

A REMARK ON RATIONAL CHEREDNIK ALGEBRAS AND DIFFERENTIAL OPERATORS ON THE CYCLIC QUIVER

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ABSTRACT. We show that the spherical subalgebra $U_{k,c}$ of the rational Cherednik algebra associated to $S_n \wr C_\ell$, the wreath product of the symmetric group and the cyclic group of order ℓ , is isomorphic to a quotient of the ring of invariant differential operators on a space of representations of the cyclic quiver of size ℓ . This confirms a version of [EG, Conjecture 11.22] in the case of cyclic groups. The proof is a straightforward application of work of Oblomkov, [O], on the deformed Harish–Chandra homomorphism, and of Crawley–Boevey, [CB1] and [CB2], and Gan and Ginzburg, [GG], on preprojective algebras.

1. INTRODUCTION

1.1. The representation theory of symplectic reflection algebras has links with a number of subjects including algebraic combinatorics, resolutions of singularities, Lie theory and integrable systems. There is a family of symplectic reflection algebras associated to any symplectic vector space V and finite subgroup $\Gamma \leq Sp(V)$, but a simple reduction allows one to study those subgroups Γ which are generated by symplectic reflections (i.e. by elements whose set of fixed points is of codimension two in V). This essentially focuses attention on two cases:

- (1) $\Gamma = W$, a finite complex reflection group, acting on $V = \mathfrak{h} \oplus \mathfrak{h}^*$ where \mathfrak{h} is a reflection representation of W ;
- (2) $\Gamma = S_n \wr K$, where K is a finite subgroup of $SL_2(\mathbb{C})$, acting naturally on $(\mathbb{C}^2)^n$.

The representation theory in the first case is mysterious at the moment: several important results are known but there is no general theory yet. On the other hand a geometric point of view on the representation theory in the second case is beginning to emerge. A key fact is that in this case the singular space V/Γ admits a crepant resolution of singularities: the representation theory of the symplectic reflection algebra is then expected to be closely related to the resolution. In the case $\Gamma = S_n$ (i.e. K is trivial) there are two approaches to this: the first is via noncommutative algebraic geometry, [GS], the second via sheaves of differential operators, [GG]. In this paper we extend the second approach to the groups $\Gamma = \Gamma_n = S_n \wr C_\ell$.

1.2. To state the result here we need to introduce a little notation. Let Q be the cyclic quiver with ℓ vertices and cyclic orientation. Choose an extending vertex (in this case any vertex) 0 . Then let Q_∞ be the quiver obtained by adding one vertex named ∞ to Q that is joined to 0 by a single arrow.

We will consider representation spaces of these quivers. Let $\delta = (1, 1, \dots, 1)$ be the affine dimension vector of Q , and set $\epsilon = e_\infty + n\delta$, a dimension vector for Q_∞ . Let $\text{Rep}(Q, n\delta)$ and $\text{Rep}(Q_\infty, \epsilon)$ be the representation spaces of these quivers with the given dimension vectors. There is an action of

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$G = \prod_{r=0}^{\ell-1} GL_n(\mathbb{C})$ on both these spaces. In fact, the action of the scalar matrices in G is trivial on $\text{Rep}(Q, n\delta)$ (but not on $\text{Rep}(Q_\infty, \epsilon)$) so in this case the action descends to an action of $PG = G/\mathbb{C}^*$.

Let $\mathfrak{X} = \text{Rep}(Q, n\delta) \times \mathbb{P}^{n-1}$. There is an action of PG on \mathfrak{X} .

1.3. Let $D(\text{Rep}(Q_\infty, \epsilon))$ denote the ring of differential operators on the affine space $\text{Rep}(Q_\infty, \epsilon)$, $D_{\mathfrak{X}}(nk)$ the sheaf of twisted differential operators on \mathfrak{X} and $D(\mathfrak{X}, nk)$ its algebra of global sections. The group action of G (respectively PG) on $\text{Rep}(Q_\infty, \epsilon)$ (respectively \mathfrak{X}) differentiates to an action of $\mathfrak{g} = \text{Lie}(G)$ (respectively $\mathfrak{pg} = \text{Lie}(PG)$) by differential operators. This gives mappings

$$\hat{\tau} : \mathfrak{g} \longrightarrow D(\text{Rep}(Q_\infty, \epsilon)), \quad \tau : \mathfrak{pg} \longrightarrow D_{\mathfrak{X}}(nk).$$

1.4. Let $U_{k,c}$ be the spherical subalgebra of type $S_n \wr C_\ell$ (this is defined in Section 3.4).

Theorem. *For all (k, c) there are isomorphisms of algebras*

$$\left(\frac{D(\text{Rep}(Q_\infty, \epsilon))}{I_{k,c}} \right)^G \cong \left(\frac{D(\mathfrak{X}, nk)}{I_c} \right)^{PG} \cong U_{k,c},$$

where $I_{k,c}$ is the left ideal of $D(\text{Rep}(Q_\infty, \epsilon))$ generated by $(\hat{\tau} - \chi_{k,c})(\mathfrak{g})$ and I_c is the left ideal of $D(\mathfrak{X}, nk)$ generated by $(\tau - \chi_c)(\mathfrak{pg})$ for suitable characters $\chi_{k,c} \in \mathfrak{g}^*$ and $\chi_c \in \mathfrak{pg}^*$ (which are defined in Section 4).

Note that it is a standard fact that the left hand side is an algebra. The proof of the theorem has two parts. One part constructs a filtered homomorphism from the left hand side to the right hand side using as its main input the work of Oblomkov, [O]. The other part proves that the associated graded homomorphism is an isomorphism and is a simple application of results of Crawley-Boevey, [CB1] and [CB2], and of Gan–Ginzburg, [GG].

1.5. We give an application of this result in Section 4. For related pairs (k, c) and (k', c') we construct “shift functors”

$$U_{k,c}\text{-mod} \longrightarrow U_{k',c'}\text{-mod}$$

using differential operators. We expect these to be a useful tool in the representation theory of Cherednik algebras, deserving of careful study.

1.6. While writing this down, we were informed that the general version of [EG, Conjecture 11.22] has been proved in [EGGO]. That result is more general than the work presented here and requires a new approach and ideas to overcome problems that simply do not arise for the case $\Gamma = S_n \wr C_\ell$.

2. QUIVERS

2.1. Once and for all fix integers ℓ and n . We assume that both are greater than 1. Set $\eta = \exp(2\pi i/\ell)$.

2.2. Let Q be the cyclic quiver with ℓ vertices and cyclic orientation. Choose an extending vertex (in this case any vertex) 0. Then let Q_∞ be the quiver obtained by adding one vertex named ∞ to Q that is joined to 0 by a single arrow. Let \overline{Q} and \overline{Q}_∞ denote the double quivers of Q and Q_∞ respectively, obtained by inserting an arrow a^* in the opposite direction to every arrow a in the quiver.

We will consider representation spaces of these quivers. Let $\delta = (1, 1, \dots, 1)$ be the affine dimension vector of Q , and set $\epsilon = e_\infty + n\delta$, a dimension vector for Q_∞ . Recall that

$$\text{Rep}(Q, n\delta) = \bigoplus_{r=0}^{\ell-1} \text{Mat}_n(\mathbb{C}) = \{(X_0, X_1, \dots, X_{\ell-1}) : X_r \in \text{Mat}_n(\mathbb{C})\} = \{(X)\}$$

and

$$\text{Rep}(Q_\infty, \epsilon) = \bigoplus_{r=0}^{\ell-1} \text{Mat}_n(\mathbb{C}) \oplus \mathbb{C}^n = \{(X_0, X_1, \dots, X_{\ell-1}, i) : X_r \in \text{Mat}_n(\mathbb{C}), i \in \mathbb{C}\} = \{(X, i)\}.$$

Let $G = \prod_{r=0}^{\ell-1} GL_n(\mathbb{C})$ be the base change group. If $g = (g_0, \dots, g_{\ell-1})$ then g acts on $\text{Rep}(Q, n\delta)$ by

$$g \cdot (X_0, X_1, \dots, X_{\ell-1}) = (g_0 X_0 g_1^{-1}, g_1 X_1 g_2^{-1}, \dots, g_{\ell-1} X_{\ell-1} g_0^{-1})$$

and on $\text{Rep}(Q_\infty, \epsilon)$ by

$$g \cdot (X_0, X_1, \dots, X_{\ell-1}, i) = (g_0 X_0 g_1^{-1}, g_1 X_1 g_2^{-1}, \dots, g_{\ell-1} X_{\ell-1} g_0^{-1}, g_0 i).$$

The action of the scalar subgroup \mathbb{C}^* is trivial in the first action (but not the second), so we can consider the first action as a PG -action where $PG = G/\mathbb{C}^*$. Let \mathfrak{g} and \mathfrak{pg} be the Lie algebras of G and PG respectively.

2.3. Let $\mathfrak{h}^{\text{reg}} \subset \mathbb{C}^n$ be the affine open subvariety consisting of points $x = (x_1, \dots, x_n)$ such that

- (i) if $i \neq j$ then $x_i \neq \eta^m x_j$ for all $m \in \mathbb{Z}$,
- (ii) for each $1 \leq i \leq n$ $x_i \neq 0$.

This is the subset of \mathbb{C}^n on which $\Gamma_n = S_n \wr C_\ell$ acts freely.

2.4. We can embed $\mathfrak{h}^{\text{reg}}$ into $\text{Rep}(Q, n\delta)$ by first considering a point $x = (x_1, \dots, x_n) \in \mathfrak{h}^{\text{reg}}$ as a diagonal matrix $X = \text{diag}(x_1, \dots, x_n)$ and then sending this to $\underline{X} = (X, X, \dots, X)$. We denote the image of $\mathfrak{h}^{\text{reg}}$ in $\text{Rep}(Q, n\delta)$ by \mathcal{S} .

Let T_Δ be the subgroup of G with elements (T, T, \dots, T) where T is a diagonal matrix in $GL_n(\mathbb{C})$. Then T_Δ is the stabiliser of \mathcal{S} . So consider the mapping

$$\pi : G/T_\Delta \times \mathfrak{h}^{\text{reg}} \longrightarrow \text{Rep}(Q, n\delta)$$

given by $\pi(gT_\Delta, x) = g \cdot \underline{X}$. If we let G act on $G/T_\Delta \times \mathfrak{h}^{\text{reg}}$ by left multiplication then π is a G -equivariant mapping.

Lemma. π is an étale mapping with covering group Γ_n . In fact its image $\text{Rep}(Q, n\delta)^{\text{reg}}$ is open in $\text{Rep}(Q, n\delta)$ and we have an isomorphism

$$\omega : G/T_\Delta \times_{\Gamma_n} \mathfrak{h}^{\text{reg}} \longrightarrow \text{Rep}(Q, n\delta)^{\text{reg}}.$$

Proof. Let $\mathcal{S} = \{\underline{X} : x \in \mathfrak{h}^{\text{reg}}\}$ and set $N_G(\mathcal{S}) = \{g \in G : g \cdot \mathcal{S} = \mathcal{S}\}$ and $Z_G(\mathcal{S}) = \{g \in G : g \cdot \underline{X} = \underline{X} \text{ for all } \underline{X} \in \mathcal{S}\}$.

Suppose $g \cdot \underline{X} = \underline{Y}$ for some $\underline{X}, \underline{Y} \in \mathcal{S}$. This implies that for each $0 \leq i \leq \ell - 1$

$$g_i \text{diag}(x)^\ell g_i^{-1} = \text{diag}(y)^\ell.$$

The hypotheses on $\mathfrak{h}^{\text{reg}}$ imply that both $\text{diag}(x)^\ell$ and $\text{diag}(y)^\ell$ are regular semisimple in \mathbb{C}^n . Two such elements are conjugate if and only if $g_i \in N_{GL_n(\mathbb{C})}(T) = T \cdot S_n$ where T is the diagonal subgroup of $GL_n(\mathbb{C})$. So there exists $\sigma \in S_n$ such that for all i we have $g_i = t_i \sigma$ for some $t_i \in T$, and for all $1 \leq r \leq n$ we have that $x_{\sigma(r)}^\ell = y_r^\ell$. Hence $x_{\sigma(r)} = \eta^{m_r} y_r$ for some $m_r \in \mathbb{Z}$. Now we find that $\underline{Y} = g \cdot \underline{X}$ implies that $\text{diag}(y_r) = t_i t_{i+1}^{-1} \text{diag}(\eta^{m_r} y_r)$. Since $y_r \neq 0$ this shows that $t_{i+1} = \text{diag}(\eta^{m_r}) t_i$ for each i . Hence we find that $gT_\Delta = (\sigma, \text{diag}(\eta^{m_r})\sigma, \dots, \text{diag}(\eta^{m_r})^{\ell-1}\sigma)T_\Delta$.

In particular, if $\underline{X} = \underline{Y}$ we see from above that each $m_r = 0$, so that $Z_G(\mathcal{S}) = T_\Delta$. Thus the group Γ_n is isomorphic to $N_G(\mathcal{S})/Z_G(\mathcal{S})$ via the homomorphism that sends $(\eta^{m_1}, \dots, \eta^{m_r})\sigma$ to $(\sigma, \text{diag}(\eta^{m_r})\sigma, \dots, \text{diag}(\eta^{m_r})^{\ell-1}\sigma)T_\Delta$.

Now suppose that $\pi(gT_\Delta, x) = \pi(hT_\Delta, y)$. Then $(h^{-1}g) \cdot \underline{X} = \underline{Y}$ and so we see that $h^{-1}g \in N_G(\mathcal{S})$. This shows that π is the composition

$$G/T_\Delta \times \mathfrak{h}^{\text{reg}} \longrightarrow G/T_\Delta \times_{\Gamma_n} \mathfrak{h}^{\text{reg}} \xrightarrow{\sim} \text{Rep}(Q, n\delta)^{\text{reg}}.$$

The first mapping factors out the action of Γ_n , and since Γ_n acts freely on $\mathfrak{h}^{\text{reg}}$ this is an étale mapping. Hence, to finish the lemma, it suffices to show that $\text{Rep}(Q, n\delta)^{\text{reg}}$ is open in $\text{Rep}(Q, n\delta)$.

We claim first that $\text{Rep}(Q, n\delta)^{\text{reg}}$ is the set O of representations of Q which decompose into n simple modules of dimension δ and whose endomorphism ring is n -dimensional. To prove this observe that any element of $\text{Rep}(Q, n\delta)^{\text{reg}}$ is isomorphic to a representation of the form \underline{X} and so it decomposes into the n indecomposable modules $\underline{X}_1, \dots, \underline{X}_n$ of dimension δ where $\underline{X}_i = (x_i, x_i, \dots, x_i)$ (the condition $x_i \neq 0$ implies simplicity). Now the representation \underline{X}_i is isomorphic to the representation $(1, 1, \dots, 1, x_i^\ell)$. By hypothesis $x_i^\ell \neq x_j^\ell$ so we deduce that the representations \underline{x}_i are pairwise non-isomorphic which ensures that the endomorphism ring of \underline{X} is n -dimensional. This proves the inclusion $\text{Rep}(Q, n\delta)^{\text{reg}} \subseteq O$. On the other hand, if V belongs to O then $V = V_1 \oplus \dots \oplus V_n$ where each V_i is isomorphic to a representation $(1, 1, \dots, 1, \nu_i)$ for some non-zero scalars ν_i . Moreover, since $\dim \text{End}(V) = n$ the ν_i must be pairwise distinct. Now, let η_i be an ℓ -th root of ν_i . Then V_i is isomorphic to (η_i, \dots, η_i) . Therefore V is isomorphic to the representation \underline{X} where $x = (\eta_1, \dots, \eta_n)$.

Now we must show that O is open in $\text{Rep}(Q, n\delta)$. We use first the fact that the canonical decomposition of the vector $n\delta$ is $\delta + \delta + \dots + \delta$, [Scho, Theorem 3.6]. This means that the representations of $\text{Rep}(Q, n\delta)$ whose indecomposable components all have dimension δ form an open set. Now, consider the morphism f from $\text{Rep}(Q, \delta)$ to \mathbb{C} which sends the representation $(\lambda_1, \dots, \lambda_\ell)$ to the product $\lambda_1 \dots \lambda_\ell$. The open set $f^{-1}(\mathbb{C}^*)$ consists of the simple representations of dimension vector δ . Therefore the subset of $\text{Rep}(Q, n\delta)$ consisting of representations which decompose as the sum of n simple representations of dimension vector δ is open. On the other hand, the function from $\text{Rep}(Q, n\delta)$ to \mathbb{N} which sends a representation V to $\dim \text{End}(V)$ is upper semi-continuous. Thus $\{V : \dim \text{End}(V) \leq n\}$ is an open set in $\text{Rep}(Q, n\delta)$. Intersecting these two sets shows that O is open, as required. \square

2.5. Now we're going to move from Q to Q_∞ . So let's start with the following

$$\{([gT_\Delta, x], i) : g_0^{-1}i \text{ is a cyclic vector for } \text{diag}(x)\} \subset (G/T_\Delta \times_{\Gamma_n} \mathfrak{h}^{\text{reg}}) \times \mathbb{C}^n.$$

By applying $\omega^{-1} \times \text{id}_{\mathbb{C}^n}$ this corresponds to an open subset of $\text{Rep}(Q, n\delta) \times \mathbb{C}^n = \text{Rep}(Q_\infty, \epsilon)$. Call that set U_∞ . This is a G -invariant open set since the G -action on triples is given by

$$h \cdot ([gT_\Delta, x], i) = ([hgT_\Delta, x], h_0i)$$

so $g_0^{-1}i$ is cyclic for $\text{diag}(x)$ if and only if $(h_0g_0)^{-1}h_0i$ is cyclic for $\text{diag}(x)$. Observe too that U_∞ is an affine variety. Indeed it is defined by the non-vanishing of the morphism

$$s : (G/T_\Delta \times_{\Gamma_n} \mathfrak{h}^{\text{reg}}) \times \mathbb{C}^n \longrightarrow \mathbb{C}$$

which sends $([gT_\Delta, x], i)$ to $(g_0^{-1}i) \wedge \text{diag}(x) \cdot (g_0^{-1}i) \wedge \cdots \wedge \text{diag}(x)^{n-1} \cdot (g_0^{-1}i)$.

Lemma. *The G -action on U_∞ is free and projection onto the second component*

$$\pi_2 : U_\infty \longrightarrow \mathfrak{h}^{\text{reg}}/\Gamma_n$$

is a principal G -bundle.

Proof. Suppose that $h \cdot ([gT_\Delta, x], i) = ([gT_\Delta, x], i)$. Then $[g^{-1}hgT_\Delta, x] = [T_\Delta, x]$, so by Lemma 2.4 $g^{-1}hg \in T_\Delta$.

We have that $h_0i = i$. Setting $i' = g_0^{-1}i$ implies that $g_0^{-1}h_0g_0i' = i'$. By hypothesis i' is a cyclic vector for $\text{diag}(x)$. So in the standard basis i' decomposes as $\sum \lambda_j e_j$ where each λ_j is non-zero. Therefore the only diagonal matrix that fixes i' is the identity element. In other words $g_0^{-1}h_0g_0 = I_n$. Since $g^{-1}hg \in T_\Delta$ this implies that $g^{-1}hg = \text{id}$. Thus $h = \text{id}$ and this proves that the action is free.

It remains to prove that each fibre of π_2 is a G -orbit. So take $([gT_\Delta, x], i) \in \pi_2^{-1}([x])$. This equals $g \cdot ([T_\Delta, x], g_0^{-1}i)$. Now $g_0^{-1}i$ is a cyclic vector for $\text{diag}(x)$ so it has the form $\sum \lambda_j e_j$ with each λ_j non-zero. Let $t = \text{diag}(\lambda_1, \dots, \lambda_n)$ and consider $\underline{t} = (t, \dots, t) \in T_\Delta$. We have

$$([gT_\Delta, x], i) = g\underline{t}t^{-1}([T_\Delta, x], g_0^{-1}i) = g\underline{t}([T_\Delta, x], \sum_{j=1}^n e_j).$$

This proves that each fibre of π_2 is indeed a G -orbit. □

2.6. Consider the representation space for the doubled quiver \overline{Q}_∞ :

$$\text{Rep}(\overline{Q}_\infty, \epsilon) = \{(X_0, \dots, X_{\ell-1}, Y_0, \dots, Y_{\ell-1}, i, j) : X_r, Y_r \in \text{Mat}_n(\mathbb{C}), i \in \mathbb{C}, j \in \mathbb{C}^*\} = \{(X, Y, i, j)\}.$$

We can naturally identify it with $T^* \text{Rep}(Q_\infty, \epsilon)$. The group G acts on the base and hence on the total space of the cotangent bundle. The resulting moment map

$$\mu : \text{Rep}(\overline{Q}_\infty, \epsilon) \longrightarrow \mathfrak{g}^* \cong \mathfrak{g}$$

is given by

$$\mu(X, Y, i, j) = [X, Y] + ij.$$

Theorem (Gan–Ginzburg, Crawley–Boevey). *Let $\mu^{-1}(0)$ denote the scheme-theoretic fibre of μ .*

- (1) $\mu^{-1}(0)$ is reduced, equidimensional and a complete intersection.
- (2) The moment map μ is flat.
- (3) $\mathbb{C}[\mu^{-1}(0)]^G \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\Gamma_n}$.

Proof. (i) This is [GG, Theorem 3.2.3].

(ii) This follows from [CB1, Theorem 1.1] and the dimension formula in [GG, Theorem 3.2.3(iii)].

(iii) This is [CB2, Theorem 1.1] □

2.7. Let $\mathfrak{X} = \{(X, i) \in \text{Rep}(Q, n\delta) \times \mathbb{P}^{n-1}\}$. This space is the quotient of the (quasi-affine) open subvariety

$$U = \{(X, i) : i \neq 0\} \subset \text{Rep}(Q_\infty, \epsilon)$$

by the scalar group \mathbb{C}^* . Thus there is an action of PG on \mathfrak{X} .

Since

$$T^* \mathbb{P}^{n-1} = \{(i, j) : i \neq 0, ji = 0\} / \mathbb{C}^*$$

we have

$$T^* \mathfrak{X} = \{(X, Y, i, j) \in \text{Rep}(\overline{Q}_\infty, \epsilon) : i \neq 0, ji = 0\} / \mathbb{C}^*.$$

The PG action on \mathfrak{X} gives rise to a moment map

$$\mu_{\mathfrak{X}} : T^* \mathfrak{X} \longrightarrow \mathfrak{pg}^* \cong \mathfrak{pg}.$$

Let

$$\mu_{\mathfrak{X}}^{-1}(0) = \{(X, Y, i, j) \in \text{Rep}(\overline{Q}_\infty, \epsilon) : i \neq 0, ji = 0, [X, Y] + ij = 0\} / \mathbb{C}^*$$

denote the scheme theoretic fibre of 0.

Proposition. *There is an isomorphism $\mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]^{PG} \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\Gamma_n}$.*

Proof. Consider the G -equivariant open subvariety of $\mu^{-1}(0)$ given by the non-vanishing of i . The variety $\mu^{-1}(0)$ is determined by the conditions $[X, Y] + ij = 0$, so if we take the trace of this equation then we see that $0 = \text{Tr}(ij) = \text{Tr}(ji) = ji$. Thus we see that $\{(X, Y, i, j) \in \text{Rep}(\overline{Q}_\infty, \epsilon) : i \neq 0, ji = 0\} \cap \mu^{-1}(0)$ is an open subvariety of $\mu^{-1}(0)$ so in particular reduced by Theorem 2.6(1). Hence factoring out by the action of $\mathbb{C}^* \leq G$ shows that $\mu_{\mathfrak{X}}^{-1}(0)$ is reduced and that there is a PG -equivariant morphism

$$\mu_{\mathfrak{X}}^{-1}(0) \longrightarrow \mu^{-1}(0) // \mathbb{C}^*.$$

This induces an algebra map

$$\alpha : \mathbb{C}[\mu^{-1}(0)]^G \longrightarrow \mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]^{PG}.$$

We now follow some of the proof of [GG, Lemma 6.3.2]. Write \mathcal{O}_1 for the conjugacy class of rank one nilpotent matrices in $\mathfrak{gl}(n)$, and let $\overline{\mathcal{O}}_1$ denote the closure of \mathcal{O}_1 in $\mathfrak{gl}(n)$. The moment map $v : T^* \mathbb{P}^{n-1} \longrightarrow \mathfrak{gl}(n)^* \cong \mathfrak{gl}(n)$ that sends (i, j) to ij gives a birational isomorphism $T^* \mathbb{P}^{n-1} \longrightarrow \overline{\mathcal{O}}_1$. Let $J \subset \mathbb{C}[\mathfrak{gl}(n)] = \mathbb{C}[Z]$ be the ideal generated by all 2×2 minors of the matrix Z and also by the trace function. Then J is a prime ideal whose zero scheme is $\overline{\mathcal{O}}_1$ and the pullback morphism $v^* : \mathbb{C}[\mathfrak{gl}(n)]/J \longrightarrow \mathbb{C}[T^* \mathbb{P}^{n-1}]$ is a graded isomorphism.

Now the moment map $\mu_{\mathfrak{X}} : T^* \mathfrak{X} \longrightarrow \mathfrak{g}^*$ factors as the composite

$$T^* \mathfrak{X} = T^* \text{Rep}(Q, n\delta) \times T^* \mathbb{P}^{n-1} \longrightarrow T^* \text{Rep}(Q, n\delta) \times \overline{\mathcal{O}}_1 \xrightarrow{\theta} \mathfrak{pg}^*$$

where the first mapping is $\text{id} \times v$ and the second mapping θ sends (X, Y, Z) to $[X, Y] + Z_0$ where Z_0 indicates that we place the matrix Z on the copy of $\mathfrak{gl}(n)$ associated to vertex 0. We have a graded algebra isomorphism

$$\mathbb{C}[T^* \text{Rep}(Q, n\delta)] \otimes \mathbb{C}[\mathfrak{gl}(n)]/J \longrightarrow \mathbb{C}[T^* \mathcal{X}].$$

Now write $\mathbb{C}[X, Y, Z] = \mathbb{C}[T^* \text{Rep}(Q, n\delta) \times \mathfrak{gl}(n)]$, and let $\mathbb{C}[X, Y, Z]/([X, Y] + Z_0)$ denote the ideal in $\mathbb{C}[X, Y, Z]$ generated by all matrix entries of the ℓ matrices $[X, Y] + Z_0$. Let \mathbf{I} denote the ideal $\mathbb{C}[X, Y, Z]/([X, Y] + Z_0) + \mathbb{C}[X, Y] \otimes J \subset \mathbb{C}[X, Y, Z]$. From the above we have

$$\mathbb{C}[\mu_{\bar{\mathbf{x}}}^{-1}(0)] \cong \mathbb{C}[T^* \text{Rep}(Q, n\delta) \times \bar{\mathbf{O}}_1]/\mathbb{C}[T^* \text{Rep}(Q, n\delta) \times \bar{\mathbf{O}}_1]\theta^*(\mathfrak{gl}(n)) = \mathbb{C}[X, Y, Z]/\mathbf{I}.$$

Define an algebra homomorphism $r : \mathbb{C}[X, Y, Z] \longrightarrow \mathbb{C}[X, Y]$ by sending $P \in \mathbb{C}[X, Y, Z]$ to the function $(X, Y) \mapsto P(X, Y, -[X, Y]_0)$. Obviously r induces an isomorphism $\mathbb{C}[X, Y, Z]/\mathbb{C}[X, Y, Z]/([X, Y] + Z_0) \cong \mathbb{C}[X, Y]/I_1$ where I_1 is the ideal of $\mathbb{C}[\text{Rep}(\bar{Q}, n\delta)] = \mathbb{C}[X, Y]$ generated by the elements

$$\sum_{h(a)=i} X_a X_{a^*} - \sum_{t(a)=i} X_{a^*} X_a$$

for all i not equal to zero. Observe that the linear function $P : (X, Y, Z) \mapsto \text{Tr} Z = \text{Tr}([X, Y] + Z_0)$ belongs to the ideal $\mathbb{C}[X, Y, Z]/([X, Y] + Z_0)$. We deduce that the mapping r sends $\mathbb{C}[X, Y] \otimes J$ to the ideal generated by

$$\text{rank}\left(\sum_{h(a)=0} X_a X_{a^*} - \sum_{t(a)=0} X_{a^*} X_a\right) \leq 1.$$

Thus we obtain algebra isomorphisms

$$\mathbb{C}[\mu_{\bar{\mathbf{x}}}^{-1}(0)] \cong \mathbb{C}[X, Y, Z]/\mathbf{I} \cong \mathbb{C}[T^* \text{Rep}(Q, n\delta)]/I_2$$

where I_2 is ideal generated by the elements

$$\sum_{h(a)=i} X_a X_{a^*} - \sum_{t(a)=i} X_{a^*} X_a$$

for all $1 \leq i \leq \ell - 1$, and

$$\text{rank}\left(\sum_{h(a)=0} X_a X_{a^*} - \sum_{t(a)=0} X_{a^*} X_a\right) \leq 1.$$

By [LP, Theorem 1] the G -invariant (respectively PG -invariant) elements of $\mathbb{C}[\text{Rep}(\bar{Q}_\infty, \epsilon)]$ (respectively $\mathbb{C}[\text{Rep}(\bar{Q}, n\delta)]$) are generated by traces along oriented cycles. Since all oriented cycles in \bar{Q} are oriented cycles in \bar{Q}_∞ we have a surjective composition of algebra homomorphisms

$$(2.7.1) \quad \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\Gamma_n} \cong \mathbb{C}[\mu^{-1}(0)]^G \longrightarrow \mathbb{C}[\mu_{\bar{\mathbf{x}}}^{-1}(0)]^{PG} \longrightarrow \left(\frac{\mathbb{C}[\text{Rep}(\bar{Q}, n\delta)]}{I_2}\right)^{PG},$$

where the first isomorphism is Theorem 2.6(3). The left hand side is a domain of dimension $2 \dim \mathfrak{h}$, so to see that the mapping is an isomorphism it suffices to prove that the right hand side also has dimension $2 \dim \mathfrak{h}$.

Let I_3 be the ideal of $\mathbb{C}[\text{Rep}(\bar{Q}, n\delta)]$ generated by the elements

$$\sum_{h(a)=i} X_a X_{a^*} - \sum_{t(a)=i} X_{a^*} X_a$$

for all i . This is the ideal of the zero fibre of the moment map for the PG -action on $\text{Rep}(\overline{Q}, n\delta)$. This ideal contains I_2 since the rank condition on the matrices is implied by the commutator condition. So there is a surjective mapping

$$\frac{\mathbb{C}[\text{Rep}(\overline{Q}, n\delta)]^{PG}}{I_2^{PG}} \longrightarrow \frac{\mathbb{C}[\text{Rep}(\overline{Q}, n\delta)]^{PG}}{I_3^{PG}}.$$

We do not know yet whether the right hand side is reduced or not, but by [CB2, Theorem 1.1] the reduced quotient of the right hand side is the ring of functions of the variety $(\mathfrak{h} \oplus \mathfrak{h}^*)/\Gamma_n$. As this variety has dimension $2 \dim \mathfrak{h}$ we deduce that the composition in (2.7.1) is an isomorphism, and hence that

$$\mathbb{C}[\mu_{\overline{x}}^{-1}(0)]^{PG} \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\Gamma_n}.$$

□

Remark. In passing let us note that the commutativity of the following diagram

$$\begin{array}{ccccc} \mathbb{C}[T^* \text{Rep}(Q, n\delta)] & \xrightarrow{\iota} & \mathbb{C}[T^* \text{Rep}(Q, n\delta)] \otimes \mathbb{C}[T^* \mathbb{P}^{n-1}] & \longrightarrow & \mathbb{C}[\mu_{\overline{x}}^{-1}(0)] \\ & & \downarrow v^* & & \downarrow \iota \\ & & \mathbb{C}[T^* \text{Rep}(Q, n\delta)] \otimes \mathbb{C}[\overline{\mathcal{O}}_1] & \xrightarrow{r} & \mathbb{C}[T^* \text{Rep}(Q, n\delta)]/I_2 \\ & \searrow & & \nearrow & \\ & & & & \text{pr} \end{array}$$

where $\iota(f) = f \otimes 1$, shows that $\text{im } \iota$ maps surjectively onto $\mathbb{C}[\mu_{\overline{x}}^{-1}(0)]$.

3. DIFFERENTIAL OPERATORS

3.1. Symplectic reflection algebras. Let C_ℓ be the cyclic subgroup of $SL_2(\mathbb{C})$ generated by $\sigma = \text{diag}(\eta, \eta^{-1})$. The vector space $V = (\mathbb{C}^2)^n$ admits an action of $S_n \wr C_\ell = S_n \times (C_\ell)^n$: $(C_\ell)^n$ acts by extending the natural action of C_ℓ on \mathbb{C}^2 , whilst S_n acts by permuting the n copies of \mathbb{C}^2 . For an element $\gamma \in C_\ell$ and an integer $1 \leq i \leq n$ we write γ_i to indicate the element $(1, \dots, \gamma, \dots, 1) \in C_\ell^n$ which is non-trivial in the i -th factor.

3.2. The elements $S_n \wr C_\ell$ whose fixed points are a subspace of codimension two in V are called symplectic reflections. In this case their conjugacy classes are of two types:

(S) the elements $s_{ij} \gamma_i \gamma_j^{-1}$ where $1 \leq i, j \leq n$, $s_{ij} \in S_n$ is the transposition that swaps i and j , and $\gamma \in C_\ell$.

(C_ℓ) the elements γ_i for $1 \leq i \leq n$ and $\gamma \in C_\ell \setminus \{1\}$.

There is a unique conjugacy class of type (S) and $\ell - 1$ of type (C_ℓ) (depending on the non-trivial element we choose from C_ℓ). We will consider a conjugation invariant function from the set of symplectic reflections to \mathbb{C} . We can identify it with a pair (k, c) where $k \in \mathbb{C}$ and c is an $\ell - 1$ -tuple of complex numbers: the function sends elements from (S) to k and the elements $(\sigma^m)_i$ to c_m .

3.3. There is a symplectic form on V which is induced from n copies of the standard symplectic form ω on \mathbb{C}^2 . If we pick a basis $\{x, y\}$ for \mathbb{C}^2 such that $\omega(x, y) = 1$ then we can extend this naturally to a basis $\{x_i, y_i : 1 \leq i \leq n\}$ of V such that the x 's and the y 's form Lagrangian subspaces and $\omega(x_i, y_j) = \delta_{ij}$. We let TV denote the tensor algebra on V : with our choice of basis this is just the free algebra on generators x_i, y_i for $1 \leq i \leq n$. The symplectic reflection algebra $H_{k,c}$ associated to $S_n \wr C_\ell$ is the quotient of $TV * (S_n \wr C_\ell)$ by the following relations:

$$\begin{aligned} x_i x_j &= x_j x_i, & y_i y_j &= y_j y_i & \text{for all } 1 \leq i, j \leq n \\ y_i x_i - x_i y_i &= 1 + \frac{k}{2} \sum_{j \neq i} \sum_{\gamma \in C_\ell} s_{ij} \gamma_i \gamma_j^{-1} + \sum_{\gamma \in C_\ell \setminus \{1\}} c_\gamma \gamma_i & \text{for } 1 \leq i \leq n \\ y_i x_j - x_j y_i &= -\frac{k}{2} \sum_{m=0}^{\ell-1} \eta^m s_{ij} (\sigma^m)_i (\sigma^m)_j^{-1} & \text{for } i \neq j. \end{aligned}$$

(NB: my k is $-k$ for Oblomkov.)

3.4. **The spherical algebra.** The symmetrising idempotent of the group algebra $\mathbb{C}(S_n \wr C_\ell)$ is

$$e = \frac{1}{|S_n \wr C_\ell|} \sum_{w \in S_n \wr C_\ell} w.$$

The subalgebra $eH_{k,c}e$ is denoted by $U_{k,c}$ and called the *spherical algebra*. It will be our main object of study.

3.5. **Rings of differential operators.** Recall the definition of \mathfrak{X} from 2.7. Let $D_{\mathfrak{X}}(nk)$ denote the sheaf of twisted differential operators on \mathfrak{X} and let $D(\mathfrak{X}, nk)$ be its algebra of global sections. This is simply the tensor product $D(\text{Rep}(Q, n\delta)) \otimes D_{\mathbb{P}^{n-1}}(nk)$. (The twisted differential operators on \mathbb{P}^{n-1} can be defined as follows. Let $A_n = \mathbb{C}[x_1, \dots, x_n, \partial_1, \dots, \partial_n]$ be the n -th Weyl algebra. This is a graded algebra with $\deg(x_i) = 1$ and $\deg(\partial_i) = -1$. The degree zero component is the subring generated by the operators $x_i \partial_j$ which, under the commutator, generate the Lie algebra $\mathfrak{gl}(n)$. Call this subring R . Let $\mathbf{E} = \sum_{i=1}^n x_i \partial_i \in R$ be the Euler operator. Then $D(\mathbb{P}^{n-1}, nk)$ is the quotient of R by the two-sided ideal generated by $\mathbf{E} - nk$.)

The group action of PG on \mathfrak{X} differentiates to an action of \mathfrak{pg} on \mathfrak{X} by differential operators. This gives a mapping

$$(3.5.1) \quad \tau : \mathfrak{pg} \longrightarrow D_{\mathfrak{X}}(nk).$$

(One way to understand this is to start back with $U \subset \text{Rep}(Q_\infty, \epsilon)$ and look at the G action on U . Differentiating the G -action gives an action of \mathfrak{g} by differential operators on U , $\hat{\tau} : \mathfrak{g} \longrightarrow D_U$. Since \mathbb{C}^* acts trivially on $\text{Rep}(Q, n\delta)$ and by scaling on $i \in \text{Rep}(Q_\infty, \epsilon)$ we find that $\hat{\tau}(\text{id}) = 1 \otimes \mathbf{E}$ where $\text{id} = (I_n, I_n, \dots, I_n) \in \mathbb{C} \subset \mathfrak{g}$. Thus we get an action of \mathfrak{pg} on $(D_U/D_U(1 \otimes \mathbf{E} - nk))^{\mathbb{C}^*} = D_{\mathfrak{X}}(nk)$.)

3.6. Recall the Lie algebra $\mathfrak{g} = \text{Lie}(G)$ and its quotient $\mathfrak{pg} = \text{Lie}(PG)$ which is simply $\mathfrak{g}/\mathbb{C} \cdot \text{id}$ where $\text{id} = (I_n, \dots, I_n) \in \mathfrak{g}$. Let $\chi_c : \mathfrak{g} \longrightarrow \mathbb{C}$ send an element $(X) = (X_0, \dots, X_{\ell-1}) \in \mathfrak{g}$ to

$$\chi_c(X) = \sum_{r=0}^{\ell-1} C_r \text{Tr}(X_r)$$

where $C_r = \ell^{-1}(1 - \sum_{m=1}^{\ell-1} \eta^{mr} c_m)$ for $1 \leq r \leq \ell - 1$ and $C_0 = \ell^{-1}(1 - \ell - \sum_{m=1}^{\ell-1} c_m)$. Observe that

$$\chi_c(\text{id}) = \text{Tr}(I_n) \sum_{r=0}^{\ell-1} C_r = n \sum_{r=0}^{\ell-1} \sum_{m=0}^{\ell-1} -\eta^{rm} c_m = 0.$$

In particular χ_c is actually a character of \mathfrak{pg} .

Let $\chi_k : \mathfrak{g} \rightarrow \mathbb{C}$ send an element $(X) = (X_0, \dots, X_{\ell-1})$ to $\chi_k(c) = k \text{Tr}(X_0)$.

We will be regularly using the character $\chi_{k,c} \in \mathfrak{g}^*$ defined by $\chi_{k,c} = \chi_c + \chi_k$.

3.7. Let us recall Oblomkov's deformed Harish–Chandra homomorphism, [O]. By Lemma 2.4 $\mathcal{S} = \omega(\mathfrak{h}^{\text{reg}}/\Gamma_n)$ is a subset of $\text{Rep}(Q, n\delta)^{\text{reg}}$ which is a slice for the PG -action on $\text{Rep}(Q, n\delta)$. Let

$$W'_k = (y_1 \dots y_n)^{-k} \mathbb{C}_{(0)}[y_1^{\pm 1}, \dots, y_n^{\pm 1}],$$

a space of multivalued functions on $(\mathbb{C}^*)^n$. The Lie algebra \mathfrak{g} acts on W'_k by projection onto its 0-th summand $\mathfrak{gl}(n)$, and then by the natural action of $\mathfrak{gl}(n)$ on polynomials (so E_{ij} acts as $y_i \partial / \partial y_j$). With this action the identity matrix in $\mathfrak{gl}(n)$ becomes the Euler operator \mathbf{E} which acts by multiplication by $-nk$. Thus we can make W'_k a \mathfrak{pg} -module by twisting W'_k by the character χ_k since then id acts trivially. If we call this module W_k then $W_k = W'_k \otimes \chi_k$. Now define Fun' to be the space of functions on $\text{Rep}(Q, n\delta)$ of the form

$$f = \tilde{f} \prod_{i=0}^{\ell-1} \det(X_i)^{r_i}$$

where \tilde{f} is a rational function on $\text{Rep}(Q, n\delta)^{\text{reg}}$ regular on \mathcal{S} , $r_i = \sum_{j=0}^i C_j + \sigma$ and $\sigma = \ell^{-1} \sum_{s=0}^{\ell-1} s C_s$. Then $(Fun' \otimes W_k)^{\mathfrak{pg}}$ is a space of (\mathfrak{pg}, χ_c) -semiinvariant functions defined on a neighbourhood of \mathcal{S} which take values in W_k . This space is a free $\mathbb{C}[\mathfrak{h}^{\text{reg}}]^{\Gamma_n}$ -module of rank 1, the isomorphism being given by restriction to \mathcal{S} . (Note that the determinant of an element of the form (X, \dots, X) is $\det(X)^{\sum r_i} = 1$ as $\sum r_i = 0$.) Any \mathfrak{pg} -invariant differential operator, D , acts on such a function, f . Oblomkov defines his homomorphism to be the restriction of $D(f)$ to \mathcal{S} .

3.8. We can view the above procedure in terms of $\text{Rep}(Q_\infty, \epsilon)$. Thanks to Lemma 2.5 we use $\mathcal{S}_\infty = \mathcal{S} \times (1, \dots, 1) \in U_\infty$ as a slice for the G -action. The space $\mathcal{S} \times (\mathbb{C}^*)^n$ is a closed subset of U_∞ since the condition that i be cyclic for $\text{diag}(x_1, \dots, x_n)$ is equivalent to $i \in (\mathbb{C}^*)^n$. Thus functions on a neighbourhood of \mathcal{S}_∞ in U_∞ can be identified with functions from a neighbourhood of \mathcal{S} taking values in functions on $(\mathbb{C}^*)^n$. In particular, we can consider elements on $(Fun' \otimes W_k)^{\mathfrak{pg}}$ first as $(\mathfrak{g}, \chi_{k,c})$ -semiinvariant functions from a neighbourhood of \mathcal{S} taking values in W'_k and hence as $(\mathfrak{g}, \chi_{k,c})$ -semiinvariant functions on an open set in a neighbourhood of \mathcal{S}_∞ . We can apply any element of $D \in D(U_\infty)^\mathfrak{g}$ to these $(\mathfrak{g}, \chi_{k,c})$ -semiinvariant functions and then restrict to \mathcal{S}_∞ to get a homomorphism

$$\mathfrak{F}_{k,c} : D(U_\infty)^\mathfrak{g} \rightarrow D(\mathfrak{h}^{\text{reg}}/\Gamma_n).$$

3.9. Since $\text{Rep}(Q_\infty, \epsilon) = \text{Rep}(Q, n\delta) \times \mathbb{C}^n$ there is a mapping

$$\mathfrak{G} : D(\text{Rep}(Q, n\delta))^{\mathfrak{pg}} \rightarrow D(U_\infty)^\mathfrak{g}$$

which sends $D \in D(\text{Rep}(Q, n\delta))^{\mathfrak{pg}}$ to $(D \otimes 1)$. Oblomkov's homomorphism is $\mathfrak{F}_{k,c} \circ \mathfrak{G}$.

3.10. Differentiating the G -action on U_∞ gives a Lie algebra homomorphism $\hat{\tau} : \mathfrak{g} \longrightarrow \text{Vect}(U_\infty)$ which we extend to an algebra map

$$\hat{\tau} : U(\mathfrak{g}) \longrightarrow D(U_\infty).$$

By Lemma 2.5 U_∞ is a principal G -bundle over $\mathfrak{h}^{\text{reg}}/\Gamma_n$, so (a generalisation of) [Schw, Corollary 4.5] shows that the kernel of $\mathfrak{F}_{k,c}$ is $(D(U_\infty)(\hat{\tau} - \chi_{k,c})(\mathfrak{g}))^{\mathfrak{g}}$. Moreover, since the finite group Γ_n acts freely on $\mathfrak{h}^{\text{reg}}$ we can identify $D(\mathfrak{h}^{\text{reg}}/\Gamma_n)$ with $D(\mathfrak{h}^{\text{reg}})^{\Gamma_n}$.

3.11. Recall that

$$D_{\mathfrak{X}}(nk) \cong \left(\frac{D_U}{D_U(\hat{\tau} - \chi_k)(\mathbb{C} \cdot \text{id})} \right)^{\mathbb{C}^*}.$$

Hence we have

$$(3.11.1) \quad \left(\frac{D_U}{D_U(\hat{\tau} - \chi_{k,c})(\mathfrak{g})} \right)^G \cong \left(\frac{D_{\mathfrak{X}}(nk)}{D_{\mathfrak{X}}(nk)(\tau - \chi_c)(\mathfrak{pg})} \right)^{PG},$$

where $U = \{(X, i) : i \neq 0\} \subset \text{Rep}(Q_\infty, n\delta)$ as in 2.7. Now we consider the restriction mapping $D_U \longrightarrow D(U_\infty)$. Composing the global sections of the above isomorphism with this restriction and the homomorphism $\mathfrak{F}_{k,c}$ gives

$$\mathfrak{R}'_{k,c} : \left(\frac{D(\mathfrak{X}, nk)}{D(\mathfrak{X}, nk)(\tau - \chi_c)(\mathfrak{pg})} \right)^{PG} \longrightarrow D(\mathfrak{h}^{\text{reg}})^{\Gamma_n}.$$

3.12. Let

$$\delta_{k,c}(x) = \delta^{-k-1} \delta_\Gamma^{\mathfrak{g}}$$

where $\delta = \prod_{1 \leq i < j \leq n} (x_i^\ell - x_j^\ell)$ and $\delta_\Gamma = \prod_{i=1}^n x_i$. Define a twisted version of $\mathfrak{R}'_{k,c}$ above

$$\mathfrak{R}_{k,c}(D) = \delta_{k,c}^{-1} \circ \mathfrak{R}'_{k,c}(D) \circ \delta_{k,c}$$

for any differential operator D .

3.13. Our main result is the following.

Theorem. *For all values of k and c , the homomorphism $\mathfrak{R}_{k,c}$ has image $\text{im } \theta_{k,c}$. In particular we have an isomorphism*

$$\theta_{k,c}^{-1} \circ \mathfrak{R}_{k,c} : \left(\frac{D(\mathfrak{X}, nk)}{D(\mathfrak{X}, nk)(\tau - \chi_c)(\mathfrak{pg})} \right)^{PG} \xrightarrow{\sim} U_{k,c}.$$

Proof. Let us abuse notation by writing $U_{k,c}$ for the image of $U_{k,c}$ in $D(\mathfrak{h}^{\text{reg}})^{\Gamma_n}$ under $\theta_{k,c}$.

Since $\mathfrak{X} = \text{Rep}(Q, n\delta) \times \mathbb{P}^{n-1}$ there is a mapping

$$D(\text{Rep}(Q, n\delta))^{PG} \longrightarrow D(\mathfrak{X}, nk)^{PG} \longrightarrow D(\mathfrak{h}^{\text{reg}})^{\Gamma_n}$$

which sends $D \in D(\text{Rep}(Q, n\delta))^{PG}$ to $\mathfrak{R}_{k,c}(D \otimes 1)$. Recall τ from (3.5.1). Since $\text{gr } \tau = \mu_{\mathfrak{X}}^*$ we have an inclusion $\text{gr}(D(\mathfrak{X}, nk))\mu_{\mathfrak{X}}^*(\mathfrak{pg}) \subseteq \text{gr}(D(\mathfrak{X}, nk)(\tau - \chi_c)(\mathfrak{pg}))$. This gives a graded surjection

$$p : \left(\frac{\text{gr } D(\mathfrak{X}, nk)}{\text{gr}(D(\mathfrak{X}, nk))\mu_{\mathfrak{X}}^*(\mathfrak{pg})} \right)^{PG} \longrightarrow \text{gr} \left(\frac{D(\mathfrak{X}, nk)}{D(\mathfrak{X}, nk)(\tau - \chi_c)(\mathfrak{pg})} \right)^{PG}.$$

By Remark 2.7 the composition

$$\text{gr } D(\text{Rep}(Q, n\delta))^{PG} \longrightarrow \text{gr } D(\mathfrak{X}, nk)^{PG} \longrightarrow \left(\frac{\text{gr } D(\mathfrak{X}, nk)}{\text{gr}(D(\mathfrak{X}, nk))\mu_{\mathfrak{X}}^*(\mathfrak{pg})} \right)^{PG} \longrightarrow \text{gr} \left(\frac{D(\mathfrak{X}, nk)}{D(\mathfrak{X}, nk)(\tau - \chi_c)(\mathfrak{pg})} \right)^{PG} \quad \blacksquare$$

is surjective. Thus the homomorphism

$$D(\mathrm{Rep}(Q, n\delta))^{PG} \longrightarrow \left(\frac{D(\mathfrak{X}, nk)}{D(\mathfrak{X}, nk)(\tau - \chi_c)(\mathfrak{pg})} \right)^{PG}$$

is also surjective. In particular, by 3.9 this implies that the image of $\mathfrak{R}_{k,c}$ equals the image of Oblomkov's Harish–Chandra homomorphism, which, by [O, Theorem 2.5], is $U_{k,c}$.

Thus we have a filtered surjective homomorphism

$$\mathfrak{R}_{k,c} : \left(\frac{D(\mathfrak{X}, nk)}{D(\mathfrak{X}, nk)(\tau - \chi_c)(\mathfrak{pg})} \right)^{PG} \longrightarrow U_{k,c}.$$

Thus the dimension of the left hand side is at least $2 \dim \mathfrak{h} = \dim U_{k,c}$. By Proposition 2.7

$$\left(\frac{\mathrm{gr} D(\mathfrak{X}, nk)}{\mathrm{gr}(D(\mathfrak{X}, nk))\mu_{\mathfrak{X}}^*(\mathfrak{pg})} \right)^{PG} \cong \mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]^{PG} \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\Gamma_n}.$$

Hence p is a surjection from a domain of dimension $2 \dim \mathfrak{h}$ onto an algebra of dimension at least $2 \dim \mathfrak{h}$ and is hence an isomorphism. Thus $(D(\mathfrak{X}, nk)/D(\mathfrak{X}, nk)(\tau - \chi_c)(\mathfrak{pg}))^{\mathfrak{pg}}$ is a domain of dimension $2 \dim \mathfrak{h}$. This implies that $\mathfrak{R}_{k,c}$ is an isomorphism. \square

4. APPLICATION: SHIFT FUNCTORS

4.1. The Holland-Schwarz Lemma. We want to understand the space

$$\frac{D(\mathrm{Rep}(Q_\infty, \epsilon))}{D(\mathrm{Rep}(Q_\infty, \epsilon))(\hat{\tau} - \chi_{k,c})(\mathfrak{g})}.$$

As we observed in the proof of Theorem 3.13 there is a natural surjective homomorphism

$$(4.1.1) \quad \frac{\mathrm{gr} D(\mathrm{Rep}(Q_\infty, \epsilon))}{\mathrm{gr} D(\mathrm{Rep}(Q_\infty, \epsilon))\mu^*(\mathfrak{g})} \longrightarrow \mathrm{gr} \left(\frac{D(\mathrm{Rep}(Q_\infty, \epsilon))}{D(\mathrm{Rep}(Q_\infty, \epsilon))(\hat{\tau} - \chi_{k,c})(\mathfrak{g})} \right).$$

It turns out that this is an isomorphism.

Lemma (Schwarz, Holland). *The homomorphism (4.1.1) is an isomorphism of $\mathbb{C}[T^* \mathrm{Rep}(Q_\infty, \epsilon)]$ -modules. \blacksquare*

Proof. This is [H, Lemma 2.2] since, by Theorem 2.6(2), the moment map μ is flat. \square

4.2. This lets us prove the first part of the isomorphism in the statement of Theorem 1.4.

Lemma. *There is an algebra isomorphism*

$$\left(\frac{D(\mathrm{Rep}(Q_\infty, \epsilon))}{D(\mathrm{Rep}(Q_\infty, \epsilon))(\hat{\tau} - \chi_{k,c})(\mathfrak{g})} \right)^G \longrightarrow \left(\frac{D(\mathfrak{X}, nk)}{D(\mathfrak{X}, nk)(\tau - \chi_c)(\mathfrak{pg})} \right)^{PG}$$

Proof. We have a natural \mathfrak{pg} -equivariant mapping

$$D(\mathrm{Rep}(Q_\infty, \epsilon))^{\mathbb{C}^*} \longrightarrow D_U^{\mathbb{C}^*} \longrightarrow D_{\mathfrak{X}}(nk)$$

which induces a homomorphism

$$D(\mathrm{Rep}(Q_\infty, \epsilon))^G \longrightarrow \left(\frac{D(\mathfrak{X}, nk)}{D(\mathfrak{X}, nk)(\tau - \chi_c)(PG)} \right)^{\mathfrak{pg}}.$$

This is surjective since, as we observed in the proof of Theorem 3.13, the image of $D(\mathrm{Rep}(Q, n\delta))^{PG} \subset D(\mathrm{Rep}(Q_\infty, \epsilon))^G$ spans the right hand side. By (3.11.1) the kernel of this homomorphism includes the ideal $(D(\mathrm{Rep}(Q, \infty), \epsilon)(\hat{\tau} - \chi_{k,c})(\mathfrak{g}))^G$. Hence we have a surjective homomorphism

$$(4.2.1) \quad \left(\frac{D(\mathrm{Rep}(Q_\infty, \epsilon))}{D(\mathrm{Rep}(Q, \infty), \epsilon)(\hat{\tau} - \chi_{k,c})(\mathfrak{g})} \right)^G \longrightarrow \left(\frac{D(\mathfrak{X}, nk)}{D(\mathfrak{X}, nk)(\tau - \chi_c)(\mathfrak{pg})} \right)^{PG}.$$

By Lemma 4.1 and Proposition 2.7 there is an isomorphism

$$\left(\text{gr} \frac{D(\text{Rep}(Q_\infty, \epsilon))}{D(\text{Rep}(Q_\infty, \epsilon))(\hat{\tau} - \chi_{k,c})(\mathfrak{g})} \right)^G \cong \left(\frac{\text{gr } D(\text{Rep}(Q_\infty, \epsilon))}{\text{gr } D(\text{Rep}(Q_\infty, \epsilon))\mu^*(\mathfrak{g})} \right)^G = \mathbb{C}[\mu^{-1}(0)]^G = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\Gamma_n}.$$

This shows that the algebra on the left is a domain of dimension of $2 \dim \mathfrak{h}$ and so (4.2.1) is also injective, as required. \square

4.3. Shifting. The previous two lemmas provide us with an interesting series of bimodules. Given a character Λ of G we define

$$B_{k,c}^\Lambda = \left(\frac{D(\text{Rep}(Q_\infty, \epsilon))}{D(\text{Rep}(Q_\infty, \epsilon))(\hat{\tau} - \chi_{k,c})(\mathfrak{g})} \right)^\Lambda$$

to be the set of (G, Λ) -semiinvariants. Thanks to Lemma 4.2 and Theorem 3.13 this is a right $U_{k,c}$ -module. Now observe that if $x \in \mathfrak{g}$ and $D \in D(\text{Rep}(Q_\infty, \epsilon))^\Lambda$ then

$$[\tau(x), D] = \lambda(x)D$$

where $\lambda = d\Lambda$. It follows that $B_{k,c}^\Lambda$ is also a left $(D(\text{Rep}(Q_\infty, \epsilon))/D(\text{Rep}(Q_\infty, \epsilon))(\hat{\tau} - \chi_{k,c} - \lambda)(\mathfrak{g}))^G$ -module. So tensoring sets up a *shift functor*

$$S_{k,c}^\Lambda : \left(\frac{D(\text{Rep}(Q_\infty, \epsilon))}{D(\text{Rep}(Q_\infty, \epsilon))(\hat{\tau} - \chi_{k,c})(\mathfrak{g})} \right)^G \text{-mod} \longrightarrow \left(\frac{D(\text{Rep}(Q_\infty, \epsilon))}{D(\text{Rep}(Q_\infty, \epsilon))(\hat{\tau} - \chi_{k,c} - \lambda)(\mathfrak{g})} \right)^G \text{-mod}.$$

4.4. The character group of G is isomorphic to \mathbb{Z}^ℓ via

$$(i_0, \dots, i_{\ell-1}) \mapsto ((g_0, \dots, g_{\ell-1}) \mapsto \prod_{r=0}^{\ell-1} \det(g_r)^{i_r}).$$

Corresponding to the standard basis element ϵ_i is the character χ_i of \mathfrak{g} which sends $X \in \mathfrak{g}$ to $\text{Tr}(X_i)$.

Lemma. *The bimodule $B_{k,c}^{\epsilon_i}$ above is a $(U_{k,c}, U_{k',c'})$ -bimodule where $k' = k + 1$ and $c' = c + (1 - \eta^{-i}, 1 - \eta^{-2i}, \dots, 1 - \eta^{-(\ell-1)i})$.*

Proof. Recall that (k, c) corresponds to the character of \mathfrak{g} we called $\chi_{k,c}$ which is defined as

$$\chi_{k,c}(X) = (C_0 + k) \text{Tr}(X_0) + \sum_{j=1}^{\ell-1} C_j \text{Tr}(X_j),$$

where $C_r = \ell^{-1}(1 - \sum_{m=1}^{\ell-1} \eta^{mr} c_m)$ for $1 \leq r \leq \ell - 1$ and $C_0 = \ell^{-1}(1 - \ell - \sum_{m=1}^{\ell-1} c_m)$. We need to calculate (k', c') so that $\chi_{k,c} + \chi_i = \chi_{k',c'}$. So we have

$$(\chi_{nk,c} + \chi_i)(X) = (C_0 + k) \text{Tr}(X_0) + \text{Tr}(X_i) + \sum_{j=1}^{\ell-1} C_j \text{Tr}(X_j) = (C'_0 + k') \text{Tr}(X_0) + \sum_{j=1}^{\ell-1} C'_j \text{Tr}(X_j).$$

Calculation shows that $k' = k + 1$ and that if $i = 0$ then $C'_j = C_j$ and otherwise

$$C'_j = C_j + \begin{cases} -1 & \text{if } j = 0 \\ 1 & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases}$$

These unpack to give $c'_m = c_m + 1 - \eta^{-mi}$. \square

4.5. **Question.** Thus for each $0 \leq i \leq \ell - 1$ we have a *shift functor*

$$S_i : U_{k,c}\text{-mod} \longrightarrow U_{k+1,c'}\text{-mod}$$

where c' is as above. When is this an equivalence of categories?

Remarks. (1) *We have been able to prove this is an equivalence when (k, c) can be reached from $(0, 0)$ by shifting.*

(2) *Shift functors are also constructed in [BC] and [V]. Hopefully they agree with the functors here.*

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