for the conjecture by showing that the $CM_{\mathbf{c}}$-partition agrees with the conjectural description of the $KL_{\mathbf{L}}$-partition for $W = G(2,1,n)$, the Weyl group of type B_n, given in [1, Section 4.2].

2. Blocks for $W = G(\ell, 1, n)$

2.1. Let ℓ and n be positive integers. Let μ_ℓ be the group of ℓ-th roots of unity in $\mathbb{C}$ with generator σ and let $\mathfrak{S}_n$ be the symmetric group on n letters. Let W be the wreath product $G(\ell,1,n) = \mu_\ell \wr \mathfrak{S}_n = (\mu_\ell)^n \rtimes \mathfrak{S}_n$ acting naturally on $\mathfrak{h} = \mathbb{C}^n$.

2.2. Let $\mathcal{P}(n)$ denote the set of partitions of n and $\mathcal{P}(\ell, n)$ the set of ℓ-multipartitions of n. The set $\mathsf{Irr}\, W$ can be identified naturally with $\mathcal{P}(\ell, n)$ so that the trivial representation corresponds to $((n), \emptyset, \dots, \emptyset)$, e.g. [8, Theorem 4.4.3]. Given an element $\mathbf{s} \in \mathbb{Z}_0^\ell = \{(s_1, \dots, s_\ell) \in \mathbb{Z}^\ell : s_1 + \cdots + s_\ell = 0\}$ there is an associated ℓ-core (a partition from which no ℓ-hooks can be removed). The inverse of the process assigning to a partition its ℓ-core and ℓ-quotient defines a bijection

$$\mathbb{Z}_0^\ell \times \coprod_n \mathcal{P}(\ell, n) \longrightarrow \coprod_n \mathcal{P}(n), \qquad (\mathbf{s}, \boldsymbol{\lambda}) \mapsto \tau_{\mathbf{s}}(\boldsymbol{\lambda}). \tag{2}$$

A detailed discussion of this can be found in [8, Section 2.7] or [6, Section 6].

2.3. The Young diagram of a partition λ will always be justified to the northwest (one of the authors is English); we will label the box in the pth row and qth column of λ by s_{pq}. With this convention the residue of s_{pq} is defined to be congruence class of $p - q$ modulo ℓ. Recall that s_{pq} is said to be *j-removable* for some $0 \le j \le \ell - 1$ if it has residue j and if $\lambda \setminus \{s_{pq}\}$ is the Young diagram of another partition, a *predecessor* of λ. We say that s_{pq} is *j-addable* to $\lambda \setminus \{s_{pq}\}$.

Let $J \subseteq \{0, \dots, \ell - 1\}$. We define the *$J$-heart* of λ to be the sub-partition of λ which is obtained by removing as often as possible j-removable boxes with $j \in J$ from λ and its predecessors. A subset of $\mathcal{P}(n)$ whose elements are the partitions with a given J-heart is called a *J-class*.

2.4. We will use the "stability parameters" of [6] $\boldsymbol{\theta}(\mathbf{c}) = (\theta_0, \dots, \theta_{\ell-1})$ defined by $\theta_k = -\delta_{0k} c_{(i,j)} + \sum_{t=1}^{\ell-1} \eta^{tk} c_{\sigma^t}$ for $0 \le k \le \ell - 1$, η a primitive l-th root of unity and an arbitrary transposition $(i,j) \in \mathfrak{S}_n$: they contain the same information as $\mathbf{c}$. Following [6, Theorem 4.1] we set $\Theta = \{(\theta_0, \dots, \theta_{\ell-1}) \in \mathbb{Q}^\ell\}$ and $\Theta_1 = \{\boldsymbol{\theta} \in \Theta : \theta_0 + \cdots + \theta_{\ell-1} = 1\}$.

Let $\tilde{\mathfrak{S}}_\ell$ denote the affine symmetric group with generators $\{\sigma_0, \ldots, \sigma_{\ell-1}\}$. It acts naturally on Θ by $\sigma_j \cdot (\theta_0, \ldots, \theta_{\ell-1}) = (\theta_0, \ldots, \theta_{j-1} + \theta_j, -\theta_j, \theta_j + \theta_{j+1}, \ldots, \theta_{\ell-1})$. This restricts to the affine reflection representation on Θ_1: the *walls* of Θ_1 are the reflecting hyperplanes and the *alcoves* of Θ_1 are the connected components of (the real extension of) $\Theta_1 \setminus \{\text{walls}\}$. Let A_0 be the alcove containing the point $\ell^{-1}(1, \ldots, 1)$: its closure is a fundamental domain for the action of $\tilde{\mathfrak{S}}_\ell$ on Θ_1. The stabiliser of a point $\boldsymbol{\theta} \in \overline{A}_0$ is a standard parabolic subgroup of $\tilde{\mathfrak{S}}_\ell$ generated by simple reflections $\{\sigma_j : j \in J\}$ for some subset $J \subseteq \{0, \ldots, \ell-1\}$. We call this subset the *type* of $\boldsymbol{\theta}$. The type of an arbitrary point $\boldsymbol{\theta} \in \Theta_1$ is defined to be the type of its conjugate in $\overline{A}_0$.

2.5. We have an isomorphism $\tilde{\mathfrak{S}}_\ell \cong \mathbb{Z}_0^\ell \times \mathfrak{S}_\ell$ with $\mathfrak{S}_\ell = \langle \sigma_1, \ldots, \sigma_{\ell-1} \rangle$ and the elements of $\mathbb{Z}_0^\ell$ corresponding to translations by elements of the dual root lattice $\mathbb{Z}R^\vee$. The symmetric group $\mathfrak{S}_\ell$ acts on $\mathcal{P}(\ell, n)$ by permuting the partitions comprising an ℓ-multipartition.

Theorem. *Assume that $\boldsymbol{\theta}(\mathbf{c}) \in \Theta_1$, so that $\boldsymbol{\theta}(\mathbf{c})$ had type J and belongs to $(\mathbf{s}, w) \cdot \overline{A}_0$ for some $(\mathbf{s}, w) \in \tilde{\mathfrak{S}}_\ell$. Then $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathsf{Irr}\, W = \mathcal{P}(\ell, n)$ belong to the same block of $\overline{H}_{\mathbf{c}}$ if and only if $\tau_{\mathbf{s}}(w \cdot \boldsymbol{\lambda})$ and $\tau_{\mathbf{s}}(w \cdot \boldsymbol{\mu})$ belong to the same J-class. In other words, the $CM_{\mathbf{c}}$-partition is governed by the J-classes.*

Proof. Rescaling gives an isomorphism between $\overline{H}_{\mathbf{c}}$ and $\overline{H}_{\mathbf{c}/2}$ so we can replace $\mathbf{c}$ by $\mathbf{c}/2$. By [7, 5.4] we must show that the baby Verma modules $M_{\mathbf{c}/2}(\boldsymbol{\lambda})$ and $M_{\mathbf{c}/2}(\boldsymbol{\mu})$ give rise to the same closed point of $\Upsilon^*_{\mathbf{c}/2}(0)$ if and only if $\tau_{\mathbf{s}}(w \cdot \boldsymbol{\lambda})$ and $\tau_{\mathbf{s}}(w \cdot \boldsymbol{\mu})$ have the same J-class. But the closed points of $\Upsilon^*_{\mathbf{c}/2}(0)$ correspond to the $\mathbb{C}^*$-fixed points of $X_{\mathbf{c}/2}$ under the action induced from the grading on $H_{\mathbf{c}/2}$ which assigns degree 1, respectively -1 and 0, to non-zero elements of $\mathfrak{h}$, respectively $\mathfrak{h}^*$ and W. By [6, Theorem 3.10] these agree with the $\mathbb{C}^*$-fixed points on the affine quiver variety $\mathcal{X}_{\boldsymbol{\theta}(\mathbf{c})}(n)$ and hence, thanks to [6, Equation (3)] to the $\mathbb{C}^*$-fixed points on the Nakajima quiver variety $\mathcal{M}_{\boldsymbol{\theta}(\mathbf{c})}(n)$. Now the result follows since the combinatorial description of these fixed points in [6, Proposition 8.3(i)] is exactly the one in the statement of the theorem. $\square$

2.6. **Remarks.** (1) The assumption $\boldsymbol{\theta}(\mathbf{c}) \in \Theta_1$ imposes two restrictions. First it places a rationality condition on the entries of $\mathbf{c}$; guided by corresponding results for Hecke algebras, [2, Theorem 1.1], we hope that this is not really a serious restriction. Second it forces $c_{(i,j)} \neq 0$; if $c_{(i,j)} \neq 0$ then we can rescale to produce an isomorphism $H_{\mathbf{c}} \cong H_{\lambda \mathbf{c}}$ and hence ensure $\theta_0 + \cdots + \theta_{\ell-1} = 1$.

(2) A generic choice of $\boldsymbol{\theta}(\mathbf{c}) \in \Theta_1$ will have type $J = \emptyset$. The corresonding $CM_{\mathbf{c}}$-partition will then be trivial and thus $X_{\mathbf{c}}$ will be smooth, [3, Corollary 1.14(i)].

3. The case $W = G(2,1,n)$

3.1. We now focus on the situation where $W = G(2,1,n)$, the Weyl group of type B_n. Here there are two conjugacy classes of reflections s and t, containing (i,j) and σ respectively. We will always assume that $\mathbf{c} = (c_s, c_t) \in \mathbb{Q}^2$ has the property that $c_s, c_t \neq 0$. Corresponding to the two group homomorphisms $\epsilon_1, \epsilon_2 : W \to \mathbb{C}^*$, $\epsilon_k(i,j) = (-1)^k$ for all $(i,j) \in s$ and $\epsilon_k(\sigma) = (-1)^{k+1}$, there exist algebra isomorphisms

$H_{(c_s,c_t)} \cong H_{(-c_s,c_t)}$ and $H_{(c_s,c_t)} \cong H_{(c_s,-c_t)}$, [5, 5.4.1]. So, without loss of generality, we may assume that $\mathbf{c} \in \mathbb{Q}^2_{>0}$.

3.2. There is a conjectural description of the two-sided cells in [1, Section 4.2] which we recall very briefly; more details can be found in both [loc.cit] and [10].

We assume $\mathbf{L}(s) = a, \mathbf{L}(t) = b$ with $a, b \in \mathbb{Q}_{>0}$ and set $d = b/a$. If $d \notin \mathbb{Z}$ then the partition is conjectured to be trivial, [1, Conjecture A(c)]. If $d = r+1 \in \mathbb{Z}$ then let $\mathcal{P}_r(n)$ be the set of partitions of size $\frac{1}{2}r(r+1)+2n$ with 2-core $(r, r-1, \ldots, 1)$. A domino tableau T on $\lambda \in \mathcal{P}_r(n)$ is a filling of the Young diagram of λ with 0's in the 2-core and then n dominoes in the remaining boxes, each labelled by a distinct integer between 1 and n which are weakly increasing both vertically and horizontally. There is a process called *moving through an open cycle* which leads to an equivalence relation on the set of domino tableaux. This in turn leads to an equivalence relation on partitions in $\mathcal{P}_r(n)$ where λ and μ are related if there is a sequence of partitions $\lambda = \lambda_0, \lambda_1, \ldots, \lambda_{s-1}, \lambda_s = \mu$ such that for each $1 \leq i \leq s$, λ_{i-1} and λ_i are the underlying shapes of some domino tableaux related by moving through an open cycle. The equivalence classes of this relation are called r-cells. [1, Conjecture D] conjectures that the two-sided cells are in natural bijection with the r-cells.

3.3. The result of this section is the following.

Theorem. *Under the bijection* (2) *the* $CM_{\mathbf{c}}$*-partition of* $\mathsf{Irr}\, W$ *is identified with the above conjectural description of the* $KL_{\mathbf{L}}$*-partition for* $\mathbf{L}(s) = c_s, \mathbf{L}(t) = c_t$.

This theorem shows that core-quotient algorithm provides a natural identification of the $CM_{\mathbf{c}}$-partition and the conjectural $KL_{\mathbf{L}}$-partition. We do not know in general whether Lusztig's conjectured mapping from $\mathsf{Irr}\, W$ to the $KL_{\mathbf{L}}$-cells is given by this algorithm.

There are special cases where [1, Conjecture D] has been checked – for instance the asymptotic case $c_t > (n-1)c_s$, [1, Remark 1.3] – and thus in those cases we really do get a natural identification between the $CM_{\mathbf{c}}$-classes and $KL_{\mathbf{c}}$-cells.

3.4. We will need the following technical lemma to prove the theorem.

Lemma. *Let* $\lambda \in \mathcal{P}_r(n)$ *and set* $j = r$ *modulo* 2 *with* $j \in \{0,1\}$*. Suppose that* s_{pq} *is a* j*-removable box and* s_{tu} *is a* j*-addable box such that* $p \geq t$ *and* $q \leq u$ *and there are no other* j*-addable or* j*-removable boxes,* s_{vw}*, with* $p \geq v \geq t$ *and* $q \leq w \leq u$*. Then there is a domino tableau* T *of shape* λ *and an open cycle* c *of* T *such that the shape of the domino obtained by moving through* c *is obtained by replacing* s_{pq} *with* s_{tu}*.*

Proof. We use the notation of [10, Sections 2.1 and 2.3] freely. We consider the rim ribbon which begins at s_{pq} and ends at $s_{t,u-1}$. We claim that this rim ribbon can be paved by dominoes. In fact this is a general property of a ribbon connecting a box, s, of residue j and a box, e, of residue $j+1$. Let R be such a ribbon. If R contains only two boxes then $R = \{s, e\}$ so it is clear. In general the box adjacent to s, say s_{ad}, has residue $j+1$ so that $R \setminus \{s, s_{\mathrm{ad}}\}$ is a ribbon of smaller length and so the result follows by induction. In our situation we can specify more. Starting at s_{pq} we tile our rim ribbon, R, as far as possible with vertical dominoes up to and including $D = \{s_{p-m+1,q}, s_{p-m,q}\}$ where $s_{p-m,q}$ has residue $j+1$. If $s_{p-m,q} = s_{t,u-1}$ we have finished our tiling. Otherwise $s_{p-m,q+1} \in R$ so the square $s_{p-m,q+1}$ will be j-removable unless $\{s_{p-m,q+1}, s_{p-m,q+2}\} \subseteq R$. We now tile with as many horizontal dominoes as possible until we get to $E = \{s_{p-m,q+k-1}, s_{p-m,q+k}\}$ with $s_{p-m,q+k}$ having residue $j+1$. If $s_{p-m,q+k} = s_{t,u-1}$ then our tiling stops. Otherwise we must have the next domino as $F = \{s_{p-m-1,q+k}, s_{p-m-2,q+k}\}$ to avoid $s_{p-m,q+k+1}$ being j-addable. We can now repeat this process to obtain our tiling of R. From this description we obtain the following consequences. Let $s_{vw} \in R$ have residue $j+1$ and suppose that $s_{vw} \neq s_{t,u-1}$. Then

(i) The domino which contains s_{vw} is either of the form $\{s_{v+1,w}, s_{vw}\}$ or $\{s_{v,w-1}, s_{vw}\}$;

(ii) If $s_{v-1,w+1} \in \lambda$ then $s_{v,w+1} \in R$. Furthermore, if s_{vw} is contained in a horizontal domino then $s_{v-1,w+1} \notin R$;

(iii) If $s_{v-1,w+1} \notin \lambda$ then $s_{v-1,w} \in R$.

Let R denote the rim ribbon above and suppose it can be tiled by t dominoes. Let μ be the shape $\lambda \setminus R$. In particular μ contains $\frac{1}{2}r(r+1) + 2(n-t)$ squares. By the previous paragraph and [8, Lemma 2.7.13], μ is a Young diagram with the same 2-core as λ and so there exists a $T' \in \mathcal{P}_r(n-t)$ with shape μ. Take such a T' filled with the numbers 1 to $n-t$. Now add R to T'. We can tile R by dominoes by the previous paragraph and we fill the dominoes with the numbers $n-t+1, \ldots, n$ where the filling is weakly increasing on the rows and columns of R. This gives a domino tableau $T = T' \cup R$ of shape λ.

We claim that $R \subseteq T$ is an open cycle, and that when we move through this cycle we remove s_{pq} from T and add s_{tu}. This will prove the lemma.

As we have seen in (i) a domino $D \subseteq R$ is either of the form $D = \{s_{vw}, s_{v+1,w}\}$ or $D = \{s_{v,w-1}, s_{vw}\}$ with s_{vw} having residue $j+1$. In the case that $D = \{s_{vw}, s_{v+1,w}\}$ we have to study the square $s_{v-1,w+1}$ to calculate D'. One of two things can happen. If this box does not belong to T then $D' = \{s_{vw}, s_{v-1,w}\}$ and $D' \subseteq R$ by (iii). Otherwise $s_{v-1,w+1}$ does belong to T. In this situation the box is not in the rim so is filled with a lower value than D and so $D' = \{s_{vw}, s_{v,w+1}\}$. In particular, by (ii) above either $D' \subseteq R$ or $D' = \{s_{t,u-1}, s_{tu}\}$.

Now suppose $D = \{s_{v,w-1}, s_{vw}\}$. If the square $s_{v-1,w+1}$ is not in T then $D' = \{s_{v-1,w}, s_{vw}\}$ and $D' \subseteq R$ by (iii). If $s_{v-1,w+1}$ is in T then it is not in the rim by (ii) and so is filled with a value lower than that of D. Thus $D' = \{s_{vw}, s_{v,w+1}\}$ and $D' \subseteq R$ unless $s_{vw} = s_{t,u-1}$, in which case $D' = \{s_{t,u-1}, s_{tu}\}$.

It is now clear R that is a cycle and moving through this cycle changes the shape of λ by removing s_{pq} and adding s_{tu}. □

3.5. **Proof of Theorem 3.3.** We have $\boldsymbol{\theta}(\mathbf{c}) = (-c_s + c_t, -c_t)$ and by rescaling, see Remark 2.6(1), we consider $\boldsymbol{\theta}'(\mathbf{c}) = (1 - \frac{c_t}{c_s}, \frac{c_t}{c_s}) \in \Theta_1$. The action of $\tilde{\mathfrak{S}}_2$ on Θ_1 is given by $\sigma_0 \cdot (\theta_0, \theta_1) = (-\theta_0, \theta_1 + 2\theta_0)$ and $\sigma_1 \cdot (\theta_0, \theta_1) = (\theta_0 + 2\theta_1, -\theta_1)$. The walls are $\{(d, -d+1) \in \Theta_1 : d \in \mathbb{Z}\}$; they are of type $\{0\}$ if $d \in 2\mathbb{Z}$ and of type $\{1\}$ if $d \in 1 + 2\mathbb{Z}$. The fundamental alcove is $A_0 = \{(d, -d+1) : 0 < d < 1\}$, and the alcove $A_r = \{(d, -d+1) : r < d < r+1\}$ is then labelled by either $((\frac{r}{2}, \frac{-r}{2}), e) \in \mathbb{Z}_0^2 \times \mathfrak{S}_2$ or $((\frac{-r+1}{2}, \frac{r-1}{2}), \sigma_1) \in \mathbb{Z}_0^2 \times \mathfrak{S}_2$ depending on whether r is even or odd.

If $\frac{c_t}{c_s} \notin \mathbb{Z}$ then $\boldsymbol{\theta}'(\mathbf{c})$ has type $\emptyset$ and the $CM_{\mathbf{c}}$-partition of $\mathsf{Irr}\, W$ is trivial by Theorem 2.5 since $\emptyset$-classes are all singletons: this agrees with the conjectured triviality of the two-sided cells in this case. Thus we may assume that $r = \frac{c_t}{c_s} - 1 \in \mathbb{Z}_{\geq 0}$. Then $\boldsymbol{\theta}'(\mathbf{c}) = (-r, r+1)$ will be in the closure of two alcoves, A_{-r-1} and A_{-r}. We consider the latter. Let $\mathbf{s}$ be the element in $\mathbb{Z}_0^2$ coming from the labelling of A_{-r}; then$\tau_{\mathbf{s}}$ of (2) produces a bijection between $\mathsf{Irr}\, W = \mathcal{P}(2, n)$ and $\mathcal{P}_r(n)$. If we set $j = r$ modulo 2 and $J = \{j\}$, the content of the theorem is simply the assertion that the r-cells in $\mathcal{P}_r(n)$ consist of the partitions in $\mathcal{P}_r(n)$ with the same J-heart.

Let us show first that if $\lambda, \mu \in \mathcal{P}_r(n)$ have the same J-heart, say ρ, then they belong to the same r-cell. The J-heart has no j-removable boxes, but we can construct the partition μ from ρ by adding, say, t j-addable boxes. Now let ν be the partition obtained from ρ by adding t j-addable boxes as far left as possible. We note that by [8, Theorem 2.7.41] $\nu \in \mathcal{P}_r(n)$. Of course μ and ν could be the same, but usually they will be different. Now we apply Lemma 3.4 again and again to ν, taking first the rightmost j-removable box from ν to the position of the rightmost j-removable box on μ and then repeating with the next j-removable box on the successor of ν. We continue until we have obtained a partition with shape μ. By Lemma 3.4, this process is obtained by moving through open cycles. On the other hand, we can perform this operation in the opposite direction to move from λ to ν via open cycles (for this we use the same algorithm and the fact that for a cycle c, moving through c twice takes us back where we started, [4, Proof of Proposition 1.5.31]). It follows that λ and μ belong to the same r-cell.

Finally, we need to see that if $\lambda, \mu \in \mathcal{P}_r(n)$ belong to the same r-cell, then they have the same J-heart. For this it is enough to assume that μ is the shape of a tableau obtained by moving through an open cycle on a tableau of shape λ. But in this case the underlying shapes differ only in some j-removable boxes, [10, Section 2.3] and so they necessarily have the same J-heart. $\square$

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