On Harish-Chandra bimodules of rational Cherednik algebras at regular parameter values.

Christopher Charles Spencer

Doctor of Philosophy
University of Edinburgh
August 21, 2012

Abstract

This thesis deals with Harish-Chandra bimodules of rational Cherednik algebras $H_{\mathbf{k}}$ at regular parameter values $\mathbf{k}$, that is those $\mathbf{k}$ for which $H_{\mathbf{k}}$ is a simple algebra. Rational Cherednik algebras can be associated to any reflection representation of a complex reflection group Γ.
The second chapter presents a review of some important results regarding rational Cherednik algebras and their category $\mathcal{O}_{\mathbf{k}}$, which will be frequently used throughout.
The third chapter contains basic results about Harish-Chandra bimodules and the structure of the category $\mathcal{HC}_{\mathbf{k}}$ of Harish-Chandra bimodules, many of which are new in the context of complex reflection groups but have known analogues for real reflection groups at integral parameter values by work of Berest-Etingof-Ginzburg in [BEG03b]. In particular we show that if $\mathbf{k}$ is regular, then $\mathcal{HC}_{\mathbf{k}}$ is a semisimple tensor category and is equivalent to a tensor-closed subcategory of modules of the associated Hecke algebra. Using work of I. Losev in [Los11a], we also deduce that $\mathcal{HC}_{\mathbf{k}}$ is equivalent as a tensor category to $\mathrm{rep}_{\mathbb{C}}(\Gamma/N_{\mathbf{k}})$, the representation category of a quotient of the complex reflection group Γ. This extends previous results for the case of integral $\mathbf{k}$. We manage to obtain some numerical consequences for the presentation of the Hecke algebra of Γ, which is linked to $H_{\mathbf{k}}$ and $\mathcal{O}_{\mathbf{k}}$ via the $\mathrm{KZ}_{\mathbf{k}}$-functor.
The fourth chapter again is a review of standard results on Morita equivalences between rings and integral shift functors giving Morita equivalences between rational Cherednik algebras and their tensor categories of Harish-Chandra bimodules at different regular parameter values. The case of integral parameter values $\mathbf{k}$ is discussed briefly, going back to Berest-Etingof-Ginzburg in [BEG03b] and Berest-Chalykh in [BC09]. The fifth chapter gives a complete description of $N_{\mathbf{k}}$ and its dependence on $\mathbf{k}$ (still for $\mathbf{k}$ regular) for the case that Γ is cyclic.
In chapter 6 we deal with finite-dimensional Harish-Chandra bimodules of rational Cherednik algebras associated to cyclic groups, compute the quiver of that category and derive a criterion for wildness of $\mathcal{HC}_{\mathbf{k}}$ in the cyclic case.
Chapter 7 finally extends a classification of the structure of $\mathcal{HC}_{\mathbf{k}}$ as a tensor category for regular $\mathbf{k}$ to finite irreducible Coxeter groups.

Declaration

I do hereby declare that this thesis was composed by myself and that the work described within is my own, except where explicitly stated otherwise.

Christopher Charles Spencer

Acknowledgements

My thanks will have to go first and foremost to my supervisor Iain Gordon. His constant encouragement, support, insight and countless suggestions are what made this thesis possible. I am very grateful for all that he has done for me which was so much more than could have been expected. I would equally like to thank my second supervisor Tom Lenagan for his support and encouragement over the past years. I am grateful for my friends, old and new, who have made my time in Edinburgh both enjoyable and worthwhile: George, Steven, Pedro, Gwyn, Chris and many more than I could name here.
I would also like to thank the Hausdorff Institute in Bonn and the organisers of the Trimester Program "On the Interaction of Representation Theory with Geometry and Combinatorics" for their hospitality over the 4 months of my visit as well as EPSRC who funded me during my PhD.

Of course my final thanks and deepest gratitude are reserved for my family, who I know I can rely upon at any time and who have always stood by my side.

Contents

Conventions

All our modules will be left modules unless explicitly stated otherwise. The real and imaginary parts of a complex number z will be denoted $\Re z$ and $\Im z$ respectively.

We will only be considering rational Cherednik algebras at "$t = 1$" and the phrase "complex reflection group" will always be understood to mean an "irreducible complex reflection group such that the dimension of the associated Hecke algebra is equal to the size of the group". The only exception to this is Section 2.1 in Chapter 2. In the statement of several results we will often say that Γ is a complex reflection group with $\dim_{\mathbb{C}} \mathcal{H}_{\mathbf{q}}(\Gamma) = \#\Gamma$, this is understood to be meant to emphasise this convention and omission of this phrase does not indicate that the conditions on Γ are dropped.

To ease notation we will often not distinguish between the standard modules $\Delta(\lambda)$ for $H_{\mathbf{k}}$ and the modules $\mathbf{e}\Delta(\lambda)$ of the spherical subalgebra $U_{\mathbf{k}}$ and denote them both by the symbol $\Delta(\lambda)$ where there is no risk of confusion. This will apply in particular throughout Chapter 3.

For any category $\mathcal{C}$ we will often write "$X \in \mathcal{C}$" as a shorthand for "X is an object of $\mathcal{C}$."

We will use μ_m to denote the group of m-th roots of unity in $\mathbb{C}^*$ and $\mathbb{Z}_m$ and $\mathbb{Z}/m\mathbb{Z}$ will both be used to denote the cyclic group of order m.

Chapter 1

Introduction

1.1 Rational Cherednik algebras

As the title already suggests, in this thesis we will investigate the category of Harish-Chandra bimodules over rational Cherednik algebras at regular parameter values. In order for this introduction to be comprehensible it makes sense to briefly put these concepts into a context and explain why they are of interest to researchers.

Rational Cherednik algebras were first introduced by Etingof-Ginzburg in [EG02] and are among the most prominent examples of symplectic reflection algebras. Rational Cherednik algebras interact with many other areas of mathematics, for example symplectic resolutions such as Hilbert schemes, quasi-invariants of complex reflection groups, quantum cohomology (conjecturally) and algebraic combinatorics. It is therefore not surprising that they have generated increasing interest amongst researchers.
For our preferred definition see Definition 2.2.3, this will look different to the definition we give here but will lead to the same algebras by Theorem 2.3.9:

Definition 1.1.1. *([EG02]) Let Γ be an irreducible complex reflection group with reflection representation $\mathfrak{h}$ and regular part $\mathfrak{h}^{reg}$:*

$$\mathfrak{h}^{reg} = \{y \in \mathfrak{h} \mid \mathrm{Stab}_\Gamma(y) = \{1\}\}.$$

The rational Cherednik algebra

$$H_{\mathbf{k}} := H_{\mathbf{k}}(\Gamma, \mathfrak{h})$$

is a subalgebra of $\mathcal{D}(\mathfrak{h}^{reg})\#\Gamma$: It is generated by

1. $\mathbb{C}[\mathfrak{h}]$ *inside* $\mathcal{D}(\mathfrak{h}^{reg})$,
2. $\mathbb{C}\Gamma$,
3. *Dunkl operators* $T_{y,\mathbf{k}} \in \mathcal{D}(\mathfrak{h}^{reg})\#\Gamma$ *for each* $y \in \mathfrak{h}$.

Here $\mathbf{k}$ is a collection of complex numbers (one for each conjugacy class of reflections in Γ). The tuple $\mathbf{k}$ are the "parameters" of $H_{\mathbf{k}}$ and the choice of parameters has a fundamental influence on the structure and representation theory of $H_{\mathbf{k}}$ as they determine the behaviour and properties of the Dunkl operators $T_{\mathbf{k}}$. Whilst we do not need to know the details of what Dunkl operators are in this introduction, they are written out in Definition 2.3.10. In fact Definition 2.3.10

describes a faithful representation of $H_{\mathbf{k}}$ and it is this representation that we have given here as a provisional definition of $H_{\mathbf{k}}$. We will record one definition that will be important for us later on and already appears in the name of the thesis, namely that of regular parameters, see also Definition 2.4.1,

Definition 1.1.2. *Fix a complex reflection group Γ. A choice of parameters $\mathbf{k}$ is called regular if the rational Cherednik algebra $H_{\mathbf{k}}$ is simple. We denote by $\operatorname{Reg}(\Gamma)$ the set of regular parameter values associated to a complex reflection group Γ.*

Rational Cherednik algebras are in many ways similar to semisimple Lie algebras and ideas from Lie theory have often led to useful concepts in the study of rational Cherednik algebras. It is therefore not surprising that one of the most successful concepts in Lie theory plays a central role in the representation theory of rational Cherednik algebras, namely that of a category $\mathcal{O}_{\mathbf{k}}$, i.e. studying a category of modules with particularly "nice" properties. For rational Cherednik algebras, the category $\mathcal{O}_{\mathbf{k}}$ is defined as follows, see also Definition 2.3.2:

Definition 1.1.3. *The category $\mathcal{O}_{\mathbf{k}}$ consists of those finitely generated $H_{\mathbf{k}}$-modules M such that for every $m \in M$ and every Dunkl operator $T_{y,\mathbf{k}}$ there exists $N \in \mathbb{N}$ such that $T_{y,\mathbf{k}}^N m = 0$.*

That this really leads to an interesting category can be seen in many different ways, for example consider Lemma 2.3.3 and Theorem 2.3.13.

One of the most important properties of $\mathcal{O}_{\mathbf{k}}$ is certainly the fact that it is a highest weight category with set of weights $\operatorname{Irr}(\Gamma)$, the irreducible finite-dimensional representations of the complex reflection group Γ. The full definition of a highest weight category can be found in [CPS88], but in particular it means that there is a 1-1 correspondence between the isomorphism classes of simple objects in $\mathcal{O}_{\mathbf{k}}$ and the set $\operatorname{Irr}(\Gamma)$ and for any $\lambda \in \operatorname{Irr}(\Gamma)$ we denote by $L(\lambda)$ the associated simple object in $\mathcal{O}_{\mathbf{k}}$. We further have a set of distinguished "standard modules" which very much play the role of Verma modules in Lie theory (for more about the Lie-theoretic category $\mathcal{O}$ see for example [Hum08]). The standard modules are again in 1-1 correspondence with $\operatorname{Irr}(\Gamma)$ and for each $\lambda \in \operatorname{Irr}(\Gamma)$ we denote by $\Delta(\lambda)$ the associated standard module. It can be defined via tensor product construction (see Definition 2.3.5) as in Lie theory, namely we have

$$\Delta(\lambda) := H_{\mathbf{k}} \otimes_{\mathbb{C}[\mathfrak{h}^*]\#\Gamma} \lambda.$$

see Definition 2.3.5. Actually we should be writing $\Delta_{\mathbf{k}}(\lambda)$ to emphasize the dependence of the module structure of $\Delta(\lambda)$ on the choice of $\mathbf{k}$.

Note that the structure of $\Delta(\lambda)$ as a $\mathbb{C}[\mathfrak{h}]\#\Gamma$-module is independent of $\mathbf{k}$ and as such we always have an isomorphism

$$\Delta(\lambda) \cong \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} \lambda$$

see Proposition 2.3.7.

The module $\Delta(\lambda)$ has a simple head, in fact this is just $L(\lambda)$ see Theorem 2.3.11. There is a close connection between the standard modules, the categorical structure of $\mathcal{O}_{\mathbf{k}}$ and the regular parameters, the next statement can be deduced from Proposition 2.4.3 and Theorem 2.4.21:

Proposition 1.1.4. *Let Γ be a complex reflection group. The following are equivalent*

1. $\Delta_{\mathbf{k}}(\lambda)$ *is simple for all* $\lambda \in \mathrm{Irr}(\Gamma)$
2. $\mathbf{k}$ *is regular, i.e.* $H_{\mathbf{k}}$ *is simple*
3. *the category* $\mathcal{O}_{\mathbf{k}}$ *is semisimple*

The Harish-Chandra bimodules we will study in this thesis have a close interplay with the category $\mathcal{O}_{\mathbf{k}}$ and it is this interaction that will be one of the main tools to use in the study of Harish-Chandra bimodules. First, let us finally give a definition of these bimodules (see Definition 3.1.2):

Definition 1.1.5. *([BEG03b], Definition 3.2) An* $H_{\mathbf{k}}$*-bimodule* V *is called Harish-Chandra if*

1. *it is finitely generated*
2. *for each* $a \in \mathbb{C}[\mathfrak{h}]^{\Gamma} \cup \mathbb{C}[\mathfrak{h}^*]^{\Gamma}$ *the action of* $a \otimes 1 - 1 \otimes a$ *is nilpotent on each* $v \in V$*.*

The category of $H_{\mathbf{k}}$*-Harish-Chandra bimodules is denoted* $\mathcal{HC}_{\mathbf{k}}$*.*

First introduced in [BEG03b] in relation to quasi-invariants, Harish-Chandra bimodules have increasingly gained attention, see for example [Los11a]. Berest-Etingof-Ginzburg also show in [BEG03b] that any Harish-Chandra bimodule is in fact finitely generated as a module on the left or the right, see Lemma 3.1.5:

Lemma 1.1.6. *([BEG03b], Lemma 3.3 (iii)) Let* V *be Harish-Chandra, then* V *is finitely generated as a left or right* $H_{\mathbf{k}}$*-module.*

One of the reasons I began my research on Harish-Chandra bimodules was the fact that they induce endofunctors on category $\mathcal{O}_{\mathbf{k}}$, see Proposition 3.1.7 and Proposition 3.1.8

Proposition 1.1.7.

1. *If* $M \in \mathcal{O}_{\mathbf{k}}$ *and* $V \in \mathcal{HC}_{\mathbf{k}}$*, then* $V \otimes_{H_{\mathbf{k}}} M$ *is in* $\mathcal{O}_{\mathbf{k}}$*.*
2. *([BEG03b], Lemma 8.3) If* U, V *are Harish-Chandra then the tensor product* $U \otimes_{H_{\mathbf{k}}} V$ *is again Harish-Chandra.*

So objects of $\mathcal{HC}_{\mathbf{k}}$ give us (right exact) functors $\mathcal{O}_{\mathbf{k}} \to \mathcal{O}_{\mathbf{k}}$ via the tensor product and the composition of two such functors arises in the same way; this suggests we should study the tensor category $(\mathcal{HC}_{\mathbf{k}}, \otimes_{H_{\mathbf{k}}})$. In Lie theory, Lie-theoretic Harish-Chandra bimodules have been used to construct interesting derived functors on the Lie-theoretic category $\mathcal{O}$, see for example [MM10] and equivalences between "slices" of $\mathcal{O}$ and certain categories of Lie-theoretic Harish-Chandra bimodules have been known for a long time, see for example Chapter 6 in [Jan79] for the details of constructing these equivalences.

Whilst it was not possible to replicate these Lie-theoretic results, we will obtain some results on the categorical structure of $\mathcal{HC}_{\mathbf{k}}$ in the case of regular parameters. As $\mathcal{O}_{\mathbf{k}}$ is semisimple in this case, the induced tensor functors are not particularly interesting from the point of view of studying $\mathcal{O}_{\mathbf{k}}$ or even $D^b(\mathcal{O}_{\mathbf{k}})$, yet this interplay is still important in determining the structure of $\mathcal{HC}_{\mathbf{k}}$.

1.2 Semisimplicity of $\mathcal{HC}_{\mathbf{k}}$ and a corollary from Losev's work

Key tools in our study of Harish-Chandra bimodules will be two functors $\mathcal{L}$ and $\mathcal{T}$ which we will describe next and which are inspired by Lie theory (again see [Jan79]).

Definition 1.2.1. *For a $H_{\mathbf{k}}$-bimodule V we denote by V_{fin} the subset of all vectors $v \in V$ such that every element $a \in \mathbb{C}[\mathfrak{h}]^{\Gamma} \cup \mathbb{C}[\mathfrak{h}^*]^{\Gamma}$ acts* ad*-nilpotently on v.*

The bimodule V_{fin} is Harish-Chandra if it is finitely generated. As $H_{\mathbf{k}}$ is Noetherian (see Proposition 2.2.8), this will for example always be the case when V is already finitely generated as a bimodule. For us however, the most important cases will be less clear-cut and will require some work and a key result from [BEG03b] about the behaviour of Harish-Chandra bimodules under localisation. For the next definition, see Definition 3.4.5 and Definition 3.4.13:

Definition 1.2.2. *(see Formula 8.9 in [BEG03b]) For any $M \in \mathcal{O}_{\mathbf{k}}$ we define a functor*

$$\begin{aligned} \mathcal{L}_M(\bullet) : \mathcal{O}_{\mathbf{k}} &\to H_{\mathbf{k}} \otimes_{\mathbb{C}} H_{\mathbf{k}}^{op} - \mathrm{Mod} \\ N &\mapsto (\mathrm{Hom}_{\mathbb{C}}(M, N))_{fin} \end{aligned}$$

For any two modules $M, N \in \mathcal{O}_{\mathbf{k}}$ we further set

$$\mathcal{L}(M, N) = (\mathrm{Hom}_{\mathbb{C}}(M, N))_{fin}\,.$$

This functor is left exact as $\mathrm{Hom}_{\mathbb{C}}(M, \bullet)$ and $(\bullet)_{fin}$ are and in fact $\mathcal{L}_M$ will give us Harish-Chandra bimodules when $M = \Delta(\mathrm{triv})$, see Theorem 3.4.8

Proposition 1.2.3. *(see also Proposition 5.7.1 in [Los11a]) For regular parameter values $\mathbf{k}$, the functor $\mathcal{L}_{\Delta(\mathrm{triv})}(\bullet)$ is a functor*

$$\mathcal{O}_{\mathbf{k}} \to \mathcal{HC}_{\mathbf{k}}.$$

In particular the spaces $\mathcal{L}(\Delta(\mathrm{triv}), M)$ are finitely generated for any $M \in \mathcal{O}_{\mathbf{k}}$.

The proof is via a localisation argument and in fact will hold in more generality for the spaces $(\mathrm{Hom}_{\mathbb{C}}(M, N))_{fin} = \mathcal{L}(M, N)$. Indeed, it holds when we drop the requirement that $\mathbf{k}$ be regular, which has been shown by Losev in [Los11a], Proposition 5.7.1.

The functor $\mathcal{L}_{\Delta(\mathrm{triv})}$ is of key importance in what will follow, as we can see in the next result (compare Theorem 3.4.8)

Proposition 1.2.4. *Suppose that $\mathbf{k}$ is regular, then for every simple Harish-Chandra bimodule L there is $\lambda \in \mathrm{Irr}(\Gamma)$ such that L is isomorphic to $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda))$.*

The other functor we will be needing is a functor from $\mathcal{HC}_{\mathbf{k}} \to \mathcal{O}_{\mathbf{k}}$, see also Definition 3.4.13:

Definition 1.2.5. *For any $M \in \mathcal{O}_{\mathbf{k}}$ we define a functor*

$$\begin{aligned} \mathcal{T}_M : \mathcal{HC}_{\mathbf{k}} &\to \mathcal{O}_{\mathbf{k}} \\ V &\mapsto V \otimes_{H_{\mathbf{k}}} M \end{aligned}$$

The two functors $\mathcal{L}_{\Delta(\mathrm{triv})}(\bullet)$ and $\mathcal{T}_{\Delta(\mathrm{triv})}(\bullet)$ are closely related, see Proposition 3.4.14:

Proposition 1.2.6. *The functor $\mathcal{T}_{\Delta(\mathrm{triv})}$ is left adjoint to $\mathcal{L}_{\Delta(\mathrm{triv})}$.*

This will prove to be important in one of our major results. What is crucial for us is a result due to Goodearl and Zhang, Lemma 1.3 in [GZ05], which we quote as Theorem 3.3.23. For this, we will need to recall the notion of a progenerator (which we assume to be known for the rest of the thesis)

Definition 1.2.7. *1. Let R be a ring and $P \in R-\mathrm{Mod}$. P is called projective if*

$$\bullet \mapsto \mathrm{Hom}_R(P, \bullet)$$

is an exact functor $R-\mathrm{Mod} \to \mathbb{Z}-\mathrm{Mod}$.

2. A module $M \in R-\mathrm{Mod}$ is called a generator if for any $N \in R-\mathrm{Mod}$ there exists a set I such that there is a surjection

$$\bigoplus_{i\in I} M \twoheadrightarrow N.$$

3. A finitely generated module which is both projective and a generator is a progenerator.

Now we can quote the following result:

Theorem 1.2.8. *([GZ05], Lemma 1.3)) Let A and B be Noetherian simple rings and V an $A-B$-bimodule such that ${}_AV$ and V_B are Noetherian. Then ${}_AV$ and V_B are progenerators in $A-\mathrm{Mod}$ and $\mathrm{Mod}-B$ respectively.*

As we already know that any Harish-Chandra bimodule is finitely generated on the left and right, they are certainly Noetherian on the left and right as $H_{\mathbf{k}}$ is Noetherian. If $\mathbf{k}$ is regular, $H_{\mathbf{k}}$ is moreover simple, so we have the following corollary (see also Theorem 3.4.1)

Corollary 1.2.9. *Let $\mathbf{k}$ be regular, then any Harish-Chandra bimodule is a progenerator in the category of left or right $H_{\mathbf{k}}$-modules. In particular, if $\mathbf{k}$ is regular, every Harish-Chandra bimodule is projective when viewed as a left or right $H_{\mathbf{k}}$-module.*

Using this result we can deduce that the functor $\mathcal{T}_{\Delta(\mathrm{triv})}$ is in fact exact and thus $\mathcal{L}_{\Delta(\mathrm{triv})}$ must take injective objects to injective objects. As $\mathcal{O}_{\mathbf{k}}$ is semisimple if $\mathbf{k}$ is regular, every standard module will be injective in this case. As every simple Harish-Chandra bimodule is given by the image of a standard module under $\mathcal{L}_{\Delta(\mathrm{triv})}$, every simple Harish-Chandra bimodule is injective. Thus we have shown Theorem 3.4.16 in the thesis:

Theorem 1.2.10. *Suppose that $\mathbf{k} \in \mathrm{Reg}(\Gamma)$. Then $\mathcal{HC}_{\mathbf{k}}$ is semisimple.*

Although this clarifies the structure of $\mathcal{HC}_{\mathbf{k}}$ as an Abelian category, the structure of $(\mathcal{HC}_{\mathbf{k}}, \otimes_{H_{\mathbf{k}}})$ is still undetermined. A special case of this was first done by Berest-Etingof-Ginzburg in Theorem 8.5 and Proposition 8.11 in [BEG03b], where they proved that if Γ is a real reflection group and $\mathbf{k}$ is integral, then there is an equivalence of tensor categories

$$(\mathcal{HC}_{\mathbf{k}}, \otimes_{H_{\mathbf{k}}}) \cong (\mathrm{rep}_{\mathbb{C}}\Gamma, \otimes) .$$

This was in principle extended to complex reflection groups by Berest-Chalykh with Proposition 4.3 and Theorem 8.2 in [BC09]).

A generalisation of this for regular parameters can be derived from work of Losev, namely Theorem 3.4.6 in [Los11a] which we cite as Theorem 3.2.4. We can use this to show our Theorem 3.2.6

Theorem 1.2.11. *Let $\mathbf{k} \in \mathrm{Reg}(\Gamma)$. Then for some normal subgroup $N_{\mathbf{k}} \subseteq \Gamma$ there is an equivalence of categories*

$$(\mathcal{HC}_{\mathbf{k}}, \otimes_{H_{\mathbf{k}}}) \cong (\mathrm{rep}_{\mathbb{C}}(\Gamma/N_{\mathbf{k}}), \otimes) .$$

To prove this, we use the result by Losev to identify $\mathcal{HC}_{\mathbf{k}}$ with a tensor-closed full subcategory of $(\mathrm{rep}_{\mathbb{C}}(\Gamma), \otimes)$ which is closed under taking subquotients, i.e. it is closed under taking irreducible components. It then follows from classical representation theory of finite groups that any such subcategory is equivalent to the representation category of a quotient of the group see Lemma 3.2.5.

1.3 Interactions with the Hecke algebra

We will be able to provide an alternative proof of the equivalence $(\mathcal{HC}_{\mathbf{k}}, \otimes_{H_{\mathbf{k}}}) \cong (\mathrm{rep}_{\mathbb{C}}(\Gamma/N_{\mathbf{k}}), \otimes)$ using Hecke algebra techniques. For this, we first need to recall the KZ-functor and how this links $H_{\mathbf{k}}$ to Hecke algebras. Hecke algebras can be associated to any complex reflection group. We will need to take a detour to braid groups to define the Hecke algebras; as in Definition 2.4.8 we follow Section B of [BMR98] and make the following definition:

Definition 1.3.1. *Let Γ be a complex reflection group with reflection representation $\mathfrak{h}$, recall that $\mathfrak{h}^{reg} = \{y \in \mathfrak{h} \mid \mathrm{Stab}_\Gamma(y) = \{1\}\}$. The braid group B_Γ is the fundamental group $\pi_1(\mathfrak{h}^{reg}/\Gamma)$.*

The braid group B_Γ is an infinite group which surjects onto Γ and the Hecke algebra is a quotient of the group algebra of B_Γ. Recall that as Γ is a complex reflection group with reflection representation $\mathfrak{h}$ say, we have an associated set $\mathcal{A}$ of reflection hyperplanes. For any hyperplane $H \in \mathcal{A}$, we denote by Γ_H the pointwise stabiliser of H and we set $m_H = \#\Gamma_H$, the order of this stabiliser. Further, we have an action of Γ on $\mathcal{A}$ by translation, and we will denote the image of $H \in \mathcal{A}$ under the action of $\gamma \in \Gamma$ by H^γ.

Definition 1.3.2. *Let Γ be a complex reflection group with irreducible reflection representation $\mathfrak{h}$ and B_Γ the associated braid group. Choose a set of parameters*

$$\mathbf{q} = \{q_{H,j} \in \mathbb{C}^* \mid H \in \mathcal{A}, 1 \leq j \leq m_H - 1, q_{H,j} = q_{H^\gamma,j} \forall \gamma \in \Gamma\}$$

then define the Hecke algebra $\mathcal{H}_q(\Gamma)$ to be the quotient of $\mathbb{C}B_\Gamma$ by the relations

$$(T-1)\prod_{j=1}^{m_H-1}(T - e^{\frac{-2\pi i j}{m_H}} q_{H,j}) = 0$$

where T is an s-generator of the monodromy around H (where s is the reflection around H with determinant $e^{\frac{2\pi i}{m_H}}$).

What an "s-generator of the monodromy around H" is, is explained in full detail in [BMR98], for a brief overview we refer to Definitions 2.4.10 and 2.4.12. However, in all cases we will consider later, the minimal number of generators of $\mathcal{H}_{\mathbf{q}}(\Gamma)$ will be equal to the rank of Γ, i.e. the dimension $\dim_{\mathbb{C}} \mathfrak{h}$.

It should be said here that throughout we will restrict attention to complex reflection groups Γ such that

$$\dim_{\mathbb{C}} \mathcal{H}_{\mathbf{q}}(\Gamma) = \#\Gamma$$

for any choice of $\mathbf{q}$. This is the case for all but finitely many complex reflection groups and in particular holds for finite Weyl groups and groups of type $G(m,1,n) = \mu_m \wr S_n$.
We can give some examples of Hecke algebras, see also Example 2.4.16:

Example 1.3.3. *1. As a first example, let us consider $\mathcal{H}_{\mathbf{q}}(S_n)$ where S_n acts on $\mathbb{C}^{n-1}$. To construct this action, we let S_n act on $\mathbb{C}^n$ via permutations and then identify $\mathfrak{h}$ with the S_n-invariant complement of the copy of the trivial representation in $\mathfrak{h}$. We have one reflection hyperplane associated to each transposition $(ij) \in S_n$, however these are all products of the neighbour transpositions $s_i = (i, i+1), i = 1, \dots, n-1$. Thus these will give generators T_i of the Hecke algebra. The presentation of S_n in terms of the s_i is*

$$S_n = \langle s_i, i = 1, \dots, n-1 \mid s_i^2 = 1, s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, s_i s_j = s_j s_i \text{ if } |i-j| > 1 \rangle$$

There is only one hyperplane orbit under the action of S_n and so the Hecke algebra will depend on one parameter q and has presentation

$$\mathcal{H}_{\mathbf{q}}(S_n) = \langle T_i \; i = 1, \dots, n-1 \mid T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, T_i T_j = T_j T_i \text{ if } |i-j| > 1, (T-1)(T+q) = 0 \rangle.$$

2. *Now take $\Gamma = \mu_m$, the group of m-th roots of unity with the natural action on $\mathbb{C}$. There is only one reflection hyperplane, namely $\{0\}$ and thus $\mathcal{H}_{\mathbf{q}}(\mu_m)$ is generated by one element T. We have $m-1$ independent parameters $q_1, \dots, q_{m-1}$ and the Hecke algebra has the presentation*

$$\mathcal{H}_{\mathbf{q}}(\mu_m) = \langle T \mid (T-1) \prod_{j=1}^{m-1} (T - e^{-\frac{2\pi i j}{m}} q_j) = 0 \rangle$$

and is a quotient of the polynomial algebra $\mathbb{C}[T]$.

3. *Let $I_2(n)$ denote the dihedral group of order $2n$. If n is odd, the Hecke algebra $\mathcal{H}_{\mathbf{q}}(I_2(n))$ has one independent parameter q and has the presentation*

$$\mathcal{H}_{\mathbf{q}}(I_2(n)) = \langle T_1, T_2 \mid \underbrace{T_1 T_2 \dots T_1}_{n \text{ factors}} = \underbrace{T_2 T_1 \dots T_2}_{n \text{ factors}}, (T_i - 1)(T_i + q) = 0 \rangle.$$

If n is even, the Hecke algebra $\mathcal{H}_{\mathbf{q}}(I_2(n))$ has two independent parameters q_1, q_2 and has the presentation

$$\mathcal{H}_{\mathbf{q}}(I_2(n)) = \langle T_1, T_2 \mid \underbrace{T_1 T_2 \dots T_1}_{n \text{ factors}} = \underbrace{T_2 T_1 \dots T_2}_{n \text{ factors}}, (T_i - 1)(T_i + q_i) = 0 \rangle.$$

The $\mathrm{KZ}_{\mathbf{k}}$-functor is a functor $\mathcal{O}_{\mathbf{k}}(\Gamma) \to \mathcal{H}_{\mathbf{q}}(\Gamma) - \mathrm{mod}$ where $\mathbf{q}$ is defined via $q_{H,j} = e^{2\pi i k_{H,j}}$, see Definition 2.4.17 and the preceding remarks. At regular parameter values it is given as follows: Localising to $\mathfrak{h}^{reg}$, the standard modules $\Delta(\lambda)$ become vector bundles $\mathcal{M}_\lambda = \Delta(\lambda)|_{\mathfrak{h}^{reg}}$ on $\mathfrak{h}^{reg}$. The action of the Dunkl operators gives $\mathcal{M}_\lambda$ the structure of a $\mathcal{D}(\mathfrak{h}^{reg})$-module, i.e. each bundle $\mathcal{M}_\lambda$ carries a flat connection, the KZ-connection. Further, each $\mathcal{M}_\lambda$ is Γ-equivariant and so corresponds to a vector bundle with a flat connection on $\mathfrak{h}^{reg}/\Gamma$. This has regular singularities and so by the Riemann-Hilbert correspondence corresponds to a local system on $\mathfrak{h}^{reg}/\Gamma$ and ultimately to a finite-dimensional representation $M(\lambda)$ of the fundamental group $\pi_1(\mathfrak{h}^{reg}/\Gamma) = B_\Gamma$. This then satisfies the Hecke relations at parameters $q_{H,j} = e^{2\pi i k_{H,j}}$.

In fact the following can be deduced from Theorem 5.14 in [GGOR03] and Lemma 2.3 in [Val06]:

Corollary 1.3.4. *([GGOR03]) The functor $\mathrm{KZ}_{\mathbf{k}} : \mathcal{O}_{\mathbf{k}} \to \mathcal{H}_{\mathbf{q}} - \mathrm{mod}$ is an equivalence of categories if and only if $\mathbf{k}$ is regular. In particular if $\mathbf{k} \in \mathrm{Reg}(\Gamma)$, then $\mathcal{H}_{\mathbf{q}}(\Gamma)$ is semisimple.*

This is Corollary 2.4.19. The existence of non-trivial Harish-Chandra bimodules in the regular case gives strong restrictions on the possible values of $\mathbf{k}$ and $\mathbf{q}$. We deduce this from the next theorem, Theorem 3.4.17

Theorem 1.3.5. *Let $\mathbf{k} \in \mathrm{Reg}(\Gamma)$ and $\lambda, \sigma \in \mathrm{Irr}(\Gamma)$. Suppose that $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda)) \neq 0$. Then*

$$\mathrm{KZ}_{\mathbf{k}}(\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda)) \otimes_{U_{\mathbf{k}}} \Delta(\sigma)) \cong \mathrm{KZ}_{\mathbf{k}}(\Delta(\lambda)) \otimes_{\mathbb{C}} \mathrm{KZ}_{\mathbf{k}}(\Delta(\sigma))$$

The latter tensor product is taken in the category of B_Γ-modules and the left-hand side the tells us that this again factors through the Hecke algebra $\mathcal{H}_{\mathbf{q}}(\Gamma)$. We can give an example to illustrate the utility of this result:

Example 1.3.6. *Consider the symmetric group of order n acting on $\mathfrak{h}_n^0 = \mathbb{C}^{n-1}$. The associated rational Cherednik algebra has one independent parameter k and so does the Hecke algebra $\mathcal{H}_{\mathbf{q}}(S_n)$. If a non-trivial Harish-Chandra bimodule V exists, it gives rise to a non-trivial $\mathcal{H}_{\mathbf{q}}$-module M such that $M \otimes M$ is again an $\mathcal{H}_{\mathbf{q}}$-module. As M is non-trivial, we can find a non-zero $m \in M$ with $T_i m = -qm$ for some i. Then $T_i(m \otimes m) = q^2(m \otimes m)$ by definition of the tensor product and as this has to be an $\mathcal{H}_{\mathbf{q}}$-module again, we must conclude that*

$$q^2 = 1 \text{ or } q^2 = -q.$$

Therefore we must have

$$q \in \{-1, 0, 1\}.$$

By construction we can immediately discount the case $q = 0$ and the case $q = -1$ leads to a non-semisimple Hecke algebra which is contradicts our hypothesis that $\mathbf{k} \in \mathrm{Reg}(\Gamma)$. Thus we are forced to conclude that

$$q = 1 \implies e^{2\pi i k} = 1 \implies k \in \mathbb{Z}.$$

This case has been dealt with by Berest-Etingof-Ginzburg in [BEG03b] and we can conclude that if $\mathbf{k}$ is regular then $H_{\mathbf{k}}(S_n)$ has non-trivial Harish-Chandra bimodules if and only if $\mathbf{k}$ is integral, in which case we have an equivalence

$$(\mathcal{HC}_{\mathbf{k}}, \otimes_{H_{\mathbf{k}}}) \cong (\mathrm{rep}_{\mathbb{C}} S_n, \otimes)\,.$$

Using similar computations we can show that the existence of non-trivial Harish-Chandra bimodules at regular parameters implies that the parameters $\mathbf{q}$ of the Hecke algebra $\mathcal{H}_{\mathbf{q}}(\Gamma)$ are such that $\mathcal{H}_{\mathbf{q}}(\Gamma)$ surjects onto $\mathbb{C}(\Gamma/N_{\mathbf{k}})$ for some $N_{\mathbf{k}} \trianglelefteq \Gamma$ and that every $\mathcal{H}_{\mathbf{q}}(\Gamma)$-module coming from a Harish-Chandra bimodule is still a module over this quotient. The precise formulation and proof of this is in Theorem 3.4.21. Thus we have

Theorem 1.3.7. *Let $\mathbf{k} \in \mathrm{Reg}(\Gamma)$. Then for some normal subgroup $N_{\mathbf{k}} \trianglelefteq \Gamma$ there is an equivalence of categories $(\mathcal{HC}_{\mathbf{k}}, \otimes_{H_{\mathbf{k}}}) \cong (\mathrm{rep}_{\mathbb{C}}(\Gamma/N_{\mathbf{k}}), \otimes)\,.$*

1.4 Determining $N_{\mathbf{k}}$ and explicit construction of Harish-Chandra bimodules

It is now a natural question to ask whether one can describe the map $\mathrm{Reg}(\Gamma) \ni \mathbf{k} \mapsto N_{\mathbf{k}} \trianglelefteq \Gamma$ explicitly, i.e. how to determine the structure of $\mathcal{HC}_{\mathbf{k}}$ depending on the parameter. This will occupy most of the remainder of the thesis, in particular Chapter 5 and Chapter 7.

In Chapter 5, we describe the correspondence $\mathbf{k} \mapsto N_{\mathbf{k}}$ for the case that Γ is cyclic. For this, we should introduce another definition, namely Definition 5.2.1

Definition 1.4.1. *We denote by $\Xi_{\mathbf{k}}(\Gamma)$ the set of isomorphism classes of simple Harish-Chandra bimodules over $H_{\mathbf{k}}(\Gamma)$.*

As the structure of a cyclic group is uniquely determined by its order, the question of the structure of $\mathcal{HC}_{\mathbf{k}}$ is one of determining the cardinality of $\Xi_{\mathbf{k}}$. For cyclic groups and regular parameter values $\mathbf{k}$ we find an explicit criterion to determine when non-trivial Harish-Chandra bimodules exist, the particular statement here is a slight variation of Corollary 5.2.16.

Proposition 1.4.2. *Let $\mathbf{k} = (k_0 = 0, k_1, \ldots, k_{m-1})$ be a choice of parameters for Γ a cyclic group of order m and $\mathbf{q} = (q_0 = 1, q_1 = e^{2\pi i k_1}, \ldots, q_{m-1} = e^{2\pi i k_{m-1}})$ the parameters of the associated Hecke algebra. Set $\tilde{q}_j = e^{\frac{-2\pi i j}{m}} q_j$. Then $\Xi_{\mathbf{k}}$ has order divisible by d if and only if there exists a partition of $\{\tilde{q}_j\}$ into m/d disjoint subsets*

$$Q_s = \{\tilde{q}_{s,0}, \ldots, \tilde{q}_{s,d-1}\}, s = 0, \ldots, \frac{m}{d} - 1$$

such that the sets

$$\{1, \tilde{q}_{s,0}^{-1}\tilde{q}_{s,1}, \ldots, \tilde{q}_{s,0}^{-1}\tilde{q}_{s,d-1}\}$$

consists of all d-th roots of unity. In particular if such a d is maximal, this is the order of $\Xi_{\mathbf{k}}$ and $\Xi_{\mathbf{k}}$ can have any order dividing m.

It is instructive to see how these Harish-Chandra bimodules can be explicitly constructed for cyclic groups, as a similar construction seems to be possible in the general case, for example for dihedral groups. We first need to introduce three types of functors which we will need in this construction. One of these are integral shift functors which exist for all complex reflection groups and go back to ideas of Heckman-Opdam. We will not go into details of the construction here, but just state the result we need which is Theorem 4.2.10, compare also Proposition 8.11 in [BEG03b]:

Theorem 1.4.3. *Suppose that $\mathbf{k} \in \mathrm{Reg}(\Gamma)$ and $\mathbf{k}'$ is a choice of parameters for Γ such that $\mathbf{k} - \mathbf{k}'$ is integral. Then $\mathbf{k}' \in \mathrm{Reg}(\Gamma)$ and there is an equivalence of tensor categories*

$$(\mathcal{HC}_{\mathbf{k}}(H_{\mathbf{k}}), \otimes_{H_{\mathbf{k}}}) \cong \left(\mathcal{HC}_{\mathbf{k}'}(H_{\mathbf{k}'}), \otimes_{H_{\mathbf{k}'}}\right)$$

We will refer to these functors as "integral shift functors."

The second functor is a Morita equivalence between $H_{\mathbf{k}}$ and a related algebra which is computationally easier to handle and will play a key role in the construction of Harish-Chandra bimodules that does not have a clear analogue on the level of rational Cherednik algebras. This

Morita equivalence does not only exist for cyclic groups but for arbitrary complex reflection groups and we will introduce it in this generality. This is Definition 2.2.7.

Definition 1.4.4. *Let*

$$\mathbf{e} = \frac{1}{\#\Gamma} \sum_{\gamma \in \Gamma} \gamma \in \mathbb{C}\Gamma \subset H_{\mathbf{k}}$$

be the trivial idempotent of the complex reflection group Γ. *We then define the spherical subalgebra* $U_{\mathbf{k}}$ *of* $H_{\mathbf{k}}$ *to be*

$$U_{\mathbf{k}} = \mathbf{e} H_{\mathbf{k}} \mathbf{e}.$$

We can define versions of both $\mathcal{HC}_{\mathbf{k}}$ and $\mathcal{O}_{\mathbf{k}}$ for $U_{\mathbf{k}}$ in the obvious manner. We give the definition of $U_{\mathbf{k}}$-Harish-Chandra bimodules as in Definition 3.1.2, this was first done in [BEG03b], Definition 3.2.

Definition 1.4.5. *We define Harish-Chandra bimodules of* $U_{\mathbf{k}}$ *to be those* $U_{\mathbf{k}}$*-bimodules that are finitely generated and on which every* $a \in \mathbb{C}[\mathfrak{h}]^{\Gamma}\mathbf{e} \cup \mathbb{C}[\mathfrak{h}^*]^{\Gamma}\mathbf{e}$ *acts locally nilpotently. We will also denote the category of such modules by* $\mathcal{HC}_{\mathbf{k}}$. *When we need to distinguish between the two categories of Harish-Chandra bimodules, we will denote them by* $\mathcal{HC}_{\mathbf{k}}(H)$ *and* $\mathcal{HC}_{\mathbf{k}}(U)$ *respectively.*

It is a standard result from ring theory that for any ring R and any idempotent $e \in R$ with $ReR = R$ the two rings R and eRe are Morita equivalent. Certainly at regular parameter values we have $H_{\mathbf{k}}\mathbf{e}H_{\mathbf{k}} = H_{\mathbf{k}}$ by simplicity of $H_{\mathbf{k}}$ and we arrive at a slightly less general version of Theorem 3.2.3:

Theorem 1.4.6. *Suppose that* $\mathbf{k} \in \mathrm{Reg}(\Gamma)$. *Then there exists a Morita equivalence between* $H_{\mathbf{k}}$ *and* $U_{\mathbf{k}}$ *which descends to an equivalence*

$$\mathcal{O}_{\mathbf{k}}(H_{\mathbf{k}}) \cong \mathcal{O}_{\mathbf{k}}(U_{\mathbf{k}})$$

and gives an equivalence

$$(\mathcal{HC}_{\mathbf{k}}(H_{\mathbf{k}}), \otimes_{H_{\mathbf{k}}}) \cong (\mathcal{HC}_{\mathbf{k}}(U_{\mathbf{k}}), \otimes_{U_{\mathbf{k}}}).$$

Moreover, this equivalence preserves the tensor action of $\mathcal{HC}_{\mathbf{k}}$ *on* $\mathcal{O}_{\mathbf{k}}$.

That $U_{\mathbf{k}}$ is particularly well suited to study Harish-Chandra bimodules might be due to the following result, which is Theorem 2.4.2 and can be found for example as Theorem 4.6 in [BEG03b] or Lemma 4.11 in [Val06]

Theorem 1.4.7. *Let* $\mathbf{k} \in \mathrm{Reg}(\Gamma)$, *then the spherical subalgebra* $U_{\mathbf{k}} = \mathbf{e}H_{\mathbf{k}}\mathbf{e}$ *is generated as an algebra by* $\mathbb{C}[\mathfrak{h}]^{\Gamma}\mathbf{e} \cup \mathbb{C}[\mathfrak{h}^*]^{\Gamma}\mathbf{e}$.

Using these two types of functors, the integral shift functors and the Morita equivalence between $H_{\mathbf{k}}$ and $U_{\mathbf{k}}$, we can already give a concise description of the construction of Harish-Chandra bimodules in the case of integral parameters. We shall digress briefly to explain this construction as it will serve to motivate some of the ideas used in other cases: We can work with the spherical subalgebra and assume without loss of generality that the parameters $\mathbf{k}$ are all zero. Then the Dunkl operator $T_{y,\mathbf{0}}$ is just the usual differential operator induced by $y \in \mathfrak{h}$ and thus we can identify

$$U_{\mathbf{0}} \cong \mathcal{D}(\mathfrak{h})^{\Gamma}.$$

This embeds $U_{\mathbf{0}}$ into the algebra of differential operators $\mathcal{D}(\mathfrak{h})$ and under this embedding $\mathcal{D}(\mathfrak{h})$ will be a Harish-Chandra bimodule over $U_{\mathbf{0}}$. As $U_{\mathbf{0}}$ is identified with the invariants of $\mathcal{D}(\mathfrak{h})$ under the action of Γ, each isotypic component of $\mathcal{D}(\mathfrak{h})$ will still be a Harish-Chandra bimodule over $U_{\mathbf{0}}$. Thus, for each irreducible representation of Γ we obtain a Harish-Chandra bimodule over $U_{\mathbf{0}}$, using the interpretation as bimodules of differential operators. It is not difficult to show that these are all non-isomorphic and to calculate that this assignment gives an equivalence of categories between $\mathcal{HC}_{\mathbf{0}}$ and $\mathrm{rep}_{\mathbb{C}}\Gamma$. Using the integral shift functors, we can then extend this result to any integral $\mathbf{k}$.

The third type of functors rests on a construction of an action of symmetric groups on the parameters of rational Cherednik algebras of the complex reflection groups of types $G(m,1,n)$ which gives isomorphisms between the rational Cherednik algebras (and the spherical subalgebras of rational Cherednik algebras) associated to parameters in the same orbit. This isomorphism preserves the grading element and the group invariants and will thus in particular give Morita equivalences between the categories of Harish-Chandra bimodules with parameters in the same orbit.

In the context of cyclic groups of order m, we obtain an action of S_{m-1} on the parameters $\mathbf{k} = (k_1, \ldots, k_{m-1})$. To see that this indeed gives isomorphisms of spherical subalgebras, we can use the fact that the spherical subalgebras associated to cyclic groups are algebras of a type first studied by Smith and Hodges, see [Smi90] and [Hod93] respectively. In particular, they have 3 generators with relations essentially depending on one polynomial $\chi_{\mathbf{k}}(t)$ in one variable, which is determined by the parameters $\mathbf{k}$. The action of S_{m-1} leaves the polynomial invariant and so will give an isomorphism between spherical subalgebras. We can summarise this in a proposition, which is Corollary 5.2.11.

Proposition 1.4.8. *There is an action of S_{m-1} on the space of parameters $\mathbf{k} = (k_1, \ldots, k_{m-1})$ of a cyclic group of order m given on the transpositions $(i, i+1)$ by*

$$(i, i+1) \cdot \mathbf{k} := (k_1, \ldots, k_{i-1}, k_{i+1} - \frac{1}{m}, k_i + \frac{1}{m}, k_{i+2}, \ldots, k_{m-1}).$$

For $w \in S_{m-1}$ there is an isomorphism $U_{\mathbf{k}} \cong U_{w(\mathbf{k})}$ given by

$$\mathbf{e}x^m\mathbf{e} \mapsto \mathbf{e}x^m\mathbf{e}$$
$$\mathbf{e}(T_{y,\mathbf{k}})^m\mathbf{e} \mapsto \mathbf{e}(T_{y,w(\mathbf{k})})^m\mathbf{e}$$
$$\mathbf{e}\underline{h}\mathbf{e} \mapsto \mathbf{e}\underline{h}\mathbf{e}$$

This gives an equivalence

$$\left(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}}\right) \left(\mathcal{HC}_{w(\mathbf{k})}, \otimes_{U_{w(\mathbf{k})}}\right).$$

The element $\mathbf{e}\underline{h}\mathbf{e}$ corresponds to a deformed Euler operator. As $\mathbf{e}x^m\mathbf{e}$ and $\mathbf{e}(T_{y,\mathbf{k}})^m\mathbf{e}$ already generate $U_{\mathbf{k}}$ for regular $\mathbf{k}$, it is not necessary to specify the isomorphism on $\mathbf{e}\underline{h}\mathbf{e}$ and we will not define $\mathbf{e}\underline{h}\mathbf{e}$ or use its properties in this introduction.

Armed with these functors, we can now reduce the construction of non-trivial Harish-Chandra bimodules over $H_{\mathbf{k}}$ to the task of constructing non-trivial Harish-Chandra bimodules over $U_{\mathbf{k}}$, which is easier. Suppose that W is a cyclic group of order m and let us fix a divisor $d \mid m$.

From the criterion we can deduce a set of possible set of regular parameter values $\mathbf{k}$ at which $\Xi_{\mathbf{k}}$ might have order d, recall that this just means $\mathcal{HC}_{\mathbf{k}}(W)$ has d distinct isomorphism classes of simple objects. We set $r = m/d$ and let $N \trianglelefteq W$ be the unique subgroup of W of order r so that W/N has order d.
Using the integral shift functors and the S_{m-1}-action, we can restrict ourselves to a unique distinguished parameter $\mathbf{k}_0$. By inspection we can then find parameters $\mathbf{b}$ for N such that we have an embedding of algebras

$$U_{\mathbf{k}_0}(W) \hookrightarrow U_{\mathbf{b}}(N).$$

This embedding makes $U_{\mathbf{b}}(N)$ into a Harish-Chandra bimodule over $U_{\mathbf{k}_0}(W)$. We further have an action of W/N on $U_{\mathbf{b}}(N)$ which is induced from the action of W on the reflection representation. Under this action we have

$$U_{\mathbf{b}}(N)^{W/N} \cong U_{\mathbf{k}_0}(W)$$

and so each W/N-isotypic component of $U_{\mathbf{b}}(N)$ will still be a Harish-Chandra bimodule over $U_{\mathbf{k}_0}(W)$. Thus we obtain d distinct Harish-Chandra bimodules over $U_{\mathbf{k}_0}(W)$ and the same will then hold true for all parameters in the orbit of $\mathbf{k}_0$ under the action of the integral shift functors and the action of S_{m-1}.

A similar strategy will be employed to construct the non-trivial Harish-Chandra bimodules for dihedral groups $I_2(n)$ with n even in Chapter 7, whereas for finite Coxeter groups with just one conjugacy class of reflections the Hecke algebra techniques of Example 1.3.6 will be useful. Let us explain this next.

So let Γ be a finite Coxeter group with just one conjugacy class of reflections - these are the finite Coxeter groups of types $A_n, D_n, I_2(n)$ for n odd and the groups E_6, E_7, E_8, H_3 and H_4. The groups of type A_n are just the symmetric groups we have dealt with in Example 1.3.6 and the groups $I_2(n)$ as already said are the dihedral groups. In fact, the arguments employed in Example 1.3.6 will work for these groups in just the same way, to deduce that for any of these finite Coxeter groups we can only have non-trivial Harish-Chandra bimodules at regular parameter values when these parameter values are integral, when again we will obtain an equivalence with the representation category of the group, see Theorem 7.2.4.

Theorem 1.4.9. *Let Γ be an irreducible finite Coxeter group with one conjugacy class of reflections and let $\mathbf{k} \in \mathrm{Reg}(\Gamma)$. Then $\mathcal{HC}_{\mathbf{k}}(\Gamma)$ is non-trivial if and only if $\mathbf{k} \in \Lambda(\Gamma)$ and $(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}}) \cong (\mathrm{rep}_{\mathbb{C}}(\Gamma), \otimes)$.*

It remains to deal with the case of irreducible finite Coxeter groups with two conjugacy classes of reflections. These are the Coxeter groups of type $BC_n, I_2(n)$ for n even, and F_4. The techniques discussed to construct the non-trivial Harish-Chandra bimodules in the cyclic case work here with slight modifications.
Let us consider the example of dihedral groups $I_2(n)$ with even n to illustrate: As $I_2(n)$ for even n has two conjugacy classes of reflections, the associated rational Cherednik algebra and the associated Hecke algebra will both have two independent parameters, say k_1, k_2 and q_1, q_2 (viewing these dihedral groups as irreducible finite Coxeter groups). Using the restrictions on the parameters of the associated Hecke algebra, we can deduce that if non-trivial Harish-Chandra bimodules exist, then at least one of the parameters of the rational Cherednik algebra

has to be integral and using the integral shift functors we may assume that this parameter is 0. We can discount the case that both parameters are integral as we have already dealt with this, so we may assume that one of k_1, k_2 is zero with the other non-integral and $\mathbf{k} \in \mathrm{Reg}(I_2(n))$. Further, we have an equivalence of tensor categories $\mathcal{HC}_{(k_1,k_2)} \cong \mathcal{HC}_{(k_2,k_1)}$.

The last equivalence enables us to assume that the parameter k_1 equals 0 and this enables us to embed

$$H_{\mathbf{k}'}(I_2(n/2)) \hookrightarrow H_{\mathbf{k}}(I_2(n))$$

where $\mathbf{k}' = (k_2, k_2)$ if $n/2$ is even and $\mathbf{k}' = k_2$ if $n/2$ is odd. The embedding gives an isomorphism

$$H_{\mathbf{k}}(I_2(n)) \cong H_{\mathbf{k}'}(I_2(n/2))\#\mathbb{Z}_2$$

where $H_{\mathbf{k}'}(I_2(n/2))\#\mathbb{Z}_2$ is the smash product of $H_{\mathbf{k}'}(I_2(n/2))$ and $\mathbb{Z}_2$, the cyclic group of order 2. The action of $\mathbb{Z}_2$ on $H_{\mathbf{k}'}(I_2(n/2))$ is induced from the action of $I_2(n)$ on $H_{\mathbf{k}}(I_2(n))$. The embedding of $H_{\mathbf{k}'}(I_2(n/2))$ in $H_{\mathbf{k}}(I_2(n))$ gives an action of $I_2(n)$ on $H_{\mathbf{k}'}(I_2(n/2)))$ and on $U_{\mathbf{k}'}(I_2(n/2))$. This factors through an action of $\mathbb{Z}_2 \cong I_2(n)/I_2(n/2)$ and the algebra of invariants is again isomorphic to $U_{\mathbf{k}}(I_2(n))$:

$$U_{\mathbf{k}'}(I_2(n/2))^{\mathbb{Z}_2} \cong U_{\mathbf{k}}(I_2(n)).$$

Thus as before $U_{\mathbf{k}'}(I_2(n/2))$ is Harish-Chandra over $U_{\mathbf{k}}(I_2(n))$ and decomposing it into $\mathbb{Z}_2$-isotypic components gives all simple Harish-Chandra bimodules for this case. So in summary we have sketched a proof of Corollary 7.2.8:

Theorem 1.4.10. *Let n be even and $\mathbf{k} = (k_1, k_2) \in I_2(n)$. Then*

1. *if neither k_1 nor k_2 is integral, there is an equivalence $(\mathcal{HC}_{\mathbf{k}}, \otimes_{H_{\mathbf{k}}}) \cong (\mathrm{Vect}_{\mathbb{C}}, \otimes)$.*
2. *if exactly one of k_1, k_2 is integral, there is an equivalence $(\mathcal{HC}_{\mathbf{k}}, \otimes_{H_{\mathbf{k}}}) \cong (\mathrm{rep}_{\mathbb{C}}\mathbb{Z}_2, \otimes)$.*
3. *if both k_1, k_2 are integral, there is an equivalence $(\mathcal{HC}_{\mathbf{k}}, \otimes_{H_{\mathbf{k}}}) \cong (\mathrm{rep}_{\mathbb{C}} I_2(n), \otimes)$.*

To adapt this approach to the groups of type BC_n and F_4 only minor modifications are necessary. Hecke algebra arguments show that if W is a finite irreducible Coxeter group with 2 conjugacy classes of reflections then one of the parameters k_1, k_2 must be integral. We set $\{a, b\} = \{1, 2\}$ and assume that $k_a \in \mathbb{Z}$ and as before using the integral shift functors we may assume that this parameter is 0. We denote by W_b the subgroup of W generated by all reflections in W conjugate to a simple reflection associated to the parameter k_b. The analogue of the embedding $H_{\mathbf{k}'}(I_2(n/2)) \hookrightarrow H_{\mathbf{k}}(I_2(n))$ will be to consider the subalgebra $R_{\mathbf{k}}(W_b)$ generated by $\mathfrak{h}, \mathfrak{h}^*$ and W_b. With $\mathbf{e}_b$ the principal idempotent of $\mathbb{C}W_b$ we then obtain an action of W/W_b on $\mathbf{e}_b R_{\mathbf{k}}(W_b)\mathbf{e}_b$ via conjugation in $H_{\mathbf{k}}(W)$. As before we have an isomorphism $(\mathbf{e}_b R_{\mathbf{k}}(W_b)\mathbf{e}_b)^{W/W_b} \cong U_{\mathbf{k}}(W)$ and we can decompose $\mathbf{e}_b R_{\mathbf{k}}(W_b)\mathbf{e}_b$ into W/W_b-isotypic components to obtain $\#\mathrm{Irr}(W/W_b)$ simple non-isomorphic Harish-Chandra bimodules over $U_{\mathbf{k}}(W)$. A Hecke algebra calculation then shows that we have an equivalence $\left(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}(W)}\right) \cong (\mathrm{rep}_{\mathbb{C}}(W/W_b), \otimes)$. This is Theorem 7.2.6. Using results of Maxwell in [Max98], see Theorem 7.1.15, we can then refine this to get a classification of $\mathcal{HC}_{\mathbf{k}}$ for groups of type BC_n and F_4:

Theorem 1.4.11. *Let W be of type BC_n and $\mathbf{k} \in \mathrm{Reg}(W)$.*

1. *If $k_1 \in \mathbb{Z}, k_2 \notin \mathbb{Z}$ then $(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}}) \cong (\mathrm{rep}_{\mathbb{C}}\mathbb{Z}_2, \otimes)$*
2. *If $k_1 \notin \mathbb{Z}, k_2 \in \mathbb{Z}$ then $(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}}) \cong (\mathrm{rep}_{\mathbb{C}} S_n, \otimes)$*

Let W be of type F_4 and $\mathbf{k} \in \mathrm{Reg}(W)$.

1. *If $k_1 \in \mathbb{Z}, k_2 \notin \mathbb{Z}$ then $(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}}) \cong (\mathrm{rep}_{\mathbb{C}} S_3, \otimes)$*
2. *If $k_1 \notin \mathbb{Z}, k_2 \in \mathbb{Z}$ then $(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}}) \cong (\mathrm{rep}_{\mathbb{C}} S_3, \otimes)$*

1.5 Finite-dimensional bimodules for cyclic groups

Finally, in a slight thematic deviation, this thesis will also briefly consider special Harish-Chandra bimodules at non-regular parameter values for cyclic groups, namely the finite-dimensional ones. This is not entirely unrelated to our previous work: Theorem 3.4.6 in [Los11a] shows that it is natural to stratify the category of Harish-Chandra bimodules in a way that relates $\mathcal{HC}_{\mathbf{k}}$ to the stratification of $(\mathfrak{h} \times \mathfrak{h}^*)/\Gamma$ by symplectic leaves, see [BG03]. If Γ is cyclic, this corresponds to considering finite-dimensional Harish-Chandra bimodules and the quotient of $\mathcal{HC}_{\mathbf{k}}$ by its subcategory of finite-dimensional bimodules. Thus t understand $\mathcal{HC}_{\mathbf{k}}$ of a cyclic group fully for all parameters, it will suffice to understand finite-dimensional Harish-Chandra bimodules, infinite-dimensional Harish-Chandra bimodules and how extensions between these two classes are possible.

The starting point for our work on finite-dimensional bimodules is the simple observation that for W a cyclic group of order m every finite-dimensional $H_{\mathbf{k}}(W)$-bimodule is Harish-Chandra. In Chapter 6 we compute the quiver with relations of the category of finite-dimensional $H_{\mathbf{k}}(W)$-bimodules and thereby the quiver with relations of finite-dimensional Harish-Chandra bimodules. Using this, we derive a simple criterion to determine when the category of finite-dimensional bimodules is wild. This is Proposition 6.6.7 and states the following:

Proposition 1.5.1. *Let W be a finite cyclic group. Then the category of finite-dimensional $H_{\mathbf{k}}(W)$-Harish-Chandra bimodules is wild if and only if $\mathcal{O}_{\mathbf{k}}$ has a block of length 4 or greater.*

Clearly the category $\mathcal{HC}_{\mathbf{k}}$ of all Harish-Chandra bimodules will also be wild when its subcategory of finite-dimensional Harish-Chandra bimodules is.

We will briefly run through the calculation necessary to determine the quiver with relations, we start with a variant of Definition 6.4.1.

Definition 1.5.2. *We denote by $\mathrm{fd}(\mathcal{HC}_{\mathbf{k}})$ the category of finite-dimensional $H_{\mathbf{k}}$-Harish-Chandra bimodules, and $H_{\mathbf{k}} - \mathrm{fin}$ will denote the category of finite-dimensional $H_{\mathbf{k}}$-modules.*

It is not difficult to show that $H_{\mathbf{k}} - \mathrm{fin}$ is a full subcategory of $\mathcal{O}_{\mathbf{k}}$ by noting that every finite-dimensional $H_{\mathbf{k}}$-module M possesses an internal grading, i.e. M has a grading induced by decomposing it into generalised eigenspaces for a certain element $\underline{\mathrm{h}} \in H_{\mathbf{k}}$. The action of every Dunkl operator $T_{y,\mathbf{k}}$ then changes the eigenvalue of any generalised eigenvector of M and by finite-dimensionality we must then have that $T_{y,\mathbf{k}}^N$ acts as 0 on M for some suitably large $N \in \mathbb{N}$. We will use the inclusion

$$H_{\mathbf{k}} - \mathrm{fin} \hookrightarrow \mathcal{O}_{\mathbf{k}}$$

as a tool to investigate the structure of $H_{\mathbf{k}} - \mathrm{fin}$ and from this derive information about $\mathrm{fd}(\mathcal{HC}_{\mathbf{k}})$.

Crawley-Boevey and Holland investigated the algebras $H_{\mathbf{k}}(W)$ and $U_{\mathbf{k}}(W)$ in [CBH98] using quiver-theoretic methods and in particular constructed Morita equivalences between $H_{\mathbf{k}}(W)$ and

$H_{\mathbf{k}'}(W)$ for related parameter values - in fact it turns out that these equivalences are essentially the same as those constructed by Losev for the rational Cherednik algebras associated to the groups $G(m,1,n)$. Crawley-Boevey and Holland then show that for every parameter $\mathbf{k}$ there exists a parameter $\mathbf{k}'$ such that $H_{\mathbf{k}}(W)$ and $H_{\mathbf{k}'}(W)$ are Morita equivalent and every finite-dimensional simple module has dimension 1. We verify that this equivalence takes $\mathcal{O}_{\mathbf{k}}$ to $\mathcal{O}_{\mathbf{k}'}$ and $H_{\mathbf{k}}-\text{fin}$ to $H_{\mathbf{k}'}-\text{fin}$ and are then able to do explicit calculations assuming that all finite-dimensional simple modules have dimension 1.

Under this hypothesis, it is not difficult to compute the projective indecomposables in $\mathcal{O}_{\mathbf{k}}$ using work of Guay in [Gua03] - they are certain generalised standard modules which we introduce in Definition 6.2.16.

Definition 1.5.3. *Suppose $\lambda \in \text{Irr}(\Gamma)$ and $n \in \mathbb{N}$. The generalised standard module of degree n associated to λ is defined to be*

$$\Delta^{(n)}(\lambda) := H_{\mathbf{k}} \otimes_{\mathbb{C}[\mathfrak{h}^*]\#\Gamma} \left((\mathbb{C}[\mathfrak{h}^*]/\mathbb{C}[\mathfrak{h}^*]^n_+) \otimes \lambda \right).$$

We can then prove that the projective indecomposables in $H_{\mathbf{k}}-\text{fin}$ are quotients of the projective indecomposables in $\mathcal{O}_{\mathbf{k}}$ (in fact, $H-\mathbf{k}-\text{fin}$ is just the category of finitely generated modules over a finite-dimensional quotient of $H_{\mathbf{k}}(W)$) and the explicit description of the projective modules in $\mathcal{O}_{\mathbf{k}}$ as generalised standard modules allows us to easily describe the projective indecomposables in $H_{\mathbf{k}}-\text{fin}$ as well as the relations on the quivers. The next result is an amalgamation of Corollary 6.3.20 and Proposition 6.4.17.

Proposition 1.5.4. *Let W be a finite cyclic group of order m*

1. *The quiver of any block of $\mathcal{O}_{\mathbf{k}}$ has the form*

$$\begin{tikzcd} \lambda_1 \arrow[r, bend left, "a_1"] & \lambda_2 \arrow[l, bend left, "b_1"] \arrow[r, bend left, "a_2"] & \lambda_3 \arrow[l, bend left, "b_2"] \arrow[r, bend left, "a_3"] & \cdots \arrow[l, bend left, "b_3"] \arrow[r, bend left, "a_{n-1}"] & \lambda_n \arrow[l, bend left, "b_{n-1}"] \end{tikzcd}$$

for some $n \in \mathbb{N}$ and is subject to the relations

$$a_{n-1}b_{n-1} = 0 \text{ and}$$
$$a_r b_r = b_{r+1}a_{r+1} \text{ for } 1 \leq r \leq n-2.$$

2. *The quiver of any block of $H_{\mathbf{k}}-\text{fin}$ has the form*

$$\begin{tikzcd} \lambda_2 \arrow[r, bend left, "\bar{a}_2"] & \lambda_3 \arrow[l, bend left, "\bar{b}_2"] \arrow[r, bend left, "\bar{a}_3"] & \cdots \arrow[l, bend left, "\bar{b}_3"] \arrow[r, bend left, "\bar{a}_{n-1}"] & \lambda_n \arrow[l, bend left, "\bar{b}_{n-1}"] \end{tikzcd}$$

for some $n \in \mathbb{N}, n \geq 2$ and is subject to the relations

$$\bar{a}_{n-1}\bar{b}_{n-1} = 0,$$
$$\bar{b}_2\bar{a}_2 = 0,$$
$$\bar{a}_r\bar{b}_r = \bar{b}_{r+1}\bar{a}_{r+1} \text{ for } 2 \leq r \leq n-2.$$

The quivers and relations describing blocks of $\text{fd}(\mathcal{HC}_{\mathbf{k}})$ can be deduced from this description, they will be of the form

$$
\begin{tikzcd}
\lambda_2 \arrow[r, bend left, "a_2"] \arrow[d, bend left] & \lambda_3 \arrow[l, bend left, "b_2"] \arrow[r, bend left, "a_3"] \arrow[d, bend left] & {} \arrow[l, bend left, "b_3"] \arrow[d, bend left] & \cdots & \lambda_{n-1} \arrow[r, bend left, "a_{n-1}"] \arrow[d, bend left] & \lambda_n \arrow[l, bend left, "b_{n-1}"] \arrow[d, bend left] \\
\lambda_2 \arrow[r, bend left, "a_2"] \arrow[d, bend left] \arrow[u, bend left] & \lambda_3 \arrow[l, bend left, "b_2"] \arrow[r, bend left, "a_3"] \arrow[d, bend left] \arrow[u, bend left] & {} \arrow[l, bend left, "b_3"] \arrow[d, bend left] \arrow[u, bend left] & \cdots & \lambda_{n-1} \arrow[r, bend left, "a_{n-1}"] \arrow[d, bend left] \arrow[u, bend left] & \lambda_n \arrow[l, bend left, "b_{n-1}"] \arrow[d, bend left] \arrow[u, bend left] \\
\vdots \arrow[r, bend left, "a_2"] \arrow[d, bend left] \arrow[u, bend left] & \vdots \arrow[l, bend left, "b_2"] \arrow[r, bend left, "a_3"] \arrow[d, bend left] \arrow[u, bend left] & {} \arrow[l, bend left, "b_3"] \arrow[d, bend left] \arrow[u, bend left] & \cdots & \vdots \arrow[r, bend left, "a_{n-1}"] \arrow[d, bend left] \arrow[u, bend left] & \vdots \arrow[l, bend left, "b_{n-1}"] \arrow[d, bend left] \arrow[u, bend left] \\
\lambda_2 \arrow[r, bend left, "a_2"] \arrow[u, bend left] & \lambda_3 \arrow[l, bend left, "b_2"] \arrow[r, bend left, "a_3"] \arrow[u, bend left] & {} \arrow[l, bend left, "b_3"] \arrow[u, bend left] & \cdots & \lambda_{n-1} \arrow[r, bend left, "a_{n-1}"] \arrow[u, bend left] & \lambda_n \arrow[l, bend left, "b_{n-1}"] \arrow[u, bend left]
\end{tikzcd}
$$

with the appropriate relations coming from Proposition 1.5.4 and imposing the additional condition that each square commutes. By a result of Ringel [Rin75], we can show that this quiver with relations is wild for $n \geq 4$.

Chapter 2

The Rational Cherednik Algebra

2.1 Generalities on complex reflection groups

We will begin with a well-known definition that is nevertheless fundamental to our subject, namely we will start by defining complex reflection groups:

Definition 2.1.1. *Let Γ be a finite group and $\mathfrak{h}$ a finite-dimensional complex vector space endowed with a faithful Γ-action. We set*

$$\mathcal{S} := \{\gamma \in \Gamma \mid \operatorname{rk}(\gamma - \mathrm{id}_{\mathfrak{h}}) = 1\}$$

and refer to the elements of $\mathcal{S}$ as complex reflections. We say that Γ is a complex reflection group or that Γ acts by complex reflections if $\mathcal{S}$ generates Γ. The space $\mathfrak{h}$ will be called a reflection representation of Γ.

It is important to note here that in the following chapters and sections we shall always assume that Γ is irreducible (see Definition 2.1.8), unless otherwise stated!
It is important to note that a group Γ might have multiple reflection representations and even multiple irreducible reflection representations.

Definition 2.1.2. *Let Γ be a complex reflection group with reflection representation $\mathfrak{h}$. As before we set*

$$\mathcal{S} = \{s \in \Gamma \mid \operatorname{rk}(s - \mathrm{id}_{\mathfrak{h}}) = 1\}$$

and further

$$H_s = \ker(s - \mathrm{id}_{\mathfrak{h}}) \text{ for } s \in \mathcal{S}$$

$$\mathcal{A} := \{H_s \mid s \in \mathcal{S}\},$$

the set of reflection hyperplanes. Moreover, for any $H \in \mathcal{A}$ we set

$$\Gamma_H := \operatorname{Stab}_{\Gamma}(H)$$

the pointwise stabiliser of the hyperplane H and we denote

$$m_H := \#\Gamma_H.$$

Γ acts on $\mathcal{A}$ by translation and we denote image of the hyperplane $H \in \mathcal{A}$ under $\gamma \in \Gamma$ by H^{γ}.

Because the definition of a complex reflection group depends on the existence of a reflection representation $\mathfrak{h}$ we should be thinking of the pair $(\Gamma, \mathfrak{h})$ as one object - as we shall see later the group Γ may have several reflection representations and these have different properties. However we shall often suppress $\mathfrak{h}$ from our notation and assume that a reflection representation $\mathfrak{h}$ for Γ has been fixed. Let us now turn to some examples.
A special case of complex reflection groups are those that arise from suitable actions on real vector spaces:

Definition 2.1.3. *Let Γ be a finite group and V a finite-dimensional real vector space endowed with a faithful Γ-action. As before, we have a set of reflections*

$$\mathcal{S} := \{\gamma \in \Gamma \mid \operatorname{rk}(\gamma - \operatorname{id}_V) = 1\}$$

and if $\mathcal{S}$ generates Γ we call Γ a real reflection group with reflection representation V.

Lemma 2.1.4. *Let Γ be a real reflection group with reflection representation V. Then Γ acts on $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{R}} V$ by complex reflections and thus is a complex reflection group with reflection representation $\mathfrak{h}$.*

Proof. We only have to show that any element $s \in \Gamma$ acting as a reflection on V still acts as a complex reflection on $\mathfrak{h}$. If $v \in V$ is such that $s(v) = v$ then both $1 \otimes v, i \otimes v \in \mathfrak{h}$ are fixed by the action of s on $\mathfrak{h}$. Hence

$$\dim_{\mathbb{C}}(s - \operatorname{id}_{\mathfrak{h}})(\mathfrak{h}) = \dim_{\mathbb{R}}(s - \operatorname{id}_V) = 1.$$

□

In a slight abuse of terminology, we will refer to complex reflection groups arising from real reflection groups still as real reflection groups.

Example 2.1.5. *Let us consider the case of cyclic groups more closely. Suppose that Γ is cyclic of order m say. We may identify Γ with μ_m, the group of m-th complex roots of unity and can let this act on $\mathfrak{h} := \mathbb{C}$ via multiplication. This makes $\mathfrak{h}$ a reflection representation of μ_m and consequently μ_m is a complex reflection group: We have $\mathcal{S} = \mu_m \setminus \{1\}$ and so clearly W is generated by its set of reflections. We further have $\mathcal{A} = \{\{0\}\}$ and $(\mu_m)_{\{0\}} = \mu_m$.*

Example 2.1.6. *1. Let us consider $\Gamma = S_n$ acting on $\mathfrak{h}_n$, a complex n-dimensional vector space with basis $e_1, \ldots, e_n$ and S_n acting by permuting the basis vectors. If $(ij) \in S_n$ is a transposition then*

$$(ij) \cdot e_k = \begin{cases} e_j \text{ if } k = i \\ e_i \text{ if } k = j \\ e_k \text{ else} \end{cases}.$$

So $((ij) - \operatorname{id}_{\mathfrak{h}_n})(\mathfrak{h}_n)$ is spanned by $e_i - e_j$ and thus (ij) is a complex reflection. Therefore $\{(ij)\}_{1 \leq i,j \leq n} \subseteq \mathcal{S}$ and hence $\mathcal{S}$ certainly generates S_n. We conclude that the symmetric groups too are complex reflection groups with $\mathfrak{h}_n$ as a possible reflection representation.
We have already seen that each transposition is a reflection and now we will aim to show the converse, i.e. that any element of S_n acting as a reflection on $\mathfrak{h}_n$ is a transposition. Take $w \in S_n$ and suppose this fulfils $\operatorname{rk}(w - \operatorname{id}_{\mathfrak{h}_n}) = 1$. Suppose that at least three elements

of $\{1, 2, \ldots, n\}$ are moved by w, say i, j, k. Then $e_{wi} - e_i$, $e_{wj} - e_j$ and $e_{wk} - e_k$ are in the image of $w - \mathrm{id}_{\mathfrak{h}_n}$ and thus the image has at least dimension 2, contradicting our assumption that w was in $\mathcal{S}$. Thus w can move at most 2 elements of $\{1, \ldots, n\}$ and thus is either a transposition or the identity and the identity is obviously not a reflection. Hence

$$\mathcal{S} = \{(ij) \in S_n \mid i, j \in \{1, \ldots, n\}\} \subset S_n.$$

The reflection hyperplane associated to the transposition $(ij) \in S_n$ is then given by

$$H_{(ij)} := \left\{ \sum_k \lambda_k e_k \in \mathfrak{h}_n \mid \lambda_i = \lambda_j \right\}$$

The stabiliser subgroup $(S_n)_{(ij)}$ associated to the reflection hyperplane $H_{(ij)}$ is given by $\{1, (ij)\}$ and is cyclic of order 2.

2. *We now set $\mathfrak{h}_n^0 = \{\sum_i \lambda_i e_i \in \mathfrak{h}_n \mid \sum_i \lambda_i = 0\} \subseteq \mathfrak{h}_n$. This is a hyperplane in $\mathfrak{h}_n$ and is preserved under the action of S_n on $\mathfrak{h}_n$. We will show that this too is a reflection representation of S_n. For this, note that a basis of $\mathfrak{h}_n^0$ is given by $e'_k = e_k - e_{k+1}$ for $k = 1, \ldots, n-1$ and the transposition $(i, i+1) \in S_n$ acts on this basis as follows:*

$$(i, i+1) \cdot {e'}_k = \begin{cases} {e'}_{k-1} + {e'}_k & \text{if } i = k-1 \\ -{e'}_k & \text{if } i = k \\ {e'}_k + {e'}_{k+1} & \text{if } i = k+1 \\ {e'}_k & \text{if } i \neq k-1, k, k+1 \end{cases}.$$

Hence the image of $(i, i+1) - \mathrm{id}_{\mathfrak{h}_n^0}$ is spanned by e'_i and therefore the transpositions $\{(i, i+1)\}_{1 \leq i \leq n-1}$ act as reflections on $\mathfrak{h}_n^0$. So $\{(i, i+1)\}_{1 \leq i \leq n-1} \subset \mathcal{S}$ and it is well known that these generate S_n, making $\mathfrak{h}_n^0$ into another reflection representation of S_n.

Note that in both cases, S_n is a real reflection group.

Example 2.1.7. *Finite Coxeter groups are real reflection groups (see Definition 7.1.1 for the definition of a finite Coxeter group).*

These will be the most important examples of complex reflection groups in what follows.

We have seen in the example of S_n that a reflection representation of a complex (or even real) reflection group is not necessarily unique. Indeed, if Γ is a complex reflection group with reflection representation $\mathfrak{h}$ and letting triv denote the trivial representation of Γ, then $\mathfrak{h} \bigoplus \oplus_{i=1}^n \mathrm{triv}$ will again be a reflection representation for any value of n. Imposing an irreducibility condition will eliminate such "silly examples":

Definition 2.1.8. *Let Γ be a complex reflection group and $\mathfrak{h}$ a reflection representation. Then we say that Γ is an irreducible complex reflection group if $\mathfrak{h}$ is an irreducible $\mathbb{C}\Gamma$-module.*

Definition 2.1.9. *Let Γ be an irreducible complex reflection group with reflection representation $\mathfrak{h}$. Then the dimension of $\mathfrak{h}$ is referred to as the rank of Γ:*

$$\mathrm{rank}(\Gamma) = \dim_{\mathbb{C}} \mathfrak{h}.$$

The irreducible complex reflection groups have been classified, see [ST54] and Theorem 2.1.12.

As any complex reflection group is isomorphic to a product of irreducible complex reflection groups, this gives a complete classification of all complex reflection groups. An important class will be introduced next:

Definition 2.1.10. *Choose integers $m, p, n \geq 1$ with $p \mid m$. We let $G(m,p)$ be the subgroup of μ_m^n consisting of elements of the form $(\zeta^{r_1}, \dots, \zeta^{r_n})$ with $\sum r_i \equiv 0 \mod p$ where ζ is a primitive m-th root of unity. Then we define*

$$G(m,p,n) = G(m,p) \rtimes S_n.$$

The groups $G(m,p,n)$ are complex reflection groups acting on $\mathbb{C}^n$ and may be identified with the groups of generalised permutation matrices whose non-zero entries are ζ^{r_i} as above.

Example 2.1.11. *1. The group $G(1,1,n)$ is the symmetric group S_n.*

2. The group $G(m,1,1)$ is the cyclic group of order m.

3. The group $G(2,2,2)$ is the Klein four-group.

4. The group $G(m,1,n)$ is the wreath product $\mu_m \wr S_n$ with the action of S_n on $(\mathbb{Z}_m)^n$ being given by permutation.

5. For any $d \in \mathbb{N}, G(d,d,2)$ is the dihedral group of order $2d$.

The classification of irreducible complex reflection groups as presented in [Bro10] then reads as follows:

Theorem 2.1.12. *(Section I and Table VII in [ST54], Theorem 1.15 in [Bro10]) Let Γ be an irreducible complex reflection group acting on $\mathfrak{h}$. Then one of the following is true*

1. There exist $m,p,n \in \mathbb{N}$ with $m \geq 2$, $p \mid m$ and $n \geq 1$ such that $\Gamma \cong G(m,p,n)$ and $\mathfrak{h} = \mathbb{C}^n$.

2. There exists $n \geq 2$ such that $\Gamma = S_n$ and $\mathfrak{h} = \mathbb{C}^{n-1}$.

3. Γ is one of 34 exceptional groups.

Finally we shall give a nice result linking a complex reflection group to the structure of its set of invariants due to Shephard-Todd [ST54]. The implication (1) $\implies$ (2) was also shown by Chevalley [Che55] in a uniform manner rather than the case-by-case approach of [ST54].

Theorem 2.1.13. *(Theorem 5.1 (c) in [ST54], Theorem A in [Che55]) Let Γ be a finite group acting faithfully on a finite-dimensional complex vector space $\mathfrak{h}$. Then the following are equivalent:*

1. Γ is generated by complex reflections, i.e. Γ is a complex reflection group with reflection representation $\mathfrak{h}$

2. $\mathbb{C}[\mathfrak{h}]^\Gamma$ is a polynomial algebra

A proof of this can also be found in Michel Broué's book [Bro10] where this result is listed as Theorem 4.1 and also includes some additional characterisations.

2.2 Rational Cherednik Algebras

We have seen in the previous examples of complex reflection groups that the stabiliser subgroups of reflection hyperplanes have turned out to be cyclic groups. That this is indeed the case for any complex reflection group is a known result for which the author does not know of a suitable reference and therefore we supply a proof.

Lemma 2.2.1. *Let Γ be a complex reflection group with reflection representation $\mathfrak{h}$ and let $H \subseteq \mathfrak{h}$ be a reflection hyperplane. Then the pointwise stabiliser of H, denoted Γ_H, is cyclic.*

Proof. By definition the fixed-point space of any element of Γ_H has dimension at least $\dim_{\mathbb{C}} \mathfrak{h} - 1$ and so every non-trivial element of Γ_H must be a complex reflection and an element $s \in \Gamma_H$ is trivial precisely when $\det_{\mathfrak{h}}(s) = 1$. Thus,

$$\det{}_{\mathfrak{h}} : \Gamma_H \to \mathbb{C}^*$$

is an injection and so Γ_H is cyclic. □

We need to recall a standard construction from ring theory.

Definition 2.2.2. *Let R be a $\mathbb{C}$-algebra and G a group acting linearly on R. We denote by $R\#\Gamma$ the vector space $R \otimes_{\mathbb{C}} \Gamma$ with algebra structure given by*

$$(r \otimes g)(s \otimes h) = rg(s) \otimes gh$$

and refer to this as the smash product of R and G.

Now we have sufficient terminology and knowledge to introduce the rational Cherednik algebras already mentioned in the title of the chapter.

Definition 2.2.3. *Let Γ be a complex reflection group with irreducible reflection representation $\mathfrak{h}$. For each $\Gamma_H, H \in \mathcal{A}$ we denote by $\mathbf{e}_{H,i}, i = 0, \ldots, m_H - 1$ the idempotent basis of $\mathbb{C}\Gamma_H$, where $\mathbf{e}_{H,i}$ is the idempotent corresponding to the character $\det_{\mathfrak{h}}^{-i}$ of Γ_H:*

$$\mathbf{e}_{H,j} = \frac{1}{m_H} \sum_{\gamma \in \Gamma_H} \det{}_{\mathfrak{h}}^{j}(\gamma)\gamma.$$

We fix complex numbers

$$t \in \mathbb{C}, \mathbf{k} := \{k_{H,j} \in \mathbb{C} \mid j = 1, \ldots, m_H - 1, H \in \mathcal{A}, k_{H,j} = k_{H^{w_\gamma},j} \forall \gamma \in \Gamma\}$$

and we further set

$$k_{H,0} = 0$$

and agree to identify $k_{H,i} = k_{H,i+m_H}$ for notational convenience.
Finally, for each $H \in \mathcal{A}$ we choose a non-zero vector $v_H \in \mathfrak{h}$ such that v_H spans $\mathrm{im}(s - \mathrm{id}_{\mathfrak{h}})$ where s is chosen such that $H = H_s$, and we choose $\alpha_H \in \mathfrak{h}^*$ such that $\alpha_H(H) = 0$ and $\alpha_H(v_H) \neq 0$. The rational Cherednik algebra $H_{(t,\mathbf{k})} = H_{(t,\mathbf{k})}(\Gamma, \mathfrak{h})$ associated to Γ and $\mathfrak{h}$ at parameters $(t, \mathbf{k})$ is the quotient of

$$T(\mathfrak{h} \oplus \mathfrak{h}^*)\#\Gamma$$

by the relations

$$(*)\ [y, x] = t\langle y, x\rangle + \sum_{H \in \mathcal{A}} \frac{\langle y, \alpha_H\rangle\langle v_H, x\rangle}{\langle \alpha_H, v_H\rangle} \sum_{i=0}^{m_H - 1} m_H(k_{H,i} - k_{H,i+1})\mathbf{e}_{H,i}$$

$$[x, x'] = 0$$

$$[y, y'] = 0$$

for all $y, y' \in \mathfrak{h}, x, x' \in \mathfrak{h}^*$.

Some points need to be addressed: Although the choice of v_H and thus also α_H seems somewhat arbitrary and dependent on s, we can show that the relations $(*)$ are independent of the particular choice of v_H and α_H and s: By Lemma 2.2.1 the stabiliser group Γ_H is cyclic and and generated by $\gamma \in \Gamma$ say. By definition, γ is again a complex reflection and as it has finite order in $\mathrm{End}_{\mathbb{C}}(\mathfrak{h})$ it is diagonalisable. Choose an eigenbasis $e_1, \ldots, e_n$ of γ such that γ is represented by a matrix $\gamma = \mathrm{diag}(\rho, 1, \ldots, 1)$ with $\rho \in \mathbb{C}$ a root of unity. Choose $s, s' \in \mathcal{S}$ with $H_s = H_{s'} = H$. Both s, s' are in Γ_H and so we have $i, j \in \mathbb{N}$ such that $s = \gamma^i, s' = \gamma^j$ say. Then we see that any choice of v_H must be a multiple of e_1 so that any two distinct choices are scalar multiples of each other. Similarly, any two choices of α_H must be scalar multiples of each other and it is then clear that $(*)$ is invariant under taking scalar multiples of v_H and α_H.

The above definition of Cherednik algebras makes sense even in the case that $\mathfrak{h}$ is not irreducible, however the resulting algebras have not been studied much and we will not consider them.

One key result in the theory of rational Cherednik algebras - and one that we will be using many times implicitly - is the fact that they fulfil a PBW-type theorem. This was first shown by Etingof and Ginzburg:

Theorem 2.2.4. *([EG02], Theorem 1.3) For any choice of parameter, the rational Cherednik algebra contains* $\mathbb{C}[\mathfrak{h}], \mathbb{C}[\mathfrak{h}^*]$ *and* $\mathbb{C}\Gamma$ *as subalgebras. We have an isomorphism of vector spaces*

$$H_{(t,\mathbf{k})} \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}^*] \otimes_{\mathbb{C}} \mathbb{C}\Gamma \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{h}].$$

In particular, a basis of $H_{\mathbf{k}}$ *is given by the elements* $p(y)\gamma q(x)$ *where* $q(x)$ *is a monomial in* $\mathbb{C}[\mathfrak{h}]$, $p(y)$ *is a monomial in* $\mathbb{C}[\mathfrak{h}^*]$ *and* $\gamma \in \Gamma$.

The behaviour of rational Cherednik algebras differs greatly depending on whether $t = 0$ or $t \neq 0$. For example if $t = 0$, the rational Cherednik algebra is a finite module over its centre, whereas in the case $t \neq 0$ the centre is trivial, consisting solely of the scalars (see e.g. Theorem 3.3 in [EG02] to deduce the first claim and Proposition 7.2 for both claims in [BG03]). In this thesis we will only be concerned with the case $t \neq 0$ and will refer exclusively to this case from now on. For any $\lambda \in \mathbb{C}^*$ we have an isomorphism $H_{(t,\mathbf{k})} \xrightarrow{\sim} H_{(\lambda^2 t, \lambda^2 \mathbf{k})}$ given by mapping

$$x \mapsto \frac{1}{\lambda}x,\ y \mapsto \frac{1}{\lambda}y,\ \gamma \mapsto \gamma$$

and thus we may and will from now on assume that $t = 1$. We will also simplify our notation from $H_{(1,\mathbf{k})}$ to $H_{\mathbf{k}}$ and as we have already done usually drop the reference to Γ and $\mathfrak{h}$ if there is no danger of confusion.

In much of the following, a special element of $H_{\mathbf{k}}$ will play an important role. We will denote it by $\underline{\mathrm{h}}$ and call it the "grading element" for reasons that will become clear later.

Definition 2.2.5. *Let $\{y_i\}_i$ be a basis of $\mathfrak{h}$ and $\{x_i\}_i$ a dual basis of $\mathfrak{h}^*$. Set*

$$\begin{aligned} H_{\mathbf{k}} \ni \underline{\mathrm{h}} &= -\sum_i x_i y_i - \frac{1}{2}\dim_{\mathbb{C}}\mathfrak{h} + \sum_{H\in\mathcal{A}}\sum_{j,l=0}^{m_H-1}(1-m_H\delta_{j,l})k_{H,l}\mathbf{e}_{H,j} \\ &= -\sum_i x_i y_i - \frac{1}{2}\dim_{\mathbb{C}}\mathfrak{h} - \sum_{H\in\mathcal{A}}\sum_{j=0}^{m_H-1} m_H k_{H,j}\mathbf{e}_{H,j} + \sum_{H\in\mathcal{A}}\sum_{j=0}^{m_H-1} k_{H,j}. \end{aligned}$$

This is independent of the choice of $\{y_i\}_i$ and we will refer to this as the grading element of $H_{\mathbf{k}}$.

Proposition 2.2.6. *([GGOR03], Section 3.1) For $x\in\mathfrak{h}^*$, $y\in\mathfrak{h}$ and $\gamma\in\Gamma$ the following holds in $H_{\mathbf{k}}$:*

$$[\underline{\mathrm{h}}, x] = -x,\ [\underline{\mathrm{h}}, y] = y \text{ and } [\underline{\mathrm{h}}, \gamma] = \gamma.$$

Moreover, $\mathfrak{h}$ is independent of the choice of dual bases $\{x_i\}, \{y_i\}$.

Proof. The argument is by direct computation, and as this is not done in [GGOR03] we provide it for the reader's convenience. Suppose that $x\in\mathfrak{h}$, then we find

$$\begin{aligned} [\underline{\mathrm{h}}, x] &= [-\sum_i x_i y_i - \sum_{H\in\mathcal{A}}\sum_{j=0}^{m_H-1} m_H k_{H,j}\mathbf{e}_{H,j}, x] \\ &= -\sum_i x_i[y_i, x] - \sum_{H\in\mathcal{A}}\sum_{j=0}^{m_H-1} m_H k_{H,j}[\mathbf{e}_{H,j}, x] \\ &= -\sum_i x_i\left(\langle y_i, x\rangle + \sum_{H\in\mathcal{A}}\frac{\langle y_i,\alpha_H\rangle\langle v_H, x\rangle}{\langle\alpha_H, v_H\rangle}\sum_{i=0}^{m_H-1} m_H(k_{H,i}-k_{H,i+1})\mathbf{e}_{H,i}\right) \\ &- \sum_{H\in\mathcal{A}}\sum_{j=0}^{m_H-1} m_H k_{H,j}[\mathbf{e}_{H,j}, x] \end{aligned}$$

Because $\{y_i\}$ and $\{x_i\}$ are dual bases, we have

$$v = \sum_i y_i\langle v, x_i\rangle \text{ and } f = \sum_i x_i\langle y_i, f\rangle$$

for all $v\in\mathfrak{h}$ and $f\in\mathfrak{h}^*$. Thus in particular we have $x=\sum_i x_i\langle y_i, x\rangle$ and $\alpha_H = \sum_i x_i\langle y_i,\alpha_H\rangle$ and therefore we can simplify

$$[\underline{\mathrm{h}}, x] = -x - \sum_{H\in\mathcal{A}}\alpha_H\frac{(x, v_H)}{(\alpha_H, v_H)}m_H\sum_{j=0}^{m_H-1}(k_{H,j}-k_{H,j+1})\mathbf{e}_{H,j} - \sum_{H\in\mathcal{A}} m_H\sum_{j=0}^{m_H-1} k_{H,j}[\mathbf{e}_{H,j}, x].$$

We will next calculate the commutator $[\mathbf{e}_{H,j}, x]$. For this, we note that we may set $x = x_H + \lambda\alpha_H$, where $x_H|_H \equiv 0$ and $\lambda\in\mathbb{C}$. Then we find that

$$\begin{aligned} \frac{(x, v_H)}{(\alpha_H, v_H)} &= \frac{(x_H+\lambda\alpha_H, v_H)}{(\alpha_H, v_H)} \\ &= \frac{(x_H, v_H)}{(\alpha_H, v_H)} + \lambda\frac{\alpha_H, v_H)}{(\alpha_H, v_H)} \\ &= \lambda \end{aligned}$$

and using this we compute

$$
\begin{aligned}
[\mathbf{e}_{H,j}, x] &= [\frac{1}{m_H} \sum_{\gamma \in \Gamma_H} \det|_{\mathfrak{h}}(\gamma)^j \gamma, x] \\
&= \frac{1}{m_H} \sum_{\gamma \in \Gamma_H} \det|_{\mathfrak{h}}(\gamma)^j \gamma x - x\mathbf{e}_{H,j} \\
&= \frac{1}{m_H} \sum_{\gamma \in \Gamma_H} \det|_{\mathfrak{h}}(\gamma)^j \gamma(x_H + \lambda\alpha_H) - (x_H + \lambda\alpha_H)\mathbf{e}_{H,j} \\
&= \frac{1}{m_H} \sum_{\gamma \in \Gamma_H} \det|_{\mathfrak{h}}(\gamma)^j (x_H\gamma + \lambda \det|_{\mathfrak{h}}(\gamma)^{-1}\alpha_H w) - (x_H + \lambda\alpha_H)\mathbf{e}_{H,j} \\
&= x_H\mathbf{e}_{H,j} + \lambda\alpha_H\mathbf{e}_{H,j-1} - x_H\mathbf{e}_{H,j} - \lambda\alpha_H\mathbf{e}_{H,j} \\
&= \frac{(x, v_H)}{(\alpha_H, v_H)}\alpha_H\mathbf{e}_{H,j-1} - \frac{(x, v_H)}{(\alpha_H, v_H)}\alpha_H\mathbf{e}_{H,j}.
\end{aligned}
$$

Thus we find

$$
\begin{aligned}
[\underline{\mathrm{h}}, x] &= -x - \sum_{H \in \mathcal{A}} \alpha_H \frac{(x, v_H)}{(\alpha_H, v_H)} m_H \sum_{j=0}^{m_H-1} (k_{H,j} - k_{H,j+1})\mathbf{e}_{H,j} \\
&- \sum_{H \in \mathcal{A}} m_H \sum_{j=0}^{m_H-1} k_{H,j} \left(\frac{(x, v_H)}{(\alpha_H, v_H)}\alpha_H\mathbf{e}_{H,j-1} - \frac{(x, v_H)}{(\alpha_H, v_H)}\alpha_H\mathbf{e}_{H,j} \right) \\
&= -x - \sum_{H \in \mathcal{A}} m_H \sum_{j=0}^{m_H-1} (-k_{H,j+1}) \frac{(x, v_H)}{(\alpha_H, v_H)}\alpha_H + k_{H,j}(\frac{(x, v_H)}{(\alpha_H, v_H)}\alpha_H\mathbf{e}_{H,j-1} \\
&= -x.
\end{aligned}
$$

The calculation that $[\underline{\mathrm{h}}, y] = y$ is similar.

To show that $[\underline{\mathrm{h}}, \gamma] = 0$ for all $\gamma \in \Gamma$ and that $\underline{\mathrm{h}}$ is independent of the choice of dual bases, we need only note the following facts: The element $\sum_i x_i y_i$ is invariant under the action of γ by the classical representation theory of finite group - indeed if G is a finite group and V an irreducible finite-dimensional G-representation with basis $\{v_i\}$, then $\sum_i v_i \otimes v_i^* \in V \otimes_{\mathbb{C}} V^*$ will span a copy of the trivial representation of G and$\sum_i v_i \otimes v_i^*$ must be independent of the choice of basis as $V \otimes_{\mathbb{C}} V^*$ contains only one copy of the trivial representation by character theory. Thus $[\sum_i x_i y_i, \gamma] = 0$ and clearly $[-\frac{1}{2}\dim_{\mathbb{C}} \mathfrak{h} + \sum_{H \in \mathcal{A}} \sum_{j=0}^{m_H-1} k_{H,j}, \gamma] = 0$. The element $\sum_{H \in \mathcal{A}} \sum_{j=0}^{m_H-1} m_H k_{H,j}\mathbf{e}_{H,j} \in \mathbb{C}\Gamma$ is a sum over elements of conjugacy classes since $k_{H,j} = k_{H^\gamma,j}$ and thus in fact lies in $Z(\mathbb{C}\Gamma)$. Hence

$$[\underline{\mathrm{h}}, \gamma] = 0.$$

□

Definition 2.2.7. *Let*

$$\mathbf{e} = \frac{1}{\#\Gamma} \sum_{\gamma \in \Gamma} \gamma \in \mathbb{C}\Gamma \subset H_{\mathbf{k}}$$

be the trivial idempotent of the complex reflection group Γ. *We then define the spherical subalgebra* $U_{\mathbf{k}}$ *of* $H_{\mathbf{k}}$ *to be*

$$U_{\mathbf{k}} = \mathbf{e}H_{\mathbf{k}}\mathbf{e}.$$

Note that although this is a subset of $H_{\mathbf{k}}$ *closed under addition, subtraction and multiplication.*

The unit elements are distinct with 1 *being the unit of* $H_{\mathbf{k}}$ *and* $\mathbf{e}$ *being the unit of* $U_{\mathbf{k}}$ *(in other words, the inclusion of* $U_{\mathbf{k}}$ *into* $H_{\mathbf{k}}$ *does not intertwine the scalars and is therefore not strictly speaking an algebra homomorphism).*
We will refer to the element $\mathbf{e}\underline{\mathrm{h}}\mathbf{e}$ *as the grading element of* $U_{\mathbf{k}}$.

We can state some well-known properties (see e.g. [Gor08]) of the algebras $H_{\mathbf{k}}$ and $U_{\mathbf{k}}$ which we will be using frequently. We give a proof for the reader's convenience.

Proposition 2.2.8. *For any complex reflection group and any parameter* $\mathbf{k}$ *the following holds:*

1. *The algebra* $H_{\mathbf{k}}$ *is Noetherian.*
2. *The algebra* $U_{\mathbf{k}}$ *is a Noetherian domain.*

Proof. We will employ a standard argument from ring theory which we will explain briefly: Suppose that R is a ring with a non-negative filtration

$$F_0S \subseteq F_1S \subseteq F_2S \subseteq \ldots \subseteq S.$$

This just means that each F_nS is a subgroup of the additive group of S with $F_iSF_jS \subseteq F_{i+j}S$ and $\bigcup_{n\geq 0} F_nS$. We can then associate a graded ring $\mathrm{gr}S$ to S (more accurately to the pair $(S, \mathcal{F})$ with $\mathcal{F} = \{F_nS \mid n \geq 0\}$ and we should write $\mathrm{gr}_{\mathcal{F}}S$) by setting

$$\mathrm{gr}_nS = S_n/S_{n-1} \text{ and } \mathrm{gr}S = \oplus_n \mathrm{gr}_nS$$

with $S_{-1} = 0$. Multiplication in $\mathrm{gr}S$ is defined as follows: Consider $a \in F_iS, b \in F_jS$ then set

$$(a + F_{i-1}S)(b + F_{j-1}S) = ab + F_{i+j-1}S$$

and extend linearly. Details of the construction are for example in [MR01], Paragraph 6 in Chapter 1. One can deduce properties of S from those of $\mathrm{gr}S$, in particular if $\mathrm{gr}S$ is a domain then so is S (Proposition 1.6.6 in [MR01]) and if $\mathrm{gr}S$ is Noetherian S is too (this can be deduced from Theorem 1.6.9 in [MR01]).

The algebra $H_{\mathbf{k}}$ has a filtration which places $\mathfrak{h}, \mathfrak{h}^*$ in degree 1 and Γ in degree 0. The associated graded algebra is isomorphic to $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]\#\Gamma$ by the PBW theorem 2.2.4 and thus $H_{\mathbf{k}}$ is Noetherian. This induces a filtration on $U_{\mathbf{k}} = \mathbf{e}H_{\mathbf{k}}\mathbf{e}$ with associated graded algebra $\mathbf{e}(\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]\#\Gamma)\mathbf{e} = \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]^\Gamma$. This is a Noetherian domain and hence so is $U_{\mathbf{k}}$. □

There is another distinguished element δ of $H_{\mathbf{k}}$ that we will introduce now, the discriminant:

Definition 2.2.9. *We set*

$$\delta = \prod_{H \in \mathcal{A}} \alpha_H.$$

Then we define $\mathfrak{h}^{reg} = \{y \in \mathfrak{h} \mid \delta(y) \neq 0\}$ *so that* $\mathbb{C}[\mathfrak{h}^{reg}] = \mathbb{C}[\mathfrak{h}][\delta^{-1}]$ *and* $\mathcal{D}(\mathfrak{h}^{reg}) = \mathcal{D}(\mathfrak{h})[\delta^{-1}]$.

Lemma 2.2.10. *The element* δ *generates a left and right Ore set in* $H_{\mathbf{k}}$.

Proof. This follows from Theorem 4.9 in [KL00] once we have argued that δ and its powers are non zero-divisors and that the maps $H_{\mathbf{k}} \to H_{\mathbf{k}}$, $f_{\delta^n} : a \mapsto \delta^n a - a\delta^n$ are locally nilpotent. Choose a basis x_i of $\mathfrak{h}^*$, y_i of $\mathfrak{h}$, using the PBW theorem, it suffices to show local nilpotence of f_{δ^n} on monomials in the x_i, y_i and elements of Γ. The element δ^n clearly commutes with all $x_i's$.

It spans a one-dimensional representation of Γ which can be seen by arguing as in the proof of Lemma 3.1 (a) in [Ste64] which in fact shows that $\mathbb{C}\delta$ transforms as $\det|_{\mathfrak{h}^*}$. It follows from the commutation relations defining $H_{\mathbf{k}}$ that f_{δ^n} lowers the degrees of any element of $\mathbb{C}[\mathfrak{h}^*]$. That no power of δ is a zero-divisor in $H_{\mathbf{k}}$ follows from the PBW Theorem, Theorem 2.2.4. □

Lemma 2.2.11. *The action of Γ descends to $\mathfrak{h}^{reg}$ and this action is free.*

Proof. Corollary 1.6 in [Ste64] states that

$$\mathfrak{h}^{reg} = \{y \in \mathfrak{h} \mid \mathrm{Stab}_{\Gamma}(y) = \{1\}\}$$

and everything follows. □

2.3 Category $\mathcal{O}_{\mathbf{k}}$

It is often the case in Lie theory, that it is a fruitful approach to restrict attention to categories of "nice" (but still fairly general) modules. Often these categories exhibit very interesting properties and behaviour. The same approach pays off for rational Cherednik algebras and the category $\mathcal{O}_{\mathbf{k}}$ that is the main object of interest is indeed easy to define.

Definition 2.3.1. *Let R be a ring, we denote by $R-\mathrm{Mod}$ the category of left R-modules and by $R-\mathrm{mod}$ the category of finitely-generated left R-modules.*

Definition 2.3.2. *The category $\mathcal{O}_{\mathbf{k}} = \mathcal{O}_{\mathbf{k}}(\Gamma, \mathfrak{h})$ is the full subcategory of $H_{\mathbf{k}}-\mathrm{mod}$ consisting of those modules on which the elements of $\mathbb{C}[\mathfrak{h}^*]_+$ act locally nilpotently. Here $\mathbb{C}[\mathfrak{h}^*]_+$ refers to the strictly positively graded component of the graded ring $\mathbb{C}[\mathfrak{h}^*]$ (i.e. those polynomials without constant term).*

To see that this category is indeed interesting, we will show that it already contains all finite-dimensional $H_{\mathbf{k}}$-modules:

Lemma 2.3.3. *Let M be a finite-dimensional $H_{\mathbf{k}}$-module. Then M is an object in $\mathcal{O}_{\mathbf{k}}$.*

Proof. Clearly M is finitely generated as it is finite-dimensional. It remains to prove the local nilpotency condition. Note that by the Jordan Normal Form theorem, M must decompose into a sum of generalised eigenspaces under the action of $\underline{\mathrm{h}}$ and let $a_1, \ldots, a_d$ be such a generalised eigenbasis. Multiplication by an element of $\mathbb{C}[\mathfrak{h}^*]_+$ will then strictly increase the real part of the generalised eigenvalues of the a_i and since there are only finitely many, we must conclude that any element of $\mathbb{C}[\mathfrak{h}^*]_+$ acts locally nilpotently. □

Definition 2.3.4. *For any group G we denote by $\mathrm{Irr}(G)$ the set of (equivalence classes of) irreducible, complex G-representations.*

Definition 2.3.5. *If Γ is a complex reflection group and $\lambda \in \mathrm{Irr}(\Gamma)$ then we denote by $\Delta(\lambda)$ the $H_{\mathbf{k}}$-module*

$$\Delta(\lambda) := H_{\mathbf{k}} \otimes_{\mathbb{C}[\mathfrak{h}^*]\#\Gamma} \lambda$$

where the regular function $p \in \mathbb{C}[\mathfrak{h}^*]$ acts on λ via multiplication by $p(0)$. Any $H_{\mathbf{k}}$-module isomorphic to a module of the form $\Delta(\lambda)$ for some $\lambda \in \mathrm{Irr}(\Gamma)$ will be referred to as a standard module.

Note that this definition makes sense for an arbitrary (not necessarily irreducible) representation of Γ and occasionally we will expand our notation to include this case. However the terminology of standard modules will be strictly limited to the case of modules derived from irreducible representations. The standard modules are important examples of highest-weight modules, which we will define next:

Definition 2.3.6. *1. If M is a $H_{\mathbf{k}}$-module and $m \in M$ is a generalised $\underline{\mathrm{h}}$-eigenvector with generalised $\underline{\mathrm{h}}$-eigenvalue $\alpha \in \mathbb{C}$, then α is referred to as a "weight" and m is a "weight vector".*

2. *A highest weight module of $H_{\mathbf{k}}$ is a module M which has a basis of weight vectors such that the set of the real parts of all weights is bounded above.*
3. *A highest weight (of a highest weight module) is a weight whose real part is an upper bound of the set of all weights of that module. A generalised $\underline{\mathrm{h}}$-eigenvector whose weight is a highest weight is called a "highest weight vector."*
4. *If M is a highest weight module, the vector space spanned by all highest weight vectors in M is called the highest weight space.*

This definition is similar to the case of a semisimple Lie-algebra $\mathfrak{g}$ where we require a highest-weight module to be a cyclic module generated by a weight vector which is annihilated by any element of the sum of the positive root spaces of $\mathfrak{g}$, see e.g [Jan79] for an extensive development of the theory of highest weight modules and their role in the representation theory of $\mathfrak{g}$. In this context, the standard modules are analogues of the Verma modules from Lie theory. more on highest-weight theory of Lie algebras is developed in [Jan83]. The next result is again well-known and as the author does not know of a suitable reference a proof is given.

Proposition 2.3.7. *For any $\lambda \in \mathrm{Irr}(\Gamma)$, the module $\Delta(\lambda)$ is in $\mathcal{O}_{\mathbf{k}}$. As a $\mathbb{C}[\mathfrak{h}]\#\Gamma$-representation, we have an isomorphism*

$$\Delta(\lambda) \cong \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} \lambda$$

and in particular a basis of $\Delta(\lambda)$ is given by $\underline{\mathrm{x}} \otimes_{\mathbb{C}} f_i$, where $\underline{\mathrm{x}}$ is a monic monomial in $\mathbb{C}[\mathfrak{h}]$ and $\{f_i\}$ is a basis of λ . Further, the module $\Delta(\lambda)$ is a highest weight module, whose highest weight space is $1 \otimes \lambda \subseteq \Delta(\lambda)$ and its highest weight is

$$\kappa_{\mathbf{k}}(\lambda) = -\frac{1}{2}\dim_{\mathbb{C}} \mathfrak{h} - \left(\sum_{H \in \mathcal{A}} \sum_{j=0}^{m_H - 1} m_H k_{H,j} \mathbf{e}_{H,j} \right) |_{\lambda} + \sum_{H \in \mathcal{A}} \sum_{j=0}^{m_H - 1} k_{H,j} ..$$

Here $\left(\sum_{H\in\mathcal{A}} \sum_{j=0}^{m_H-1} m_H k_{H,j} \mathbf{e}_{H,j}\right) |_{\lambda}$ denotes the scalar by which $\left(\sum_{H\in\mathcal{A}} \sum_{j=0}^{m_H-1} m_H k_{H,j} \mathbf{e}_{H,j}\right) \in Z(\mathbb{C}\Gamma)$ acts on the representation λ.

Proof. Follows from Theorem 2.2.4 and direct computation. $\square$

Proposition 2.3.8. *(Lemma 3.2.1 and Proposition 3.5 in [Lev90]) Let Γ be a complex reflection group with reflection representation $\mathfrak{h}$. There is a filtered isomorphism*

$$\mathcal{D}(\mathfrak{h}^{reg})^{\Gamma} \cong \mathcal{D}(\mathfrak{h}^{reg}/\Gamma).$$

We can give an example for the usefulness of the standard modules by using them to prove the following well-known result:

Theorem 2.3.9. *([EG02], Proposition 4.5) For any choice of* $\mathbf{k}$ *we have an injection*

$$\theta_{\mathbf{k}} : H_{\mathbf{k}} \hookrightarrow \mathcal{D}(\mathfrak{h}^{reg})\#\Gamma$$

defined by

$$\begin{aligned} x &\mapsto x \\ y &\mapsto \partial_y - \sum_{H\in\mathcal{A}} \frac{\alpha_H(y)}{\alpha_H} \sum_{j=0}^{m_H-1} m_H k_{H,j}\mathbf{e}_{H,j} \\ \gamma &\mapsto \gamma \end{aligned}$$

Localising at $\{\delta^i\}_i$, $\theta_{\mathbf{k}}$ *becomes an isomorphism* $H_{\mathbf{k}}|_{\mathfrak{h}^{reg}} := H_{\mathbf{k}}[\delta^{-1}] \cong \mathcal{D}(\mathfrak{h}^{reg})\#\Gamma$*. The map* $\theta_{\mathbf{k}}$ *descends to an injection*

$$U_{\mathbf{k}} \hookrightarrow \mathcal{D}(\mathfrak{h}^{reg})^{\Gamma} \cong \mathcal{D}(\mathfrak{h}^{reg}/\Gamma)$$

and again gives an isomorphism upon localisation.

Proof. We have an isomorphism of vector spaces

$$\Delta(\mathrm{triv}) \cong \mathbb{C}[\mathfrak{h}]$$

given by mapping $p(x) \otimes \mathrm{triv} \mapsto p(x)$. Thus the action of $H_{\mathbf{k}}$ on $\Delta(\mathrm{triv})$ gives a map

$$H_{\mathbf{k}} \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}[\mathfrak{h}])$$

and this is just $\theta_{\mathbf{k}}$ as described above. The fact that $\theta_{\mathbf{k}}$ is an isomorphism upon localisation follows by noting that $\theta|_{\mathfrak{h}^{reg}}(H|_{\mathfrak{h}^{reg}})$ contains $\mathbb{C}[\mathfrak{h}^{reg}]$- as inverting δ inverts each of the α_H - and therefore also contains all derivations ∂_y for $y \in \mathfrak{h}$, i.e. derivations of the form $\bullet \mapsto \langle \bullet, y\rangle$. But these sets generate $\mathcal{D}(\mathfrak{h}^{reg})$.
The statements for $U_{\mathbf{k}}$ follow similarly, using the fact that the action of Γ on $\mathfrak{h}^{reg}$ is free and that $\mathfrak{h}^{reg}/\Gamma$ is smooth. □

Definition 2.3.10. *This representation of* $H_{\mathbf{k}}$ *as a subalgebra of* $\mathcal{D}(\mathfrak{h}^{reg})\#\Gamma$ *given by* $\theta_{\mathbf{k}}$ *is referred to as the "Dunkl representation" and the operators*

$$T_{y,\mathbf{k}} := \partial_y - \sum_{H\in\mathcal{A}} \frac{\alpha_H(y)}{\alpha_H} \sum_{j=0}^{m_H-1} m_H k_{H,j}\mathbf{e}_{H,j} \text{ with } y \in \mathfrak{h}$$

are called "Dunkl operators". The representation of $U_{\mathbf{k}}$ *as a subalgebra of* $\mathcal{D}(\mathfrak{h}^{reg})^{\Gamma}$ *or* $\mathcal{D}(\mathfrak{h}^{reg}/\Gamma)$ *given by restricting* $\theta_{\mathbf{k}}$ *will be referred to as the Dunkl representation of* $U_{\mathbf{k}}$*.*

The category $\mathcal{O}_{\mathbf{k}}$ has many nice representation-theoretic properties and behaves similarly to its Lie-theoretic counterpart. The essence of this is captured in the following theorem

Theorem 2.3.11. *For any choice of parameter* $\mathbf{k}$*, the category* $\mathcal{O}_{\mathbf{k}}$ *is a highest weight-category in the sense of Cline, Parshall and Scott (see [CPS88]). Its set of weights is* $\mathrm{Irr}(\Gamma)$ *with* $\lambda \prec_{\mathbf{k}} \mu$ *if and only if* $\kappa_{\mathbf{k}}(\mu) - \kappa_{\mathbf{k}}(\lambda) \in \mathbb{N}$*.*

1. *Its standard objects are exactly the* $\Delta(\lambda)$ *with* $\lambda \in \mathrm{Irr}(\Gamma)$*.*
2. *Each* $\Delta(\lambda)$ *has a unique simple head* $L(\lambda)$*.*

3. *The $L(\lambda)$ form a complete set of representatives of the isomorphism classes of simple objects in $\mathcal{O}_{\mathbf{k}}$.*
4. *Each $L(\lambda)$ has an indecomposable projective cover $P(\lambda)$ and an indecomposable injective envelope $I(\lambda)$.*
5. *Each $P(\lambda)$ has a filtration by standard modules $\Delta(\mu_i)$. Amongst these, $\Delta(\lambda)$ occurs precisely once and all remaining μ_i are strictly greater than λ in the ordering $\prec_{\mathbf{k}}$*
6. *Each module in $\mathcal{O}_{\mathbf{k}}$ has finite length.*

Proof. This is essentially all contained in Theorem 2.19 of [GGOR03] and standard facts about highest weight categories, see [CPS88]. In more detail, we refer to several results in [GGOR03], namely Proposition 2.11 for (2) and (3), Theorem 2.19 for the claim that $\mathcal{O}_{\mathbf{k}}$ is highest weight and (1). The statements concerning projective covers is Corollary 2.10 although they can also be found in Guay's earlier work [Gua03]. The existence of injective envelopes follows form the existence of projective covers and the naive duality functor of Section 4.2 in [GGOR03], see also Proposition 4.7. The finite length claim can be deduced from Corollary 2.8. □

Corollary 2.3.12. *BGG reciprocity holds for $\mathcal{O}_{\mathbf{k}}$: Let $\lambda, \mu \in \mathrm{Irr}(\Gamma)$, then*

$$[P(\lambda) : \Delta(\mu)] = [\Delta(\mu) : L(\lambda)].$$

Here $[P(\lambda) : \Delta(\mu)]$ counts the times the standard object $\Delta(\mu)$ occurs in a filtration of $P(\lambda)$ by standard objects and $[\Delta(\mu) : L(\lambda)]$ counts the number of times the simple object $L(\lambda)$ occurs in a Jordan Hölder series of $\Delta(\mu)$.

Proof. This is Theorem 1.2 in [Gua03] (which is proved as Theorem 4.2). Alternatively, we refer to Proposition 3.3 and the reciprocity formulas following Theorem 2.19 in [GGOR03] which hold in arbitrary highest weight categories. □

Another result of interest about the interplay between the structure of $\mathcal{O}_{\mathbf{k}}$ and the algebraic structure of $H_{\mathbf{k}}$ is the following result of Ginzburg. Recall that a primitive ideal of a ring is an ideal that is the annihilator of a simple module.

Theorem 2.3.13. *(Theorem 2.3 and Corollary 6.6 in [Gin03]) Let I be a primitive ideal of the algebra $H_{\mathbf{k}}$. Then I is the annihilator of a simple module in $\mathcal{O}_{\mathbf{k}}$.*

The importance of this theorem - sometimes referred to as "Ginzburg's Duflo theorem" - is that although $H_{\mathbf{k}}$ will have many simple modules that are not in $\mathcal{O}_{\mathbf{k}}$, we can obtain all primitive ideals of $H_{\mathbf{k}}$ by considering only the finitely (!) many simple modules in $\mathcal{O}_{\mathbf{k}}$. We will have use for this result for example in Proposition 2.3.20.
We can define an analogue of category $\mathcal{O}_{\mathbf{k}}(H) = \mathcal{O}_{\mathbf{k}}$ of $H_{\mathbf{k}}$ for the spherical subalgebra $U_{\mathbf{k}}$

Definition 2.3.14. *The category $\mathcal{O}_{\mathbf{k}}(U)$ for $U_{\mathbf{k}}$ is the full subcategory of finitely-generated $U_{\mathbf{k}}$-modules N such that any $p \in \mathbb{C}[\mathfrak{h}^*]_+^{\Gamma}$ acts locally nilpotently on N.*

Definition 2.3.15. *For any group G acting linearly on a finite-dimensional complex vector space V, we define the coinvariant ring $\mathbb{C}[V]^{\mathrm{co}G}$ as follows:*

$$\mathbb{C}[V]^{\mathrm{co}G} := \frac{\mathbb{C}[V]}{\langle \mathbb{C}[V]_+^G \rangle}$$

the quotient of $\mathbb{C}[V]$ by the ideal of G-invariant functions with no constant term.

The following result regarding the structure of the coinvariant rings of complex reflection groups is well-known:

Proposition 2.3.16. *(Theorem 4.1 in [Bro10]) Suppose* Γ *is a complex reflection group with reflection representation* $\mathfrak{h}$*. Then* $\mathbb{C}[\mathfrak{h}]^{\mathrm{co}\Gamma}$ *is isomorphic to* $\mathbb{C}\Gamma$ *as a* Γ*-representation - in particular* $\dim_{\mathbb{C}} \mathbb{C}[\mathfrak{h}]^{\mathrm{co}\Gamma} < \infty$ *- and it carries a natural grading as a quotient of the graded algebra* $\mathbb{C}[\mathfrak{h}]$ *by a homogeneous polynomial.*

Corollary 2.3.17. *Let* Γ *be a a complex reflection group with reflection representation* $\mathfrak{h}$*. Take any* $p \in \mathbb{C}[\mathfrak{h}]_+$*. Then there exists* $n \in \mathbb{N}_0$ *such that* $p^n \in \mathbb{C}[\mathfrak{h}]\mathbb{C}[\mathfrak{h}]_+^\Gamma$*.*

Proof. Suppose that $p \notin \mathbb{C}[\mathfrak{h}]_+^\Gamma$. Then since $p(0) = 0$, p is not in the zero-graded component of the coinvariant ring. Since $\mathbb{C}[\mathfrak{h}]^{\mathrm{co}\Gamma}$ is finite-dimensional and graded, we can then deduce that $p^n = 0$ in $\mathbb{C}[\mathfrak{h}]^{\mathrm{co}\Gamma}$ for some $n \in \mathbb{N}_0$ and thus $p^n \in \mathbb{C}[\mathfrak{h}]\mathbb{C}[\mathfrak{h}]_+^\Gamma$. $\square$

We will need this result shortly.

Definition 2.3.18. *For any parameter value* $\mathbf{k}$ *we have functors*

$$\begin{aligned} E_{\mathbf{k}} : H_{\mathbf{k}} - \mathrm{Mod} &\to U_{\mathbf{k}} - \mathrm{Mod} \\ M &\mapsto \mathbf{e}M \\ &\text{and} \\ T_{\mathbf{k}} : U_{\mathbf{k}} - \mathrm{Mod} &\to H_{\mathbf{k}} - \mathrm{Mod} \\ N &\mapsto H_{\mathbf{k}}\mathbf{e} \otimes_{U_{\mathbf{k}}} N \end{aligned}$$

Definition 2.3.19. *We call a choice of parameters* $\mathbf{k}$ *aspherical if for some* $\lambda \in \mathrm{Irr}(\Gamma)$ *we have* $\mathbf{e}L_{\mathbf{k}}(\lambda) = 0$*, i.e. if multiplication by the trivial group idempotent annihilates a simple module in* $\mathcal{O}_{\mathbf{k}}$*. We denote by* $\mathrm{Asp}(\Gamma) = \mathrm{Asp}(\Gamma, \mathfrak{h})$ *the set of all aspherical parameters.*

Proposition 2.3.20. *Let* Γ *be a complex reflection group. The following are equivalent:*

1. $\mathbf{k} \notin \mathrm{Asp}(\Gamma)$
2. $H_{\mathbf{k}}\mathbf{e}H_{\mathbf{k}} = H_{\mathbf{k}}$
3. *the multiplication map induces an isomorphism* $\mu : H_{\mathbf{k}}\mathbf{e} \otimes_{U_{\mathbf{k}}} \mathbf{e}H_{\mathbf{k}} \xrightarrow{\sim} H_{\mathbf{k}}$

Proof. (1) $\implies$ (2): Suppose that $H_{\mathbf{k}}\mathbf{e}H_{\mathbf{k}} \neq H_{\mathbf{k}}$. Then $H_{\mathbf{k}}\mathbf{e}H_{\mathbf{k}}$ is a proper 2-sided ideal of $H_{\mathbf{k}}$ and contained in a primitive ideal $\mathfrak{a}$. By Ginzburg's Duflo Theorem 2.3.13, $\mathfrak{a}$ is the annihilator of a simple object in $\mathcal{O}_{\mathbf{k}}$ and thus $\mathbf{e}L = 0$ for some simple object in $\mathcal{O}_{\mathbf{k}}$, contradicting our hypothesis. So we must have $H_{\mathbf{k}}\mathbf{e}H_{\mathbf{k}} = H_{\mathbf{k}}$.
(2) $\implies$ (3): By hypothesis the map $\mu : H_{\mathbf{k}}\mathbf{e} \otimes_{U_{\mathbf{k}}} \mathbf{e}H_{\mathbf{k}} \to H_{\mathbf{k}}$ is surjective. Injectivity follows by taking $\sum_i r_i\mathbf{e} \otimes_{U_{\mathbf{k}}} \mathbf{e}s_i \in \ker(\mu)$ and $\sum_j u_j\mathbf{e} \otimes_{U_{\mathbf{k}}} \mathbf{e}v_j$ with $\mu(\sum_j u_j\mathbf{e} \otimes_{U_{\mathbf{k}}} \mathbf{e}v_j) = \sum_j u_j\mathbf{e}v_j = 1$. Then

$$\begin{aligned} \sum_i r_i\mathbf{e} \otimes_{U_{\mathbf{k}}} \mathbf{e}s_i &= \sum_{j,i} r_i\mathbf{e} \otimes_{U_{\mathbf{k}}} \mathbf{e}s_i u_j \mathbf{e}v_j \\ &= \sum_{j,i} r_i\mathbf{e}s_i u_j\mathbf{e} \otimes_{U_{\mathbf{k}}} \mathbf{e}v_j \\ &= \left(\sum_i r_i\mathbf{e}s_i\right)(u_j\mathbf{e} \otimes_{U_{\mathbf{k}}} \mathbf{e}v_j) \\ &= 0 \end{aligned}$$

and hence $\ker(\mu) = 0$ and μ is an isomorphism. We will meet variants of this argument several times.
(3) $\implies$ (1): The hypothesis shows that the functor $H_{\mathbf{k}} - \mathrm{Mod} \to U_{\mathbf{k}} - \mathrm{Mod}$ given by

$$M \mapsto H_{\mathbf{k}}\mathbf{e} \otimes_{U_{\mathbf{k}}} \mathbf{e}M \cong H_{\mathbf{k}}\mathbf{e} \otimes_{U_{\mathbf{k}}} \mathbf{e}H \otimes_{H_{\mathbf{k}}} M$$

is equivalent to the identity functor of $H_{\mathbf{k}} - \mathrm{Mod}$. Thus if $M \neq 0$ then $\mathbf{e}M$ must necessarily be non-zero and so $\mathbf{k}$ cannot be aspherical. □

Example 2.3.21. *1. If $\mathbf{k}$ is chosen such that $H_{\mathbf{k}}$ is simple, it is not aspherical.*

2. *The set of aspherical parameter values for $H_{k_1}(\mathbb{Z}_2)$ is $\mathrm{Asp}(\mathbb{Z}_2) = \{-\frac{1}{2}\}$. This can be checked by hand by computing the dimension of $\mathbf{e}L_{k_1}(\mathrm{triv})$ and $\mathbf{e}L_{k_1}(\mathrm{sgn})$.*
3. *The set of aspherical parameter values $\mathbf{k} = (k_0, k_1, \ldots, k_{m-1})$ of $H_{\mathbf{k}}(\mathbb{Z}_m)$ is*

$$\mathrm{Asp}(\mathbb{Z}_m) = \{\mathbf{k} \in 0 \times \mathbb{C}^{m-1} \mid \delta_{0,p} - \delta_{0,q} + \frac{p-q}{m} - (k_p - k_q) = 0 \text{ for some } 0 \leq p, q \leq m-1\}$$

see Proposition 6.2.2.

Theorem 2.3.22. *1. $E_{\mathbf{k}}$ is exact and $T_{\mathbf{k}}$ is right exact.*

2. *$E_{\mathbf{k}}$ and $T_{\mathbf{k}}$ restrict to functors between $H_{\mathbf{k}} - \mathrm{mod}$ and $U_{\mathbf{k}} - \mathrm{mod}$.*
3. *$E_{\mathbf{k}}$ and $T_{\mathbf{k}}$ restrict to functors between $\mathcal{O}_{\mathbf{k}}$ and $\mathcal{O}_{\mathbf{k}}(U)$.*
4. *$E_{\mathbf{k}} \circ T_{\mathbf{k}} \cong \mathrm{Id}_{U_{\mathbf{k}}}$.*
5. *Suppose that $\mathbf{k}$ is a non-aspherical parameter. Then $E_{\mathbf{k}}$ is an equivalence of categories between $H_{\mathbf{k}} - \mathrm{Mod}$ and $U_{\mathbf{k}} - \mathrm{Mod}$ with inverse $T_{\mathbf{k}}$ and they restrict to an equivalence between $\mathcal{O}_{\mathbf{k}}$ and $\mathcal{O}_{\mathbf{k}}(U)$.*

Proof. The first two statements are obvious, as is (4). We will now show that if $\mathbf{k} \notin \mathrm{Asp}\Gamma$ the functors $E_{\mathbf{k}}$ and $T_{\mathbf{k}}$ give equivalences between $H_{\mathbf{k}} - \mathrm{Mod}$ and $U_{\mathbf{k}} - \mathrm{Mod}$. As $\mathbf{k}$ is not aspherical, we have that $H_{\mathbf{k}} \cong H_{\mathbf{k}}\mathbf{e} \otimes_{U_{\mathbf{k}}} \mathbf{e}H_{\mathbf{k}}$ and thus $T_{\mathbf{k}} \circ E_{\mathbf{k}} \cong \mathrm{Id}_{H_{\mathbf{k}}}$and by (4) we have $E_{\mathbf{k}} \circ T_{\mathbf{k}} \cong \mathrm{Id}_{U_{\mathbf{k}}}$. So $T_{\mathbf{k}}$ and $E_{\mathbf{k}}$ indeed give equivalences.
It remains to check that these functors preserve the categories $\mathcal{O}_{\mathbf{k}}$ and $\mathcal{O}_{\mathbf{k}}(U)$. This is immediate for $E_{\mathbf{k}}$ (since set-theoretically $E_{\mathbf{k}}(N) \subseteq N$) so let us consider $T_{\mathbf{k}}$ and $N \in \mathcal{O}_{\mathbf{k}}(U)$. Then let us take $p \in \mathbb{C}[\mathfrak{h}^*]_+$ and $v \in N$. We first want to show that p acts nilpotently on $\mathbf{e} \otimes v \in H_{\mathbf{k}}\mathbf{e} \otimes_{U_{\mathbf{k}}} N$. Choose $n_0 \in \mathbb{N}_0$ such that $p^{n_0} \in \mathbb{C}[\mathfrak{h}^*]\mathbb{C}[\mathfrak{h}^*]_+^{\Gamma}$ with

$$p^{n_0} = \sum_{i=1}^{k} q_i p_i$$

for some $k \in \mathbb{N}$ and $q_i \in \mathbb{C}[\mathfrak{h}^*]$ and $p_i \in \mathbb{C}[\mathfrak{h}^*]_+^{\Gamma}$. Further there exists $n_1 \in \mathbb{N}_0$ such that $p_i^{n_1}\mathbf{e}v = 0$ as $N \in \mathcal{O}_{\mathbf{k}}(U)$. Then

$$\begin{aligned} p^{n_0 n_1 k} &= \left(\sum_{i=1}^{k} q_i p_i\right)^{k n_1} \\ &= \sum_{i_1 + i_2 + \ldots + i_k = k n_1} \binom{k n_1}{i_1, \ldots, i_k} \prod_{t=1}^{k} (q_t p_t)^{i_t} \end{aligned}$$

Now as $i_1 + i_2 + \ldots + i_k = k n_1$ at least one i_t must always be greater than or equal to n_1.

Hence indeed

$$p^{n_0 n_1 k}(\mathbf{e} \otimes v) = 0.$$

Multiplication by an element of $H_{\mathbf{k}}$ will preserve the nilpotency of the action of p. To see this, we can argue as follows: Take $r \in H_{\mathbf{k}}$ and $a \in \mathbb{C}[\mathfrak{h}^*]_+^\Gamma$ then for some suitably large $d \in \mathbb{N}$ we have

$$\operatorname{ad}(a)^d(r) = 0$$

see Proposition 3.1.3 which is independent of this result. By Proposition 3.3.2 we must then have $\operatorname{ad}(a^j)^d(r) = 0$ for any $j \in \mathbb{N}$ too and thus

$$a^{jd}r = -\sum_{i=0}^{d-1} \binom{d}{i} (-1)^{d-i} a^{ji} r a^{j(d-i)}.$$

Choose $j \in \mathbb{N}$ large enough so that $a^j v = 0$, then

$$\begin{aligned} a^{jd} r\mathbf{e} \otimes v &= -\sum_{i=0}^{d-1} \binom{d}{i} (-1)^{d-i} a^{ji} r a^{j(d-i)} \mathbf{e} \otimes v \\ &= -\sum_{i=0}^{d-1} \binom{d}{i} (-1)^{d-i} a^{ji} r\mathbf{e} \otimes a^{j(d-i)} v \\ &= 0 \end{aligned}$$

where we are using $a\mathbf{e} = \mathbf{e}a\mathbf{e}$ for $a \in \mathbb{C}[\mathfrak{h}^*]_+^\Gamma$ again. Now we can argue as before since for any $p \in \mathbb{C}[\mathfrak{h}^*]_+$ a suitable power will lie in $\mathbb{C}[\mathfrak{h}^*]\mathbb{C}[\mathfrak{h}^*]_+^\Gamma$. Hence p will act nilpotently on any $r\mathbf{e} \otimes v$ for $r \in H_{\mathbf{k}}$. As these span $T_{\mathbf{k}}(N)$, p acts locally nilpotently on $T_{\mathbf{k}}(N)$ and so $T_{\mathbf{k}}(N) \in \mathcal{O}_{\mathbf{k}}$. $\square$

To better illustrate some of the concepts introduced here, we will work out a small example and consider the case of $\Gamma = \{1, -1\}$ the cyclic group of order 2.

Example 2.3.23. *As usual for cyclic groups, we assume μ_2 acts on a one-dimensional vector space $\mathfrak{h} = \mathbb{C}x$, namely we let s act via multiplication by -1. Then $\{0\}$ is the unique reflection hyperplane, the only isotropic subgroup of $\mathbb{Z}_2$ is $\mathbb{Z}_2$ itself and the group idempotents are $\mathbf{e} = \frac{1}{2}(1+s)$ and $\mathbf{e}_- = \frac{1}{2}(1-s)$. Letting x, y be dual bases of $\mathfrak{h}$ and $\mathfrak{h}^*$ respectively, the defining relations of $H_{\mathbf{k}}$ as a quotient of $T(\mathfrak{h} \oplus \mathfrak{h}^*)\#\mathbb{Z}_2$ can then be reduced to*

$$\begin{aligned} [y, x] &= 1 + 2\left((k_0 - k_1)\mathbf{e} + (k_1 - k_0)\mathbf{e}_-\right) \\ &= 1 + 2\left((-\frac{k_1}{2})(1+s) + (\frac{k_1}{2})(1-s)\right) \\ &= 1 + ((-k_1)(1+s) + (k_1)(1-s)) \\ &= 1 + ((-k_1 - k_1 s + k_1 - k_1 s) \\ &= 1 - 2k_1 s. \end{aligned}$$

The group invariants $\mathbb{C}[\mathfrak{h}]^{\mathbb{Z}_2}$ and $\mathbb{C}[\mathfrak{h}^*]^{\mathbb{Z}_2}$ are generated by x^2 and y^2 respectively, and by direct computation the spherical subalgebra U_{k_1} will be spanned by elements of the form

$$\mathbf{e}x^{2a}y^{2b}\underline{h}^c\mathbf{e}.$$

We will use this to exhibit U_{k_1} as isomorphic to a quotient of the enveloping algebra $U(\mathfrak{sl}_2)$,

thus identifying $\mathcal{O}_{\mathbf{k}}$ with a subcategory of category $\mathcal{O}$ of $\mathfrak{sl}_2$ (if k_1 is not aspherical).
First we claim that $U_{\mathbf{k}}$ is generated by $\mathbf{e}y^2\mathbf{e}$ and $\mathbf{e}x^2\mathbf{e}$ and the identity element. For this, we calculate

$$\begin{aligned}
[\mathbf{e}y^2\mathbf{e}, \mathbf{e}x^2\mathbf{e}] &= \mathbf{e}[y^2, x^2]\mathbf{e} \\
&= \mathbf{e}\left(yx[y,x] + y[y,x]x + x[y,x]y + [y,x]xy\right)\mathbf{e} \\
&= \mathbf{e}(yx(1-2k_1s) + y(1-2k_1s)x + x(1-2k_1s)y + (1-2k_1s)xy)\mathbf{e} \\
&= (1-2k_1)\mathbf{e}yx\mathbf{e} + (1+2k_1)\mathbf{e}xy\mathbf{e} + (1+2k_1)\mathbf{e}yx\mathbf{e} + (1-2k_1)\mathbf{e}xy\mathbf{e} \\
&= 2\mathbf{e}(yx+xy)\mathbf{e} \\
&= -4\mathbf{e}\underline{h}\mathbf{e}
\end{aligned}$$

and using the description of elements the claim about generation follows. Recall the definition of $\mathfrak{sl}_2$ in terms of generators and relations: As a vector space, $\mathfrak{sl}_2$ has a basis E, H, F with the commutation relations

$$[E,F] = H, \;\; [H,E] = 2E, \;\; [H,F] = -2F$$

and its universal enveloping algebra $U(\mathfrak{sl}_2)$ is generated by E, F, H subject to the same relations as above. So we can define a map

$$\begin{aligned}
f : U(\mathfrak{sl}_2) &\to U_{k_1} \\
E &\mapsto \frac{-1}{2}\mathbf{e}y^2\mathbf{e} \\
F &\mapsto \frac{1}{2}\mathbf{e}x^2\mathbf{e} \\
H &\mapsto \mathbf{e}\underline{h}\mathbf{e}
\end{aligned}$$

This is clearly surjective and we will next identify the kernel of f. Recall that the Casimir element Ω of $U(\mathfrak{sl}_2)$ is given by

$$\Omega = H^2 - 2H + 4EF$$

and that this generates $Z(U(\mathfrak{sl}_2))$, the centre of the enveloping algebra. Under f, Ω is mapped to

$$\begin{aligned}
f(\Omega) &= (\mathbf{e}\underline{h}\mathbf{e})^2 - 2\mathbf{e}\underline{h}\mathbf{e} - \mathbf{e}y^2x^2\mathbf{e} \\
&= \mathbf{e}(\frac{-1}{2}(xy+yx))^2\mathbf{e} + \mathbf{e}(yx+xy)\mathbf{e} - \mathbf{e}yxyx\mathbf{e} - \mathbf{e}y(1+2k_1s)x\mathbf{e} \\
&= \frac{1}{4}\mathbf{e}(2yx - 1 + 2k_1s)^2\mathbf{e} + \mathbf{e}(2yx - 1 + 2k_1s)\mathbf{e} - \mathbf{e}yxyx\mathbf{e} - (1+2k_1)\mathbf{e}yx\mathbf{e} \\
&= \frac{1}{4}\mathbf{e}(2yx - 1 + 2k_1)^2\mathbf{e} + \mathbf{e}(2yx - 1 + 2k_1)\mathbf{e} - \mathbf{e}yxyx\mathbf{e} - (1+2k_1)\mathbf{e}yx\mathbf{e} \\
&= \frac{1}{4}\mathbf{e}(4yxyx - 4(1-2k_1)yx + (1-2k_1)^2)\mathbf{e} + 2\mathbf{e}yx\mathbf{e} - (1-2k_1)\mathbf{e} - \mathbf{e}yxyx\mathbf{e} - (1+2k_1)\mathbf{e}yx\mathbf{e} \\
&= \mathbf{e}(yxyx - (1-2k_1)yx + \frac{1}{4}(1-2k_1)^2)\mathbf{e} + 2\mathbf{e}yx\mathbf{e} - (1-2k_1)\mathbf{e} - \mathbf{e}yxyx\mathbf{e} - (1+2k_1)\mathbf{e}yx\mathbf{e} \\
&= \mathbf{e}yxyx\mathbf{e} - \mathbf{e}yxyx\mathbf{e} - (1-2k_1)\mathbf{e}yx\mathbf{e} + 2\mathbf{e}yx\mathbf{e} - (1+2k_1)\mathbf{e}yx\mathbf{e} + \frac{1}{4}(1-2k_1)^2\mathbf{e} - (1-2k_1)\mathbf{e} \\
&= \frac{1}{4}\left((2k_1+1)^2 - 2\right)\mathbf{e}
\end{aligned}$$

So the ideal $(\Omega - (2k_1+1)^2/4)$ *will lie in the kernel of* f *To see that* f *gives an isomorphism*

$$f : \frac{U(\mathfrak{sl}_2)}{(\Omega - ((2k_1+1)^2 - 2)/4)} \to U_{k_1(\mathbb{Z}_2)}$$

we can note that that both have the same GK-dimension and that any quotient of the form $\frac{U(\mathfrak{sl}_2)}{(\Omega - \alpha}$ *with* $\alpha \in \mathbb{C}$ *will be a domain by Theorem 4.15 in [Maz09]. Thus* f *induces a functor*

$$f_* : U_{k_1} - \mathrm{Mod} \to U(\mathfrak{sl}_2) - \mathrm{Mod}$$

and f_* *identifies* $U_{k_1} - \mathrm{Mod}$ *with those* $U(\mathfrak{sl}_2)$*-modules on which* Ω *via the scalar* $k_1^2 + k_1 - \frac{1}{4}$*. Further,* f_* *induces and equivalence between* $\mathcal{O}_{k_1}(U_{k_1}(\mathbb{Z}_2))$ *and the category of* $U(\mathfrak{sl}_2)$*-modules fulfilling the conditions that*

- *they are finitely generated*
- E *acts locally nilpotently*
- H *acts generalised semisimply on them*
- Ω *acts as multiplication by* $\left(\left((2k_1+1)^2 - 2\right)/4\right.$.

This is one version of $\mathcal{O}$ *for* $\mathfrak{sl}_2$*. In particular we can see that the case of the aspherical parameter value* $k_1 = -\frac{1}{2}$ *corresponds to the case where the* $\mathfrak{sl}_2$*-modules in question have weight* $-\varrho$*, where* ϱ *denotes one half of the sum of positive roots of* $\mathfrak{sl}_2$*.*

2.4 The KZ-functor and regular parameter values

The category $\mathcal{O}_{\mathbf{k}}$ offers a way to relate the representation theory of $H_{\mathbf{k}}$ with that of certain finite-dimensional Hecke algebras $\mathcal{H}_{\mathbf{q}}(\Gamma)$. Both rational Cherednik algebras and Hecke algebras are of independent interest and so any connection between them is of great utility as well as being quite unexpected. The relationship between them is given by an exact functor from $\mathcal{O}_{\mathbf{k}}$ to finite-dimensional modules over $\mathcal{H}_{\mathbf{q}}(\Gamma)$ called the KZ-functor and for generic choice of $\mathbf{k}$ this will even be an equivalence.

Definition 2.4.1. *We call a choice of parameters* $\mathbf{k}$ *regular and write* $\mathbf{k} \in \mathrm{Reg}(\Gamma)$ *if the category* $\mathcal{O}_{\mathbf{k}}$ *is semisimple.*

We have already defined another important set of parameters, namely the aspherical ones and we have seen that in the non-aspherical case the algebras $H_{\mathbf{k}}$ and $U_{\mathbf{k}}$ are Morita equivalent as are the categories $\mathcal{O}_{\mathbf{k}}$ and $\mathcal{O}_{\mathbf{k}}(U)$. A the algebra $U_{\mathbf{k}}$ is easier to approach computationally and will prove to be a crucial tool for us in all of the following chapters, it is important for us to know how the two sets $\mathrm{Asp}(\Gamma)$ and $\mathrm{Reg}(\Gamma)$ relate. We will be able to answer this shortly in Corollary 2.4.22.

The spherical subalgebra has a very nice set of generators in case $\mathbf{k}$ is regular:

Theorem 2.4.2. *([BEG03b] Theorem 4.6 or [Val06] Lemma 4.11) Let* $\mathbf{k} \in \mathrm{Reg}(\Gamma)$*, then the spherical subalgebra* $U_{\mathbf{k}} = \mathbf{e}H_{\mathbf{k}}\mathbf{e}$ *is generated as an algebra by* $\mathbb{C}[\mathfrak{h}]^{\Gamma}\mathbf{e} \cup \mathbb{C}[\mathfrak{h}^*]^{\Gamma}\mathbf{e}$*.*

Proposition 2.4.3. *(see e.g. Corollary 4.1 in [Gua03]) The category* $\mathcal{O}_{\mathbf{k}}$ *is semisimple if and only if the standard module* $\Delta_{\mathbf{k}}(\lambda)$ *is simple for any* $\lambda \in \mathrm{Irr}(\Gamma)$*.*

Proof. We can deduce this from BGG reciprocity, Corollary 2.3.12. Let us suppose that all standard modules are simple, then BGG reciprocity tells us that all standard modules are projective and hence the simple objects in $\mathcal{O}_{\mathbf{k}}$ are all projective. Thus $\mathcal{O}_{\mathbf{k}}$ is semisimple. Conversely, if $\mathcal{O}_{\mathbf{k}}$ is semisimple, then all simple objects are projective. For any standard module $\Delta(\lambda)$ we have $\dim_{\mathbb{C}} \mathrm{End}_{\mathcal{O}_{\mathbf{k}}}(\Delta(\lambda)) = 1$ (independent of the choice of $\mathbf{k}$) and thus the standard modules are always indecomposable. Together with projectivity of the simple objects we can deduce that the standard objects are simple. □

The next result will show that a "generic parameter" is regular and can be found in many articles concerning rational Cherednik algebra. As the author could not find a suitable reference a proof is supplied.

Corollary 2.4.4. *The set* $\mathrm{Reg}(\Gamma)$ *is dense in the analytic topology of* $\prod_{H\in\mathcal{A}/\Gamma}\prod_{j=1}^{m_H-1}\mathbb{C}$.

Proof. Each $\kappa_{\mathbf{k}}(\lambda)$ is linear as a function of the parameters and so for any $z \in \mathbb{Z}$ and $\lambda, \mu \in \mathrm{Irr}(\Gamma)$ we have an affine hyperplane

$$H_{\lambda,\mu,z} = \left\{ \mathbf{k} \in \prod_{H\in\mathcal{A}/\Gamma}\prod_{j=1}^{m_H-1}\mathbb{C} \mid \kappa_{\mathbf{k}}(\lambda) - \kappa_{\mathbf{k}}(\mu) = z \right\}.$$

Then the set

$$\mathcal{R} = \prod_{H\in\mathcal{A}/\Gamma}\prod_{j=1}^{m_H-1}\mathbb{C} \setminus \bigcup_{\substack{\lambda,\mu\in\mathrm{Irr}(\Gamma)\\ n\in\mathbb{N}_0}} H_{\lambda,\mu,n}$$

is dense in $\prod_{H\in\mathcal{A}/\Gamma}\prod_{j=1}^{m_H-1}\mathbb{C}$ as it is the complement of a countable union of affine hyperplanes and for any $\mathbf{k} \in \mathcal{R}$ no two elements of $\mathrm{Irr}(\Gamma)$ are comparable. But by Theorem 2.3.11 and BGG reciprocity this implies that $\Delta(\lambda)$ is simple for all $\lambda \in \mathrm{Irr}(\Gamma)$. So by Proposition 2.4.3, $\mathcal{O}_{\mathbf{k}}$ is semisimple. □

Definition 2.4.5. *We denote by* $\mathcal{O}_{\mathbf{k}}^{tor}$ *the full subcategory* $\mathcal{O}_{\mathbf{k}}$ *consisting of those modules* M *with* $M|_{\mathfrak{h}^{reg}} = 0$.

Recall that $L(\lambda)$ is the simple object in $\mathcal{O}_{\mathbf{k}}$ associated to λ, $P(\lambda)$ is the indecomposable projective cover of $L(\lambda)$ and $I(\lambda)$ isn the indecomposable injective hull of $L(\lambda)$.

Proposition 2.4.6. *([GGOR03], Proposition 5.21) For* $\lambda \in \mathrm{Irr}(\Gamma)$*, the following are equivalent:*

1. $L(\lambda)|_{\mathfrak{h}^{reg}} \neq 0$.
2. $L(\lambda)$ *is a submodule of a standard module.*
3. $P(\lambda)$ *is injective.*
4. $I(\lambda)$ *is projective*

Proposition 2.4.7. *([Val06], Lemma 2.3) The category* $\mathcal{O}_{\mathbf{k}}$ *is semisimple if and only if* $\mathcal{O}_{\mathbf{k}}^{tor} = 0$.

Proof. A proof is easy to give using Proposition 2.4.3, which states that $\mathcal{O}_{\mathbf{k}}$ is semisimple if and only if each standard module $\Delta(\lambda)$ is simple. So if $\mathcal{O}_{\mathbf{k}}$ is semisimple, each simple module is trivially a submodule of a standard module and so by Proposition 2.4.7 $L|_{\mathfrak{h}^{reg}} \neq 0$ for any simple object in $\mathcal{O}_{\mathbf{k}}$ and thus $\mathcal{O}_{\mathbf{k}}^{tor} = 0$.
Conversely suppose that $\mathcal{O}_{\mathbf{k}}^{tor} = 0$. An inductive argument using the ordering of $\mathrm{Irr}(\Gamma)$ can then be used to argue that each standard module has to be projective, implying that all standard modules are simple by BGG reciprocity. □

We now turn our attention to further objects derived from complex reflection groups Γ, namely the (topological) braid groups. We follow the exposition in [BMR98] and [Chl09].

Definition 2.4.8. *We suppose that Γ is a complex reflection group with reflection representation $\mathfrak{h}$. Recall that $\mathcal{A}$ denotes the set of reflection hyperplanes and that $\mathfrak{h}^{reg} = h \setminus \bigcup_{H\in\mathcal{A}} H$. As $\mathfrak{h}$ is a complex vector space and $\mathfrak{h}^{reg}$ is the complement of a finite set of hyperplanes, the space $\mathfrak{h}^{reg}$ is path-connected as is the orbit space $\mathfrak{h}^{reg}/\Gamma$. We will define two associated groups, the braid group B_Γ and the pure braid group P_Γ:*

1. *The braid group B_Γ is*
$$B_\Gamma := \pi_1(\mathfrak{h}^{reg}/\Gamma, *).$$
2. *The pure braid group P_Γ is defined similarly by*
$$P_\Gamma = \pi_1(\mathfrak{h}^{reg}, *).$$

It is known that the groups Γ, B_Γ and P_Γ are all related via a short exact sequence:

Proposition 2.4.9. *(see Formula 2.10 in [BMR98]) There is an exact sequence of groups*
$$1 \to P_\Gamma \to B_\Gamma \to \Gamma \to 1$$
with the maps $B_\Gamma \to \Gamma$ induced by the projection $\mathfrak{h}^{reg} \to \mathfrak{h}^{reg}/\Gamma$.

Proof. It is a standard result from algebraic topology that if a finite group G acts freely on a path connected, locally path-connected Hausdorff space X, then G^{op} is isomorphic to the quotient $\pi(X/G, p(x))/p_*\pi(X, x)$ and the image $p_*\pi(X, x)$ in $\pi(X/G, p(x))$ is isomorphic to $\pi(X, x)$. Here, p is the natural projection map $X \to X/G$, see e.g. [Hat], Proposition 1.40 and the following remarks as well as Proposition 1.3 in loc. cit. . Note that the isomorphisms mentioned there are in fact maps to the opposite group G^{op}, which is of course itself isomorphic to G.
The action of Γ on $\mathfrak{h}^{reg}$ is free by Lemma 2.2.11, from which it follows that the only sets of $\mathfrak{h}$ with non-trivial (pointwise) stabilisers are intersections of reflection hyperplanes. Clearly $\mathfrak{h}^{reg}$ is Hausdorff, path-connected and locally path connected, so we indeed have an exact sequence
$$1 \to \pi(\mathfrak{h}^{reg}) \xrightarrow{p_*} \pi(\mathfrak{h}^{reg}/\Gamma) \to \Gamma^{op} \to 1$$
□

It will help to explain this surjection explicitly: Fix $[y] \in \mathfrak{h}^{reg}/\Gamma$, where $[\bullet]$ stands for the Γ-orbit of $\bullet$ then $p^{-1}([y]) = \Gamma \cdot y \subset \mathfrak{h}^{reg}$. Note that $\Gamma \cdot y$ contains $\#\Gamma$ distinct elements, hence any element $\gamma \in \Gamma$ is uniquely determined by $\gamma(y) \in \Gamma \cdot y$. Take a path $\alpha \in \pi(\mathfrak{h}^{reg}/\Gamma, [y])$ and let $\tilde{\alpha}$ denote a lift of α to $\mathfrak{h}^{reg}$ starting at y. Then α is mapped to the element of Γ taking y to $\tilde{\alpha}(1)$
We will now turn our attention to determining a set of generators for B_Γ. Recall that for any $H \in \mathcal{A}$ we denote by m_H the order of the cyclic group Γ_H. Elements of Γ_H are uniquely determined by their determinant and $\det_{\mathfrak{h}}$ gives an isomorphism from Γ_H onto its image, the group μ_{m_H} of m_H-th roots of unity.

Definition 2.4.10. *We denote by s_H the element of Γ_H with determinant $e^{\frac{2\pi i}{m_H}}$ and will refer to s_H as a distinguished reflection for the purposes of this section.*

We also set

$$L_H = \mathrm{im}(s_h - \mathrm{id}_{\mathfrak{h}})$$

then $\mathfrak{h} = H \oplus L_H$ and we have projection maps

$$pr_H : \mathfrak{h} \to H, pr_{L_H} : \mathfrak{h} \to L_H.$$

For any $t \in [0,1]$ we may then set

$$s_H^t := pr_H + e^{\frac{2\pi i t}{m_h}} pr_{L_H} \in \mathrm{GL}(\mathfrak{h})$$

and note that this gives a path from $\mathrm{id}_{\mathfrak{h}} \to s_H$ in $\mathrm{GL}(\mathfrak{h})$. For any $y \in \mathfrak{h}$ we can obtain from this a path $\sigma_{H,y}$ in $\mathfrak{h}$ by simply setting

$$\sigma_{H,y} : [0,1] \to \mathfrak{h},\ t \mapsto s_H^t(y).$$

Let us fix a basepoint $y_0 \in \mathfrak{h}^{reg}$ and consider any path $\gamma : [0,1] \to \mathfrak{h}^{reg}$ starting at y_0 and ending at some y_H. If we choose y_H "close to H and far away from the other hyperplanes" the path

$$\sigma_{H,\gamma}(t) := s_H(\gamma^{-1}(t)) \cdot \sigma_{H,y_H} \cdot \gamma(t)$$

lies in $\mathfrak{h}^{reg}$ and its homotopy class is independent of our choice of y_H. The path $\sigma_{H,\gamma}$ has starting point y_0 and endpoint $s_H(y_0)$ and thus gives rise to a loop in $\mathfrak{h}^{reg}/\Gamma$ and so to an element $s_{H,\gamma}$ of the braid group B_Γ. Note that if $\gamma\tilde{\gamma}'$ are homotopic paths $[0,1] \to \mathfrak{h}^{reg}$ then $\sigma_{H,\gamma}\tilde{\sigma}_{H,\gamma'}$ will be homotopic as well.

Lemma 2.4.11. *([BMR98], Lemma 2.14 (1)) Under the map $B_\Gamma \to \Gamma^{op}$ the element $s_{H,\gamma}$ is mapped to the reflection s_H.*

Definition 2.4.12. *The paths $s_{H,\gamma}$ thus constructed are referred to as distinguished braid reflections around H. A generator of monodromy around H is a distinguished braid reflection derived from a distinguished reflection (see Definition 2.4.10).*

Theorem 2.4.13. *([BMR98], Theorem 2.17, and Theorem 2.28 in [Bro01] with Theorem 2.27 in [BMR98]) The group B_Γ is generated by one generator of monodromy $s_{H,\alpha}$ for each hyperplane $H \in \mathcal{A}$. In fact, B_Γ is generated by at most $\dim_{\mathbb{C}} \mathfrak{h} + 1$ generators of monodromy.*

Full details concerning these constructions can be found in [BMR98] and a description of the braid groups in terms of generators and relations can be read off from Table 5 of [BMR98]. We will not be using the second assertion.

Definition 2.4.14. *Let Γ be a complex reflection group with irreducible reflection representation $\mathfrak{h}$ and B_Γ the associated braid group. Choose a set of parameters*

$$\mathbf{q} = \{q_{H,j} \in \mathbb{C}^* \mid H \in \mathcal{A}, 1 \leq j \leq m_H - 1, q_{H,j} = q_{H^\gamma,j} \forall \gamma \in \Gamma\}$$

then define the Hecke algebra $\mathcal{H}_q(\Gamma)$ to be the quotient of $\mathbb{C}B_\Gamma$ by the relations

$$(*)(T-1)\prod_{j=1}^{m_H-1}(T - e^{\frac{2\pi i j}{m_H}} q_{H,j})$$

where T is an s-generator of the monodromy around H (where s is the reflection around H with determinant $e^{\frac{2\pi i}{m_H}}$).

Note that the relations $(T-1)\prod_{j=1}^{m_H-1}(T-e^{\frac{2\pi ij}{m_H}}q_{H,j})$ define $\mathcal{H}_{\mathbf{q}}(\Gamma)$ as a quotient of $\mathbb{C}B_\Gamma$ and therefore the T_i additionally fulfil the appropriate braid relations for Γ. In fact, the braid relations and the relations $(T-1)\prod_{j=1}^{m_H-1}(T-e^{\frac{2\pi ij}{m_H}}q_{H,j})$ suffice to give a presentation of $\mathcal{H}_{\mathbf{q}}(\Gamma)$.

Theorem 2.4.15. *([BMR98], Proposition 4.22) Let Γ be a complex reflection group with $\dim_{\mathbb{C}}\mathcal{H}_{\mathbf{q}}(\Gamma)=\#\Gamma$ for all $\mathbf{q}$. Then $\mathcal{H}_{\mathbf{q}}(\Gamma)$ has a presentation with either $\operatorname{rank}(\Gamma)$ or $\operatorname{rank}(\Gamma)+1$ generators and this number equals the number of minimal generators of Γ. The relations are given by $(*)$ for each generator and the appropriate braid relations. In particular, if Γ is a real reflection group or of type $G(m,1,n)$ then the number of generators of $\mathcal{H}_{\mathbf{q}}(\Gamma)$ is equal to the rank of Γ*

A full description of the generators and relations for all complex reflection groups Γ with $\dim_{\mathbb{C}}\mathcal{H}_{\mathbf{q}}(\Gamma)=\#\Gamma$ can be found in Tables 1-4 of [BMR98].

Example 2.4.16.

1. *As a first example, let us consider $\mathcal{H}_{\mathbf{q}}(S_n)$ where S_n acts on $\mathfrak{h}_n^0=\mathbb{C}^{n-1}$ as in Example 2.1.6. We have one reflection hyperplane associated to each transposition (ij), however these are all products of the neighbour transpositions $s_i=(i,i+1), i=1,\dots,n-1$. Thus the $(i,i+1)$ will give generators T_i of the Hecke algebra. The presentation of S_n in terms of the s_i is*

$$S_n=\langle s_i, i=1,\dots,n-1 \mid s_i^2=1, s_is_{i+1}s_i=s_{i+1}s_is_{i+1}, s_is_j=s_js_i \text{ if } |i-j|>1\rangle$$

   There is only one hyperplane orbit under the action of S_n and so the Hecke algebra will depend on one parameter q and has presentation

$$\mathcal{H}_{\mathbf{q}}(S_n)=\mathbb{C}\langle T_i, i=1,\dots,n-1 \mid T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}, T_iT_j=T_jT_i \text{ if } |i-j|>1, (T-1)(T+q)=0\rangle.$$

2. *Now take $\Gamma=\mu_m$, the group of m-th roots of unity with the natural action on $\mathbb{C}$. There is only one reflection hyperplane, namely $\{0\}$ and thus $\mathcal{H}_{\mathbf{q}}(\mu_m)$ is generated by one element T. We have $m-1$ independent parameters $q_1,\dots,q_{m-1}$ and the Hecke algebra has the presentation*

$$\mathcal{H}_{\mathbf{q}}(\mu_m)=\mathbb{C}\langle T \mid (T-1)\prod_{j=1}^{m-1}(T-e^{-\frac{2\pi ij}{m}}q_j)=0\rangle$$

   and is a quotient of the polynomial algebra $\mathbb{C}[T]$.

The construction of the $\mathrm{KZ}_{\mathbf{k}}$-functor is comparatively easy to understand in the case of regular parameter values: Then the simple objects in $\mathcal{O}_{\mathbf{k}}$ are precisely the standard modules by Proposition 2.4.3 and so we only need to construct $\mathrm{KZ}_{\mathbf{k}}(\Delta(\lambda))$. We follow the exposition of [Val06], Section 1.3.5:

Localising to $\mathfrak{h}^{reg}$ we turn $\Delta(\lambda)|_{\mathfrak{h}^{reg}}$ into a $H_{\mathbf{k}}|_{\mathfrak{h}^{reg}}\cong\mathcal{D}(\mathfrak{h})\#\Gamma$-module. As a $\mathbb{C}[\mathfrak{h}^{reg}]\#\Gamma$-module, $\Delta(\lambda)|_{\mathfrak{h}^{reg}}$ is isomorphic to $\mathbb{C}[\mathfrak{h}^{reg}]\otimes\lambda$ and so we can regard $\Delta(\lambda)|_{\mathfrak{h}^{reg}}$ as a Γ-equivariant vector bundle with a flat, algebraic connection over $\mathfrak{h}^{reg}$. This connection is determined by the action of the Dunkl operators $T_{\mathbf{k},y}$ on $\Delta(\lambda)$ and is called the KZ-connection. It has regular singularities by [GGOR03], Proposition 5.7. As $\Delta(\lambda)|_{\mathfrak{h}^{reg}}$ is Γ-invariant, it corresponds to a vector bundle

$\mathcal{M}_\lambda$ on $\mathfrak{h}^{reg}/\Gamma$ and $\mathcal{M}_\lambda$ carries a flat connection induced from the connection on $\Delta(\lambda)|_{\mathfrak{h}^{reg}}$. This will still have regular singularities. Therefore, by the Riemann-Hilbert correspondence the bundle $\mathcal{M}_\lambda$ with the KZ-connection corresponds to a finite-dimensional representation $M(\lambda)$ of the braid group $B_\Gamma = \pi_1(\mathfrak{h}^{reg}/\Gamma)$. We then have

$$\mathrm{KZ}_{\mathbf{k}}(\Delta_{\mathbf{k}}(\lambda)) = M(\lambda).$$

It is shown in Theorem 4.12 in [BMR98] and Theorem 5.13 in [GGOR03] that this fulfils the Hecke relation at parameters

$$q_{H,j} = e^{2\pi i k_{H,j}}$$

and so we can regard $\mathrm{KZ}_{\mathbf{k}}$ as a functor

$$\mathrm{KZ}_{\mathbf{k}} : \mathcal{O}_{\mathbf{k}} \to \mathcal{H}_{\mathbf{q}} - \mathrm{mod}.$$

Details of the construction of the $\mathrm{KZ}_{\mathbf{k}}$-functor and of how to extend $\mathrm{KZ}_{\mathbf{k}}$ to parameters $\mathbf{k}$ that are not necessarily regular can be found in Sections 5.2 and 5.3 of [GGOR03] or Section 4 of the notes [Ari07] by Ariki.

Definition 2.4.17. *The* KZ*-functor*

$$\mathrm{KZ}_{\mathbf{k}} : \mathcal{O}_{\mathbf{k}} \to \mathcal{H}_{\mathbf{q}} - \mathrm{mod}$$

is the functor which is described above for regular parameter values and extended to non-regular values.

It is clear from the description of the $\mathrm{KZ}_{\mathbf{k}}$-functor that we could analogously define a "spherical $\mathrm{KZ}_{\mathbf{k}}$ functor" from $\mathcal{O}_{\mathbf{k}}(U)$ to B_Γ-representations and that at least for aspherical parameter values both the $\mathrm{KZ}_{\mathbf{k}}$-functor and the spherical $\mathrm{KZ}_{\mathbf{k}}$-functor agree in the sense that $\mathrm{KZ}_{\mathbf{k}}(M) = \mathrm{KZ}_{\mathbf{k}}^{spherical}(\mathbf{e}M)$ for any $M \in \mathcal{O}_{\mathbf{k}}$. We will be using this freely in Chapter 3, most notably in Theorem 3.4.17.

Proposition 2.4.18. *(Theorem 5.14 in [GGOR03]) The functor* $\mathrm{KZ}_{\mathbf{k}} : \mathcal{O}_{\mathbf{k}} \to \mathcal{H}_{\mathbf{q}} - \mathrm{mod}$ *descends to an equivalence*

$$\mathrm{KZ}_{\mathbf{k}} : \mathcal{O}_{\mathbf{k}}/\mathcal{O}_{\mathbf{k}}^{tor} \cong \mathcal{H}_{\mathbf{q}} - \mathrm{mod}.$$

Corollary 2.4.19. *If* $\mathbf{k} \in \mathrm{Reg}(\Gamma)$*, then* $\mathrm{KZ}_{\mathbf{k}} : \mathcal{O}_{\mathbf{k}} \to \mathcal{H}_{\mathbf{q}} - \mathrm{mod}$ *is an equivalence of categories.*

Proof. By Proposition 2.4.7, $\mathcal{O}_{\mathbf{k}}^{tor} = 0$ for regular parameter values. □

Finally it turns out that a double centraliser property holds which describes the relationship between $\mathcal{O}_{\mathbf{k}}$ and $\mathcal{H}_{\mathbf{q}}(\Gamma)$. Recall that $L(\lambda)$ is the simple object in $\mathcal{O}_{\mathbf{k}}$ associated to $\lambda \in \mathrm{Irr}(\Gamma)$ and that $P(\lambda)$ is the projective cover of $L(\lambda)$.

Theorem 2.4.20. *([GGOR03], Theorems 5.15 and 5.16) Suppose that* Γ *is a complex reflection group such that* $\dim_{\mathbb{C}} \mathcal{H}_{\mathbf{q}(\mathbf{k})}(\Gamma) = \#\Gamma$*. We set*

$$P_{KZ} = \oplus_{\lambda \in \mathrm{Irr}(\Gamma)} P(\lambda)^{\oplus \dim_{\mathbb{C}} \mathrm{KZ}_{\mathbf{k}}(L(\lambda))} \in \mathcal{O}_{\mathbf{k}}$$

and we let $Q \in \mathcal{O}_{\mathbf{k}}$ *be any progenerator.*

1. *There is an isomorphism of algebras* $\mathcal{H}_{\mathbf{q}} \cong \mathrm{End}_{\mathcal{O}_{\mathbf{k}}}(P_{KZ})^{opp}$.

2. *There is an equivalence of categories* $\mathcal{O}_{\mathbf{k}} \cong \mathrm{End}_{\mathcal{H}_{\mathbf{q}}}(\mathrm{KZ}_{\mathbf{k}}(Q))^{opp} - \mathrm{mod}$.

Theorem 2.4.21. *(See Theorem 2.1 in [Val06]) Suppose that* Γ *is a complex reflection group such that* $\dim_{\mathbb{C}} \mathcal{H}_{\mathbf{q}(\mathbf{k})}(\Gamma) = \#\Gamma$. *Then the following are equivalent:*

1. $H_{\mathbf{k}} = H_{\mathbf{k}}(\Gamma, \mathfrak{h})$ *is a simple ring.*
2. $\mathcal{O}_{\mathbf{k}}$ *is completely reducible.*
3. $\mathcal{H}_{\mathbf{q}}(\Gamma)$ *is semisimple.*

In particular, $\mathbf{k}$ *is regular if and only if* $H_{\mathbf{k}}$ *is simple.*

Proof. The proof is now easy: (1) $\implies$ (2) : Suppose $H_{\mathbf{k}}$ is a simple algebra and consider $M \in \mathcal{O}_{\mathbf{k}}^{tor}$. As $M|_{\mathfrak{h}^{reg}} = 0$ every simple subquotient of M must lie in $\mathcal{O}^{tor}$ too and thus we may assume that M is a simple object in $\mathcal{O}_{\mathbf{k}}$ or the zero module. As such, it is a quotient of a standard module and so finitely generated as a $\mathbb{C}[\mathfrak{h}]$-module. Some power of the discriminant δ must then annihilate a generating set of M as a $\mathbb{C}[\mathfrak{h}]$-module and so this power will in fact lie in $\mathrm{Ann}_{H_{\mathbf{k}}}(M)$. Thus $\mathrm{Ann}_{H_{\mathbf{k}}}(M)$is non-zero and as $H_{\mathbf{k}}$ is simple, we have $\mathrm{Ann}_{H_{\mathbf{k}}}(M) = H_{\mathbf{k}}$ and $M = 0$, hence $\mathcal{O}_{\mathbf{k}}^{tor} = 0$. This implies that $\mathcal{O}_{\mathbf{k}}$ is completely reducible by Proposition 2.4.7.
(2) $\implies$ (3) : By Corollary 2.4.19 the category $\mathcal{H}_{\mathbf{q}}(\Gamma) - \mathrm{mod}$ is semisimple.
(3) $\implies$ (1) : As $\mathcal{H}_{\mathbf{q}}(\Gamma)$ is semisimple, so is $(\mathrm{End}_{\mathcal{H}_{\mathbf{q}}(\Gamma)}(Q_{\mathbf{k}}))^{opp}$ with Q any progenerator of $\mathcal{H}_{\mathbf{q}} - \mathrm{mod}$. Hence $\mathcal{O}_{\mathbf{k}}$ is semisimple by Theorem 2.4.20. By Ginzburg's Duflo Theorem 2.3.13, $H_{\mathbf{k}}$ has no primitive ideals and is simple as an algebra. □

The restriction $\dim_{\mathbb{C}} \mathcal{H}_{\mathbf{k}} = \#\Gamma$ is necessary for the above proof and luckily this is a mild restriction on Γ - indeed it has been shown that for an irreducible complex reflection group Γ, $\mathcal{H}_q(\Gamma)$ always has dimension $\#\Gamma$ unless Γ is one of a finite number of exceptional groups (and for these the statement is conjectured to be true). In the reducible case, we will have $\dim_{\mathbb{C}} \mathcal{H}_q(\Gamma) = \#\Gamma$ if no irreducible component of Γ is one of those exceptions. In particular for any finite Coxeter group and any complex reflection group of type $G(m, p, n)$ the dimension of the associated Hecke algebra will be equal to the order of the group.

Corollary 2.4.22. *Let* Γ *be a complex reflection group such that* $\dim_{\mathbb{C}} \mathcal{H}_{\mathbf{q}}(\Gamma) = \#\Gamma$. *Then* $\mathrm{Reg}(\Gamma) \cap \mathrm{Asp}(\Gamma) = \emptyset$.

Proof. By Theorem 2.4.21 we can identify the set $\mathrm{Reg}(\Gamma)$ with those parameters at which $H_{\mathbf{k}}$ is simple. The statement follows from Proposition 2.3.20 (2). □

Chapter 3

Harish-Chandra Bimodules and Basic Results

3.1 Harish-Chandra Bimodules

We can now turn our attention to our main objects of study - Harish-Chandra bimodules of rational Cherednik algebras. All references in this chapter to the paper [BEG03b] by Berest-Etingof-Ginzburg apply to the case of real reflection groups. We will start with two definitions and an example:

Definition 3.1.1. *Let A be an algebra and V an $A-A$ bimodule. For $a \in A$ we define the adjoint action of a on V to be the map $v \mapsto av - va$ and write this as $\mathrm{ad}(a)(v)$. We say that a acts locally ad-nilpotently on V if for any $v \in V$ there is $N_v \in \mathbb{N}$ such that $\mathrm{ad}(a)^{N_v}(v) = 0$.*

Definition 3.1.2. *([BEG03b], Definition 3.2) A Harish-Chandra bimodule is a finitely generated $H_{\mathbf{k}} - H_{\mathbf{k}}$ bimodule V, such that any $a \in \mathbb{C}[\mathfrak{h}]^\Gamma \cup \mathbb{C}[\mathfrak{h}^*]^\Gamma$ acts locally ad-nilpotently on V. We denote the category of Harish-Chandra bimodules of $H_{\mathbf{k}}$ by $\mathcal{HC}_{\mathbf{k}}$.*
Similarly, we can define Harish-Chandra bimodules of $U_{\mathbf{k}}$ by requiring that every $a \in \mathbb{C}[\mathfrak{h}]^\Gamma \mathbf{e} \cup \mathbb{C}[\mathfrak{h}^*]^\Gamma \mathbf{e}$ act locally nilpotently. We will also denote the category of such modules by $\mathcal{HC}_{\mathbf{k}}$. When we need to distinguish between these two categories, we will denote them by $\mathcal{HC}_{\mathbf{k}}(H)$ and $\mathcal{HC}_{\mathbf{k}}(U)$ respectively.

We will usually only concern ourselves with $U_{\mathbf{k}}$-Harish-Chandra bimodules. This has several computational advantages and will give us corresponding results on $H_{\mathbf{k}}$-Harish-Chandra bimodules too, provided our parameters $\mathbf{k}$ are not aspherical, recall Definition 2.3.19 and see Theorem 3.2.3.

Proposition 3.1.3. *([BEG03b], Lemma 3.3 (v)) For any choice of parameters, the algebra $U_{\mathbf{k}}$ is a Harish-Chandra bimodule.*

Proof. We will repeat the argument given in [BEG03b]: Take $a \in \mathbb{C}[\mathfrak{h}]^\Gamma$. $H_{\mathbf{k}}$ has a positive filtration given by placing $\mathbb{C}[\mathfrak{h}^*]$ in degree 1 and $\mathbb{C}[\mathfrak{h}], \mathbb{C}\Gamma$ in degree 0. Then $\mathrm{ad}(a)$ strictly decreases the grading on $H_{\mathbf{k}}$ and must thus act locally nilpotently (note that to show that $\mathrm{ad}(a)$ annihilates elements of $\mathbb{C}[\mathfrak{h}]\#\mathbb{C}\Gamma$ we need to use the fact that it is Γ-invariant). For $a \in \mathbb{C}[\mathfrak{h}^*]^\Gamma$ argue similarly by placing $\mathbb{C}[\mathfrak{h}]$ in degree 1 and $\mathbb{C}[\mathfrak{h}^*], \mathbb{C}\Gamma$ in degree 0. This descends to $U_{\mathbf{k}}$. $\square$

Other examples of Harish-Chandra bimodules can be found easily for non-regular parameter values:

Lemma 3.1.4. *Let F be a finite-dimensional bimodule over $U_{\mathbf{k}}$, then F is Harish-Chandra. If $\mathfrak{a} \subset U_{\mathbf{k}}$ is a two-sided ideal of $U_{\mathbf{k}}$ it is a Harish-Chandra bimodule.*

Proof. F is manifestly finitely generated as a bimodule and it remains to check for local ad-nilpotency. Consider the linear map $T := \mathrm{ad}(\mathbf{e}\underline{h}\mathbf{e})$ on F (T is the adjoint action map of $\mathbf{e}\underline{h}\mathbf{e}$). We can decompose F into a direct sum of generalised T-eigenspaces

$$F = \oplus_\lambda F_\lambda$$

say, where F_λ denotes the generalised eigenspace with eigenvalue $\lambda \in \mathbb{C}$. The adjoint action of $a \in \mathbb{C}[\mathfrak{h}]^\Gamma \mathbf{e}$ will decrease the T-eigenvalue of any generalised eigenvector and by finite-dimensionality upon iterating this action we must eventually obtain zero, i.e. the adjoint action of any $a \in \mathbb{C}[\mathfrak{h}]^\Gamma \mathbf{e}$ is locally nilpotent. The same argument shows that the adjoint action of any $a \in \mathbb{C}[\mathfrak{h}^*]^\Gamma \mathbf{e}$ is locally nilpotent on F and hence F is Harish-Chandra.
To show that $\mathfrak{a} \subset U_{\mathbf{k}}$ is Harish -Chandra we need only concern ourselves with the question of finite generation, but as $U_{\mathbf{k}}$ is Noetherian this is clear. □

Note that this result is similar to the situation for $\mathcal{O}_{\mathbf{k}}$ and finite-dimensional $U_{\mathbf{k}}$-modules. We will study the finite-dimensional Harish-Chandra bimodules of $U_{\mathbf{k}}(\mu_m)$ in more detail in chapter 6

Lemma 3.1.5. *([BEG03b], Lemma 3.3 (iii)) Let V be Harish-Chandra. Then V is finitely generated when regarded as a module on either side.*

Proof. The proof again can be found in but we can give a separate proof too: It will suffice to show that V is finitely generated on the left say, the arguments for finite generation on the right will be the same. First, we choose bases $e_1, \dots, e_{\#\Gamma}$ of $\mathbb{C}[\mathfrak{h}]^{\mathrm{co}\Gamma}$ and $f_1, \dots, f_{\#\Gamma}$ of $\mathbb{C}[\mathfrak{h}^*]^{\mathrm{co}\Gamma}$ where we may assume that $1 = e_1 = f_1$. Next, since Γ is a complex reflection group, $\mathbb{C}[\mathfrak{h}]^\Gamma$ and $\mathbb{C}[\mathfrak{h}^*]^\Gamma$ are both polynomial rings, with algebraically independent generators $q_1, \dots, q_n \in \mathbb{C}[\mathfrak{h}^*]^\Gamma$ and $p_1, \dots, p_n \in \mathbb{C}[\mathfrak{h}]^\Gamma$ say. Without loss of generality we shall assume that V is cyclic as a bimodule with generator v (the argument is completely analogous for the general case). We can now choose $d_1 \in \mathbb{N}$ such that

$$\mathrm{ad}^{d_1}(p_i)(v) = 0$$

for each i, j and choose $d_2 \in \mathbb{N}$ such that

$$\mathrm{ad}^{d_2}(q_j)(vp_i^s e_a) = 0$$

where $0 \leq s \leq d_1 - 1$ and $a = 1, \dots, \#\Gamma$. We now claim that

$$\mathcal{V} = \{vp_i^s e_a q_j^r f_b \gamma\}$$

is a generating set of V as a left module, where γ runs through Γ, $1 \leq a, b \leq \#\Gamma$, $1 \leq s \leq d_1$ and $1 \leq r \leq d_2$ (in particular, v lies in this enlarged set). To show that this indeed generates ${}_{H_{\mathbf{k}}}V$, we first note that by the PBW theorem $H_{\mathbf{k}}$ is isomorphic as a vector space to $\mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} [\mathfrak{h}^*] \otimes_{\mathbb{C}} \mathbb{C}\Gamma$. It will suffice to show that for any choice of i, s, a and $z \in \mathbb{C}[\mathfrak{h}^*]$ the element $vp_i^s e_a z$ lies in the left span of $\mathcal{V}$.

lies in the left span of $\{vp_i^s e_a q_j^r f_b \gamma\}$. So choose $z \in \mathbb{C}[\mathfrak{h}^*]$ and again without loss of generality we may assume that $z = f_b q_j^n$ for some b, j, n. If $n \leq d_2$ we are done by definition of $\mathcal{V}$. If not, note that since $\mathrm{ad}^{d_2}(q_j)(vp_i^s e_a) = 0$ the element $vp_i^s e_a q_j^{d_2}$ lies in the left span of $\mathcal{V}$. As this is regardless of the choice of i, s and a, we can deduce that $vp_i^s e_a z$ indeed lies in the left span of $\mathcal{V}$ and we are done. $\square$

Lemma 3.1.6. *([BEG03b], Lemma 3.3 (i)) The category $\mathcal{HC}_{\mathbf{k}}$ is Abelian and closed under extensions.*

Proof. To show that $\mathcal{HC}_{\mathbf{k}}$ is Abelian, we need to show that quotients, submodules and finite direct sums of Harish-Chandra bimodules are again Harish-Chandra. As $U_{\mathbf{k}}$ is Noetherian, it is clear that submodules of Harish-Chandra bimodules are again finitely generated and hence Harish-Chandra. That quotients of Harish-Chandra bimodules are Harish-Chandra is obvious and if V_1, V_2 are Harish-Chandra so is $V_1 \oplus V_2$. Now let V be a bimodule such that

$$0 \to V_1 \to V \to V_2 \to 0$$

is exact. Take $v \in V$ and $a \in \mathbb{C}[\mathfrak{h}]^\Gamma \mathbf{e} \cup \mathbb{C}[\mathfrak{h}^*]^\Gamma \mathbf{e}$. Due to the isomorphism $V/V_1 \cong V_2$ we see that there exists $n \in \mathbb{N}_0$ such that $\mathrm{ad}(a)^n(v) \in V_1$ and as V_1 is Harish-Chandra, a acts ad-nilpotently on $\mathrm{ad}(a)^n(v)$ and hence on v. Thus V is Harish-Chandra. $\square$

We can now prove our next result

Proposition 3.1.7. *Let $V \in \mathcal{HC}_{\mathbf{k}}$ and $N \in \mathcal{O}_{\mathbf{k}}$. Then $V \otimes_{U_{\mathbf{k}}} N$ is again a module in $\mathcal{O}_{\mathbf{k}}$.*

Proof. The tensor product $V \otimes_{U_{\mathbf{k}}} N$ will again be finitely generated as N is a finitely generated module and V is finitely generated on both sides. Pick $v \in V, z \in N$ and $a \in \mathbb{C}[\mathfrak{h}^*]^\Gamma \mathbf{e}$. We can choose $n_1 \in \mathbb{N}_0$ such that $\mathrm{ad}(a)^{n_1}(v) = 0$ and $n_2 \in \mathbb{N}_0$ such that $a^{n_2} z = 0$. Then

$$\begin{aligned} a^{n_1 n_2}(v \otimes z) &= (a^{n_2})^{n_1}(v \otimes z) \\ &= -\sum_{i=0}^{n_1-1} (-1)^i \binom{n_1}{i} a^{in_2} v a^{(n_1-i)n_2} \otimes z \\ &= -\sum_{i=0}^{n_1-1} (-1)^i \binom{n_1}{i} a^{in_2} v \otimes a^{(n_1-i)n_2} z \\ &= 0 \end{aligned}$$

$\square$

Thus Harish-Chandra bimodules give rise to a collection of right exact functors $\mathcal{O}_{\mathbf{k}} \to \mathcal{O}_{\mathbf{k}}$. The following two results will show that the property of being Harish-Chandra is preserved under some standard constructions, namely taking morphisms of left or right modules and tensor products:

Proposition 3.1.8. *([BEG03b], Lemma 8.3) Let V_1, V_2 be two Harish-Chandra bimodules. Then $V_1 \otimes_{U_{\mathbf{k}}} V_2$ is again Harish-Chandra.*

Proof. The tensor product $V_1 \otimes_{U_{\mathbf{k}}} V_2$ is again finitely generated as V_1 is finitely generated on the left and V_2 is finitely generated on the right, so that if $v_1^1, \ldots v_n^1$ is a left generating set of V_1 and $v_1^2, \ldots, v_k^2$ is a right generating set of V_2, then $v_i^1 \otimes_{U_{\mathbf{k}}} v_j^2$ is a generating set of $V_1 \otimes_{U_{\mathbf{k}}} V_2$ as a bimodule. There only remains to check local ad-nilpotence. If $a \in \mathbb{C}[\mathfrak{h}]^\Gamma \cup \mathbb{C}[\mathfrak{h}^*]^\Gamma$ and $v_i \in V_i$ for $i = 1, 2$ then we have

$$\mathrm{ad}(a)(v_1 \otimes v_2) = \mathrm{ad}(a)(v_1) \otimes v_2 + v_1 \otimes \mathrm{ad}(a)(v_2).$$

So if $n_1, n_2 \in \mathbb{N}_0$ are such that $\mathrm{ad}(a)^{n_i}(v_i) = 0$ then

$$\begin{aligned}\mathrm{ad}(a)^{n_1+n_2}(v_1 \otimes v_2) &= \sum_{i=0}^{n_1+n_2} \binom{n_1+n_2}{i} \mathrm{ad}(a)^{n_1+n_2-i}(v_1) \otimes \mathrm{ad}(a)^i(v_2) \\ &= \sum_{i=0}^{n_2} \binom{n_1+n_2}{i} \mathrm{ad}(a)^{n_1+n_2-i}(v_1) \otimes \mathrm{ad}(a)^i(v_2) \\ &+ \sum_{i=n_1+1}^{n_2} \binom{n_1+n_2}{i} \mathrm{ad}(a)^{n_1+n_2-i}(v_1) \otimes \mathrm{ad}(a)^i(v_2) \\ &= 0.\end{aligned}$$

So the tensor product is Harish-Chandra. $\square$

In particular we can deduce that a composition of two functors $\mathcal{O}_{\mathbf{k}} \to \mathcal{O}_{\mathbf{k}}$ induced by tensor products with Harish-Chandra bimodules is again of the same form, i.e. it comes from taking a tensor product with a suitable Harish-Chandra bimodule

Proposition 3.1.9. *Let L, V be Harish-Chandra bimodules, then* $\mathrm{Hom}_{U_{\mathbf{k}}-\mathrm{mod}}(L, V)$ *is again Harish-Chandra.*

Proof. As both L and V are finitely-generated as left modules, so is $\mathrm{Hom}_{U_{\mathbf{k}}-\mathrm{mod}}(L, V)$. Moreover, it carries a canonical bimodule structure given by

$$(r \cdot f)(m) = f(mr) \text{ and } (f \cdot r)(m) = f(m)r$$

for all $r \in H_{\mathbf{k}}$ and $m \in L$. Thus for $a \in \mathbb{C}[\mathfrak{h}]^\Gamma \mathbf{e} \cup \mathbb{C}[\mathfrak{h}^*]^\Gamma \mathbf{e}$ we have

$$\mathrm{ad}(a)(f)(m) = f(ma) - f(m)a$$

and in particular

$$\mathrm{ad}(a)(f)(sm) = f(sma) - f(sm)a = s\,(\mathrm{ad}(a)(f)(m))$$

and

$$\mathrm{ad}(a)(f)(m+n) = \mathrm{ad}(a)(f)(m) + \mathrm{ad}(a)(f)(n)$$

Let $m_1, \dots, m_k$ be a generating set of L as a left module. If $a \in \mathbb{C}[\mathfrak{h}]^\Gamma \mathbf{e} \cup \mathbb{C}[\mathfrak{h}^*]^\Gamma \mathbf{e}$ we can choose $N_1 = N_1(a) \in \mathbb{N}$ (depending on a) such that

$$\mathrm{ad}(a)^{N_1}(m_i) = 0$$

for all $i = 1, \dots, k$. Similarly, we can choose $N_2 = N_2(a) \in \mathbb{N}$ such that

$$\mathrm{ad}(a)^{N_2}(f(m_i a^j)) = 0$$

for all $i = 1, \dots, k$ and $0 \le j \le N_1 - 1$. Note that we have

$$\mathrm{ad}(a)^N(f)(m) = \sum_{p=0}^{N}(-1)^{N-p}\binom{N}{p} f(ma^{N-p})a^p$$

so in particular

$$\begin{aligned}
\mathrm{ad}(a)^{N_1}(f)(m) &= \sum_{p=0}^{N_1}(-1)^p\binom{N_1}{p} f(ma^{N-p})a^p \\
&= f(ma^{N_1}) + \sum_{p=1}^{N_1}(-1)^p\binom{N_1}{p} f(ma^{N-p})a^p \\
&= \sum_{p=1}^{N_1}(-1)^p\binom{N_1}{p} f(ma^{N-p})a^p \\
&- \sum_{p=0}^{N_1-1}(-1)^p\binom{N_1}{p} f(a^{N_1-p}ma^p) \\
&= -\sum_{p=1}^{N_1}(-1)^{N_1-p}\binom{N_1}{N_1-p} f(a^pma^{N_1-p}) \\
&+ \sum_{p=1}^{N_1}(-1)^p\binom{N_1}{p} f(ma^{N_1-p})a^p \\
&= -\sum_{p=1}^{N_1}(-1)^{N_1+p}\binom{N_1}{p} a^p f(ma^{N_1-p}) \\
&+ \sum_{p=1}^{N_1}(-1)^p\binom{N_1}{p} f(ma^{N_1-p})a^p
\end{aligned}$$

Choosing N_1 even then allows us to simplify this to

$$\begin{aligned}
\mathrm{ad}(a)^{N_1}(f)(m) &= \sum_{p=1}^{N_1}(-1)^p\binom{N_1}{p}\left(f(ma^{N_1-p})a^p - f(a^pma^{N_1-p})\right) \\
&= -\sum_{p=1}^{N_1}(-1)^p\binom{N_1}{p}\mathrm{ad}(a^p)\left(f(ma^{N_1-p})\right)
\end{aligned}$$

Thus we also find

$$\mathrm{ad}(a)^{2N_1}(f)(m) = \mathrm{ad}(a)^{N_1}\mathrm{ad}(a)^{N_1} = \sum_{p,q=1}^{N_1}(-1)^{p+q}\binom{N_1}{p}\binom{N_1}{q}\mathrm{ad}(a^q)\mathrm{ad}(a^p)\left(f(ma^{2N_1-p-q})\right)$$

But whenever $r \ge N_1$ we can write

$$ma^r = \sum_{j=0}^{N_1-1}\beta_j(a)ma^j$$

for some polynomials $\beta_j(a)$ in a. Thus

$$\begin{aligned}\mathrm{ad}(a)^{rN_1}(f)(m) &= \sum_{p_1,\dots,p_r=1}^{N_1} (-1)^{p_1+\dots+p_r}\binom{N_1}{p_1}\dots\binom{N_1}{p_r}\mathrm{ad}(a^{p_r})\dots\mathrm{ad}(a^{p_1})\left(f(ma^{2N_1-p_1-\dots-p_r})\right)\\ &= \sum_{j=0}^{N_1-1}\sum_{q_1,\dots,q_r=1}^{N_1}\beta_{q_1,\dots,q_r}(a)\mathrm{ad}(a^{q_r})\dots\mathrm{ad}(a^{q_1})\left(f(ma^j)\right)\end{aligned}$$

And thus for $r = N_2$ we can deduce that

$$\mathrm{ad}(a^{q_{N_2}})\dots\mathrm{ad}(a^{q_1})\left(f(ma^j)\right) = 0$$

as $\mathrm{ad}(a)^{N_2}(f(ma^j)) = 0$ for any $0 \leq j \leq N_2 - 1$.

Thus $\mathrm{Hom}_{U_{\mathbf{k}}-\mathrm{mod}}(L, V)$ is again Harish-Chandra. □

3.2 Basic Results on $(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}})$ I

Definition 3.2.1. *For a ring (or $\mathbb{C}$-algebra) R we denote by $\mathrm{BMod}(R)$ the category of R-R -bimodules and by $\mathrm{bmod}(R)$ the category of finitely generated R-R-bimodules. If R is a $\mathbb{C}$-algebra we assume that the right and left action of the scalars coincide with each other.*

Having dealt with the basic structure of $\mathcal{HC}_{\mathbf{k}}$ in the preceding section (e.g. showing that it is a full Serre subcategory of $\mathrm{bmod}(U_{\mathbf{k}})$) we will now turn our attention to the structure of $\mathcal{HC}_{\mathbf{k}}$ as a tensor category and how it interacts with $\mathcal{O}_{\mathbf{k}}$. The very first basic result shows that for regular parameter values, the categories $\mathcal{HC}_{\mathbf{k}}(H)$ and $\mathcal{HC}_{\mathbf{k}}(U)$ are indeed equivalent as tensor categories, much like the categories $\mathcal{O}_{\mathbf{k}}(H)$ and $\mathcal{O}_{\mathbf{k}}(U)$ are equivalent.
We make a definition analogous to Definition 2.3.18:

Definition 3.2.2. *For any parameter value $\mathbf{k}$ we have functors*

$$\begin{aligned} E_{\mathbf{k}}^{\sharp} : \mathrm{BMod}(H_{\mathbf{k}}) &\to \mathrm{BMod}(U_{\mathbf{k}}) \\ V &\mapsto \mathbf{e}V\mathbf{e} \end{aligned}$$

and

$$\begin{aligned} T_{\mathbf{k}}^{\sharp} : \mathrm{BMod}(U_{\mathbf{k}}) &\to \mathrm{BMod}(H_{\mathbf{k}}) \\ L &\mapsto H_{\mathbf{k}}\mathbf{e} \otimes_{U_{\mathbf{k}}} L \otimes_{U_{\mathbf{k}}} \mathbf{e}H_{\mathbf{k}} \end{aligned}$$

Just as Theorem 2.3.22 we have the following result:

Theorem 3.2.3. *1. $E_{\mathbf{k}}^{\sharp}$ is exact and $T_{\mathbf{k}}^{\sharp}$ is right exact.*

2. *$E_{\mathbf{k}}^{\sharp}$ and $T_{\mathbf{k}}^{\sharp}$ restrict to functors between $\mathrm{bmod}(H_{\mathbf{k}})$ and $\mathrm{bmod}(U_{\mathbf{k}})$.*
3. *$E_{\mathbf{k}}^{\sharp}$ and $T_{\mathbf{k}}^{\sharp}$ restrict to functors between $\mathcal{HC}_{\mathbf{k}}(H_{\mathbf{k}})$ and $\mathcal{HC}_{\mathbf{k}}(U_{\mathbf{k}})$.*
4. *$E_{\mathbf{k}}^{\sharp} \circ T_{\mathbf{k}}^{\sharp} \cong \mathrm{Id}_{\mathrm{BMod}(U_{\mathbf{k}})}$.*
5. *Suppose that $\mathbf{k}$ is a non-aspherical parameter. Then $E_{\mathbf{k}}^{\sharp}$ is an equivalence of categories between $\mathrm{BMod}(H_{\mathbf{k}})$ and $\mathrm{BMod}(U_{\mathbf{k}})$ with inverse $T_{\mathbf{k}}^{\sharp}$ and they restrict to an equivalence between $(\mathcal{HC}_{\mathbf{k}}(H_{\mathbf{k}}), \otimes_{H_{\mathbf{k}}})$ and $(\mathcal{HC}_{\mathbf{k}}(U_{\mathbf{k}}), \otimes_{U_{\mathbf{k}}})$.*
6. *The functors $E_{\mathbf{k}}, T_{\mathbf{k}}, E_{\mathbf{k}}^{\sharp}, T_{\mathbf{k}}^{\sharp}$ respect the tensor action of Harish-Chandra bimodules on the*

categories $\mathcal{O}_{\mathbf{k}}$, namely for $V \in \mathcal{HC}_{\mathbf{k}}(H_{\mathbf{k}})$ and $M \in \mathcal{O}_{\mathbf{k}}(H_{\mathbf{k}})$ we have

$$E_{\mathbf{k}}(V \otimes_{H_{\mathbf{k}}} M) \cong E^{\sharp}_{\mathbf{k}}(V) \otimes_{U_{\mathbf{k}}} E_{\mathbf{k}}(M)$$

and for $L \in \mathcal{HC}_{\mathbf{k}}(U_{\mathbf{k}})$ and $N \in \mathcal{O}_{\mathbf{k}}(U_{\mathbf{k}})$ we have

$$T_{\mathbf{k}}(L \otimes_{U_{\mathbf{k}}} N) \cong T^{\sharp}_{\mathbf{k}}(L) \otimes_{H_{\mathbf{k}}} T_{\mathbf{k}}(N).$$

Proof. Statements (1), (2) and (4) are clear as is the fact that $E^{\sharp}_{\mathbf{k}}$ restricts to a functor $\mathcal{HC}_{\mathbf{k}}(H_{\mathbf{k}}) \to \mathcal{HC}_{\mathbf{k}}(U_{\mathbf{k}})$. That $T^{\sharp}_{\mathbf{k}}$ gives a functor $\mathcal{HC}_{\mathbf{k}}(U_{\mathbf{k}}) \to \mathcal{HC}_{\mathbf{k}}(H_{\mathbf{k}})$ follows from the fact that $H_{\mathbf{k}}$ is Harish-Chandra and as $H_{\mathbf{k}} \subset H_{\mathbf{k}}\mathbf{e}$, local ad-nilpotence follows, showing (3). Proposition 2.3.20 can then be used to show that $E^{\sharp}_{\mathbf{k}}$ and $T^{\sharp}_{\mathbf{k}}$ give equivalences just as in the proof of Theorem 2.3.22.

To show that the tensor action of $\mathcal{HC}_{\mathbf{k}}$ on $\mathcal{O}_{\mathbf{k}}$ is preserved by these, we proceed similarly to before, suppose that V is Harish-Chandra and N is in $\mathcal{O}_{\mathbf{k}}$:

$$\begin{aligned} E(V \otimes_{H_{\mathbf{k}}} N) &= \mathbf{e}V \otimes_{H_{\mathbf{k}}} N \\ &= \mathbf{e}V \otimes_{H_{\mathbf{k}}} H_{\mathbf{k}}\mathbf{e} \otimes_{U_{\mathbf{k}}} \mathbf{e}H_{\mathbf{k}} \otimes_{H_{\mathbf{k}}} N \\ &= \mathbf{e}V\mathbf{e} \otimes_{U_{\mathbf{k}}} \mathbf{e}N \\ &= E(V) \otimes_{U_{\mathbf{k}}} E(N) \end{aligned}$$

and

$$\begin{aligned} T(V \otimes_{U_{\mathbf{k}}} N) &= H_{\mathbf{k}}\mathbf{e} \otimes_{U_{\mathbf{k}}} V \otimes_{U_{\mathbf{k}}} N \\ &= H_{\mathbf{k}}\mathbf{e} \otimes_{U_{\mathbf{k}}} V \otimes_{U_{\mathbf{k}}} U_{\mathbf{k}} \otimes_{U_{\mathbf{k}}} N \\ &= H_{\mathbf{k}}\mathbf{e} \otimes_{U_{\mathbf{k}}} V \otimes_{U_{\mathbf{k}}} \mathbf{e}H_{\mathbf{k}} \otimes_{H_{\mathbf{k}}} H_{\mathbf{k}}\mathbf{e} \otimes_{U_{\mathbf{k}}} N \\ &= T(V) \otimes_{H_{\mathbf{k}}} T(N) \end{aligned}$$

□

Our first important result on the tensor structure of $\mathcal{HC}_{\mathbf{k}}$ can be deduced easily from a result due to I. Losev, namely Theorem 3.4.6 in [Los11a]. Let Γ be a complex reflection group acting on $\mathfrak{h}$. This then gives rise to and action of Γ on $\mathfrak{h} \oplus \mathfrak{h}^*$. Take a subgroup $\underline{\Gamma} \subseteq \Gamma$ arising as the stabiliser of some point in $P \in \mathfrak{h} \oplus \mathfrak{h}^*$. We then have a decomposition

$$\mathfrak{h} \oplus \mathfrak{h}^* = V_0 \oplus V_+$$

where V_0 is the space of fixed points of $\underline{\Gamma}$ and V_+ is the unique $\underline{\Gamma}$-stable complement. As the action of Γ on V is diagonal, this corresponds to decomposing $\mathfrak{h}$ and $\mathfrak{h}^*$ into Γ-fixed subspaces $(\mathfrak{h}_0)$ and $\mathfrak{h}_0^*$ and Γ-invariant complements $\mathfrak{h}_+$ and $\mathfrak{h}_+^*$. Then we have

$$V_0 = \mathfrak{h}_0 \oplus \mathfrak{h}_0^* \text{ and } V_+ = \mathfrak{h}_+ \oplus \mathfrak{h}_+^*.$$

By Theorem 1.5 in [Ste64], the group $\underline{\Gamma}$ is again a complex reflection group with reflection representation $\mathfrak{h}$ (although not explicitly in the statement, it is implicit in the proof) and hence the reflections in $\underline{\Gamma}$ are precisely those reflections in Γ fixing P). Therefore, the restriction of $\underline{\Gamma}$ to $\mathfrak{h}_+$ is again a reflection representation as is the restriction of $\underline{\Gamma}$ to $\mathfrak{h}_+^*$. Denoting by $\mathbf{k}^+$ those parameters in $\mathbf{k}$ attached to reflection hyperplanes of $\underline{\Gamma}$, we can form the rational Cherednik algebra

$$\underline{H}^+_{\mathbf{k}^+} = H_{\mathbf{k}^+}(\underline{\Gamma}, \mathfrak{h}_+).$$

It was shown in [BG03], Proposition 7.4 (in Section 7.4), that the variety $(\mathfrak{h} \times \mathfrak{h}^*)/\Gamma$ possesses a natural stratification into finitely many symplectic leaves $\{\mathcal{L}\}$ which we will describe next following [BG03]:
The symplectic leaves of $(\mathfrak{h} \times \mathfrak{h}^*)/\Gamma$ are in one-to-one correspondence with the conjugacy classes of stabilisers of elements of $\mathfrak{h} \oplus \mathfrak{h}^*$. Choosing such a subgroup $\underline{\Gamma} \subseteq \Gamma$ the associated symplectic leaf $\mathcal{L}(\underline{\Gamma})$ is then given by

$$\mathcal{L}(\underline{\Gamma}) = \{b \in \mathfrak{h} \oplus \mathfrak{h}^* \mid \Gamma_b \text{ conjugate to } \underline{\Gamma}\}/\Gamma.$$

The closure of a symplectic leaf is a union of symplectic leaves:

$$\overline{\mathcal{L}(\underline{\Gamma})} = \bigcup_{\underline{\Gamma} \subseteq \Gamma' \subseteq \Gamma} \mathcal{L}(\Gamma')$$

and

$$\partial \mathcal{L} = \overline{\mathcal{L}} \setminus \mathcal{L}.$$

In [Los11a] Section 3.4 the associated variety $V(M)$ of a Harish-Chandra bimodule M is defined to be the support of the $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]^\Gamma$-module $\mathrm{gr}M$ for a suitable filtration on M, see Definition 3.4.4 and Proposition 5.4.3 in [Los11a]. It is shown in Lemma 3.3.3 in loc.cit. that $V(M)$ is a Poisson subvariety of $(\mathfrak{h} \times \mathfrak{h}^*)/\Gamma$. Ultimately, we will only need the properties of $V(M)$ stated in Theorem 3.2.4 so we will not dwell on its construction. For any subvariety $Y \subseteq (\mathfrak{h} \times \mathfrak{h}^*)/\Gamma$ we denote by $\mathcal{HC}_{\mathbf{k},Y}$ the subcategory of those Harish-Chandra bimodules whose associated variety is contained in Y.

Theorem 3.2.4. *([Los11a], Theorem 3.4.6) Let $\underline{\Gamma} \leq \Gamma$ be a stabiliser of a point of $\mathfrak{h} \oplus \mathfrak{h}^*$, $\mathcal{L} \subseteq (\mathfrak{h} \times \mathfrak{h}^*)$ the associated symplectic leaf and $\underline{H}^+_{\mathbf{k}^+} = H_{\mathbf{k}^+}(\underline{\Gamma}, V_+ \cap \mathfrak{h})$ rational Cherednik algebra of $\underline{\Gamma}$ inside $H_{\mathbf{k}}(\Gamma, \mathfrak{h})$. There exists a functor*

$$(\bullet)_\dagger : \mathcal{HC}_{\mathbf{k}}(H_{\mathbf{k}})_{\overline{\mathcal{L}}} \to {}^{N_\Gamma(\underline{\Gamma})}\mathcal{HC}^{fin}_{\mathbf{k}^+}(\underline{H}^+_{\mathbf{k}^+})$$

with the following properties

1. *$(\bullet)_\dagger$ is exact and intertwines tensor products.*
2. *The kernel of $(\bullet)_\dagger$ is $\mathcal{HC}_{\mathbf{k}}(H_{\mathbf{k}})_{\partial\mathcal{L}}$.*
3. *$\mathrm{LAnn}(M_\dagger) = \mathrm{LAnn}(M)_\dagger$ and $\mathrm{RAnn}(M_\dagger) = \mathrm{RAnn}(M)_\dagger$*
4. *$(\bullet)_\dagger$ gives an equivalence from $\frac{\mathcal{HC}_{\mathbf{k}}(H_{\mathbf{k}})_{\overline{\mathcal{L}}}}{\mathcal{HC}_{\mathbf{k}}(H_{\mathbf{k}})_{\partial\mathcal{L}}}$ to a full subcategory of ${}^{N_\Gamma(\underline{\Gamma})}\mathcal{HC}^{fin}_{\mathbf{k}^+}(\underline{H}^+_{\mathbf{k}^+})$ closed under taking tensor products and subquotients.*

Here ${}^{N_\Gamma(\underline{\Gamma})}\mathcal{HC}^{fin}_{\mathbf{k}^+}(\underline{H}^+_{\mathbf{k}^+})$ is the category of finite-dimensional $N_\Gamma(\underline{\Gamma})$-equivariant Harish-Chandra $\underline{H}^+_{\mathbf{k}^+}$-bimodules.

The following Lemma is a known result for which the author does not know of a suitable reference and therefore we supply a proof.

Lemma 3.2.5. *Let G be a finite group and $\mathcal{C}$ a full subcategory of $\mathrm{rep}_{\mathbb{C}}G$ closed under taking tensor products and subquotients (i.e. irreducible summands). Let Λ be a set of representatives of isomorphism classes of irreducible objects in $\mathcal{C}$ and $N = \bigcap_{\lambda \in \Lambda} \ker(\lambda)$. Then there is an* *equivalence*

$$\mathcal{C} \cong (\mathrm{rep}_{\mathbb{C}}(G/N), \otimes).$$

Proof. For any normal subgroup H of G the projection $\pi : G \twoheadrightarrow G/H$ gives a correspondence between G/H-representations and representation of G with kernel containing H. Thus π gives rise to an embedding

$$\pi^* : \mathrm{rep}_{\mathbb{C}}(G/H) \hookrightarrow \mathrm{rep}_{\mathbb{C}}G$$

respecting tensor products. So we will identify $\mathrm{rep}_{\mathbb{C}}(G/N)$ with $\pi^*\mathrm{rep}_{\mathbb{C}}(G/N)$, i.e. the subcategory of representations on which N acts trivially. Clearly $\mathcal{C}$ is a subcategory of $\mathrm{rep}_{\mathbb{C}}(G/N)$ in this identification and we need to show that every representation of $\mathrm{rep}_{\mathbb{C}}(G/N)$ occurs in $\mathcal{C}$. It is known from the usual representation theory of finite groups that any irreducible representation of G occurs as an irreducible summand in a suitable tensor power $\chi^{\otimes n}, n \in \mathbb{N}$ of a faithful representation χ, see for example Theorem 19.10 in [JL01] (note that Theorem 19.10 makes use of the zeroth tensor power $\chi^0 = \mathrm{triv}$, however the need for this can be eliminated by noticing that the dual $\chi^\vee$ occurs as a summand of $\chi^{\otimes n}$ for $n > 0$ and triv is a summand of $\chi \otimes \chi^\vee$).

Now note that $\chi = \bigoplus_{\lambda \in \Lambda} \lambda$ is a faithful representation of G/N with N as defined in the statement of the lemma. Now taking tensor powers and using that $\mathcal{C}$ is closed under taking irreducible components and tensor products, every representation of G/N must occur in $\mathcal{C}$ and thus we indeed have an equivalence

$$\mathcal{C} \cong (\mathrm{rep}_{\mathbb{C}}(G/N), \otimes)$$

as claimed. □

Theorem 3.2.6. *Let* $\mathbf{k} \in \mathrm{Reg}(\Gamma)$. *Then for some normal subgroup* $N_{\mathbf{k}} \subseteq \Gamma$ *there is an equivalence of categories*

$$(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}}) \cong (\mathrm{rep}_{\mathbb{C}}(\Gamma/N_{\mathbf{k}}), \otimes)\,,$$

Proof. For the proof of this theorem we will actually have to work over $H_{\mathbf{k}}$ and appeal to Morita equivalence to prove the statement for $U_{\mathbf{k}}$. Let us choose $\underline{\Gamma} = \{1\}$. The symplectic leaf $\mathcal{L}(\{1\})$ is dense in $\mathfrak{h} \oplus \mathfrak{h}^*/\Gamma$ so that $\mathcal{HC}_{\mathbf{k}}(H_{\mathbf{k}})_{\overline{\mathcal{L}}} = \mathcal{HC}_{\mathbf{k}}$. Moreover, the fixed-point space of $\{1\}$ is all of $\mathfrak{h} \oplus \mathfrak{h}^*$ and so the only $\{1\}$-stable complement is 0. Therefore the algebra $\underline{H}^+_{\mathbf{k}^+} = \mathbb{C}$ and Theorem 3.2.4 gives us a functor

$$(\bullet)_\dagger : \mathcal{HC}_{\mathbf{k}} \to \mathrm{rep}_{\mathbb{C}}\Gamma.$$

To identify the kernel, we will use Theorem 3.2.4 again: Suppose that M is in $\mathcal{HC}_{\mathbf{k}}$ and $M_\dagger = 0$. Then $\mathrm{LAnn}(M_\dagger) = \mathrm{LAnn}(M)_\dagger$ is non-trivial and so $\mathrm{LAnn}(M)$ must be non-trivial too. But $\mathbf{k}$ is regular and therefore $H_{\mathbf{k}}$ is simple, so that $\mathrm{LAnn}(M) = H_{\mathbf{k}}$ and $M = 0$. Thus $(\bullet)_\dagger$ gives an equivalence between $(\mathcal{HC}_{\mathbf{k}}, \otimes_{H_{\mathbf{k}}})$ and some full subcategory of $(\mathrm{rep}_{\mathbb{C}}\Gamma, \otimes)$ closed under taking subquotients and tensor products and is thus equivalent to $(\mathrm{rep}_{\mathbb{C}}(\Gamma/N_{\mathbf{k}}), \otimes)$ by Lemma 3.2.5. □

3.3 Interlude

For the remainder of this chapter, we will work over $U_{\mathbf{k}}$, hence any references to $\mathcal{HC}_{\mathbf{k}}$ and $\mathcal{O}_{\mathbf{k}}$ should be understood to mean $\mathcal{HC}_{\mathbf{k}}(U_{\mathbf{k}})$ and $\mathcal{O}_{\mathbf{k}}(U_{\mathbf{k}})$ unless stated otherwise.

In this section we will mostly collect together some basic results on localising bimodules with an ad-nilpotent action and recall some results on $\mathcal{D}$-modules. Localising to $\mathfrak{h}^{reg}/\Gamma$ will be an absolutely crucial tool for us for the remainder of this chapter and beyond. It is therefore prudent to make certain that we fully understand what happens to Harish-Chandra bimodules when localising and the section should be viewed in this light. Another crucial result is due to Goodearl and Zhang and is Theorem 3.3.23.

Definition 3.3.1. *Let A be a commutative algebra and M an A-bimodule. For $a \in A$ and $m \in M$ we define the adjoint action of a on m by $\mathrm{ad}(a)(m) = am - ma$ for any $m \in M$. We say that a act locally* ad*-nilpotently if for any $m \in M$ there exists $N_m \in \mathbb{N}$ such that*

$$\mathrm{ad}(a)^{N_m}(m) = 0$$

and we say that A acts locally ad*-nilpotently on M if for any $m \in M$ there exists $N \in \mathbb{N}$ such that*

$$\mathrm{ad}(a_1)\mathrm{ad}(a_2)\ldots\mathrm{ad}(a_N)(m) = 0$$

for any choice of $a_1, \ldots, a_N \in \mathbb{N}$.

Proposition 3.3.2. *Let A be a commutative, finitely generated $\mathbb{C}$-algebra and M an A-bimodule. Then the following hold:*

1. *If $a \in A$ acts locally nilpotently on $m \in M$, $\mathrm{ad}(a)^N(m) = 0$ say, then*

   $$\mathrm{ad}(a^{i_1})\ldots\mathrm{ad}(a^{i_N})(m) = 0$$

   for any choice of $i_1, \ldots, i_N \in \mathbb{N}$.
2. *If $a, b \in A$ act locally* ad*-nilpotently on $m \in M$, then ab acts locally* ad*-nilpotently on $m \in M$*
3. *If each generator of A acts locally* ad*-nilpotently on any $m \in M$, then A acts locally* ad*-nilpotently on M.*
4. *Assume that A is a subalgebra of a not necessarily commutative algebra R such that M is finitely generated as a left R-module. Assume further that A acts locally* ad*-nilpotently on M and R. If $S \subseteq A \setminus 0$ is a finitely generated multiplicatively closed subset (i.e. finitely generated as a multiplicative semigroup) and an Ore subset of R, then $S^{-1}A \subseteq S^{-1}R$ acts locally* ad*-nilpotently on $S^{-1}M$.*

Proof. 1. We shall induct on N. If $N = 1$ so that $am = ma$, we are done since then $a^i m = ma^i$ for any $i \in \mathbb{N}$. So suppose the statement is true for all $N' < N \in \mathbb{N}$ and let us suppose that $\mathrm{ad}(a)^N(m) = 0$. Choose any $i_1, \ldots, i_N \in \mathbb{N}$ and consider $\mathrm{ad}(a^{i_1})\mathrm{ad}(a^{i_2})\ldots\mathrm{ad}(a^{i_N})(m)$. Without loss of generality, we may assume that $i_1 > 1$ (else the implication is trivial since we can then write

$$\mathrm{ad}(a)\mathrm{ad}(a^{i_2})\ldots\mathrm{ad}(a^{i_N})(m) = \mathrm{ad}(a^{i_2})\ldots\mathrm{ad}(a^{i_N})\,(\mathrm{ad}(a)(m))$$

and this will be zero by the induction hypothesis as $\mathrm{ad}(a)^{N-1}(\mathrm{ad}(a)(m)) = 0)$ and note that

$$\begin{aligned}\mathrm{ad}(a^i)(v) &= a^i v - a^{i-1} va + a^{i-1} va - a^{i-2} va^2 \dots \\ &= \sum_{j=1}^{i} a^{i-j}\left(\mathrm{ad}(a)(v)\right) a^{j-1}\end{aligned}$$

so that with $v = \mathrm{ad}(a^{i_2}) \dots \mathrm{ad}(a^{i_N})(m)$ and $m' = \mathrm{ad}(a)(m)$ we can write

$$\begin{aligned}\mathrm{ad}(a^{i_1})\mathrm{ad}(a^{i_2}) \dots \mathrm{ad}(a^{i_N})(m) &= \sum_{j=1}^{i_1} a^{i-j}\left(\mathrm{ad}(a)(v)\right) a^{j-1} \\ &= \sum_{j=1}^{i_1} a^{i-j}\left(\mathrm{ad}(a^{i_2}) \dots \mathrm{ad}(a^{i_N})(m')\right) a^{j-1}\end{aligned}$$

But clearly $\mathrm{ad}(a)^{N-1}(m') = 0$ and so by the induction hypothesis,
$\sum_{j=1}^{i} a^{i-j}\left(\mathrm{ad}(a^{i_2}) \dots \mathrm{ad}(a^{i_N})(m')\right) a^{j-1} = 0$ and thus
$\mathrm{ad}(a^{i_1})\mathrm{ad}(a^{i_2}) \dots \mathrm{ad}(a^{i_N})(m) = 0$.

2. Note that by commutativity of A, the maps $\mathrm{ad}(c)$ are A-linear on the left and the right for any $c \in A$. For any $m \in M$ we have

$$\mathrm{ad}(ab)(m) = a\mathrm{ad}(b)(m) + \mathrm{ad}(a)(m)b.$$

and thus for any $N \in \mathbb{N}$ we have

$$\mathrm{ad}(ab)^N(m) = \sum_{j=0}^{N} \binom{N}{j} a^{N-j} \mathrm{ad}(a)^j \mathrm{ad}(b)^{N-j}(m) b^j.$$

Choose $N_a, N_b \in \mathbb{N}$ such that $\mathrm{ad}(a)^{N_a}(m) = 0$ and $\mathrm{ad}(b)^{N_b}(m) = 0$. For $N = N_a + N_b$ the above formula then gives

$$\begin{aligned}\mathrm{ad}(ab)^N(m) = &\sum_{j=0}^{N_a+N_b} \binom{N_a+N_b}{j} a^{N_a+N_b-j} \mathrm{ad}(a)^j \mathrm{ad}(b)^{N_a+N_b-j}(m) b^j \\ = &\sum_{j=0}^{N_a} \binom{N_a+N_b}{j} a^{N_a+N_b-j} \mathrm{ad}(a)^j \mathrm{ad}(b)^{N_a+N_b-j}(m) b^j \\ + &\sum_{j=N_a+1}^{N_a+N_b} \binom{N_a+N_b}{j} a^{N_a+N_b-j} \mathrm{ad}(a)^j \mathrm{ad}(b)^{N_a+N_b-j}(m) b^j \\ = &\ 0.\end{aligned}$$

3. For this we will use the already proved 1) and 2): Suppose $x_1, \dots, x_n$ generate A and act locally ad-nilpotently on all of M. For each $m \in M$ we choose $N_j(m) \in \mathbb{N}$ such that $\mathrm{ad}(x_j)^{N_j(m)}(m) = 0$ and set $N(m) = N_1(m) + \dots + N_n(m)$. We now claim that for any $a_1, \dots, a_N \in A$ we have
$$\mathrm{ad}(a_1) \dots \mathrm{ad}(a_N)(m) = 0.$$

It will suffice to prove this for arbitrary monomials in the x_j, so let us suppose

$$a_i = x_1^{d_{i,1}} x_2^{d_{i,2}} \cdot \ldots \cdot x_n^{d_{i,n}}$$

again using the Leibnitz rule for $\mathrm{ad}(bc)$, namely

$$\mathrm{ad}(bc)(m) = b\mathrm{ad}(c)(m) + \mathrm{ad}(b)(m)c$$

we can deduce that

$$\begin{aligned}\mathrm{ad}(a_i)(m) &= \mathrm{ad}(x_1^{d_{i,1}} x_2^{d_{i,2}} \cdot \ldots \cdot x_n^{d_{i,n}})(m) \\ &= \sum_{j=1}^{n} x_1^{d_{i,1}} \ldots x_{j-1}^{d_{i,j-1}} \mathrm{ad}(x_j^{d_{i,j}})(m) x_{j+1}^{d_{i,j+1}} \ldots x_n^{d_{i,n}}\end{aligned}$$

and so we can write

$$\mathrm{ad}(a_1)\ldots\mathrm{ad}(a_n)(m) = \sum_s p_s(x_1,\ldots,x_n)\left(\prod_{k=1}^{n}\mathrm{ad}(x_{j_k}^{d_k})\right)(m) q_s(x_1,\ldots,x_n).$$

We need to analyse the behaviour of

$$\left(\prod_{k=1}^{n}\mathrm{ad}(x_{j_k}^{d_k})\right)(m) = \mathrm{ad}(x_{j_1}^{d_1})\mathrm{ad}(x_{j_2}^{d_2})\ldots\mathrm{ad}(x_{j_n}^{d_n})(m).$$

We now claim that one x_j of the generators $x_1,\ldots,x_n$ must appear at least N_j-times. For if not, the total number of terms in the above expression would be strictly less than $N_1 + \ldots + N_n = N$ which is absurd. But then since $\mathrm{ad}(x_j)^{N_j}(m) = 0$ the product $\left(\prod_{k=1}^{n}\mathrm{ad}(x_{j_k}^{d_k})\right)(m) = 0$ and so

$$\mathrm{ad}(a_1)\ldots\mathrm{ad}(a_n)(m) = 0.$$

4. Note that $S^{-1}A$ is again a finitely generated algebra: Let $x_1,\ldots,x_n$ be generators of A and $s_1,\ldots,s_m$ generators of S as a multiplicative semigroup, then $S^{-1}A$ is generated by $x_1,\ldots,x_n$ and $s_1^{-1},\ldots,s_m^{-1}$. Similarly, $S^{-1}M$ is finitely generated as an $S^{-1}R$-module with generating set $m_1,\ldots,m_k$ say. Without loss of generality, we may assume that in fact $\{m_1,\ldots,m_k\} \subseteq M$ and that they generate M as an R-module (since $S^{-1}M \cong S^{-1}R \otimes_R M$ any finite generating set of M will fulfil these conditions). Now by part 3) it is sufficient to show that each generator of $S^{-1}A$ acts locally ad-nilpotently on any $m \in S^{-1}M$. Writing

$$m = \sum_{i=1}^{k} t_i^{-1} r_i m_i, \quad \text{with } t_i \in S, r_i \in R$$

we may without loss of generality assume that $m = t_1^{-1} r_1 m_1$ and we abbreviate $t^{-1} = t_1^{-1}$ and $r_1 = r$ and finally we set $v = m_1$. So we need to show that each generator of $S^{-1}A$ acts ad-nilpotently on $t^{-1}rv$.

We will show this separately for $x \in \{x_1,\ldots,x_n\}$ and $s^{-1} \in \{s_1^{-1},\ldots,s_m^{-1}\}$. We know that x acts locally ad-nilpotently on v as this is true before localisation and similarly it acts ad-nilpotently on r. Since $\mathrm{ad}(x)(rv) = r\mathrm{ad}(x)(v) + \mathrm{ad}(x)(r)v$ we can conclude that x acts ad-nilpotently on rv and we will only need to show that it acts ad-nilpotently on t^{-1} to conclude that it acts ad-nilpotently on $t^{-1}rv$. But x commutes with t and thus will also commute with t^{-1}. Hence x acts ad-nilpotently on $t^{-1}rv$.

To show that s^{-1} acts ad-nilpotently on $t^{-1}rv$ will follow once we know that s^{-1} acts ad-nilpotently on rv. As s acts ad-nilpotently on both r and v we can choose $N >> 0$ such that $\mathrm{ad}(s)^N(rv) = 0$ and thus

$$\sum_{j=0}^{N}(-1)^{N-j}\binom{N}{j}s^{N-j}rvs^j = 0$$

Multipliying this by s^{-N} on the left and the right gives

$$\begin{aligned}
0 &= \sum_{j=0}^{N}(-1)^{N-j}\binom{N}{j}s^{-j}rvs^{j-N} \\
&= (-1)^N\sum_{j=0}^{N}(-1)^{N-j}\binom{N}{N-j}(s^{-1})^{j-N}rv(s^{-1})^{-j} \\
&= (-1)^N\mathrm{ad}(s^{-1})^N(rv)
\end{aligned}$$

So each generator of $S^{-1}A$ acts locally ad-nilpotently on each element of $R^{-1}M$ and hence $S^{-1}A$ acts ad-nilpotently on $S^{-1}M$.

$\square$

Definition 3.3.3. *Let A be a commutative $\mathbb{C}$-algebra and M, N A-modules. For any $k \in \mathbb{N}_0$ we define recursively*

$$\mathcal{D}iff^k(M,N) := \begin{cases} \mathrm{Hom}_A(M,N) & \text{if } k = 0 \\ \{f \in \mathrm{Hom}_{\mathbb{C}}(M,N) \mid \forall a \in A : af - fa \in \mathcal{D}iff^{k-1}(M,N)\} & \text{if } k > 1 \end{cases}$$

Finally we set

$$\mathcal{D}iff_A(M,N) = \bigcup_k \mathcal{D}iff_A^k(M,N).$$

Example 3.3.4. *By definition, $\mathcal{D}iff_A(A,A)$ are just then differential operators on A. We will abbreviate this to $D_A = \mathcal{D}iff_A(A,A)$.*

Lemma 3.3.5. *Let A be a commutative $\mathbb{C}$-algebra and M, N A-modules.*

1. *$\mathcal{D}iff_A(M,M)$ is a ring under composition and contains $\mathrm{End}_A(M)$ as a subring.*
2. *$\mathcal{D}iff_A(M,N)$ is a $\mathcal{D}iff_A(N,N)$-$\mathcal{D}iff_A(M,M)$-bimodule.*
3. *$\mathcal{D}iff_A(A,N)$ is an A-D_A-bimodule.*
4. *If N is a left D_A-module, then $\mathcal{D}iff_A(A,N)$ is a $D_A - D_A$-bimodule. The D_A-action on N extends the A-action on N.*

Proof. 1) Let $f, g \in \mathcal{D}iff_A(M,M)$ and choose $N \in \mathbb{N}_0$ such that for any $a_1, \ldots, a_N \in A$ we have

$$\mathrm{ad}(a_N)\ldots\mathrm{ad}(a_1)(f) = \mathrm{ad}(a_N)\ldots\mathrm{ad}(a_1)(g) = 0.$$

We need to show that there exists $N_1 \in \mathbb{N}_0$ such that

$$\mathrm{ad}(a_{N_1})\dots\mathrm{ad}(a_1)(f\circ g)=0$$

for any $a_{N_1},\dots,a_1 \in A$. We set $N_1 = 2N$ and claim that this will suffice. Let us choose $a_{2N}\dots,a_1,a\in A$ and let us first compute for any $m\in M$

$$\begin{aligned}\mathrm{ad}(a)(f\circ g)(m) &= a(f\circ g)(m)-(f\circ g)(am)\\ &= af(g(m))-f(ag(m))+f(ag(m))-f(g(am))\\ &= \left((\mathrm{ad}(a)(f))\circ g\right)(m)+\left(f\circ \mathrm{ad}(a)(g)\right)(m)\end{aligned}$$

which we can summarise as

$$\mathrm{ad}(a)(f\circ g)=(\mathrm{ad}(a)(f))\circ g+f\circ(\mathrm{ad}(a)(g)).$$

We can use this to find

$$\mathrm{ad}(a_{2N})\dots\mathrm{ad}(a_1)(f\circ g)=\sum_{r=0}^{2N}\left(\prod_{\substack{i\in\mathcal{P}\\ \mathcal{P}\subseteq\{1,\dots,2N\},\#\mathcal{P}=r}}\mathrm{ad}(a_i)\right)(f)\circ\left(\prod_{i\in\{1,\dots,2N\}\setminus\mathcal{P}}\mathrm{ad}(a_i)\right)(g).$$

Now if $\#\mathcal{P}=r\geq N$, then $\left(\prod_{i\in\mathcal{P}}\mathrm{ad}(a_i)\right)(f)=0$ and if $\#\mathcal{P}=r<N$ then $\left(\prod_{i\in\{1,\dots,2N\}\setminus\mathcal{P}}\mathrm{ad}(a_i)\right)(g)=0$, so that

$$\mathrm{ad}(a_{2N})\dots\mathrm{ad}(a_1)(f\circ g)=0.$$

Therefore indeed $f\circ g\in \mathcal{D}iff_A(M,M)$ and this is closed under composition. The other ring axioms follow as $\mathrm{Id}_M\in\mathcal{D}iff_A(M,M)$ and $\mathrm{Hom}(M,M)$ is a ring under $\circ$.
It follows from the definition that $\mathrm{End}_A(M,M)\subseteq\mathcal{D}iff_A(M,M)$.
2) Using the same calculation as in 1), we can see that $\mathcal{D}iff_A(M,N)$ is preserved under left composition with elements of $\mathcal{D}iff_A(N,N)$ and under right composition with elements of $\mathcal{D}iff_A(M,M)$. This then gives a bimodule structure on $\mathcal{D}iff_A(M,N)$.
3) By 2), $\mathcal{D}iff_A(A,N)$ is a $\mathcal{D}iff_A(N,N)-\mathcal{D}iff_A(A,A)$-bimodule. It is immediate that $D_A=\mathcal{D}iff_A(A,A)$. For any $a\in A$ we denote by $\hat{a}$ the element of $\mathrm{End}_A(N)$ induced by scalar multiplication by a (here we are using commutativity of A). The map $\hat{\bullet}: A\to\mathrm{End}_A(N,N)\subseteq\mathcal{D}iff_A(N,N)$ given by $a\mapsto\hat{a}$ is a ring homomorphism and this gives $\mathcal{D}iff_A(A,N)$ a left A-structure. This is just the restriction of the left $\mathcal{D}iff_A(N,N)$-structure to a subring and thus preserves the bimodule structure on $\mathcal{D}iff_A(A,N)$.
4) As N is a left D_A-module, scalar multiplication with any element of D_A gives rise to an element of $\mathcal{D}iff_A(N,N)$. Then arguing similarly as in 3) we can then see that left and right composition with elements of D_A gives a D_A-D_A-bimodule structure on $\mathcal{D}iff_A(A,N)$. That this extends the $A-D_A$-structure follows from the definitions. □

Definition 3.3.6. *Let K is a commutative ring with unit and A a K-algebra. A K-derivation $\partial: A\to A$ is a K-linear map $A\to A$ such that for any $a,b\in A$ we have*

$$\partial(ab)=a\partial(b)+\partial(a)b.$$

The set of all K-derivations of A is denoted $\mathrm{Der}_K(A)$.

Example 3.3.7. *Differentiation is an $\mathbb{R}$-derivation $\mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R})$ on the algebra of infinitely often differentiable functions $\mathbb{R} \to \mathbb{R}$.*

Definition 3.3.8. *A commutative ring A is called a regular local ring if it is a Noetherian local ring and the minimal number of generators of its unique maximal ideal equals its Krull dimension.*
A commutative ring A is called regular if $A_{\mathfrak{m}}$ is a regular local ring for each maximal ideal $\mathfrak{m}$ of A.

Definition 3.3.9. *An algebra A over a field k is called affine if for some N there is a surjection $k[x_1, \dots, x_N] \twoheadrightarrow A$, i.e. if A is finitely generated commutative over k*

Although the term "affine algebra" is often used to refer to noncommutative algebras as well, we will have no need for this.
The next result gathers together some well-known properties of differential operators which can be found in many textbooks

Theorem 3.3.10. *Let A be a regular, affine $\mathbb{C}$-algebra which is a domain.*
1. *D_A is generated as a ring by A, whose elements act via scalar multiplication, and the derivations $\mathrm{Der}_{\mathbb{C}}(A)$.*
2. *For any multiplicatively closed subset $S \subseteq A$ we have $S^{-1}D_A = D_{S^{-1}A}$ and this preserves the filtration by degree of differential operators.*
3. *To define a left D_A-structure on a left A-module M it suffices to specify a Lie-algebra action of $\mathrm{Der}_{\mathbb{C}}(A)$ on M such that for $\partial \in \mathrm{Der}_{\mathbb{C}}(A), a \in A$ and $m \in M$ we have $\partial(am) = \partial(a)m + a(\partial m)$.*
4. *D_A is a simple Noetherian domain.*

Proof.
1. Using Corollary 15.5.6 (i.e. Corollary 5.6 in Chapter 15) in [MR01] we can identify D_A as the subring of $\mathrm{End}_{\mathbb{C}}(A)$ generated by A and $\mathrm{Der}_{\mathbb{C}}(A)$.
2. The second statement follows from from Lemma 15.5.4 in the same book.
3. Follows from Lemma 1.2.1 in [HTT08].
4. That D_A is a domain follows from Proposition 15.3.6 in [MR01], that D_A is simple Noetherian follows from Corollary 15.3.11 by localising at $1 \in A$.

□

It will be useful to recall from [MR01] how derivations extend to localisations: Let $\partial : A \to A$ be a derivation and let $s^{-1}a \in S^{-1}A$ with $s \in S$ and $a \in A$. We want to extend ∂ in such a way that it remains a derivation and therefore we must have (formally):

$$\partial(s^{-1}a) = \partial(s^{-1})a + s^{-1}\partial(a).$$

Thus we will need to define what we mean by $\partial(s^{-1})$ for $s \in S$. In the localisation $S^{-1}A$ we have $ss^{-1} = 1$ and thus

$$0 = \partial(1) = \partial(s)s^{-1} + s\partial(s^{-1})$$

(recall that a K-derivation vanishes on any element of K). Thus we must conclude that

$$\partial(s^{-1}) = -s^{-2}\partial(s)$$

and indeed this is well defined (see e.g. Proposition 15.1.24 and Lemma 15.5.4 in [MR01]). The next result is crucial and although it is well-known, see for example exercise 2.1.16 in [Gin], we supply a proof for the reader's convenience

Proposition 3.3.11. *Let A be a regular affine $\mathbb{C}$-algebra which is a domain and N a left D_A module which is finitely-generated and projective as an A-module. We can view N as an A-A-bimodule as A is commutative. There is an isomorphism of $D_A - D_A$-bimodules*

$$\phi : N \otimes_A D_A \xrightarrow{\sim} \mathcal{D}iff_A(A, N)$$

given by

$$\phi(n \otimes P) \mapsto (a \mapsto P(a)n).$$

The bimodule structure on $N \otimes_A D_A$ is given as follows:

$$\begin{aligned}(n \otimes P)Q &:= n \otimes (PQ)\\ a(n \otimes P) &:= (an) \otimes P\\ \partial(n \otimes P) &:= (\partial n) \otimes P + n \otimes \partial P\end{aligned}$$

for $Q \in D_A$, $a \in A$ and $\partial \in \mathrm{Der}_{\mathbb{C}}(A)$ a derivation.

Proof. Note that by Theorem 3.3.10 we indeed have a well-defined left module structure on $N \otimes_A D_A$. To check that this gives a bimodule structure we calculate with notation as before

$$\begin{aligned}(a(n \otimes P))Q &= ((an) \otimes P)Q = (an) \otimes (PQ)\\ &= a(n \otimes (PQ)) = a((n \otimes P)Q)\end{aligned}$$

and

$$\begin{aligned}\partial(n \otimes P))Q &= ((\partial n) \otimes P + n \otimes \partial P)Q\\ &= ((\partial n) \otimes (PQ) + n \otimes (\partial PQ)\\ &= \partial(n \otimes (PQ))\\ &= \partial((n \otimes P)Q).\end{aligned}$$

It remains to show the isomorphism.

We first need to check that ϕ is well-defined and that $\phi(n \otimes P)$ indeed lies in $\mathcal{D}iff_A(A, N)$. To prove well-definedness, choose $n \in N$, $x \in A$ and $P \in D_A$. Then we need to check that

$$\phi((nx) \otimes P) = \phi(n \otimes (xP)).$$

Then for any $a \in A$ we compute

$$\begin{aligned}\phi((nx) \otimes P)(a) &= P(a)xn\\ &= xP(a)n\\ &= \phi(n \otimes xP)(a)\\ &\implies \phi((nx) \otimes P) = \phi(n \otimes (xP))\end{aligned}$$

and we can conclude that ϕ is well-defined. Next, we will show that $\phi(n \otimes P) \in \mathcal{D}iff_A(A, N)$. We find

$$\begin{aligned} \mathrm{ad}(x)\left(\phi(n \otimes P)\right)(a) &= x\left(P(a)n\right) - P(xa)n \\ &= \left(xP(a) - P(xa)\right)n \\ &= \phi\left(n \otimes (\mathrm{ad}(x)(P))\right) \end{aligned}$$

Thus choosing $N \in \mathbb{N}_0$ such that for any $a_1, \dots, a_N \in A$ we have $\mathrm{ad}(a_N) \dots \mathrm{ad}(a_1)(P) = 0$ we have

$$\mathrm{ad}(a_N) \dots \mathrm{ad}(a_1)(\phi(n \otimes P)) = 0$$

and thus $\phi(n \otimes P) \in \mathcal{D}iff_A(A, N)$.

To show that ϕ is an isomorphism, we can localise to maximal ideals and show that ϕ induces an isomorphism of A-modules, see for example Proposition 3.9 in [AM94]. So let $\mathfrak{m} \triangleleft A$ be a maximal ideal of A and consider the induced map

$$\phi_{\mathfrak{m}} : A_{\mathfrak{m}} \otimes_A (N \otimes_A D_A) \to A_{\mathfrak{m}} \otimes_A (\mathcal{D}iff_A(A, N)).$$

To simplify we note that:

$$\begin{aligned} A_{\mathfrak{m}} \otimes_A (N \otimes_A D_A) &= (A_{\mathfrak{m}} \otimes_A N) \otimes_A D_A \\ &= (A_{\mathfrak{m}} \otimes_A N \otimes_A A_{\mathfrak{m}}) \otimes_A D_A \\ &= (A_{\mathfrak{m}} \otimes_A N \otimes_A A_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} A_{\mathfrak{m}}) \otimes_A D_A \\ &= (A_{\mathfrak{m}} \otimes_A N \otimes_A A_{\mathfrak{m}}) \otimes_{A_{\mathfrak{m}}} (A_{\mathfrak{m}} \otimes_A D_A) \\ &= (A_{\mathfrak{m}} \otimes_A N \otimes_A A_{\mathfrak{m}}) \otimes_{A_{\mathfrak{m}}} (A_{\mathfrak{m}} \otimes_A D_A) \\ &= (A_{\mathfrak{m}} \otimes_A N \otimes_A A_{\mathfrak{m}}) \otimes_{A_{\mathfrak{m}}} (A_{\mathfrak{m}} \otimes_A D_A \otimes_A A_{\mathfrak{m}}) \end{aligned}$$

where we have used the fact that A acts locally ad-nilpotently on each element of D_A and N to conclude that $A_{\mathfrak{m}} \otimes_A D_A = A_{\mathfrak{m}} \otimes_A D_A \otimes_A A_{\mathfrak{m}}$. Note further that by Theorem 3.3.10 $A_{\mathfrak{m}} \otimes_A D_A = D_{A_{\mathfrak{m}}}$. Thus $\phi_{\mathfrak{m}}$ is a map

$$\phi_{\mathfrak{m}} : N_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} D_{A_{\mathfrak{m}}} \to A_{\mathfrak{m}} \otimes_A (\mathcal{D}iff_A(A, N)).(\mathcal{D}iff_A(A, N))$$

Note that the latter module embeds into $\mathcal{D}iff_{A_{\mathfrak{m}}}(A_{\mathfrak{m}}, N_{\mathfrak{m}})$, to show this we can argue as in [MR01]. and so let us consider

$$\hat{\phi_{\mathfrak{m}}} : N_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} D_{A_{\mathfrak{m}}} \to \mathcal{D}iff_{A_{\mathfrak{m}}}(A_{\mathfrak{m}}, N_{\mathfrak{m}}).$$

By hypothesis, N is a finitely generated projective A-module and and hence $N_{\mathfrak{m}}$ is free as a left $A_{\mathfrak{m}}$-module as $A_{\mathfrak{m}}$ is local, this follows from Nakayama's Lemma. Choose an $A_{\mathfrak{m}}$-basis $e_1, \dots, e_k$ of $N_{\mathfrak{m}}$ and consider $f \in \mathcal{D}iff_{A_{\mathfrak{m}}}(A_{\mathfrak{m}}, N_{\mathfrak{m}})$. For any $a \in A_{\mathfrak{m}}$ we can choose unique $P_j(a) \in A_{\mathfrak{m}}$ such that

$$f(a) = \sum_j P_j(a)e_j.$$

This defines maps $P_j : A_{\mathfrak{m}} \to A_{\mathfrak{m}}$ and as $f \in \mathcal{D}iff_{A_{\mathfrak{m}}}(A_{\mathfrak{m}}, N_{\mathfrak{m}})$ there exists $N \in \mathbb{N}_0$ such that for any $a_1, \dots, a_N$ $\mathrm{ad}(a_N)\dots\mathrm{ad}(a_1)(f) = 0$. As computed in the proof of we have

$$\mathrm{ad}(a_N)\dots\mathrm{ad}(a_1)(f)(a) = \sum_j \mathrm{ad}(a_N)\dots\mathrm{ad}(a_1)(P_j)(a)e_j$$

and since $e_1, \dots, e_k$ is a basis of $N_{\mathfrak{m}}$ we conclude that for any j;

$$\mathrm{ad}(a_N)\dots\mathrm{ad}(a_1)(P_j) = 0.$$

Thus each P_j is a differential operator on $A_{\mathfrak{m}}$ and so

$$f(a) = \sum_j P_j(a)e_j = \sum_j \hat{\phi_{\mathfrak{m}}}(e_j \otimes P_j).$$

So $\hat{\phi_{\mathfrak{m}}}$ is surjective and thus so is $\phi_{\mathfrak{m}}$.
To show injectivity, we will again consider injectivity of $\phi_{\mathfrak{m}}$. Suppose that $\sum_i n_i \otimes P_i \in \ker(\phi_{\mathfrak{m}})$. We can find $a_{ij} \in A_{\mathfrak{m}}$ such that $n_i = \sum_j a_{ij}e_j$ for each i. Then we can compute

$$\phi_{\mathfrak{m}}(\sum_i n_i \otimes P_i)(a) = \sum_{i,j} P_i(a)a_{ij}e_j.$$

As the e_j are an $A_{\mathfrak{m}}$ basis of $N_{\mathfrak{m}}$, we can conclude that

$$\sum_i a_{ij}P_i(a) = 0$$

for fixed j and any $a \in A_{\mathfrak{m}}$. Thus we deduce that

$$\sum_i a_{ij}P_i = 0$$

in $D_{A_{\mathfrak{m}}}$. This shows that

$$\begin{aligned}\sum_i n_i \otimes P_i &= \sum_{i,j} a_{ij}e_j \otimes P_i \\ &= \sum_j e_j \otimes \sum_i (a_{ij}P_i) \\ &= 0\end{aligned}$$

and so $\ker(\phi_{\mathfrak{m}}) = 0$ and ϕ is injective too.
Finally, it remains to show that ϕ really gives a map of bimodules. For this, let $P, Q \in D_A$, $n \in N$, $a, x \in A$ and $\partial \in \mathrm{Der}_{\mathbb{C}}(A)$. We then find that

$$\begin{aligned}\phi((n \otimes P)Q)(a) &= \phi(n \otimes (PQ))(a) \\ &= (PQ)(a)n \\ &= P(Q(a))n\end{aligned}$$

$$\begin{aligned}(\phi(n \otimes P)Q)(a) &= \phi(n \otimes P)(Q(a)) \\ &= P(Q(a))n\end{aligned}$$

and so

$$\phi((n \otimes P)Q) = \phi(n \otimes P)Q.$$

Further

$$\begin{aligned}\phi(xn \otimes P)(a) &= P(a)xn \\ &= xP(a)n \\ &= x(\phi(n \otimes P)(a))\end{aligned}$$

and

$$\begin{aligned}\phi(\partial(n \otimes P))(a) &= \phi((\partial n) \otimes P + n \otimes (\partial P))(a) \\ &= P(a)(\partial n) + (\partial P)(a) \\ &= P(a)(\partial n) + \partial(P(a))n \\ &= P(a)(\partial n) + [\partial, P(a)]n \\ &= \partial(P(a)n) \\ &= \partial(\phi(n \otimes P)(a))\end{aligned}$$

Hence ϕ is indeed a bimodule isomorphism. □

We will later need to consider a sheaf-theoretic verison of this construction, see for example Chapter 2 of [Gin] or Chapter 1, Section 2 of [Kas03]. Hence we need to mention how these spaces behave under localisation and we have implicitly already done this in the proof of Proposition 3.3.11

Corollary 3.3.12. *Let A be a regular, affine $\mathbb{C}$-algebra, $S \subseteq A$ a multiplicatively closed set and N a projective A-module. Then*

$$S^{-1}\left(\mathcal{D}iff_A(A, N)\right) \cong \mathcal{D}iff_{S^{-1}A}(S^{-1}A, S^{-1}N)$$

as $S^{-1}A$-$D_{S^{-1}A}$-modules.

Proof. We have

$$\begin{aligned}S^{-1}\left(\mathcal{D}iff_A(A, N)\right) &\cong S^{-1}\left(N \otimes_A D_A\right) \\ &\cong (S^{-1}A) \otimes_A N \otimes_A D_A \\ &\cong (S^{-1}A) \otimes_A N \otimes_{S^{-1}A} S^{-1}A \otimes_A D_A \\ &\cong \left(S^{-1}A \otimes AN\right) \otimes_{S^{-1}A} (S^{-1}A \otimes D_A) \\ &\cong \left(S^{-1}A \otimes AN\right) \otimes_{S^{-1}A} D_{S^{-1}A}\end{aligned}$$

As by Chapter 15, Proposition 1.24 in [MR01] we have

$$S^{-1}D_A \cong D_{S^{-1}A}$$

for such A using Chapter 15, Corollary 5.6. □

The definition of differential morphisms can be extended to the setting of $\mathcal{D}$-modules and is indeed more natural there:

Definition 3.3.13. *Let Y be a smooth variety and $\mathcal{M},\mathcal{N}$ two $\mathcal{O}_Y$-modules. For any $p \in \mathbb{N}_0$ we then set*

$$\mathcal{D}iff^p_{\mathcal{O}_Y}(\mathcal{M},\mathcal{N}) = \{f \in \mathcal{H}om_{\mathbb{C}}(\mathcal{M},\mathcal{N}) \mid \mathrm{ad}(a_0)\ldots\mathrm{ad}(a_p)(f) = 0 \forall a_0,\ldots,a_p \in \mathcal{O}_Y\}$$

and

$$\mathcal{D}iff_{\mathcal{O}_Y}(\mathcal{M},\mathcal{N}) = \bigcup_{p\geq 0} \mathcal{D}iff^p(\mathcal{M},\mathcal{N}).$$

Similarly we set

$$\mathcal{D}_Y = \mathcal{D}iff_{\mathcal{O}_Y}(\mathcal{O}_Y,\mathcal{O}_Y)$$

It follows from standard results in the theory of differential operators on varieties that this indeed a sheaf on Y, we appeal to Proposition 3.3.2 and Corollary 3.3.12.

Definition 3.3.14. *Let Y be a smooth variety, we denote by Θ_Y the sheaf of vector fields on Y, i.e. Θ_Y is the tangent sheaf on Y.*

Essentially for a smooth variety $Y = \mathrm{Spec}(A)$, the sheaf Θ_Y is the sheafification of the presheaf of derivations given by $U \mapsto \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_Y(U))$ for open affine sets $U \subseteq Y$.
It is essential for us that if $Y = \mathrm{Spec}(A)$ is a smooth affine variety (over $\mathbb{C}$), the categories of $\mathcal{D}_Y$-modules and of modules over D_A are equivalent. We shall spell the result out precisely and quote it from [HTT08]:

Theorem 3.3.15. *(Proposition 1.4.3 and Proposition 1.4.4 from [HTT08]) Let $Y = \mathrm{Spec}(A)$ be a smooth affine variety. Denote by $\mathcal{D}_Y - \mathrm{Mod}_{qc}$ the category of $\mathcal{D}_Y$-modules, i.e. those modules over $\mathcal{D}_Y$ which are quasi-coherent as $\mathcal{O}_Y$-modules. We denote by $\mathcal{D}_Y - \mathrm{mod}$ the category of those objects in $\mathcal{D}_Y - \mathrm{Mod}_{qc}$ that are coherent as $\mathcal{D}_Y$-modules. Then the global sections functor $\Gamma(Y,\bullet)$ induces an equivalence*

$$\Gamma(Y,\bullet) : \mathcal{D}_Y - \mathrm{Mod}_{qc} \cong D_A - \mathrm{Mod}$$

and this restricts to an equivalence

$$\Gamma(Y,\bullet) : \mathcal{D}_Y - \mathrm{mod} \cong D_A - \mathrm{mod}.$$

Definition 3.3.16. *Let Y be a smooth variety and let $\mathcal{M},\mathcal{N}$ be $\mathcal{D}_Y$-modules. We define their tensor product $\mathcal{M} \otimes^D \mathcal{N}$ as follows. As sheaves of vector spaces*

$$\mathcal{M} \otimes^D \mathcal{N} = \mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N}$$

with the action of $\mathcal{D}_Y$ determined as follows:

$$a(m \otimes n) = am \otimes n \text{ for } a \in \mathcal{O}_Y$$
$$\partial(m \otimes n) = (\partial m) \otimes n + m \otimes (\partial n) \text{ for } \partial \in \Theta_Y.$$

All our results on the bimodules $\mathcal{D}iff_A(M,N)$ have obvious $\mathcal{D}$-module theoretic counterparts, which will in fact follow from the statements we have shown in the case of affine varieties.
We note one more thing which we shall often use in subsequent calculations:

Lemma 3.3.17. *Let $\mathcal{M}$,$\mathcal{N}$ be $\mathcal{D}_Y$-modules with Y a smooth variety. Then* $\left(\mathcal{M} \otimes^D \mathcal{D}_Y\right) \otimes_{\mathcal{D}_Y} \mathcal{N} \cong \mathcal{M} \otimes^D \mathcal{N}$

Proof. We have a map

$$\begin{aligned} \phi : \left(\mathcal{M} \otimes^D \mathcal{D}_Y\right) \otimes_{\mathcal{D}_Y} \mathcal{N} &\to \mathcal{M} \otimes^D \mathcal{N} \\ m \otimes^D P \otimes n &\mapsto m \otimes^D P(n) \end{aligned}$$

and claim that this is an isomorphism. We shall first show that this is a morphism of $\mathcal{D}_Y$-modules. By Lemma 1.2.1 in [HTT08] it suffices to check this for $a \in \mathcal{O}_Y$ and $\partial \in \Theta_Y$. Then

$$\begin{aligned} \phi(a(m \otimes^D P \otimes n)) &= \phi((am) \otimes^D P \otimes n) \\ &= (am) \otimes^D P(n) \\ &= a(m \otimes^D P(n)) \\ &= a\phi(m \otimes^D P \otimes n) \end{aligned}$$

and

$$\begin{aligned} \phi(\partial(m \otimes^D P \otimes n)) &= \phi((\partial m) \otimes^D P \otimes n + m \otimes^D (\partial P) \otimes n) \\ &= (\partial m) \otimes^D P(n) + m \otimes^D (\partial P)(n) \\ &= \partial(m \otimes^D P(n)) \\ &= \partial(\phi(m \otimes^D P \otimes n)) \end{aligned}$$

As $\mathcal{O}_Y$-modules we have

$$\begin{aligned} \mathcal{M} \otimes^D \mathcal{D}_Y \otimes_{\mathcal{D}_Y} \mathcal{N} &\cong \mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \otimes_{\mathcal{D}_Y} \mathcal{N} \\ &\cong \mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N} \end{aligned}$$

and so ϕ is an isomorphism as it is an isomorphism of $\mathcal{O}_Y$-modules. □

Next we will prove a technical lemma whose importance should nevertheless not be underestimated. We will mainly follow [HTT08], in particular Section 4 in Chapter 2, although Chapter 4 in [Kas03] is also a good reference.
First, suppose that $f : X \to Y$ is a morphism of smooth algebraic varieties. We can form the fibre product

$$X \times_Y T^*Y$$

which we can think of as the set of points $(x, \vartheta) \in X \times T^*Y$ such that $\pi_Y(\vartheta) = f(x)$ where $\pi_Y : T^*Y \to Y$ is the canonical projection. So $X \times_Y T^*Y$ depends on f despite the notation (which is standard). Further we can associate to $X \times_Y T^*Y$ two canonical morphisms

$$T^*X \xleftarrow{\rho_f} X \times_Y T^*Y \xrightarrow{\varpi_f} T^*Y$$

as follows:

- ϖ_f is the projection map $X \times_Y T^*Y \to T^*Y$.
- ρ_f is "dual to differentiation": The morphism f induces maps $d_x f : T_x X \to T_{f(x)}Y$ giving rise to dual maps $(d_x f)^* : T^*_{f(x)}Y \to T_x X$ and these form ρ_f.

We denote by $T^*_X X$ and $T^*_Y Y$ the zero sections of the cotangent bundles T^*X and T^*Y respectively. Now we set

$$T^*_X Y = \rho_f^{-1}(T^*_X X) \subseteq X \times_Y T^*Y.$$

Definition 3.3.18. *Let $f : X \to Y$ be a morphism of smooth varieties and M a coherent $\mathcal{D}_Y$-module. We say that M is non-characteristic for f if*

$$\varpi_f^{-1}(\mathrm{Ch}(M)) \cap T^*_X Y \subseteq X \times_Y T^*_Y Y.$$

Lemma 3.3.19. *Let X be a smooth variety and let $\mathcal{M}, \mathcal{N}$ be coherent $\mathcal{D}_X$-modules.*

1. *The external tensor product $\mathcal{M} \overset{D}{\boxtimes} \mathcal{N}$ (of $\mathcal{D}_X$-modules) is a coherent $\mathcal{D}_{X\times X}$-module.*
2. *If one of $\mathcal{M}, \mathcal{N}$ is $\mathcal{O}_X$-coherent, the diagonal embedding $\Delta_X : X \to X \times X$ is non-characteristic for $\mathcal{M} \overset{D}{\boxtimes} \mathcal{N}$.*

Proof. 1. That $\mathcal{M} \overset{D}{\boxtimes} \mathcal{N}$ is again coherent as a $\mathcal{D}_{X\times X}$-module follows from $\mathcal{D}_{X\times X} = \mathcal{D}_X \overset{D}{\boxtimes} \mathcal{D}_X$.

2. By Proposition 4.5 in [Kas03] we have

$$\mathrm{Ch}(\mathcal{M} \overset{D}{\boxtimes} \mathcal{N}) = \mathrm{Ch}(\mathcal{M}) \times \mathrm{Ch}(\mathcal{N})$$

and by Proposition 2.2.5 in [HTT08] we have

$$\mathrm{Ch}(\mathcal{L}) = T^*_X X$$

for any $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module $\mathcal{L}$. Without loss of generality, let us suppose that $\mathcal{M}$ is $\mathcal{O}_X$-coherent. Let $Y = X \times X$ and $f = \Delta_X$, the diagonal inclusion $X \overset{\Delta_X}{\hookrightarrow} X \times X$. We have $T^*Y = T^*(X \times X) = T^*X \times T^*X$ and

$$\varpi_f^{-1}(\mathrm{Ch}(\mathcal{M} \overset{D}{\boxtimes} \mathcal{N})) \subseteq X \times_Y (T^*_X X \times T^*X).$$

We have

$$\rho_f^{-1}(T^*_X X) = \{(v, (\lambda, -\lambda)) \in X \times_Y T^*Y\}.$$

Thus it is clear that

$$\varpi_f^{-1}(\mathrm{Ch}(\mathcal{M} \overset{D}{\boxtimes} \mathcal{N})) \cap T^*_X Y = X \times_Y (T^*_X X \times T^*X) \cap (X \times_Y \{(v, (\lambda, -\lambda))\}) \subseteq X \times_Y T^*_Y Y$$

and therefore f is non-characteristic for $\mathcal{M} \overset{D}{\boxtimes} \mathcal{N}$.

□

The reason why the concept of non-characteristic morphisms is important in the theory of $\mathcal{D}$-modules is that these morphisms interact in particularly nice ways with various $\mathcal{D}$-module theoretic functors. In particular we have the following result:

Proposition 3.3.20. *Let $f : X \to Y$ be a morphism of smooth varieties and M a coherent $\mathcal{D}$-module on Y such that f is non-characteristic for M. Then $H^0(Lf^*M)$ is a coherent $\mathcal{D}$-module on X, i.e. f^*M is coherent on X.*

Proof. This is part of Theorem 2.4.6 in [HTT08]. □

Corollary 3.3.21. *Let X be a smooth variety and let $\mathcal{M}, \mathcal{N}$ be coherent $\mathcal{D}_X$-modules. If one of $\mathcal{M}, \mathcal{N}$ is $\mathcal{O}_X$-coherent, $\mathcal{M} \otimes^D \mathcal{N}$ is again $\mathcal{D}_X$-coherent.*

Proof. As before denote by $\Delta_X : X \to X \times X$ the diagonal embedding. Then

$$\mathcal{M} \otimes^D \mathcal{N} = \Delta_X^*(\mathcal{M} \overset{D}{\boxtimes} \mathcal{N})$$

and applying Lemma 3.3.19 and Proposition 3.3.20 gives the result. □

Corollary 3.3.22. *Let X be a smooth variety and let $\mathcal{M}$ be a $\mathcal{D}_X$-coherent, $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module. Then $\mathcal{D}iff_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M})$ is $\mathcal{D}_X$-coherent as a left and as a right $\mathcal{D}_X$-module.*

Proof. As

$$\mathcal{D}iff_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M}) \cong \mathcal{M} \otimes^D \mathcal{D}_X$$

coherence as a right $\mathcal{D}_X$-module is clear, coherence as a left $\mathcal{D}_X$-module follows from Corollary 3.3.21. □

We quote the following result of Goodearl and Zhang from [GZ05]. Although it is stated as a lemma there its importance to us will justify its elevation to a theorem in this context:

Theorem 3.3.23. *([GZ05], Lemma 1.3) Let A and B be Noetherian simple rings and V an $A-B$-bimodule such that ${}_AV$ and V_B are Noetherian. Then ${}_AV$ and V_B are progenerators in $A-\mathrm{Mod}$ and $\mathrm{Mod}-B$ respectively.*

Corollary 3.3.24. *Let A and B be Noetherian simple rings and V an $A-B$-bimodule such that ${}_AV$ and V_B are Noetherian. Let M be a B-module such that $V \otimes_B M = 0$ then M=0.*

Proof. By Theorem 3.3.23 V_B is a progenerator in the category of right B-modules. So for some $n \in \mathbb{N}$ we have a surjection of right modules

$$V_B^{\oplus n} \twoheadrightarrow B_B.$$

As $\bullet \otimes_B M$ is right exact we obtain a surjection of Abelian groups

$$V^{\oplus n} \otimes_B M \cong (V \otimes_B M)^{\oplus n} \twoheadrightarrow B \otimes_B M \cong M.$$

Hence if $V \otimes_B M = 0$ we must conclude that $M = 0$ as claimed. □

Lemma 3.3.25. *Let A, B be Noetherian simple rings and V an $A-B$-bimodule such that ${}_AV$ and V_B are both Noetherian. If there exists a B-module M such that $V \otimes_B M$ is simple as an A-module, then V is simple as a bimodule.*

Proof. Assuming that V were not simple as a bimodule, we can find non-zero $A-B$-bimodules X, Y such that

$$0 \to X \to V \to Y \to 0$$

is exact. Both X and Y will be Noetherian on either side as this holds for V. By Theorem 3.3.23 above, both X_B and Y_B are progenerators in Mod-B and in particular they are faithfully flat. Thus $\mathrm{Tor}_{\mathrm{Mod}-B}(Y_B, M) = 0$ and the sequence

$$0 \to X \otimes_B M \to V \otimes_B M \to Y \otimes_B M \to 0$$

is again exact. But by hypothesis $V \otimes_B M$ is simple and so one of $Y \otimes_B M, X \otimes_B M$ must be zero. By Corollary 3.3.24 we deduce that one of X, Y has to be zero contradicting our assumption that both were non-trivial. Hence V has no proper non-trivial submodules and is therefore simple. □

Corollary 3.3.26. *Let A be a regular affine $\mathbb{C}$-algebra which is a domain and N a simple D_A-module which is finitely-generated as an A-module. Then $\mathcal{D}iff_A(A, N)$ is simple as a D_A-D_A-bimodule and a progenerator in D_A-Mod and Mod-D_A.*

Proof. By Corollary 3.3.22, $\mathcal{D}iff(A, N)$ is finitely generated on the left and on the right.
Since D_A is simple and Noetherian, we conclude from Theorem 3.3.23 that $\mathcal{D}iff_A(A, N)$ is a progenerator in D_A-Mod and Mod-D_A. To deduce simplicity as a bimodule, we will apply Lemma 3.3.25. To do this, let us evaluate $\mathcal{D}iff_A(A, N) \otimes_{D_A} A$. Note that we have a natural evaluation homomorphism $\phi : \mathcal{D}iff_A(A, N) \otimes A \to N$ and we will aim to show that this is an isomorphism. As $\mathcal{D}iff_A(A, N) \neq 0$, the evaluation map is non-zero and by simplicity it must be surjective. Since

$$\mathcal{D}iff_A(A, N) \otimes_{D_A} A \cong N \otimes_A D_A \otimes_{D_A} A \cong N$$

the map ϕ must be trivial or an isomorphism and thus we can conclude that ϕ is indeed an isomorphism. □

3.4 Basic Results on$(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}})$ II.

Much of the rest of this chapter will present an independent approach to Theorem 3.2.6 as well as an investigation into which normal subgroups $N_{\mathbf{k}}$ can occur for certain cases of Γ. For a Harish-Chandra bimodule V we denote by

$$V|_{\mathfrak{h}^{reg}/\Gamma}$$

the localisation to $\mathfrak{h}^{reg}/\Gamma$ of V as a left module. We start with several core theorems:

Theorem 3.4.1. *Suppose that V is a Harish-Chandra $U_{\mathbf{k}}$-bimodule.*

1. *There are natural isomorphisms*

$$U_{\mathbf{k}}|_{\mathfrak{h}^{reg}/\Gamma} \otimes_{U_{\mathbf{k}}} V \cong V \otimes_{U_{\mathbf{k}}} U_{\mathbf{k}}|_{\mathfrak{h}^{reg}/\Gamma} \cong U_{\mathbf{k}}|_{\mathfrak{h}^{reg}/\Gamma} \otimes_{U_{\mathbf{k}}} V \otimes_{U_{\mathbf{k}}} U_{\mathbf{k}}|_{\mathfrak{h}^{reg}/\Gamma}$$

   i.e. the localisations of V as a left-, right- or bimodule all agree.
2. *If $V|_{\mathfrak{h}^{reg}/\Gamma} \neq 0$, then $V_{\mathfrak{h}^{reg}/\Gamma}$ is a progenerator in the category of left (and right) $U_{\mathbf{k}}|_{\mathfrak{h}^{reg}/\Gamma} = \mathcal{D}(\mathfrak{h}^{reg}/\Gamma)$ modules.*
3. *Let $\mathbf{k}$ be a regular choice of parameters and V a Harish-Chandra bimodule. Then V is a progenerator in the category of left (and right) $U_{\mathbf{k}}$ modules.*

Proof. The first statement follows as in [BEG03b], see the remark preceding Corollary 3.6
For regular parameter values, the algebra $U_{\mathbf{k}}$ is simple and Noetherian and the same holds for the localisations at any parameter values. As V and thus also $V|_{\mathfrak{h}^{reg}/\Gamma}$ is finitely generated on both sides, both are Noetherian when regarded as left or right $\mathcal{D}(\mathfrak{h}^{reg}/\Gamma)$ or $U_{\mathbf{k}}$-modules respectively. The result then follows as a special case of a Theorem 3.3.23. □

Corollary 3.4.2. *1. Let V be a Harish-Chandra bimodule. If $V \otimes_{U_{\mathbf{k}}} M = 0$ for some non-zero M in $\mathcal{O}_{\mathbf{k}}$ with $M|_{\mathfrak{h}^{reg}/\Gamma} \neq 0$, then $V|_{\mathfrak{h}^{reg}/\Gamma} = 0$.*

2. If $\mathbf{k} \in \mathrm{Reg}(\Gamma)$ and V a non-zero Harish-Chandra bimodule then $V \otimes_{U_{\mathbf{k}}} M \neq 0$ for any $M \in U_{\mathbf{k}} - \mathrm{Mod}$

Proof. 1. We have

$$(V \otimes_{U_{\mathbf{k}}} M) \cong V|_{\mathfrak{h}^{reg}/\Gamma} \otimes_{U_{\mathbf{k}}|_{\mathfrak{h}^{reg}/\Gamma}} M|_{\mathfrak{h}^{reg}/\Gamma}.$$

If $V|_{\mathfrak{h}^{reg}/\Gamma} \neq 0$, then we would have $V|_{\mathfrak{h}^{reg}/\Gamma} \otimes_{U_{\mathbf{k}}|_{\mathfrak{h}^{reg}}} M|_{\mathfrak{h}^{reg}/\Gamma} \neq 0$ since $V|_{\mathfrak{h}^{reg}/\Gamma}$ would be a generator.

2. This is just Corollary 3.3.24 restated in the special case of Harish-Chandra bimodules.

□

All of this leads to a proof of the following result of Berest-Etingof-Ginzburg first shown in [BEG03b]

Theorem 3.4.3. *Let $\mathbf{k} \in \mathrm{Reg}(\Gamma)$ and V a Harish-Chandra bimodule in $\mathcal{HC}_{\mathbf{k}}$. Then $V = 0$ if and only if $V|_{\mathfrak{h}^{reg}/\Gamma} = 0$. In particular, if $f : V \to L$ is a morphism of Harish-Chandra bimodules, f is an isomorphism if and only if it becomes an isomorphism upon localisation to $\mathfrak{h}^{reg}/\Gamma$.*

Proof. The statement that $V = 0 \iff V|_{\mathfrak{h}^{reg}/\Gamma}$ is just Corollary 4.5 in [BEG03b] and the proof holds for complex reflection groups as well, though it also follows from Theorem 3.4.1: If V is nonzero, it is a generator and thus for some M in $\mathcal{O}_{\mathbf{k}}$ we have $V \otimes_{U_{\mathbf{k}}} M \neq 0$. If $V|_{\mathfrak{h}^{reg}/\Gamma} = 0$, then $(V \otimes_{U_{\mathbf{k}}} M)|_{\mathfrak{h}^{reg}/\Gamma} = 0$, but in the regular case no non-zero object in $\mathcal{O}_{\mathbf{k}}$ vanishes upon localisation. Thus $V|_{\mathfrak{h}^{reg}/\Gamma} \neq 0$.
The isomorphism statement follows using exactness of Ore localisation: Suppose that $f : V \to L$ becomes an isomorphism upon localisation. Set $K = \ker(f)$ and $Q = \mathrm{coker}(f)$. Then $K|_{\mathfrak{h}^{reg}/\Gamma} = 0$ and $Q|_{\mathfrak{h}^{reg}/\Gamma} = 0$ and thus $K = Q = 0$. □

Finally, let us state the following beautiful result due to Berest-Etingof-Ginzburg and Ginzburg:

Theorem 3.4.4. *(Corollary 3.7 in [BEG03b], Corollary 6.7 in [Gin03]) Every object in $\mathcal{HC}_{\mathbf{k}}$ has finite length even if $\mathbf{k}$ is not regular.*

We need to make some caveats though: The proof in [BEG03b] only applies to regular parameter values and is written out only for real reflection groups. The same arguments apply in the regular case of complex reflection. In [Gin03] the result is again only formulated for real reflection groups but applies equally in the case of complex reflection groups. We will give an alternative proof of Theorem 3.4.4 for regular parameter values. Another proof for all parameter values might be possible using an inductive argument on layers of the stratification of $(\mathfrak{h} \times \mathfrak{h}^*)/\Gamma$ by symplectic leaves and Theorem 3.4.6 in [Los11a], which we cite as Theorem 3.2.4 which shows in particular that the quotient category of Harish-Chandra bimodules with associated variety in the closure of a leaf by the category of Harish-Chadra bimodules with associated variety in the given leaf is finite length.

Proof. Suppose V were Harish-Chandra of infinite length and let $\Delta \in \mathcal{O}_{\mathbf{k}}$ be non-zero. By Proposition 3.1.7 the tensor product $V \otimes_{U_{\mathbf{k}}} \Delta$ is again in $\mathcal{O}_{\mathbf{k}}$ and non-zero by Corollary 3.4.2. As

V is Noetherian and of infinite length, it has quotients of arbitrary finite length (as bimodules). Suppose we have a sequence of quotients

$$V \twoheadrightarrow V_1 \twoheadrightarrow V_2 \twoheadrightarrow \dots V_\ell \twoheadrightarrow 0$$

with each V_i being non-zero and each quotient map having a non-trivial kernel. Then we have a sequence of surjections in $\mathcal{O}_{\mathbf{k}}$

$$V \otimes_{U_{\mathbf{k}}} \Delta \twoheadrightarrow V_1 \otimes_{U_{\mathbf{k}}} \Delta \twoheadrightarrow V_2 \otimes_{U_{\mathbf{k}}} \Delta \twoheadrightarrow \dots V_\ell \otimes_{U_{\mathbf{k}}} \Delta \twoheadrightarrow 0$$

and by (2) in Corollary 3.4.2 each surjection again must have a non-zero kernel. Thus $V \otimes_{U_{\mathbf{k}}} \Delta$ has length at least ℓ for arbitrary $\ell \in \mathbb{N}$. But every object in $\mathcal{O}_{\mathbf{k}}$ has finite length. This is a contradiction and V must hence have finite length as a bimodule. $\square$

Definition 3.4.5. *(see Formula 8.9 in [BEG03b]) Let $M, N \in \mathcal{O}_{\mathbf{k}}$. We define*

$$\mathcal{L}(M, N) = \{f \in \mathrm{Hom}_{\mathbb{C}}(M, N) \mid \text{ any } a \in \mathbb{C}[\mathfrak{h}]^\Gamma \mathbf{e} \cup \mathbb{C}[\mathfrak{h}^*]^\Gamma \mathbf{e} \text{ acts ad-nilpotently on } f\}$$

i.e. $\mathcal{L}(M, N)$ is the submodule of $\mathrm{Hom}_{\mathbb{C}}(M, N)$ *on which each element* $\mathbb{C}[\mathfrak{h}]^\Gamma \mathbf{e} \cup \mathbb{C}[\mathfrak{h}^*]^\Gamma \mathbf{e}$ *acts* ad-*nilpotently.*

At present we can only say that the spaces $\mathcal{L}(M, N)$ are nearly Harish-Chandra. They are $U_{\mathbf{k}}$-bimodules and fulfil the appropriate nilpotence conditions, but we do not yet know if they are finitely generated, they will be as it turns out.

Proposition 3.4.6. *Let $\mathbf{k}$ be any choice of parameters and $M, N \in \mathcal{O}_{\mathbf{k}}$. Then $\mathcal{L}(M, N)|_{\mathfrak{h}^{reg}/\Gamma}$ is naturally a sub-bimodule of the $\mathcal{D}_{\mathfrak{h}^{reg}/\Gamma}$-bimodule $\mathcal{D}iff_{\mathcal{O}_{\mathfrak{h}^{reg}/\Gamma}}(M|_{\mathfrak{h}^{reg}/\Gamma}, N|_{\mathfrak{h}^{reg}/\Gamma})$.*

Proof. For brevity, we will set $X = \mathfrak{h}^{reg}/\Gamma$ and $\mathcal{N} := N|_{\mathfrak{h}^{reg}/\Gamma}$ as well as $\mathcal{M} := M|_{\mathfrak{h}^{reg}/\Gamma}$. Recall that $\delta \in \mathbb{C}[\mathfrak{h}]$ is defined by

$$\delta = \prod_{s \in \mathcal{S}} \alpha_s.$$

and this spans a one-dimensional representation of Γ. So for some $n \in \mathbb{N}$, $\delta^n \in \mathbb{C}[\mathfrak{h}]^\Gamma$ Then

$$U_{\mathbf{k}}|_X = U_{\mathbf{k}}[\mathbf{e}\delta^{-n}\mathbf{e}].$$

Now choose $f \in \mathcal{L}(M, N)$, we need to extend this to an element of $\mathcal{D}iff_{\mathcal{O}_X}(\mathcal{N}, \mathcal{M})$. As $\mathbf{e}\delta^n\mathbf{e} \in \mathbb{C}[\mathfrak{h}]^\Gamma\mathbf{e}$ we can choose any $d >> 0$ such that $\mathrm{ad}(\mathbf{e}\delta^n\mathbf{e})^d(f) = 0$, i.e.

$$\sum_{j=0}^{d} (-1)^j \binom{d}{j} (\mathbf{e}\delta^n\mathbf{e})^{d-j} f((\mathbf{e}\delta^n\mathbf{e})^j \bullet).$$

Thus we must conclude that

$$f(\delta^{-1}\bullet) = -(\mathbf{e}\delta^n\mathbf{e})^{-d} \sum_{j=1}^{d} (-1)^j \binom{d}{j} (\mathbf{e}\delta^n\mathbf{e})^{d-j} f((\mathbf{e}\delta^n\mathbf{e})^{j-1}\bullet)$$

and we may extend f to a map between $\mathcal{M}$ and $\mathcal{N}$ via this formula. This is Lemma 5.4 in [MR01].

It remains to check that f is in $\mathcal{D}iff_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})$. But this follows from Proposition 3.3.2, as $\mathcal{O}_X(X) = \mathbb{C}[\mathfrak{h}]^\Gamma[\delta^{-n}]$. □

The next two results will be absolutely crucial for our further reasoning.

Proposition 3.4.7. *Let* $\mathbf{k} \in \mathrm{Reg}(\Gamma)$ *and* $\lambda \in \mathrm{Irr}(\Gamma)$. *If* $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda)) \neq 0$ *then there is an isomorphism of* $\mathcal{D}(\mathfrak{h}^{reg}/\Gamma)$*-bimodules*

$$\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda))|_{\mathfrak{h}^{reg}/\Gamma} \cong \mathcal{D}iff_{\mathcal{O}_X}(\mathcal{O}_X, \Delta(\lambda)|_{\mathfrak{h}^{reg}/\Gamma}).$$

Proof. For brevity, let us set $= \mathfrak{h}^{reg}/\Gamma$. By Proposition 3.4.6, $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda))|_{\mathfrak{h}^{reg}/\Gamma}$ will be a sub-bimodule of $\mathcal{D}iff_{\mathcal{O}_X}(\Delta(\mathrm{triv})|_{\mathfrak{h}^{reg}/\Gamma}, \Delta(\lambda)|_{\mathfrak{h}^{reg}/\Gamma})$. Now $\Delta(\mathrm{triv})|_{\mathfrak{h}^{reg}/\Gamma} \cong \mathcal{O}_{\mathfrak{h}^{reg}/\Gamma}$ by considering the KZ-connection on $\Delta(\mathrm{triv})$. Further, if $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda)) \neq 0$ it will contain a sub-bimodule V which is non-zero and Harish-Chandra, namely we may take V to be any non-zero finitely generated sub-bimodule of $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda))$. By Theorem 3.4.3, $V|_{\mathfrak{h}^{reg}/\Gamma} \neq 0$ and by exactness of localisation we see that $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda))|_{\mathfrak{h}^{reg}/\Gamma} \neq 0$. By Corollary 3.3.26 $\mathcal{D}iff_{\mathcal{O}_X}(\mathcal{O}_X, \Delta(\lambda)|_{\mathfrak{h}^{reg}/\Gamma})$ is a simple bimodule and the result follows. □

Theorem 3.4.8. *Let* $\mathbf{k}$ *be a regular choice of parameters and* $\lambda \in \mathrm{Irr}(\Gamma)$.

1. $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda))$ *is finitely generated as a* $U_{\mathbf{k}} - U_{\mathbf{k}}$*-bimodule and as a right or left* $U_{\mathbf{k}}$*-module. In particular,* $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda))$ *is Harish-Chandra.*
2. *If* $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda)) \neq 0$ *then* $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda))$ *is a progenerator in* $U_{\mathbf{k}} - \mathrm{Mod}$ *and* $\mathrm{Mod} - U_{\mathbf{k}}$.
3. *If* $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda) \neq 0$ *then it is simple as a bimodule.*
4. *If* V *is a simple Harish-Chandra bimodule then there* $V \cong \mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda))$ *for a suitable choice of* λ.

Proof. For ease of notation we set $X = \mathfrak{h}^{reg}/\Gamma$ and denote by $\mathcal{M}_\lambda$ the localisation $\Delta(\lambda)|_X$ of a standard module $\Delta(\lambda)$ in $\mathcal{O}_{\mathbf{k}}$. Note that as $\mathbf{k}$ is regular, $U_{\mathbf{k}}$ is a simple Noetherian domain and clearly $U_{\mathbf{k}}|_X \cong \mathcal{D}(X)$ is too, as it is the differential operator ring over a smooth variety. Clearly $\mathcal{M}_\lambda$ is a $\mathcal{D}_X$-module which is coherent as an $\mathcal{O}_X$-module, i.e. a vector bundle on X. Further we can see that it is simple since it corresponds to an irreducible representation of $B_\Gamma = \pi_1(X)$ - these are calculated via the KZ-functor and $\mathrm{KZ}(\Delta(\lambda))$ is simple whenever $\mathbf{k}$ is regular, as stated in Corollary 2.4.19.

1. Suppose that $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda)) \neq 0$ and consider $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda))|_X$. By Proposition 3.4.6 this is a right $\mathcal{D}_X = U_{\mathbf{k}}|_X$-submodule of $\mathcal{D}iff_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M}_\lambda)$. By Theorem 2.1.13, the space $\mathfrak{h}/\Gamma$ is smooth and thus so is $\mathfrak{h}^{reg}/\Gamma$. So Proposition 3.3.11 applies and we have an isomorphism $\mathcal{D}iff_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M}_\lambda) \cong \mathcal{M}_\lambda \otimes_{\mathcal{O}_X} \mathcal{D}_X$. As $U_{\mathbf{k}}$ is Noetherian, being finitely generated is the same as being Noetherian. Let us suppose that $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda))$ were not right Noetherian. Then we could find an infinite, strictly increasing chain of finitely generated submodules
$$V_1 \subset V_2 \subset \ldots \subset \mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda))$$
and upon localising we would obtain
$$V_1|_X \subset V_2|_X \subset \ldots \subset \mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda))|_X \subseteq \mathcal{D}iff_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M}_\lambda)$$
with each $V_i|_X \neq 0$ and each inclusion strict by Theorem 3.4.3 as each V_i is Harish-Chandra.

But $\mathcal{D}iff_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M}_\lambda)$ is finitely generated as a right $\mathcal{D}(X) \cong U_{\mathbf{k}}|_X$ -module and hence Noetherian, giving a contradiction. So $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda))$ must be right Noetherian and thus also finitely generated on the right. Thus it is also finitely generated as bimodule, which proves that $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda))$ is Harish-Chandra.

2. This follows from Theorem 3.4.1 using the results from 1).
3. This is an application of Lemma 3.3.25. Let us consider the tensor product $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda)) \otimes_{U_{\mathbf{k}}} \Delta(\mathrm{triv})$ and the natural evaluation map

$$\varepsilon_{\Delta(\mathrm{triv})} : \mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda)) \otimes_{U_{\mathbf{k}}} \Delta(\mathrm{triv}) \to \Delta(\lambda).$$

Upon localising to X this becomes

$$\nu : \mathcal{D}iff_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M}_\lambda) \otimes_{\mathcal{D}_X} \mathcal{O}_X \to \mathcal{M}_\lambda$$

where again ν is the natural evaluation map. Using Proposition 3.4.7, we have isomorphisms of $\mathcal{D}_X$-modules;

$$\begin{aligned} \mathcal{D}iff_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M}_\lambda) \otimes_{\mathcal{O}_X} \mathcal{O}_X &\cong \left(\mathcal{M}_\lambda \otimes^D \mathcal{D}_X\right) \otimes_{\mathcal{D}_X} \mathcal{O}_X \\ &\cong \mathcal{M}_\lambda \otimes^D \mathcal{O}_X \end{aligned}$$

and as we have an isomorphism of $\mathcal{O}_X$-modules

$$\mathcal{M}_\lambda \otimes^D \mathcal{O}_X \cong \mathcal{M}_\lambda \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong \mathcal{M}_\lambda$$

we conclude that

$$\mathcal{D}iff_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M}_\lambda) \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong \mathcal{M}_\lambda.$$

Hence ν is an isomorphism as it is a non-zero endomorphism of a simple module. So by Theorem 3.4.3 we can conclude that $\varepsilon_{\Delta(\mathrm{triv})}$ is an isomorphism and hence $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda))$ is simple as a bimodule.
4. Suppose that V is a simple Harish-Chandra $U_{\mathbf{k}} - U_{\mathbf{k}}$-bimodule. Then $V \otimes_{U_{\mathbf{k}}} \Delta(\mathrm{triv}) \neq 0$ by Corollary 3.4.2 and we set $V \otimes_{U_{\mathbf{k}}} \Delta(\mathrm{triv}) = \oplus_i \Delta(\alpha_i)$ for suitable $\alpha_i \in \mathrm{Irr}(\Gamma)$ by semisimplicity of $\mathcal{O}_{\mathbf{k}}$. The map

$$\begin{aligned} \mu : V &\to \mathcal{L}(\Delta(\mathrm{triv}), \oplus_i \Delta(\alpha_i)) \\ V &\mapsto (a \mapsto v \otimes a) \end{aligned}$$

is then a non-trivial map of bimodules and by simplicity of V it is injective. We have natural projection maps

$$\pi_i : \oplus_i \Delta(\alpha_i) \to \Delta(\alpha_i)$$

giving rise to bimodule maps

$$\pi_i : \mathcal{L}(\Delta(\mathrm{triv}), \oplus_i \Delta(\alpha_i)) \to \mathcal{L}(\Delta(\mathrm{triv}), \Delta(\alpha_i))$$

and for some i we must have $\pi_i \circ \mu \neq 0$. Thus by simplicity of V again this gives an injection

$$\pi_i \circ \mu : V \hookrightarrow \mathcal{L}(\Delta(\mathrm{triv}), \Delta(\alpha_i))$$

for some i and we set $\lambda = \alpha_i$. But by (3) the latter is simple as a bimodule and so we have an isomorphism

$$V \cong \mathcal{L}(\Delta(\text{triv}), \Delta(\lambda)).$$

$\square$

In fact we can use this method of proof to show that if $\mathbf{k} \in \text{Reg}(\Gamma)$ $\mathcal{L}(\Delta_1, \Delta_2)$ is finitely generated and hence Harish-Chandra for any two standard modules $\Delta_1, \Delta_2 \in \mathcal{O}_{\mathbf{k}}$. For this it suffices to replace the isomorphism

$$\mathcal{D}iff_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M}) \cong \mathcal{M} \otimes^D \mathcal{D}_X$$

with the isomorphism

$$\mathcal{D}iff_{\mathcal{O}_X}(\mathcal{N}_X, \mathcal{M}) \cong \mathcal{M} \otimes^D \mathcal{D}_X \otimes^D \mathcal{N}^\vee$$

where $\mathcal{N}^\vee$ is the dual bundle to $\mathcal{N}$, see for example Exercise 2.1.16 in [Gin]. We will have no use for this in the remainder of the thesis, apart from Definition 3.4.13. This is mainly for aesthetic purposes and we could restrict ourselves to $\Delta = \Delta(\text{triv})$ in Definition 3.4.13 without problems.

Note that the first claim was also proved by I. Losev in [Los11a] for all parameters:

Theorem 3.4.9. *(Proposition 5.7.1 in [Los11a]) Let $\alpha, \beta \in \text{Irr}(\Gamma)$ and $L(\alpha)$, $L(\beta)$ the associated simple objects in $\mathcal{O}_{\mathbf{k}}$. Then the space $\mathcal{L}(L(\alpha), L(\beta))$ is finitely generated as an $H_{\mathbf{k}} - H_{\mathbf{k}}$-bimodule for any choice of $\mathbf{k}$.*

Corollary 3.4.10. *Let $\mathbf{k}$ be regular. Then $\mathcal{HC}_{\mathbf{k}}$ has at most $\#\text{Irr}(\Gamma)$ simple objects.*

Proof. Immediate from Theorem 3.4.8, part 4). $\square$

Corollary 3.4.11. *Let $\mathbf{k} \in \text{Reg}(\Gamma)$. Suppose that L is a simple Harish-Chandra bimodule. Then there exists a unique $\lambda \in \text{Irr}(\Gamma)$ such that $L \cong \mathcal{L}_{\Delta(\text{triv})}(\Delta(\lambda))$ and $L \otimes_{U_{\mathbf{k}}} \Delta(\text{triv}) \cong \Delta(\lambda)$.*

Proof. Existence of λ follows from Theorem 3.4.8 and uniqueness will follow once we have shown the isomorphism $L \otimes_{U_{\mathbf{k}}} \Delta(\text{triv}) \cong \Delta(\lambda)$. But it was shown in the proof of (3) in 3.4.8 that the evaluation map is an isomorphism. $\square$

We can give an example of how we will use these results:

Lemma 3.4.12. *Let $\mathbf{k}$ be regular, then $\mathcal{L}(\Delta(\text{triv}), \Delta(\text{triv})) \cong U_{\mathbf{k}}$.*

Proof. As $\mathbf{k}$ is simple, $U_{\mathbf{k}}$ is simple as a Harish-Chandra bimodule. The natural action of $U_{\mathbf{k}}$ on $\Delta(\text{triv})$ gives an embedding

$$U_{\mathbf{k}} \hookrightarrow \mathcal{L}(\Delta(\text{triv}), \Delta(\text{triv})).$$

By Theorem 3.4.8 the latter is simple as a bimodule and thus the embedding must be an isomorphism. $\square$

We will now make our Definition 3.4.5 functorial:

Definition 3.4.13. *Let Δ be an object in $\mathcal{O}_{\mathbf{k}}$ for an arbitrary choice of parameters. We have two functors*

$$\begin{aligned} \mathcal{T}_\Delta : \mathcal{HC}_{\mathbf{k}} &\to \mathcal{O}_{\mathbf{k}} \\ V &\mapsto V \otimes_{U_{\mathbf{k}}} \Delta \end{aligned} \qquad\qquad \begin{aligned} \mathcal{L}_\Delta : \mathcal{O}_{\mathbf{k}} &\to \mathcal{HC}_{\mathbf{k}} \\ M &\mapsto \mathcal{L}(\Delta, M). \end{aligned}$$

There are canonical morphisms of functors

$$\varepsilon_\Delta : \mathcal{T}_\Delta \circ \mathcal{L}_\Delta \to Id_{\mathcal{O}}$$
$$\varepsilon_\Delta(M) : \mathcal{L}_\Delta(M) \otimes_{U_{\mathbf{k}}} \Delta \to M,\ f \otimes a \mapsto f(a)$$
$$\text{and}$$
$$\mu_\Delta : Id_{\mathcal{HC}} \to \mathcal{L}_\Delta \circ \mathcal{T}_\Delta$$
$$\mu_\Delta(V) : V \to \mathcal{L}_\Delta(V \otimes_{U_{\mathbf{k}}} \Delta)\ v \mapsto (a \mapsto v \otimes a)$$

Proposition 3.4.14. *The functor $\mathcal{T}_\Delta$ is left adjoint to $\mathcal{L}_\Delta$. The unit and counit of the adjunction are given by μ_Δ and ε_Δ respectively.*

Proof. We need to show that the composite transformations of functors

$$\mathcal{T}_\Delta \to \mathcal{T}_\Delta \circ \mathcal{L}_\Delta \circ \mathcal{T}_\Delta \to \mathcal{T}_\Delta$$

and

$$\mathcal{L}_\Delta \to \mathcal{L}_\Delta \circ \mathcal{T}_\Delta \circ \mathcal{L}_\Delta \to \mathcal{L}_\Delta$$

are the identity transformations. So let us consider any Harish-Chandra bimodule V, the first composition becomes on the level of objects

$$V \otimes_{U_{\mathbf{k}}} \Delta \longrightarrow \mathcal{L}(\Delta, V \otimes_{U_{\mathbf{k}}} \Delta) \otimes \Delta \longrightarrow V \otimes_{U_{\mathbf{k}}} \Delta$$

$$v \otimes m \longrightarrow (a \mapsto v \otimes a) \otimes m \longrightarrow v \otimes m$$

whereas for any M in $\mathcal{O}_{\mathbf{k}}$ the second reads

$$\mathcal{L}(\Delta, M) \longrightarrow \mathcal{L}(\Delta, \mathcal{L}(\Delta, M) \otimes \Delta) \longrightarrow \mathcal{L}(\Delta, M)$$

$$f \longrightarrow (a \mapsto f \otimes a) \longrightarrow (a \mapsto f(a)) = f$$

On morphisms these compositions act as follows: Suppose we have a morphism $\phi : X \to Y$ of

Harish-Chandra bimodules, this gets mapped to the following

$$\begin{array}{ccc}
X \otimes \Delta & \longrightarrow & Y \otimes \Delta \\
v \otimes m & \longrightarrow & \phi(v) \otimes m \\
\downarrow & & \downarrow \\
\mathcal{L}_\Delta(X \otimes \Delta) \otimes \Delta & \longrightarrow & \mathcal{L}_\Delta(Y \otimes \Delta) \otimes \Delta \\
(a \mapsto v \otimes a) \otimes m & \longrightarrow & (a \mapsto \phi(v) \otimes a) \otimes m \\
\downarrow & & \downarrow \\
X \otimes \Delta & \longrightarrow & Y \otimes \Delta \\
v \otimes m & \longrightarrow & \phi(v) \otimes m.
\end{array}$$

A morphism

$$\phi : M \to N$$

in $\mathcal{O}_{\mathbf{k}}$ gets mapped to

$$\begin{array}{ccc}
\mathcal{L}(\Delta, M) & \longrightarrow & \mathcal{L}(\Delta, N) \\
f & \longrightarrow & \phi \circ f \\
\downarrow & & \downarrow \\
\mathcal{L}(\Delta, \mathcal{L}(\Delta, M) \otimes \Delta) & \longrightarrow & \mathcal{L}(\Delta, \mathcal{L}(\Delta, N) \otimes \Delta) \\
(a \mapsto f \otimes a) & \longrightarrow & (a \mapsto \phi \circ f \otimes a) \\
\downarrow & & \downarrow \\
\mathcal{L}(\Delta, M) & \longrightarrow & \mathcal{L}(\Delta, N) \\
(a \mapsto f(a)) = f & \longrightarrow & (a \mapsto \phi \circ f(a)) = \phi \circ f
\end{array}$$

Thus indeed ε_Δ and μ_Δ give the counit and unit of an adjunction between the two functors $\mathcal{T}_\Delta$ and $\mathcal{L}_\Delta$. ☐

Proposition 3.4.15. *Let* $\mathbf{k} \in \mathrm{Reg}(\Gamma)$*. Then for any standard object* Δ *in* $\mathcal{O}_{\mathbf{k}}$*, the functor* $\mathcal{T}_\Delta$ *is exact.*

Proof. If

$$0 \to K \to V \to Q \to 0$$

is exact in $\mathcal{HC}_{\mathbf{k}}$ then

$$\ldots \to \mathrm{Tor}^i_{\mathrm{Mod}-U_{\mathbf{k}}}(Q, \Delta) \to K \otimes_{U_{\mathbf{k}}} \Delta \to V \otimes_{U_{\mathbf{k}}} \Delta \to Q \otimes_{U_{\mathbf{k}}} \Delta \to 0$$

is exact as a sequence of vector spaces. By Theorem 3.4.8, any non-zero object in $\mathcal{HC}_{\mathbf{k}}$ is projective on the right and hence $\mathrm{Tor}^i_{\mathrm{Mod}-U_{\mathbf{k}}}(Q, \Delta) = 0$ for $i > 0$ for any Harish-Chandra bimodule Q. So we obtain the exact sequence

$$0 \to K \otimes_{U_{\mathbf{k}}} \Delta \to V \otimes_{U_{\mathbf{k}}} \Delta \to Q \otimes_{U_{\mathbf{k}}} \Delta \to 0$$

which must be exact in $\mathcal{O}_{\mathbf{k}}$.

□

Theorem 3.4.16. *Let* $\mathbf{k} \in \mathrm{Reg}(\Gamma)$. *Then* $\mathcal{HC}_{\mathbf{k}}$ *is semisimple.*

Proof. By Theorem 3.4.4 it suffices to show that there are no extensions between simple objects in $\mathcal{HC}_{\mathbf{k}}$. Let L be a simple Harish-Chandra bimodule. Then L is isomorphic to $\mathcal{L}(\Delta(\mathrm{triv}), \Delta)$ for some standard Δ by Theorem 3.4.8. The functor $\mathcal{L}_{\Delta(\mathrm{triv})}$ has an exact left adjoint $\mathcal{T}_{\Delta(\mathrm{triv})}$ by Proposition 3.4.15 and as such it takes injectives to injectives. Since $\mathcal{O}_{\mathbf{k}}$ is semisimple, any standard object is injective and hence so is L. □

The following theorem will essentially give us our version of Theorem 3.2.6

Theorem 3.4.17. *Let* $\mathbf{k} \in \mathrm{Reg}(\Gamma)$ *and* $\lambda, \sigma \in \mathrm{Irr}(\Gamma)$. *Suppose that* $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda)) \neq 0$. *Then*

$$\mathrm{KZ}_{\mathbf{k}}(\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda)) \otimes_{U_{\mathbf{k}}} \Delta(\sigma)) \cong \mathrm{KZ}_{\mathbf{k}}(\Delta(\lambda)) \otimes \mathrm{KZ}_{\mathbf{k}}(\Delta(\sigma))$$

Proof. We set

$$X := \mathfrak{h}^{reg}/\Gamma$$

and

$$\mathcal{M}_\alpha := \Delta(\alpha)|_X$$

for $\alpha \in \mathrm{Irr}(\Gamma)$. Each $\mathcal{M}_\alpha$ is naturally a $\mathcal{D}_X$-module with the action being induced from the $U_{\mathbf{k}}$-action on $\Delta(\alpha)$ (recall from Proposition 2.3.8 and Theorem 2.3.9 that $U_{\mathbf{k}}|_X \cong \mathcal{D}(X)$). Note further that $\mathcal{M}_{\mathrm{triv}} \cong \mathcal{O}_X$ as $\mathcal{D}_X$-modules.

In a first step, we will determine $(\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda)) \otimes_{U_{\mathbf{k}}} \Delta(\sigma))\,|_X$. By Proposition 3.4.7 we have

$$\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\alpha))|_X \cong \mathcal{D}iff_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M}_\alpha)$$

for any $\alpha \in \mathrm{Irr}(\Gamma)$.

$$\begin{aligned}(\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda)) \otimes_{U_{\mathbf{k}}} \Delta(\sigma))\,|_X &\cong (\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda))|_X \otimes_{\mathcal{D}_X} \Delta(\sigma))\,|_X \\ &\cong \mathcal{D}iff_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M}_\lambda) \otimes_{\mathcal{D}_X} \mathcal{M}_\sigma \\ &\cong \left(\mathcal{M}_\lambda \otimes^D \mathcal{D}_X\right) \otimes_{\mathcal{D}_X} \mathcal{M}_\sigma \\ &\cong \mathcal{M}_\lambda \otimes^D \mathcal{M}_\sigma.\end{aligned}$$

As $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda)) \otimes_{U_{\mathbf{k}}} \Delta(\sigma) \in \mathcal{O}_{\mathbf{k}}$, the flat connection it carries when viewed as a $\mathcal{D}_X$-module again has regular singularities.

Denote by X^{an} the complex manifold underlying $X = \mathfrak{h}^{reg}/\Gamma$. As explained in Section 4.7 of [HTT08], there is a functor

$$\begin{aligned}\mathcal{D}_X - \mathrm{Mod} &\to \mathcal{D}_{X^{an}} \\ M &\mapsto M^{an}\end{aligned}$$

and we can define a de Rham functor on algebraic $\mathcal{D}_X$-modules (more accurately the bounded derived category of $\mathcal{D}_X$-modules) as on page 121 in [HTT08],

$$\begin{aligned} DR_X : D^b(\mathcal{D}_X) &\to D^b(\underline{\mathbb{C}}_X) \\ M &\mapsto DR_{X^{an}}(M^{an}) \end{aligned}$$

where $\underline{\mathbb{C}}_X$ is the constant sheaf on X^{an} and therefore $D^b(\underline{\mathbb{C}}_{X^{an}})$ is the bounded derived category of locally constant sheaves of vector spaces on X^{an}. By Corollary 5.3.10 in [HTT08], DR_X induces an equivalence of Abelian categories from $\mathrm{Conn}^{reg}(X)$ to $\mathrm{Loc}(X^{an})$.

Here $\mathrm{Conn}^{reg}(X)$ is the category of coherent $\mathcal{O}_X$-modules with a flat connection with regular singularities and $\mathrm{Loc}(X^{an})$ is the category of local systems on X^{an}, i.e. the locally constant sheaves of vector spaces on X^{an} of finite dimension. This category again has a natural tensor product, namely that of sheaves which we will denote by $\otimes_{\mathbb{C}}$. Showing that DR_X intertwines $\otimes^D$ and $\otimes_{\mathbb{C}}$ is our next aim.

We will argue as in Corollary 3.3.21 by using $\bullet \otimes^D \bullet \cong \Delta_X^*(\bullet \boxtimes^D \bullet)$ for the diagonal embedding $\Delta_X : X \to X \times X$. As shown in Proposition 3.3.20, the diagonal embedding is non-characteristic for the external tensor product of algebraic vector bundles with flat connections. By Corollary 4.3.3 in [HTT08] this allows us to deduce that

$$DR_X(L\Delta_X^*(\mathcal{M}_\lambda \boxtimes^D \mathcal{M}_\sigma)) \cong \Delta_X^{-1}(DR_{X\times X}(\mathcal{M}_\lambda \boxtimes^D \mathcal{M}_\sigma))[-\dim X].$$

Recall that by Proposition 2.2.5 in [HTT08] the characteristic variety of $\mathcal{M}_\lambda$ (and that of $\mathcal{M}_\sigma$) is the zero section of T_X^*X and hence both $\mathcal{M}_\lambda, \mathcal{M}_\sigma$ are holonomic (see Definition 2.3.6 in [HTT08]). By Proposition 4.7.8 in [HTT08] we have a canonical isomorphism

$$DR_{X\times X}(\mathcal{M}_\lambda \boxtimes^D \mathcal{M}_\sigma) \xrightarrow{\sim} DR_X(\mathcal{M}_\lambda) \boxtimes_{\mathbb{C}} DR_X(\mathcal{M}_\sigma)$$

and hence we have a functorial isomorphism

$$DR_X(L\Delta_X^*(\mathcal{M}_\lambda \boxtimes^D \mathcal{M}_\sigma)) \cong \Delta_X^{-1}(DR_X(\mathcal{M}_\lambda) \boxtimes_{\mathbb{C}} DR_X(\mathcal{M}_\sigma))[-\dim X].$$

The final remark on page 39 of [HTT08] states that we have another functorial isomorphism

$$L\Delta_X^*(\mathcal{M}_\lambda \boxtimes^D \mathcal{M}_\sigma) \cong \mathcal{M}_\lambda \otimes^D \mathcal{M}_\sigma$$

(the functor $\bullet \otimes^D \otimes$ is already exact) and we can further see that

$$\Delta_X^{-1}(DR_X(\mathcal{M}_\lambda) \boxtimes_{\mathbb{C}} DR_X(\mathcal{M}_\sigma)) \cong DR_X(\mathcal{M}_\lambda) \otimes_{\mathbb{C}} DR_X(\mathcal{M}_\sigma).$$

In summary we now have

$$DR_X(\mathcal{M}_\lambda \otimes^D \mathcal{M}_\sigma) \cong DR_X(\mathcal{M}_\lambda) \otimes_{\mathbb{C}} DR_X(\mathcal{M}_\sigma)[-\dim X]$$

and after taking cohomology $H^{-\dim X}$ (and applying Theorem 4.2.4 in [HTT08]) we arrive at the statement for the second step, namely that the equivalence

$$\mathrm{Conn}^{reg}(X) \cong \mathrm{Loc}(X^{an})$$

intertwines tensor products.

Finally, we need to prove that the if we have local systems F, G on X^{an} corresponding to representations ϕ, γ of $\pi_1(X^{an})$, then the local system $F \otimes_{\mathbb{C}} G$ corresponds to the representation $\phi \otimes \gamma$. For this we will briefly recall the construction of the equivalence between $\mathrm{rep}_{\mathbb{C}}\pi(X^{an})$ and $\mathrm{Loc}(X^{an})$ and for this we follow [Ach07]: Take a local system F and fix a basepoint $x \in X^{an}$. Let α be a loop at x and let $x = a_0, a_1, \ldots, a_n, a_{n+1} = x$ be a collection of "suitably close" points along α. Then there is a natural isomorphism $f_i : F_{a_i} \xrightarrow{\sim} F_{a_{i+1}}$given by identifying both with $F(V_i)$ where V_i is a small neighbourhood of both a_i and a_{i+1}. For any $v \in F_x$ we then set

$$[\alpha] \cdot v := (f_n \circ \ldots \circ f_0)(v)$$

where $[\alpha]$ denotes the homotopy class of α in $\pi(X^{an}, x)$.

Now considering the local system $F \otimes_{\mathbb{C}} G$ we have $(F \otimes G)_{a_i} = F_{a_i} \otimes_{\mathbb{C}} G_{a_i}$ for suitably small neighbourhoods and the isomorphisms $(F \otimes_{\mathbb{C}} G)_{a_i} \to (F \otimes_{\mathbb{C}} G)_{a_{i+1}}$ factor as $f_i \otimes g_i$ where $f_i : F_{a_i} \xrightarrow{\sim} F_{a_{i+1}}$ and $g_i : G_{a_i} \xrightarrow{\sim} G_{a_{i+1}}$ are derived as in the description of the $\pi(X^{an}, x)$-action. So then indeed we find that

$$[\alpha] \cdot (u \otimes v) = ([\alpha] \cdot u) \otimes ([\alpha] \cdot u)$$

and so $F \otimes G$ does correspond to $\phi \otimes \gamma$. □

Lemma 3.4.18. *Let Γ be a complex reflection group with $\dim_{\mathbb{C}} \mathcal{H}_{\mathbf{q}}(\Gamma) = \#\Gamma$ and suppose that $\mathbf{k} \in \mathrm{Reg}(\Gamma)$. The following holds for the parameters $\{q_{H,j}\}$ of $\mathcal{H}_{\mathbf{q}}(\Gamma)$:*

1. *For any fixed reflection hyperplane the values $1 \cup \{e^{-\frac{2\pi i j}{m_H}} q_{H,j} \mid 1 \leq j \leq m_H - 1\}$ are pairwise distinct.*
2. *If T is a generator of $\mathcal{H}_{\mathbf{q}}(\Gamma)$ arising from an s-generator of monodromy around H, so that*

$$(T - 1) \prod_{j=1}^{m_H - 1} (T - e^{-\frac{2\pi i j}{m_H}} q_{H,j}) = 0,$$

then for every $1 \leq j \leq m_H - 1$ there exists a $\mathcal{H}_{\mathbf{q}}(\Gamma)$-module M such that for some non-zero $m \in M$ we have $Tm = e^{-\frac{2\pi i j}{m_H}} q_{H,j} m$.

Proof. 1. Suppose for some hyperplane H the values $1 \cup \{e^{-\frac{2\pi i j}{m_H}} q_{H,j} \mid 1 \leq j \leq m_H - 1\}$ are not pairwise distinct, then $\mathcal{H}_{\bar{\mathbf{q}}}(\Gamma_H)$ is not semisimple and hence neither is the associated category $\mathcal{O}_{\bar{\mathbf{k}}}(\Gamma_H)$. Then by Proposition 2.4.3 there exists a non-simple standard module Δ in $\mathcal{O}_{\bar{\mathbf{k}}}(\Gamma_H)$. Thus Δ has a non-trivial quotient and as every irreducible representation of the cyclic group Γ_H is one-dimensional, Δ has rank 1 over $\mathbb{C}[\mathfrak{h}^*]$ by Proposition 2.3.7. Therefore any non-trivial quotient of Δ is finite-dimensional. It follows from Theorem 1.5 in [Ste64] that the pointwise stabiliser Γ_Y of a set $Y \subseteq \mathfrak{h}$ is generated by those reflections whose reflecting hyperplane contains Y, for an alternative proof see Theorem 4.7 in [Bro10]. Choosing a point $b \in H$ not lying on any other reflection hyperplane we see that

$$\Gamma_H = \Gamma_b.$$

By Proposition 3.23 (ii) in [BE09] there exists a module in $\mathcal{O}_{\mathbf{k}}$ whose support in $\mathfrak{h}$ equals $\{y \in \mathfrak{h} \mid \Gamma_H \subseteq \Gamma_y\} \neq \mathfrak{h}$ (this requires Γ_H to be the stabiliser of a point). But as $\mathcal{O}_{\mathbf{k}}$

is semisimple, every non-trivial module has full support leading to a contradiction.The claim follows.

2. We consider the regular representation ${}_{\mathcal{H}_{\mathbf{q}}(\Gamma)}\mathcal{H}_{\mathbf{q}}(\Gamma)$ of $\mathcal{H}_{\mathbf{q}}(\Gamma)$. Consider the Hecke relation $\prod_{j=0}^{m_H-1}(T - e^{-\frac{2\pi i j}{m_H}} q_{H,j}) = 0$ with $q_{H,0} = 1$. Then for some $a \in \{0, \ldots, m_H - 1\}$ we can consider $\prod_{\substack{j=0 \\ j\neq a}}^{m_H-1}(T - e^{-\frac{2\pi i j}{m_H}} q_{H,j})$ which will be an eigenvector of T with eigenvalue $e^{\frac{2\pi i a}{m_H}} q_{H,a}$.

□

Definition 3.4.19. *We say that a complex reflection group Γ has "property I" if the following holds: Fix a set of generators $(T_i)_i$ of $\mathcal{H}_{\mathbf{q}}(\Gamma)$ as in Theorem 2.4.15, then for any two such generators T_a, T_b whose associated reflection hyperplanes lie in the same Γ-orbit we can find an invertible element $I_{ab} \in \mathcal{H}_{\mathbf{q}}(\Gamma)$ such that*

$$I_{ab}T_a = T_b I_{ab}.$$

So property I just says that generators corresponding to hyperplanes in the same orbit are conjugate in $\mathcal{H}_{\mathbf{q}}(\Gamma)$

Lemma 3.4.20. *Suppose Γ is a finite Coxeter group or a group of type $G(m, p, n)$ or of type G_4 or G_{25} in the notation of [BMR98]. Then Γ has property I and $\dim_{\mathbb{C}} \mathcal{H}_{\mathbf{q}}(\Gamma) = \#\Gamma$.*

Proof. That each such Γ is such that $\dim_{\mathbb{C}} \mathcal{H}_{\mathbf{q}}(\Gamma) = \#\Gamma$ follows from [Ari95], Proposition 1.6 (2), for the groups $G(m, p, n)$, for Coxeter groups this follows for example from the existence of the Kazhdan-Lusztig basis see for example [Hum92] Theorem 7.9 as well as the following exercise and remark, and for the groups G_4 or G_{25} this is Satz 4.7 in [BM93]. Property I follows from case-by-case inspection of the presentations of $\mathcal{H}_{\mathbf{q}}(\Gamma)$. If Γ is a Coxeter group, we refer to Tables 7.4 and 7.5 and if Γ is of type $G(m, p, n)$ we refer to the presentation of the Hecke algebra given on page 165 of [Ari95]. For the groups G_4 or G_{25} we refer to Proposition 4.22 in [BMR98] as well as Tables 1-4 in loc.cit.. Note that in all cases we may assume that $I_{a,b}$ is a monomial in the generators. □

Theorem 3.4.21. *Let $\mathbf{k} \in \mathrm{Reg}(\Gamma)$ and suppose Γ is such that $\dim_{\mathbb{C}} \mathcal{H}_{\mathbf{q}}(\Gamma) = \#\Gamma$.*

1. *$(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}})$ is equivalent to a full subcategory of $\mathcal{H}_{\mathbf{q}}(\Gamma) - \mathrm{mod}$ closed under taking subquotients and tensor products as B_Γ-modules.*
2. *If $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda)) \neq 0$ then for any $\mathcal{H}_{\mathbf{q}}(\Gamma)$-module L the tensor product $\mathrm{KZ}_{\mathbf{k}}(\Delta(\lambda)) \otimes L$ taken as B_Γ-modules is again an $\mathcal{H}_{\mathbf{q}}(\Gamma)$-module.*
3. *If Γ has property I, then $(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}})$ is equivalent to $(\mathrm{rep}_{\mathbb{C}}(\Gamma/N_{\mathbf{k}}), \otimes)$ for some normal subgroup $N_{\mathbf{k}} \trianglelefteq \Gamma$.*

Proof. 1. Define a functor

$$\begin{aligned} F : \mathcal{HC}_{\mathbf{k}} &\to \mathcal{H}_{\mathbf{q}} - \mathrm{mod} \\ V &\mapsto \mathrm{KZ}_{\mathbf{k}}(V \otimes_{U_{\mathbf{k}}} \Delta(\mathrm{triv})) \end{aligned}$$

We claim that F identifies $\mathcal{HC}_{\mathbf{k}}$ with a full subcategory of $\mathcal{H}_{\mathbf{q}} - \mathrm{mod}$ closed under taking subquotients and tensor products as B_Γ-representations. We can write F as the composition of two functors, namely

$$F = \mathrm{KZ}_{\mathbf{k}} \circ \mathcal{T}_{\Delta(\mathrm{triv})}$$

with $\mathcal{T}_{\Delta(\mathrm{triv})}$ as defined in Definition 3.4.13. As $\mathbf{k}$ is regular, $\mathcal{T}_{\Delta(\mathrm{triv})}$ is exact by Proposition 3.4.15 and thus F is exact too. Further, both functors preserve simple objects if $\mathbf{k}$ is regular - for $\mathrm{KZ}_{\mathbf{k}}$ this follows from Corollary 2.4.19 and for $\mathcal{T}_{\Delta(\mathrm{triv})}$ this is a consequence of Corollary 3.4.11. As both categories are semisimple, F is full and faithful. Thus F embeds $\mathcal{HC}_{\mathbf{k}}$ into $\mathcal{H}_{\mathbf{q}}(\Gamma)$ as an Abelian subcategory. Next we need to show that F takes $\bullet \otimes_{U_{\mathbf{k}}} \bullet$ to $\bullet \otimes \bullet$ the tensor product of B_Γ-representations: Let V_1, V_2 be simple objects in $\mathcal{HC}_{\mathbf{k}}$ with $V_i \cong \mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda_i))$ $i = 1, 2$.

$$\begin{aligned}
F(V_1 \otimes_{U_{\mathbf{k}}} V_2) &= \mathrm{KZ}_{\mathbf{k}}(V_1 \otimes_{U_{\mathbf{k}}} V_2 \otimes_{U_{\mathbf{k}}} \Delta(\mathrm{triv})) \\
&= \mathrm{KZ}_{\mathbf{k}}(\Delta(\lambda_1)) \otimes \mathrm{KZ}_{\mathbf{k}}(V_2 \otimes_{U_{\mathbf{k}}} \Delta(\mathrm{triv})) \\
&= \mathrm{KZ}_{\mathbf{k}}(\Delta(\lambda_1)) \otimes \mathrm{KZ}_{\mathbf{k}}(\Delta(\lambda_2)) \otimes \mathrm{KZ}_{\mathbf{k}}(\Delta(\mathrm{triv})) \\
&\cong \mathrm{KZ}_{\mathbf{k}}(\Delta(\lambda_1)) \otimes \mathrm{KZ}_{\mathbf{k}}(\Delta(\lambda_2)) \\
&\cong F(V_1 \otimes_{U_{\mathbf{k}}} \Delta(\mathrm{triv})) \otimes F(V_2 \otimes_{U_{\mathbf{k}}} \Delta(\mathrm{triv}))
\end{aligned}$$

where we used that $\mathrm{KZ}_{\mathbf{k}}(\Delta(\mathrm{triv})) = \mathrm{triv} \in \mathrm{Irr}(B_\Gamma)$ and Corollary 3.4.11 to deduce that $V_i \otimes_{U_{\mathbf{k}}} \Delta(\mathrm{triv}) \cong \Delta(\lambda_i)$.

2. By Corollary 2.4.19 the categories $\mathcal{O}_{\mathbf{k}}$ and $\mathcal{H}_{\mathbf{q}}(\Gamma) - \mathrm{mod}$ are equivalent and in particular $\mathcal{H}_{\mathbf{q}}(\Gamma) - \mathrm{mod}$ is semisimple with a complete set of isomorphism classes of simple objects being given by $\{\mathrm{KZ}_{\mathbf{k}}(\Delta(\alpha)) \mid \alpha \in \mathrm{Irr}(\Gamma)\}$. The statement now follows immediately from Theorem 3.4.17.
3. We set

$$\mathrm{Irr}_{\mathbf{k}}^{\mathcal{HC}}(\Gamma) = \{\lambda \in \mathrm{Irr}(\Gamma) \mid \mathcal{L}(\Delta(\mathrm{triv})), \Delta(\lambda)) \neq 0\}$$

and for $\alpha \in \mathrm{Irr}_{\mathbf{k}}(\Gamma)$ we set

$$M(\alpha) = \mathrm{KZ}_{\mathbf{k}}(\Delta(\alpha)).$$

Then by (1) the set

$$\{M(\lambda) \mid \lambda \in \mathrm{Irr}_{\mathbf{k}}^{\mathcal{HC}}(\Gamma)\}$$

generates a subcategory $\mathcal{M}$ of $\mathcal{H}_{\mathbf{q}}(\Gamma) - \mathrm{mod}$ which is closed under tensor products (of B_Γ-modules) and in fact for any $\alpha \in \mathrm{Irr}(\Gamma)$ the tensor product $M(\lambda) \otimes M(\alpha)$ will again be a $\mathcal{H}_{\mathbf{q}}(\Gamma)$-module. Fix a set of generators T_i of $\mathcal{H}_{\mathbf{q}}(\Gamma)$ as in Theorem 2.4.15, in particular the T_i are in 1-1 correspondence with the set of reflection hyperplanes and we will from now on index the generators by the hyperplanes and write T_H instead of i . For ease of notation we also set $q_{H,0} = 1$ for any each $H \in \mathcal{A}$ so that our relations, apart from the braid relations, now read

$$\prod_{j=0}^{m_H - 1} (T_H - e^{-\frac{2\pi i j}{m_H}} q_{H,j}) = 0. \tag{$*_H$}$$

For every $H \in \mathcal{A}$ we then set

$$Q_H = \left\{ e^{-\frac{2\pi i j}{m_H}} q_{H,j} \mid \exists \lambda \in \mathrm{Irr}_{\mathbf{k}}^{\mathcal{HC}}(\Gamma) \text{ with } 0 \neq m \in M(\lambda) : T_H(m) = e^{-\frac{2\pi i j}{m_H}} q_{H,j} m \right\}$$

so that Q_H is the set of all roots of the polynomial $(*_H)$ which occur as the eigenvalue of a vector for one of the $M(\lambda)$. As $1 \in Q_H$ it is not empty.
Let us now show that for any $\gamma \in H$ the sets Q_H and Q_{H^γ} agree. This will follow from property I. Choose an invertible operator I_γ with $I_\gamma T_H = T_{H_\gamma} I_\gamma$. If $m \in M(\lambda)$ is an eigenvector of T_H with eigenvalue a say, a quick calculation shows that $I_\gamma(m)$ is an eigenvector of T_{H^γ} with eigenvalue a:

$$\begin{aligned} T_{H^\gamma}(I_\gamma(m)) &= I_\gamma(T_H(m)) \\ &= aI_\gamma(m). \end{aligned}$$

Hence indeed

$$Q_H = Q_{H^\gamma}$$

for all $\gamma \in \Gamma$.
We now claim that the set Q_H is actually a subgroup of $\mathbb{C}^*$ and hence cyclic. To show this, we will use (1). For $u, v \in Q_H$ take $m_u \in M_u$, $m_v \in M_v$ with $T_H m_u = u m_u$ $T_H m_v = v m_v$ and $M_u, M_v \in \{M(\lambda) \mid \lambda \in \mathrm{Irr}_{\mathbf{k}}^{\mathcal{HC}}(\Gamma)\}$. Then $m_u \otimes m_v \in M_u \otimes M_v$ and

$$T_H(m_u \otimes m_v) = uv(m_u \otimes m_v).$$

Thus Q_H is a multiplicatively closed finite subset of $\mathbb{C}^*$ and hence a group. In particular, Q_H is a set of roots of unity.

Next, we need to show that in fact $\#Q_H$ divides m_H. For this, choose a root of a of $(*_H)$ not in Q_H. As the tensor product $M(\lambda) \otimes M(\alpha)$ is a $\mathcal{H}_{\mathbf{q}}(\Gamma)$-module for any $\lambda \in \mathrm{Irr}_{\mathbf{k}}^{\mathcal{HC}}(\Gamma)$ and $\alpha \in \mathrm{Irr}(\Gamma)$, we can choose a module M and a non-zero vector $m \in M$ with $T_H m = am$ by Lemma 3.4.18. For any $u \in Q_H$ choose $m_u \in M_u$ as before and consider the action of T_H on $m_u \otimes m$. This shows that the product ua is again a root of $(*_H)$ for any root a of $(*_H)$ and any $u \in Q_H$. So for a_1 a root of $(*_H)$ the set

$$Q_H \cup Q_H a$$

consists of roots of $(*_H)$. Choosing another root $a_2 \notin Q_H \cup aQ_H$ we then obtain that $Q_H \cup Q_H a_1 \cup Q_H a_2$ is again a set of roots of $(*_H)$ and so for some $r \in \mathbb{N}_0$ we have that $Q_H \cup Q_H a_1 \cup \ldots \cup Q_H a_r$ is the set of roots of $(*_H)$. We claim that for $i \neq j$ the sets $Q_H a_i$ and $Q_H a_j$ are pairwise disjoint (and we set $a_0 = 1$). Suppose for $u, v \in Q_H$ we had $ua_i = va_j$, as Q_H is a group we then have $a_j = v^{-1} u a_i \in Q_H a_i$ and by construction this forces $j = i$. It is easy to see that each set has size $\#Q_H$ as each a_i is in $\mathbb{C}^*$. By Lemma 3.4.18 part (1) the set of roots of $(*_H)$ has size m_H and hence we must have

$$(r+1)\#Q_H = m_H$$

so that $Q_H \mid m_H$ and so Q_H is a subgroup of μ_{m_H}, the group of m_H-th roots of unity. Now we see that $(*_H)$ reads

$$\prod_{i=0}^{r} \prod_{q \in Q_H} (T - qa_i) = 0. \tag{$**_H$}$$

By construction, the minimal polynomial of T_H on $\mathcal{M}$ (i.e. on any $M(\lambda)$ with $\lambda \in \mathrm{Irr}^{\mathcal{HC}}_{\mathbf{k}}(\Gamma)$) is

$$\mu_H(t) = \prod_{q \in Q_H} (t - q) \qquad (***_H)$$

and this is also the minimal polynomial of T_{H^γ} for any $\gamma \in \Gamma$ as Γ has property I (we already knew that T_H acts semisimply on any object in $\mathcal{H}_{\mathbf{q}}(\Gamma) - \mathrm{mod}$ by semisimplicity of this category and Lemma 3.4.18 (1)).

Thus $\mathcal{M}$ is in fact a category of modules over the algebra $\bar{\mathcal{H}}(\Gamma)$ with generators the T_H and relations the braid relations and relations $(***_H)$ for any $H \in \mathcal{A}$. The algebra $\bar{\mathcal{H}}(\Gamma)$ is a quotient of $\mathcal{H}_{\mathbf{q}}(\Gamma)$ and also a quotient of $\mathcal{H}_{\mathfrak{z}}(\Gamma)$ for any integral parameter choice $\mathfrak{z} = (z_{H,j})_{H \in \mathcal{A}/\Gamma, 1 \leq j \leq m_H - 1}$ with each $z_{H,j} \in \mathbb{Z}$. Note that $\mathcal{H}_{\mathfrak{z}}(\Gamma) \cong \mathbb{C}\Gamma$ and thus we may regard each $M(\lambda)$ as Γ-representation. By construction the tensor product in $\mathcal{M}$ induced from B_Γ and that as Γ-representations agree and so we have identified $\mathcal{M}$ with a full subcategory of $\mathrm{rep}_{\mathbb{C}}\Gamma$ closed under taking tensor products and subquotients. The claim follows from Lemma 3.2.5.

□

Conjecture 1. *Let Γ be a complex reflection group and $\mathbf{k} \in \mathrm{Reg}(\Gamma)$. Then non-trivial Harish-Chandra bimodules over $H_{\mathbf{k}}$ exist if and only if there is at least one orbit $C \in \mathcal{A}/\Gamma$ of reflection hyperplanes such that there exists a non-trivial group Q_C of m_C-th roots of unity (where $m_C = m_H$ for any $H \in C$) such that if T is a generator of $\mathcal{H}_{\mathbf{q}}(\Gamma)$ associated to some $H \in C$ the associated Hecke relation has the form $(**_H)$.*

We will later see that this conjecture is true for cyclic groups and finite irreducible Coxeter groups. Note also that the "only if" part is clear: A non-trivial Harish-Chandra bimodule will give rise to some $M(\lambda)$ that cannot just have all eigenvalues equal to 1 as it is a simple non-trivial $\mathcal{H}_{\mathbf{q}}(\Gamma)$-module. Hence arguing as in the proof of Theorem 3.4.21 we have a Hecke relation of the form $(**_H)$ with Q non-trivial.

It is an immediate question to ask which normal subgroups $N_{\mathbf{k}}$ can occur in the description of $\mathcal{HC}_{\mathbf{k}}$ in Theorem 3.4.21 and how to describe $N_{\mathbf{k}}$ depending on $\mathbf{k} \in \mathrm{Reg}(\Gamma)$. In this chapter we can only give a generic answer

Proposition 3.4.22. *Suppose Γ has property I. For a generic choice of $\mathbf{k} \in \mathrm{Reg}(\Gamma)$ we have $N_{\mathbf{k}} = \Gamma$ and in particular, $U_{\mathbf{k}}$ is the only simple Harish-Chandra bimodule up to isomorphism.*

Proof. It follows from the proof of (3) in Theorem 3.4.21 that the existence of non-trivial Harish-Chandra bimodules at regular parameter values implies that one of the associated parameters of $q_{H,j} = e^{2\pi i k_{H,j}}$ of $\mathcal{H}_{\mathbf{q}}(\Gamma)$ must be a root of unity. Thus choosing $\mathbf{k} \in \mathrm{Reg}(\Gamma)$ such that $k_{H,j} \notin \mathbb{Q}$ for $1 \leq j \leq m_H - 1$ ensures that $\mathcal{HC}_{\mathbf{k}}$ consists solely of $U_{\mathbf{k}}$ and the zero module. Denote this set of parameters by K. It remains to show that this is a "generic choice" of parameters. From Theorem 2.3.11 we can deduce that the complement of $\mathrm{Reg}(\Gamma)$ is contained in a countable union of hyperplanes (see for example also the proof of Lemma 6.8 in [BC09]) and hence $K \cap \mathrm{Reg}(\Gamma)$ cannot be empty. As K is dense in the space of all parameters, $K \cap \mathrm{Reg}(\Gamma)$ is dense in $\mathrm{Reg}(\Gamma)$. □

We shall sketch an alternative proof of this which does not rely on property I at the end of this chapter.

Two more results follow easily from the equivalence of categories in Theorem 3.4.21

Lemma 3.4.23. *Suppose* Γ *has property* I. *Let* $\mathbf{k} \in \operatorname{Reg}(\Gamma)$ *and* $\lambda \in \operatorname{Irr}(\Gamma)$ *a on-dimensional representation. Then if* $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda)) \neq 0$ *it is an invertible bimodule.*

Proof. We set
$$V := \mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda)) \neq 0.$$
Under the equivalence of Theorem 3.4.21 V is sent to a one-dimensional representation of $\Gamma/N_{\mathbf{k}}$ and is thus an invertible object in $(\mathrm{rep}_{\mathbb{C}}(\Gamma/N_{\mathbf{k}}), \otimes)$ and thus in $(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}})$. □

Alternative proofs could use conditions on the Hecke algebra of Γ imposed by the existence of Harish-Chandra bimodules, see for example Chapter 7 or it could use Theorem 3.4.25 as explained there.

Lemma 3.4.24. *Suppose* Γ *has property* I. *Let* $\mathbf{k} \in \operatorname{Reg}(\Gamma)$ *and suppose that* α *is a one-dimensional representation of* Γ *and* $\lambda \in \operatorname{Irr}(\Gamma)$. *Then if* $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\alpha)) \neq 0$ *then* $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\alpha)) \otimes_{U_{\mathbf{k}}} \Delta(\lambda)$ *is isomorphic to* $\Delta(\mu)$ *for some* $\mu \in \operatorname{Irr}(\Gamma)$ *with* μ *of the same rank as* λ.

Proof. If $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\alpha)) \neq 0$ then it is an invertible bimodule by Lemma 3.4.23. Thus $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\alpha)) \otimes_{U_{\mathbf{k}}} \Delta(\lambda)$ is again irreducible and thus isomorphic to $\Delta(\mu)$ for some $\mu \in \operatorname{Irr}(\Gamma)$. Localising we have
$$(\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\alpha)) \otimes_{U_{\mathbf{k}}} \Delta(\lambda))\,|_X \cong \mathcal{M}_\alpha \otimes \mathcal{M}_\lambda$$
and as a $\mathcal{O}_X$-module this is clearly of rank equal to the rank of λ. Thus μ must also have rank equal to λ. □

Of course Lemma 3.4.23 and Lemma 3.4.24 also hold for complex reflection groups that do not have property I. In the proof, we just need to appeal to Theorem 3.2.6 instead of Theorem 3.4.21.
We shall finish with another useful result first established in [BEG03b] for real reflection groups and arbitrary parameter values:

Theorem 3.4.25. *(Proposition 3.8 in [BEG03b], Proposition 5.5.1 in [Los11a]) Let* $\mathbf{k} \in \operatorname{Reg}(\Gamma)$ *and* V *a Harish-Chandra bimodule. Then* V *has a basis consisting of* $\operatorname{ad}(\mathbf{e}\underline{h}\mathbf{e})$*-eigenvectors.*

Proof. It will suffice to prove this for simple Harish-Chandra bimodules by semisimplicity of $\mathcal{HC}_{\mathbf{k}}$ and thus we may suppose that $V = \mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda))$ for some $\lambda \in \operatorname{Irr}(\Gamma)$ by Theorem 3.4.8. Pick a non-zero $f \in \mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda))$ and using Proposition 3.3.2 we may suppose without loss of generality that f is an $\mathbb{C}[\mathfrak{h}]^\Gamma\mathbf{e}$-morphism. Let $\mathbf{1}$ be a $\mathbb{C}[\mathfrak{h}]^\Gamma\mathbf{e}$-basis vector of $\Delta(\mathrm{triv})$ and let v_i be a $\mathbb{C}[\mathfrak{h}]^\Gamma\mathbf{e}$ basis of $\Delta(\lambda)$; for some $p_i \in \mathbb{C}[\mathfrak{h}]^\Gamma\mathbf{e}$ we have
$$f(\mathbf{1}) = \sum_i p_i v_i$$
and this uniquely determines f. Each p_i can be written as a sum of non-zero, linearly independent monomials a_{ij}:
$$p_i = \sum_j a_{ij}$$

with a_{ij} of degree d_{ij}, i.e.

$$\mathrm{ad}(\mathbf{e}\underline{h}\mathbf{e})a_{ij} = d_{ij}a_{ij}.$$

Choose d amongst the d_{ij}, we define an $\mathbb{C}[\mathfrak{h}]^\Gamma\mathbf{e}$-morphism $\tilde{f}$ by

$$\tilde{f}(\mathbf{1}) = \sum_{\substack{i,j \\ d_{ij}=d}} a_i j v_i$$

i.e. for any monomial $p \in \mathbb{C}[\mathfrak{h}]^\Gamma\mathbf{e}$, $\tilde{f}(p\mathbf{1})$ is a projection of $f(p\mathbf{1})$ onto a graded component of $\Delta(\lambda)$. We claim that $\tilde{f}$ is again in $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda))$. For this it remains to show local ad-nilpotence for elements of $\mathbb{C}[\mathfrak{h}^*]^\Gamma\mathbf{e}$ on $\tilde{f}$. But note that distinct graded pieces of $\Delta(\lambda)$ are linearly independent. Take $b \in \mathbb{C}[\mathfrak{h}^*]^\Gamma\mathbf{e}$ and suppose $\mathrm{ad}(b)^D(f) = 0$. Then for any monomial $p \in \mathbb{C}[\mathfrak{h}]^\Gamma\mathbf{e}$ the projections of $\mathrm{ad}(b)^D(f)(p\mathbf{1})$ must vanish separately and thus

$$\mathrm{ad}(b)^D(\tilde{f})(p\mathbf{1}) = 0.$$

Thus $\tilde{f} \in \mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda))$ and it is easy to check that $\tilde{f}$ is an $\mathrm{ad}(\mathbf{e}\underline{h}\mathbf{e})$-eigenfunction since by construction it takes graded pieces of $\Delta(\mathrm{triv})$ to graded pieces of $\Delta(\lambda)$.

So $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda))$ contains at least one non-zero $\mathrm{ad}(\mathbf{e}\underline{h}\mathbf{e})$-eigenfunction and by simplicity this is a generator of $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda))$. The claim follows as $U_\mathbf{k}$ itself has an $\mathrm{ad}(\mathbf{e}\underline{h}\mathbf{e})$-eigenbasis. $\square$

The same strategy might yield a proof in the non-regular case once we weaken the statement to the existence of a generalised $\mathrm{ad}(\mathbf{e}\underline{h}\mathbf{e})$-eigenbasis. However this should also follow from the work of Losev, see the proof of Proposition 5.5.1 in [Los11a].

We will now turn to the alternative proof of Proposition 3.4.22:

Definition 2.3.6 has an immediate extension to $U_\mathbf{k}$-modules and $U_\mathbf{k}$-bimodules when replacing $\underline{h}$ with $\mathbf{e}\underline{h}\mathbf{e}$. We state this new version for convenience

Definition 3.4.26. *1. If M is a $U_\mathbf{k}$-module and $m \in M$ is a generalised $\mathbf{e}\underline{h}\mathbf{e}$-eigenvector with generalised $\mathbf{e}\underline{h}\mathbf{e}$-eigenvalue $\alpha \in \mathbb{C}$, then α is referred to as a "weight" and m is a "weight vector".*

2. *A highest weight module of $U_\mathbf{k}$ is a module M which has a basis of weight vectors such that the set of the real parts of all weights is bounded above.*
3. *A highest weight (of a highest weight module) is a weight whose real part is an upper bound of the set of all weights of that module. A generalised $\mathbf{e}\underline{h}\mathbf{e}$-eigenvector whose weight is a highest weight is called a "highest weight vector."*
4. *If M is a highest weight module, the vector space spanned by all highest weight vectors in M is called the highest weight space.*
5. *If V is a $U_\mathbf{k}$-bimodule and $v \in V$ is a generalised $\mathrm{ad}(\mathbf{e}\underline{h}\mathbf{e})$-eigenvector with generalised $\mathrm{ad}(\mathbf{e}\underline{h}\mathbf{e})$-eigenvalue $\alpha \in \mathbb{C}$, then α is referred to as a "weight" and m is a "weight vector".*
6. *We denote weights of weight vectors $m \in M$ or $v \in V$ by $\mathfrak{w}(m)$ and $\mathfrak{w}(v)$ respectively.*

Consider a standard module $\Delta(\lambda) \in \mathcal{O}_\mathbf{k}(U)$, i.e. a module of the form $\mathbf{e}\Delta(\lambda)$ with $\Delta(\lambda \in \mathcal{O}_\mathbf{k})$. As a $\mathbb{C}\Gamma$-module we have an isomorphism in $\mathcal{O}_\mathbf{k}$ (of $H_\mathbf{k}$)

$$\Delta(\lambda) \cong \mathbb{C}[\mathfrak{h}] \otimes \lambda$$

by Proposition 2.3.7. Then $\mathbf{e}\Delta(\lambda)$ is just the triv-isotypic component of $\mathbb{C}[\mathfrak{h}] \otimes \lambda$. As a Γ-representation we have $\mathbb{C}[\mathfrak{h}] = S\mathfrak{h}^*$, the symmetric algebra on $\mathfrak{h}^*$ and the Γ-action respects degrees, so that for any $i \in \mathbb{N}_0$ the degree i-component $(S\mathfrak{h}^*)_i$ is a Γ-submodule. If i_0 is minimal so that $(S\mathfrak{h}^*)_{i_0} \otimes \lambda$ has a triv-isotypic component.

Definition 3.4.27. *Recall the highest weight $\kappa_{\mathbf{k}}(\lambda)$ of $\Delta(\lambda)$ which is the eigenvalue of $\underline{\mathrm{h}}$ on $1 \otimes \lambda \subset \Delta(\lambda)$, then we denote by $\mathfrak{z}_{\mathbf{k}}(\lambda) = -i_0 + \kappa_{\mathbf{k}}(\lambda)$ the highest weight of $\mathbf{e}\Delta(\lambda)$.*

We further set

$$d = \gcd\{\text{degrees of elements of } \mathbb{C}[\mathfrak{h}^*]^\Gamma \mathbf{e}\}$$

then for any weight vector $a \in \mathbf{e}\Delta(\lambda)$ we have

$$\mathfrak{w}(a) = \mathfrak{z}_{\mathbf{k}}(\lambda) + d\mathbb{Z}.$$

Consider the Harish-Chandra bimodule $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda)) \neq 0$. We may suppose that this is generated by f which is a $\mathbb{C}[\mathfrak{h}]^\Gamma\mathbf{e}$-morphism and an $\mathrm{ad}(\mathbf{e}\underline{\mathrm{h}}\mathbf{e})$-eigenfunction. Thus its weight must be an element of

$$\mathfrak{w}(f) \in \mathfrak{z}(\lambda) - \mathfrak{z}(\mathrm{triv}) + d\mathbb{Z}$$

and as any element of $U_{\mathbf{k}}$ has weights in $d\mathbb{Z}$ we have

$$\mathfrak{w}(v) \in \mathfrak{z}(\lambda) - \mathfrak{z}(\mathrm{triv}) + d\mathbb{Z}$$

for any $v \in V$.

Lemma 3.4.28. *Let V_1, V_2 be $U_{\mathbf{k}}$-bimodules and let $v_1 \in V_1, v_2 \in V_2$ be weight vectors. Then $v_1 \otimes_{U_{\mathbf{k}}} v_2 \in V_1 \otimes_{U_{\mathbf{k}}} V_2$ is again a weight vector and its weight is*

$$\mathfrak{w}(v_1 \otimes_{U_{\mathbf{k}}} v_2) = \mathfrak{w}(v_1) + \mathfrak{w}(v_2).$$

Similarly if $M \in U_{\mathbf{k}} - \mathrm{Mod}$ and $m \in M$ is a weight vector. Then $v_1 \otimes_{U_{\mathbf{k}}} m \in V_1 \otimes_{U_{\mathbf{k}}} M$ is again a weight vector and its weight is

$$\mathfrak{w}(v_1 \otimes_{U_{\mathbf{k}}} m) = \mathfrak{w}(v_1) + \mathfrak{w}(m).$$

Proof. We calculate

$$\begin{aligned}
\mathrm{ad}(\mathbf{e}\underline{\mathrm{h}}\mathbf{e})(v_1 \otimes_{U_{\mathbf{k}}} v_2) &= \mathbf{e}\underline{\mathrm{h}}\mathbf{e}(v_1 \otimes_{U_{\mathbf{k}}} v_2) - (v_1 \otimes_{U_{\mathbf{k}}} v_2)\mathbf{e}\underline{\mathrm{h}}\mathbf{e} \\
&= (\mathbf{e}\underline{\mathrm{h}}\mathbf{e}v_1 \otimes_{U_{\mathbf{k}}} v_2 - (v_1\mathbf{e}\underline{\mathrm{h}}\mathbf{e}) \otimes_{U_{\mathbf{k}}} v_2 \\
&+ v_1 \otimes_{U_{\mathbf{k}}} (\mathbf{e}\underline{\mathrm{h}}\mathbf{e}v_2) - v_1 \otimes_{U_{\mathbf{k}}} (v_2\mathbf{e}\underline{\mathrm{h}}\mathbf{e}) \\
&= \mathfrak{w}(v_1)(v_1 \otimes_{U_{\mathbf{k}}} v_2) + \mathfrak{w}(v_2)(v_1 \otimes_{U_{\mathbf{k}}} v_2) \\
&= (\mathfrak{w}(v_1) + \mathfrak{w}(v_2))(v_1 \otimes_{U_{\mathbf{k}}} v_2).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\mathbf{e}\underline{\mathrm{h}}\mathbf{e}(v_1 \otimes_{U_{\mathbf{k}}} m) &= (\mathbf{e}\underline{\mathrm{h}}\mathbf{e}v_1) \otimes_{U_{\mathbf{k}}} m \\
&= (\mathfrak{w}(v_1)v_1 + v_1\mathbf{e}\underline{\mathrm{h}}\mathbf{e}) \otimes_{U_{\mathbf{k}}} m \\
&= \mathfrak{w}(v_1)(v_1 \otimes_{U_{\mathbf{k}}} m) + v_1 \otimes_{U_{\mathbf{k}}} \mathbf{e}\underline{\mathrm{h}}\mathbf{e}m \\
&= (\mathfrak{w}(v_1) + \mathfrak{w}(m))(v_1 \otimes_{U_{\mathbf{k}}} m).
\end{aligned}$$

□

If $\mathcal{L}(\mathbf{e}\Delta(\mathrm{triv}), \mathbf{e}\Delta(\lambda)) \neq 0$ we know that for any $\alpha \in \mathrm{Irr}(\Gamma)$ we have

$$\mathcal{L}(\mathbf{e}\Delta(\mathrm{triv}), \mathbf{e}\Delta(\lambda)) \otimes_{U_{\mathbf{k}}} \mathbf{e}\Delta(\alpha) = \oplus_i \mathbf{e}\Delta(\beta_i) \neq 0.$$

Thus if non-trivial Harish-Chandra bimodules exist (at regular parameter values), we must be able to find $\beta \in \mathrm{Irr}(\Gamma)$ for any $\alpha \in \mathrm{Irr}(\Gamma)$ such that

$$\mathfrak{z}(\beta) + d\mathbb{Z} = \mathfrak{z}(\lambda) + \mathfrak{z}(\alpha) - \mathfrak{z}(\mathrm{triv}) + d\mathbb{Z}.$$

This is not true in the generic case and we can re-prove Proposition 3.4.22.

Chapter 4

Review of Morita Equivalences and Shift Functors

4.1 Morita Equivalences

In this chapter, we will briefly review some equivalences of categories that we will be using frequently. We will begin by sketching the correct context and establishing some useful results.

Definition 4.1.1. *Let R, S be rings (or k-algebras). We say R and S are Morita equivalent if there is an equivalence of Abelian categories*

$$R - \mathrm{Mod} \to S - \mathrm{Mod}$$

(an equivalence of k-linear categories in the case of k-algebras). Any such equivalence will be called a Morita equivalence.

It is interesting and important to note that any Morita equivalence will respect finite generation of modules. To show this, we will need to give a categorical description of finite generation, which we will do next. The statement of the next result is taken from [Lam98], definition 18.2.

Lemma 4.1.2. *Let M be a (left) module over a ring T. Then M is finitely generated if and only if for for any family of submodules $(N_\alpha)_{\alpha \in A}$ such that $\bigoplus_{\alpha \in A} N_\alpha \twoheadrightarrow M$ we actually have $\bigoplus_{\beta \in B} N_\beta \twoheadrightarrow M$ for some finite subset $B \subseteq A$. Where the maps are understood to be induced from the inclusions $N_\alpha \hookrightarrow M$.*

Proof. Suppose M is finitely generated with generators $m_1, \ldots, m_n$ say. Suppose that $\bigoplus_{\alpha \in A} N_\alpha \twoheadrightarrow M$. By definition we can then write

$$m_k = n^k_{\alpha(k,1)} + \ldots + n^k_{\alpha(k,t_k)}$$

for all $k = 1, \ldots, n$ and some $\alpha(k,1), \alpha(k,2), \ldots, \alpha(k,t_k) \in A$ and $n^k_{\alpha(k,t)} \in N_{\alpha(k,t)}$ for all $k = 1, \ldots, n$ and all $t = 1, \ldots, t_k$. Thus setting

$$B = \{\alpha(k,t) \mid 1 \leq k \leq n, 1 \leq t \leq t_k\}$$

we then have shown that $\sum_{\beta \in B} N_\beta$ contains the generators of M and thus equals M.
Now suppose that M satisfies the property mentioned in the lemma. For any $m \in M$, let

$N_m = Tm$ denote the cyclic submodule generated by m. Then clearly $\bigoplus_{m\in M} N_m \twoheadrightarrow M$ and so $\bigoplus_{\beta=1}^{n} N_{m_\beta} \twoheadrightarrow M$ for some finite set $\{m_1, \dots, m_n\}$ and this is then a set of generators of M. □

Lemma 4.1.3. *Let R and S be Morita equivalent rings with $F : R-\mathrm{Mod} \to S-\mathrm{Mod}$ an equivalence. Then F restricts to an equivalence $F|_{fin} : R-\mathrm{mod} \to S-\mathrm{mod}$.*

Proof. The previous Lemma 4.1.2 characterised finite generation of a module in a categorical way and F preserves direct sums and inclusions (up to natural isomorphism). □

The following theorem is known as Watts' Theorem and is of fundamental importance for much of the theory of Morita equivalences

Theorem 4.1.4. *([Wat60], Theorem 1) Let R, S be rings and $F : R-\mathrm{Mod} \to S-\mathrm{Mod}$ a right exact functor respecting direct sums. There exists an $S-R$-bimodule ${}_SV_R$ such that there is a natural isomorphism of functors $F(\bullet) \cong V \otimes_R \bullet$.*

Proof. This fact is well-known, we repeat a proof here for completeness and follow [Rot09]. Set

$$V := F(R).$$

We will need to endow this with the structure of an $S-R$-bimodule. By definition, it is already a left S-module and we will aim to give a compatible right R-structure. For any $r \in R$ denote by ρ_r the left R-module morphism $R \to R$ given by right multiplication by r. Then $F(\rho_r)$ is an endomorphism of V as a left S-module and so defining

$$v \cdot r := F(\rho_r)(v)$$

will give the desired bimodule structure on V. It remains to show the isomorphism

$$F(M) \cong V \otimes_R M$$

for any $M \in R-\mathrm{Mod}$. For any $m \in M$, we can define a morphism of left R-modules $\rho_m : R \to M, x \mapsto xm$ (note that if $M = R$ then our two definitions of ρ_r agree). Again, we obtain a morphism of left S-modules $F(\rho_m) : V \to F(M)$. We can now define a map

$$\begin{aligned}\Phi_M : V \times M &\to F(M)\\ (v, m) &\mapsto F(\rho_m)(v)\end{aligned}$$

and claim that this descends to $V \otimes_R M$:

$$\begin{aligned}\Phi_M(vr, m) &= F(\rho_m)(vr)\\ &= F(\rho_m)(F(\rho_r)(v))\\ &= F(\rho_{rm})(v)\\ &= \Phi_M(v, rm)\end{aligned}$$

so indeed

$$\Phi_M : V \otimes_R M \to F(M).$$

Further we can check that for any R-modules M, N and any morphism $f : M \to N$ in $R-\mathrm{Mod}$ the following diagram

$$\begin{array}{ccc} V \otimes_R M & \xrightarrow{\Phi_M} & F(M) \\ {\scriptstyle 1\otimes f}\downarrow & & \downarrow{\scriptstyle F(f)} \\ V \otimes_R N & \xrightarrow{\Phi_N} & F(N) \end{array}$$

commutes: racing theT diagram through the top path we have

$$v \otimes_R m \mapsto F(\rho_m(v)) \mapsto F(f)(F(\rho_m)(v))$$

and tracing the bottom path gives

$$v \otimes m \mapsto v \otimes f(m) \mapsto F(\rho_{f(m)})(v).$$

Now $\rho_{f(m)}$ is the map $R \to N$ given by $x \mapsto xf(m)$ and as f is an R-morphism $xf(m) = f(xm)$ and hence $\rho_{f(m)} = f \circ \rho_m$. Thus $F(\rho_{f(m)}) = F(f) \circ F(\rho_m)$ by functoriality and the diagram does indeed commute and so Φ gives us a natural transformation

$$V \otimes_R \bullet \to F(\bullet).$$

It remains to show that this is an isomorphism and for this we will need to show that Φ_M is an isomorphism for any M. By definition, it clearly is an isomorphism if $M \cong R$ and as F respects directs sums, Φ_M will be an isomorphism whenever M is free. As F is right exact, this is all we will need: For an arbitrary R-module M we can find free R-modules A_0 and A_1 such that $A_1 \to A_0 \to M \to 0$ is exact. Then $F(A_1) \to F(A_0) \to F(M) \to 0$ is again exact and we have a commutative diagram

$$\begin{array}{ccccccc} V \otimes_R A_1 & \longrightarrow & V \otimes_R A_0 & \longrightarrow & V \otimes_R M & \longrightarrow & 0 \\ \downarrow{\scriptstyle \Phi_{A_1}} & & {\scriptstyle \Phi_{A_0}}\downarrow & & {\scriptstyle \Phi_M}\downarrow & & \\ F(A_1) & \longrightarrow & F(A_0) & \longrightarrow & F(M) & \longrightarrow & 0 \end{array}$$

with Φ_{A_1} and Φ_{A_0} being isomorphisms. By the 5-Lemma Φ_M then must be an isomorphism too. $\square$

Watts' Theorem in particular implies that any Morita equivalence is given by a tensor functor with a suitable bimodule.

Definition 4.1.5. *Let R, S be rings. We denote by $\mathrm{RFun}^{\oplus}(R, S)$ the category whose objects are those right exact functors $R - \mathrm{Mod} \to S - \mathrm{Mod}$ which respect direct sum and whose the morphisms are natural transformations. For any $F \in \mathrm{RFun}^{\oplus}(R, S)$ we denote by $W(F)$ the $S - R$-bimodule constructed in the proof of Theorem 4.1.4 and for any $V \in S \otimes R^{op} - \mathrm{Mod}$ we denote by $T(V)$ the induced tensor functor $R - \mathrm{Mod} \to S - \mathrm{Mod}$*

Theorem 4.1.6. *There is 1-1 correspondence between right exact functors $R-\mathrm{Mod} \to S-\mathrm{Mod}$ respecting direct sums up to natural isomorphism and $R - S$-bimodules up to isomorphism. In fact, the categories $\mathrm{RFun}^{\oplus}(R, S)$ and $S \otimes R^{op} - \mathrm{Mod}$ are equivalent with equivalences given by W and T from Definition 4.1.5*

Proof. This is essentially Chapter 2, Theorem (2.3) in [Bas68]. Bass attributes this to Eilenberg and Watts. $\square$

Definition 4.1.7. *Let k be a field and R a k-algebra. We denote by $R-\mathrm{fin}$ the full subcategory of $R-\mathrm{mod}$ consisting of those R-modules which are finite-dimensional as k-vector spaces.*

Corollary 4.1.8. *Let R and S be Morita equivalent k-algebras with k a field and $F : R-\mathrm{Mod} \to S-\mathrm{Mod}$ the equivalence. Then F restricts to an equivalence*

$$F|_{fd} : R-\mathrm{fin} \to S-\mathrm{fin}.$$

Proof. By Theorem 4.1.6 any equivalence $R-\mathrm{Mod} \to S-\mathrm{Mod}$ is given by tensoring with an invertible $S-R$-bimodule say $F(\bullet) \cong V\bullet$. As V is invertible, it is finitely generated on both sides. Indeed, we have an $R-S$-bimodule X such that

$$X \otimes_S V \cong R$$

and

$$V \otimes_R X \cong S.$$

We will show that V is finitely generated as a right R-module: Choose $v_1, \ldots, v_n \in V$ and $x_1, \ldots, x_n \in X$ such that under the isomorphism $f : V \otimes_R X \cong S$ we have $f(\sum_i v_i \otimes x_i) = 1_S \in R$. We have isomorphisms

$$\begin{aligned} V &\cong S \otimes_S V \\ &\cong V \otimes_R X \otimes_S V \\ &\cong V \otimes_R R \\ &\cong V \end{aligned}$$

For any $v \in V$ tracing its image under these maps gives

$$\begin{aligned} v &\mapsto 1_S \otimes v \\ &\mapsto \sum_i v_i \otimes x_i \otimes v \\ &\mapsto \sum_i v_i \otimes (x_i v) \\ &\mapsto \sum_i v_i(x_i v) \end{aligned}$$

where $x_i v$ denotes the image of $x_i \otimes v$ in $R \cong X \otimes_S V$. In particular we see that $v_1, \ldots, v_n$ generate V as a right R-module.

Thus for some $n \in \mathbb{N}$ we have surjections $R^{\oplus n} \twoheadrightarrow V_R$ and so if M is a finite-dimensional module in $R-\mathrm{Mod}$ we have surjections of vector spaces

$$M^n \cong R^{\oplus n} \otimes M \twoheadrightarrow V \otimes_R M$$

and as $\dim_k M^n < \infty$ it follows that $F(M) \cong V \otimes_R M$ is finite-dimensional.

By a similar argument, the inverse functor to F must also preserve finite-dimensional modules and so the equivalences restrict as desired. $\square$

Proposition 4.1.9. *Let R, S be rings. They are Morita equivalent if and only if R^{op} and S^{op} are Morita equivalent.*

Proof. Suppose that $F : R-\mathrm{Mod} \to S-\mathrm{Mod}$ is the Morita equivalence and that (up to natural isomorphism) F is given by $X \otimes_R \bullet$ where X is an invertible $S-R$-bimodule. Regarding X as an invertible $R^{op} - S^{op}$-bimodule gives the result. $\square$

Lemma 4.1.10. *Let R and S be Morita equivalent rings. The categories* $(\mathrm{BMod}(R), \otimes_R)$ *and* $(\mathrm{BMod}(S), \otimes_S)$ *are equivalent.*

Proof. Let

$$F : \mathrm{Mod}(R) \xrightarrow{\sim} \mathrm{Mod}(S)$$

and

$$G : \mathrm{Mod}(S) \xrightarrow{\sim} \mathrm{Mod}(R)$$

be quasi-inverse exact functors giving Morita equivalences between R and S. By Theorem 4.1.6, there exist bimodules ${}_SP_R$ and ${}_RQ_S$ such that there are natural isomorphisms of functors

$$F \cong_S P_R \otimes_R \bullet$$
$$G \cong_R Q_S \otimes_S \bullet$$

and moreover we have

$$P \otimes_R Q \cong_S S_S \text{ and } Q \otimes_S P \cong_R R_R.$$

We claim that the functor

$$\begin{aligned} \Psi : \mathrm{BMod}(R) &\to \mathrm{BMod}(S) \\ X &\mapsto P \otimes_R X \otimes_R Q \end{aligned}$$

is an equivalence of the corresponding bimodule categories. Indeed we can easily check that

$$\begin{aligned} \Phi : \mathrm{BMod}(S) &\to \mathrm{BMod}(R) \\ Y &\mapsto Q \otimes_S Y \otimes_S P \end{aligned}$$

is a quasi-inverse to Φ, namely we find:

$$\begin{aligned} (\Phi \circ \Psi)(X) &\cong \Phi(P \otimes_R X \otimes_R Q) \\ &\cong Q \otimes_S (P \otimes_R X \otimes_R Q) \otimes_S P \\ &\cong (Q \otimes_S P) \otimes_R X \otimes_R (Q \otimes_S P) \\ &\cong R \otimes_R X \otimes_R R \\ &\cong X \end{aligned}$$

and

$$\begin{aligned}(\Psi \circ \Phi)(Y) &\cong \Phi(Q \otimes_S Y \otimes_S P) \\ &\cong P \otimes_R (Q \otimes_S Y \otimes_S P) \otimes_R Q \\ &\cong (P \otimes_R Q) \otimes_S Y \otimes_S (P \otimes_R Q) \\ &\cong S \otimes_S Y \otimes_S S \\ &\cong Y\end{aligned}$$

and these are compatible with morphisms. So indeed Ψ and Φ are quasi-inverse exact functors of Abelian categories and hence give the desired equivalence. To show that these respect tensor products we calculate as follows: With X_1, X_2 being R-R-bimodules we have

$$\begin{aligned}\Psi(X_1 \otimes_R X_2) &= P \otimes_R (X_1 \otimes_R X_2) \otimes_R Q \\ &\cong P \otimes_R X_1 \otimes R \otimes_R X_2 \otimes_R Q \\ &\cong (P \otimes_R X_1 \otimes Q) \otimes_S (P \otimes_R X_2 \otimes_R Q) \\ &\cong \Psi(X_1) \otimes_S \Psi(X_2)\end{aligned}$$

and similarly for Φ. $\square$

Theorem 4.1.11. *Let R be a ring and P an R-module. The following are equivalent:*

1. *P is a generator,*
2. *The functor $\bullet \mapsto \mathrm{Hom}_R(P, \bullet)$ is faithful,*
3. *The submodule of R generated by $\{f(p) \mid f \in \mathrm{Hom}_R(P, R), p \in P\}$ is all of R.*
4. *R is a direct summand of a finite direct sum of copies of P.*
5. *R is a direct summand of a direct sum of copies of P.*

Proof. (1) $\implies$ (2) : We need to show that if M, N are R-modules and $f \in \mathrm{Hom}_R(M, N)$ is non-zero, there exists a morphism $\phi : P \to M$ such that the composition $f \circ \phi$ is non-zero. By hypothesis we have a surjection

$$\Phi : \bigoplus_{i \in I} P \twoheadrightarrow M$$

and denoting by $\alpha_i : P \hookrightarrow \bigoplus_{i \in I} P$ the inclusions we have the compositions $\phi_i = \Phi \circ \alpha_i : P \to M$. As $f \neq 0$, we can find $m \in M$ such that $f(m) \neq 0$. We must then have $p_1, \ldots, p_n \in P$ such that $\Phi(\oplus p_i) = \sum_i \phi_i(p_i) = m$ by surjectivity of Φ. Therefore not all of $\phi_1, \ldots, \phi_n$ can have image lying in the kernel of f, so suppose $\phi_j(P)$ is not contained in $\ker(f)$. Then clearly $f \circ \phi_j$ is non-zero and so $\mathrm{Hom}_R(P, \bullet)$ is a faithful functor.
(2) $\implies$ (3): We set

$$T = \{\text{submodule of } R \text{ generated by } f(p) \mid \forall f \in \mathrm{Hom}_R(P, R), \forall p \in P\}$$

and need to show that $T = R$. Suppose not, then we have a non-zero map $\pi : R \twoheadrightarrow R/T$ and thus we must have $\phi \in \mathrm{Hom}_R(P, R)$ such that $\pi \circ \phi$ is non-zero. But this is clearly impossible by definition of T. So we must have $R = T$.
(3) $\implies$ (4) : Keeping the notation just introduced, we have $T = R$ and in particular $1 \in T$.

So we have $f_1, \ldots, f_n$ and $p_1, \ldots, p_n$ with $f_1(p_1) + \ldots + f_n(p_n) = 1$. Then the map

$$\oplus_{i=1}^n f_i \colon \bigoplus_{i=1}^n P \to R$$

contains 1 in the image and so must be surjective. As R is projective, it must be a direct summand of $\bigoplus_{i=1}^n P$.
(4) $\implies$ (5) : Clear.
(5) $\implies$ (1) : Take an index set I such that R is a direct summand of $\oplus_{i\in I} P$ with complement Q say: $\oplus_{i\in I} P = R \oplus Q$. Take an arbitrary module N and for any $n \in N$ define maps $f_n : R \to N$ by $r \mapsto rn$ and extend them to $\oplus_{i\in I} P$ by setting $f_n(Q) = 0$. Then

$$\oplus_{n\in N} f_n : \bigoplus_{n\in N} \bigoplus_{i\in I} P \to N$$

is surjective. □

The following result is well-known in Morita theory, we will give a full proof:

Theorem 4.1.12. *Let $P \in R-\mathrm{Mod}$ be a progenerator. Then R and $\mathrm{End}_R(P)^{op}$ are Morita equivalent with an equivalence given by regarding P as an $R-\mathrm{End}_R(P)^{op}$-bimodule and taking the tensor functor $P \otimes_{\mathrm{End}_R(P)^{op}} \bullet$.*

Proof. Set
$$E = \mathrm{End}_R(P)^{op},$$

P is an $R-E$-bimodule as it is clearly both a left R- and a left E^{op}-module and both of these actions commute by construction. We set

$$Q = \mathrm{Hom}_R(P, R)$$

and note that Q is an $E-R$-bimodule with the action of E being precomposition and the right R action given as
$$(f \cdot r)(x) = f(x)r \text{ for } x \in P, r \in R, f \in Q.$$

Further, Q is non-zero as P is a generator. Now we are able to define functors

$$\begin{aligned} F : E-\mathrm{Mod} &\to R-\mathrm{Mod} \\ N &\mapsto P \otimes_E N \\ G : R-\mathrm{Mod} &\to E-\mathrm{Mod} \\ M &\mapsto Q \otimes_R M \end{aligned}$$

To show that these are indeed inverse, we need to evaluate the tensor products $P \otimes_E Q$ and $Q \otimes_R P$.

We have maps

$$\begin{aligned}\phi : P \otimes_E Q &\to R \\ p \otimes q &\mapsto q(p) \\ \psi : Q \otimes_R P &\to E \\ q \otimes p &\mapsto (a \mapsto q(a)p)\end{aligned}$$

and by Theorem 4.1.11 the map ϕ is surjective as P is a generator. We will show it is also injective Let $\sum_i p_i \otimes q_i \in \ker \phi$ and suppose that $\sum_j u_j \otimes f_j \in \phi^{-1}(1_R)$ so $\sum_j f_j(u_j) = 1$. We calculate

$$\sum_i p_i \otimes q_i = \sum_{i,j} p_i \otimes q_i f_j(u_j).$$

Note that $q_i f_j(u_j)$ is a morphism $P \mapsto R$ given as $a \mapsto q_i(a) f_j(u_j)$. It is equal to the composition of the P-endomorphism $a \mapsto q_i(a)u_j$ with f and in the right action of E on P applying this multiplying p_i on the right with this endomorphism equals $q_i(p_i)u_j$ so that for any i, j we have the equality $p_i \otimes q_i f_j(u_j) = q_i(p_i)u_j \otimes f_j$. Thus

$$\begin{aligned}\sum_i p_i \otimes q_i &= \sum_{i,j} q_i(p_i) u_j \otimes f_j \\ &= 0\end{aligned}$$

as by hypothesis $\sum_i q_i(p_i) = 0$. So we have shown that there exists a bimodule isomorphism

$$\phi : P \otimes_E Q \to R$$

and now we need to work out that ψ is an isomorphism too. As P is finitely generated projective, there exists a collection of R-morphisms $f_1, \ldots, f_n \in \mathrm{Hom}_R(P, R)$ and elements $p_1, \ldots, p_n$ such that $\sum_i f_i(\bullet)p_i$ is the identity map on P. Then clearly $\psi(\sum_i f_i \otimes p_i) = 1_E$ and so ψ is surjective and arguing as before injectivity follows. So we have another isomorphism

$$\psi : Q \otimes_R P \to E.$$

So the functors F and G are inverse and thus equivalences. □

The next Lemma follows from Theorem 4.1.12 and explains why $H_{\mathbf{k}}$ and $U_{\mathbf{k}}$ are Morita equivalent at regular parameter values. The result is known and can be found in many textbooks on ring theory, for example it can be deduced from Proposition 5.6 in [MR01].

Lemma 4.1.13. *Let R be a ring and $e \in R$ an idempotent. Then the functor $R - \mathrm{Mod} \to eRe - \mathrm{Mod}$ given by $M \mapsto eM$ is a Morita equivalence between R and eRe if and only if $ReR = R$.*

Proof. First suppose that $ReR \neq R$. Then R/ReR is a non-zero R-module annihilated by the functor $M \mapsto eM$ and thus this functor cannot be an equivalence.

It remains to show that $ReR = R$ leads to a Morita equivalence as claimed. Let us consider the left R-module Re and explain why this is a progenerator for $R - \mathrm{Mod}$. As e is an idempotent we have a direct sum decomposition (as modules)

$$R = Re \oplus R(1-e)$$

which shows that Re is projective. For any R-module M we have an isomorphism of Abelian groups

$$\mathrm{Hom}_R(Re, M) \cong eM$$

given by taking f to $f(e)$. In particular we have

$$\mathrm{Hom}_R(Re, R) \cong eR$$

with the map $\mathrm{Hom}_R(Re, R) \to eR$ given by $f \mapsto f(e) = ef(e)$ and the inverse $eR \to \mathrm{Hom}_R(Re, R)$ given by sending $er \in eR$ to the right multiplication map by er. Thus the submodule of R generated by $\{f(re) \mid f \in \mathrm{Hom}_R(Re, R), r \in R\}$ is identical to ReR and by our assumption this is equal to R. Thus by Theorem 4.1.11 Re is a generator in $R-\mathrm{Mod}$.
We note that we have an isomorphism of rings

$$(eRe)^{op} \xrightarrow{\sim} \mathrm{End}_R(Re)$$

given by letting elements of $(eRe)^{op}$ act by right multiplication on Re. Thus R and $eRe = \mathrm{End}_R(Re)^{op}$ are Morita equivalent by Theorem 4.1.12 with equivalences given by

$$\begin{aligned} F_e : \mathrm{Mod}(R) &\xrightarrow{\sim} \mathrm{Mod}(eRe) \\ M &\mapsto eM \end{aligned}$$

$$\begin{aligned} G_e : \mathrm{Mod}(eRe) &\xrightarrow{\sim} \mathrm{Mod}(R) \\ N &\mapsto Re \otimes_{eRe} N. \end{aligned}$$

□

The methods of this chapter can be used to show the following:

Theorem 4.1.14. *1. Let $\mathcal{C}$ be an Abelian category in which arbitrary direct sums exist. If P is a progenerator in $\mathcal{C}$ then*

$$\mathrm{Hom}_{\mathcal{C}}(P, \bullet) : \mathcal{C} \to \mathrm{End}_{\mathcal{C}}(P)^{op} - \mathrm{Mod}$$

is an equivalence of Abelian categories.

2. Let $\mathcal{C}$ be an Abelian category in which all objects are Noetherian (i.e. fulfil the ascending chain condition). If P is a progenerator in $\mathcal{C}$, then

$$\mathrm{Hom}_{\mathcal{C}}(P, \bullet) : \mathcal{C} \to \mathrm{End}_{\mathcal{C}}(P)^{op} - \mathrm{mod}$$

is an equivalence of categories and $\mathrm{End}_{\mathcal{C}}(P)$ *is a Noetherian ring.*

Proof. The first part is due to Gabriel and Mitchell, and is Chapter 2, Theorem (1.3) in [Bas68]. The second is given as a an exercise in [Bas68] following Chapter 2, Theorem (1.3) and is ascribed to Lam. □

This well-known fact will underpin much of our later investigation of finite-dimensional Harish-Chandra bimodules of rational Cherednik algebras of cyclic groups.

4.2 Shift Functors

In this section we will outline the construction of certain shift functors, see e.g. [BEG03b] for the analogous construction for the case of real reflection groups. A key definition is that of integral parameter values:

Definition 4.2.1. *Let Γ be a complex reflection group with reflection representation $\mathfrak{h}$. We denote by $\Lambda := \Lambda(\Gamma)$ the set of all integral parameter values of $H_{\mathbf{k}} = H_{\mathbf{k}}(\Gamma, \mathfrak{h})$:*

$$\Lambda = \Lambda(\Gamma) = \Lambda(\Gamma, \mathfrak{h}) = \{k_{H,j} \in \mathbb{Z} \mid H \in \mathcal{A}, 0 \leq j \leq m_H - 1, k_{H,0} = 0\}.$$

We denote by $\mathbf{0}$ the zero parameters, i.e. those parameters for which $k_{H,j} = 0$ for all $H \in \mathcal{A}$ and $0 \leq j \leq m_H - 1$

Lemma 4.2.2. *(see for example Proposition 7.6 in [BC09]) For any complex reflection group Γ with $\dim_{\mathbb{C}} \mathcal{H}_{\mathbf{q}}(\Gamma) = \#\Gamma$ for all $\mathbf{q}$ we have*

$$\Lambda(\Gamma) \subset \mathrm{Reg}(\Gamma).$$

So every integral parameter is regular.

Proof. By Theorem 2.4.21 a parameter will be regular if and only if the associated Hecke algebra $\mathcal{H}_{\mathbf{q}}(\Gamma)$ with $q_{H,j} = e^{2\pi i k_{H,j}}$ is semisimple. But if $\mathbf{k} \in \Lambda$, then $q_{H,j} = 1$ for all $H \in \mathcal{A}$ and $1 \leq j \leq m_H - 1$. Then

$$\mathcal{H}_{\mathbf{q}}(\Gamma) \cong \mathbb{C}\Gamma$$

and so $\mathcal{H}_{\mathbf{q}}(\Gamma)$ is semisimple and $\mathbf{k} \in \mathrm{Reg}(\Gamma)$. ☐

We will also outline a proof of the following:

Theorem 4.2.3. *(see e.g. Proposition 8.11 and Theorem 8.10 in [BEG03b] for real reflection groups) Let Γ be a complex reflection group and $\mathbf{k}, \mathbf{k}' \in \mathrm{Reg}(\Gamma)$ with $\mathbf{k} - \mathbf{k}' \in \Lambda(\Gamma)$. Then $U_{\mathbf{k}}$ and $U'_{\mathbf{k}}$ are Morita equivalent and this equivalence descends to equivalences between $\mathcal{O}_{\mathbf{k}}$ and $\mathcal{O}_{\mathbf{k}'}$ and $(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}})$ and $\left(\mathcal{HC}_{\mathbf{k}'}, \otimes_{U_{\mathbf{k}'}}\right)$*

Variants of these Morita equivalences between $H_{\mathbf{k}}$ and $H_{\mathbf{k}'}$ were studied amongst others by R. Vale in [Val06] in the case of $\Gamma = G(m, 1, n)$ and by Berest, Etingof and Ginzburg for real reflection groups in [BEG03b] as well as Berest-Chalykh in [BC09] the case of more general complex reflection groups.

We will often in this section quote results from the paper [BC09]. It should therefore be noted that the parameters $\mathbf{k}$ used here and the parameters $\mathbf{k}^{BC}$ used in [BC09] are slightly different. This stems from the fact we denote by $\mathbf{e}_{H,j}$ the idempotent of Γ_H corresponding to the representation $\det_{\mathfrak{h}}^{-j}$ whereas in [BC09] the idempotent $\mathbf{e}^{BC}_{H,j}$ corresponds to $\det^{j}_{\mathfrak{h}}$. Thus we have the relation

$$\mathbf{e}_{H,m_H-j} = \mathbf{e}^{BC}_{H,j}$$

which implies that $k_{H,j} - k_{H,j+1} = k^{BC}_{H,m_H-j} - k^{BC}_{H,m_H-j+1}$ from which we deduce that

$$(*)\ k_j = k^{BC}_{H,1} - k^{BC}_{H,m_H-j+1} \text{ and } k^{BC}_j = k_{H,1} - k_{H,m_H-j+1}.$$

Thus while the parameters $\mathbf{k}$ corresponding to $\mathbf{k}^{BC}$ might look fairly different from the $\mathbf{k}^{BC}$, we can immediately note that if $\mathbf{k}$ and $\mathbf{k}^{BC}$ correspond to each other under the map $(*)$, then $\mathbf{k}$ is integral if and only if $\mathbf{k}^{BC}$ is integral and any difference $\mathbf{k} - \mathbf{k}'$ is integral if and only if $\mathbf{k}^{BC} - \mathbf{k}'^{BC}$ is integral. Thus the results of [BC09] concerning integral parameter values still hold although the actual calculations may change.
Recall that we may identify $H_{\mathbf{k}}$ and $U_{\mathbf{k}}$ with certain sets of differential operators via the Dunkl representation $\theta_{\mathbf{k}}$ (see Definition 2.3.10). The following was shown in [BC09]:

Proposition 4.2.4. *([BC09], Theorem 5.7 and remarks following Proposition 7.19) Let Γ be a complex reflection group acting on $\mathfrak{h}$ with $\dim_{\mathbb{C}} \mathcal{H}_{\mathbf{q}}(\Gamma) = \#\Gamma$ for all values of $\mathbf{q}$. Then if $\mathbf{k}, \mathbf{k}'$ are parameter choices such that $\mathbf{k}' - \mathbf{k} \in \Lambda(\Gamma)$, there exists a non-zero Γ-invariant differential operator $S \in \mathcal{D}(\mathfrak{h}^{reg})$ such that*

$$\theta_{\mathbf{k}'}(p)\mathbf{e}S = \mathbf{e}S\theta_{\mathbf{k}}(p)$$

for any $p \in \mathbb{C}[\mathfrak{h}^*]^\Gamma$.

Corollary 4.2.5. *Let Γ be a complex reflection group acting on $\mathfrak{h}$ with $\dim_{\mathbb{C}} \mathcal{H}_{\mathbf{q}}(\Gamma) = \#\Gamma$ for all values of $\mathbf{q}$. Then if $\mathbf{k}, \mathbf{k}'$ are parameter choices such that $\mathbf{k}' - \mathbf{k} \in \Lambda(\Gamma)$, there exists a non-zero Γ-invariant differential operator $S \in \mathcal{D}(\mathfrak{h}^{reg})$ such that*

$$\theta_{\mathbf{k}'}(\mathbf{e}p\mathbf{e})\mathbf{e}S\mathbf{e} = \mathbf{e}S\mathbf{e}\theta_{\mathbf{k}}(\mathbf{e}p\mathbf{e})$$

for any $p \in \mathbb{C}[\mathfrak{h}^*]^\Gamma$.

Proof. This is an immediate consequence of Proposition 4.2.4, we may take the same operator S. □

Using this differential operator $\mathbf{e}S\mathbf{e}$, we will construct an equivalence $(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}}) \cong (\mathcal{HC}_{\mathbf{k}'}, \otimes_{U_{\mathbf{k}'}})$ whenever $\mathbf{k} - \mathbf{k}'$ is integral and $\mathbf{k}$ or $\mathbf{k}'$ are in $\mathrm{Reg}(\Gamma)$.
Following [BEG03b], Formula 8.9, we now make the following definition:

Definition 4.2.6. *Let Γ be a complex reflection group acting on $\mathfrak{h}$ with $\dim_{\mathbb{C}} \mathcal{H}_{\mathbf{q}}(\Gamma) = \#\Gamma$ for all values of $\mathbf{q}$. For any parameters $\mathbf{k}, \mathbf{k}'$ with $\mathbf{k}' - \mathbf{k} \in \Lambda(\Gamma)$ we choose S as in Proposition 4.2.4 and set*

$${}_{\mathbf{k}'}P_{\mathbf{k}} = \theta_{\mathbf{k}'}(U_{\mathbf{k}'})\mathbf{e}S\mathbf{e}\theta_{\mathbf{k}}(U_{\mathbf{k}}) \subseteq \mathcal{D}(\mathfrak{h}^{reg}/\Gamma).$$

This is a $U_{\mathbf{k}'}$-$U_{\mathbf{k}}$-bimodule. We further define a functor

$$\begin{aligned} P_{\mathbf{k}\to\mathbf{k}'} : U_{\mathbf{k}} - \mathrm{Mod} &\to U_{\mathbf{k}'} - \mathrm{Mod} \\ M &\mapsto {}_{\mathbf{k}'}P_{\mathbf{k}} \otimes_{U_{\mathbf{k}}} M \end{aligned}$$

and a functor

$$\begin{aligned} P^\sharp_{\mathbf{k}\to\mathbf{k}'} : \mathrm{BMod}(U_{\mathbf{k}}) &\to \mathrm{BMod}(U'_{\mathbf{k}}) \\ V &\mapsto {}_{\mathbf{k}'}P_{\mathbf{k}} \otimes_{U_{\mathbf{k}}} V \otimes_{U_{\mathbf{k}}} {}_{\mathbf{k}}P_{\mathbf{k}'} \end{aligned}$$

Note that the definition of a Harish-Chandra bimodule naturally extends to $U_{\mathbf{l}}(\Gamma)$-$U_{\mathbf{k}}(\Gamma)$-bimodules as any spherical subalgebra $U_{\mathbf{k}}$ contains the subalgebras $\mathbb{C}[\mathfrak{h}]^{\Gamma}\mathbf{e}$ and $\mathbb{C}[\mathfrak{h}^*]^{\Gamma}\mathbf{e}$ regardless of the parameters.

Definition 4.2.7. *A $U_{\mathbf{k}'}$-$U_{\mathbf{k}}$-bimodule V is called Harish-Chandra if it is a finitely generated and any $a \in \mathbb{C}[\mathfrak{h}]^{\Gamma}\mathbf{e} \cup \mathbb{C}[\mathfrak{h}^*]^{\Gamma}\mathbf{e}$ acts locally* ad*-nilpotently on V. We denote the category of $U_{\mathbf{k}'} - U_{\mathbf{k}}$-Harish-Chandra bimodules by ${}_{\mathbf{k}'}\mathcal{HC}_{\mathbf{k}}$.*

These share many of the established properties of Harish-Chandra bimodules:

Proposition 4.2.8. *Let $V_2 \in {}_{\mathbf{k}'}\mathcal{HC}_{\mathbf{k}'}$,, $V_1, V_1' \in {}_{\mathbf{k}''}\mathcal{HC}_{\mathbf{k}}$ and $M \in \mathcal{O}_{\mathbf{k}}$. Then*

1. *V_1 is finitely generated as a left $U_{\mathbf{k}'}$,-bimodule and as a right $U_{\mathbf{k}}$-bimodule.*
2. *If both $\mathbf{k}', \mathbf{k} \in \mathrm{Reg}(\Gamma)$ then V_1 is a progenerator in both $U_{\mathbf{k}''} - \mathrm{Mod}$ and $\mathrm{Mod} - U_{\mathbf{k}}$.*
3. *If $\mathbf{k}, \mathbf{k}'' \in \mathrm{Reg}(\Gamma)$, a morphism $f : V_1 \to V_1'$ is an isomorphism if and only if it is an isomorphism upon localisation*
4. *$V_2 \otimes V_1$ is an object in ${}_{\mathbf{k}'}\mathcal{HC}_{\mathbf{k}}$.*
5. *$V_1 \otimes_{U_{\mathbf{k}}} M$ is an object of $\mathcal{O}_{\mathbf{k}''}$.*

Proof. As the corresponding proofs in Chapter 3, see also Lemma 3.3, Corollary 3.6 and Lemma 8.3 in [BEG03b]. □

Lemma 4.2.9. *Suppose that $\mathbf{k} \in \mathrm{Reg}(\Gamma)$ and $\mathbf{k}' - \mathbf{k} \in \Lambda(\Gamma)$. Then $\mathbf{k}' \in \mathrm{Reg}(\Gamma)$.*

Proof. This follows from Theorem 2.4.21 as the Hecke algebras associated to $H_{\mathbf{k}}$ and $H_{\mathbf{k}'}$ are isomorphic and thus $\mathcal{O}_{\mathbf{k}'}$ is simple when $\mathcal{O}_{\mathbf{k}}$ is. □

Probably the most important payoff of all of this for us is (4) in the following theorem, essentially allowing us to "shift parameters by integers" whenever we are performing calculations to determine $(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}})$. This is analogous to the situation of real reflection groups, see for example Proposition 8.11 in [BEG03b].

Theorem 4.2.10. *(compare Proposition 8.11 in [BEG03b]) Under the hypotheses of Definition 4.2.6, the bimodule ${}_{\mathbf{k}'}P_{\mathbf{k}}$ is $U_{\mathbf{k}'}$-$U_{\mathbf{k}}$ Harish-Chandra. Moreover, if $\mathbf{k}$ or $\mathbf{k}'$ are regular the following hold:*

1. *$P_{\mathbf{k}\to\mathbf{k}'}$ is a Morita equivalence between $U_{\mathbf{k}}$ and $U_{\mathbf{k}'}$ with inverse $P_{\mathbf{k}'\to\mathbf{k}}$.*
2. *$P_{\mathbf{k}\to\mathbf{k}'}$ restricts to an equivalence $\mathcal{O}_{\mathbf{k}} \to \mathcal{O}_{\mathbf{k}'}$.*
3. *$P^{\sharp}_{\mathbf{k}\to\mathbf{k}'} : \mathrm{BMod}(U_{\mathbf{k}}) \to \mathrm{BMod}(U_{\mathbf{k}'})$ is an equivalence of tensor categories with inverse $P^{\sharp}_{\mathbf{k}'\to\mathbf{k}}$.*
4. *$P^{\sharp}_{\mathbf{k}\to\mathbf{k}'}$ restricts to an equivalence $(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}}) \cong (\mathcal{HC}_{\mathbf{k}'}, \otimes_{U_{\mathbf{k}'}})$.*

Proof. The bimodule ${}_{\mathbf{k}'}P_{\mathbf{k}}$ is certainly finitely generated (indeed cyclic) and so will be Harish-Chandra once we can show that any element $a \in \mathbb{C}[\mathfrak{h}]^{\Gamma}\mathbf{e} \cup \mathbb{C}[\mathfrak{h}^*]^{\Gamma}\mathbf{e}$ acts locally ad-nilpotently on $\mathbf{e}S\mathbf{e}$. But by Proposition 4.2.4 this is true for any $a \in \mathbb{C}[\mathfrak{h}^*]^{\Gamma}\mathbf{e}$ and holds for $a \in \mathbb{C}[\mathfrak{h}]^{\Gamma}\mathbf{e}$ as S is a differential operator. Now it is clear that $P_{\mathbf{k}\to\mathbf{k}'}$ will restrict to a functor $\mathcal{O}_{\mathbf{k}} \to \mathcal{O}_{\mathbf{k}'}$ and that $P^{\sharp}_{\mathbf{k}\to\mathbf{k}'}$ will restrict to a functor $\mathcal{HC}_{\mathbf{k}} \to \mathcal{HC}_{\mathbf{k}'}$.
Upon localising ${}_{\mathbf{k}}P_{\mathbf{k}'} \otimes_{U_{\mathbf{k}'}} {}_{\mathbf{k}'}P_{\mathbf{k}}$ to $\mathfrak{h}^{reg}/\Gamma$ we obtain a surjection

$$\left({}_{\mathbf{k}}P_{\mathbf{k}'} \otimes_{U_{\mathbf{k}'}} {}_{\mathbf{k}'}P_{\mathbf{k}}\right) |_{\mathfrak{h}^{reg}/\Gamma} \twoheadrightarrow \mathcal{D}(\mathfrak{h}^{reg}/\Gamma)$$

as $\mathcal{D}(\mathfrak{h}^{reg}/\Gamma)$ is a simple ring. Hence

$$\left({}_{\mathbf{k}}P_{\mathbf{k}'} \otimes_{U_{\mathbf{k}'}} {}_{\mathbf{k}'}P_{\mathbf{k}} \otimes_{U_{\mathbf{k}}} \Delta_{\mathbf{k}}(\mathrm{triv})\right) |_{\mathfrak{h}^{reg}/\Gamma} \twoheadrightarrow \Delta_{\mathbf{k}}(\mathrm{triv}) |_{\mathfrak{h}^{reg}/\Gamma} \cong \mathcal{O}(\mathfrak{h}^{reg}/\Gamma)$$

and thus we must have a morphism in $\mathcal{O}_{\mathbf{k}}$

$$ {}_{\mathbf{k}}P_{\mathbf{k}'} \otimes_{U_{\mathbf{k}'}} {}_{\mathbf{k}'}P_{\mathbf{k}} \otimes_{U_{\mathbf{k}}} \Delta_{\mathbf{k}}(\mathrm{triv}) \twoheadrightarrow \Delta_{\mathbf{k}}(\mathrm{triv}) $$

giving a non-zero morphism

$$ {}_{\mathbf{k}}P_{\mathbf{k}'} \otimes_{U_{\mathbf{k}'}} {}_{\mathbf{k}'}P_{\mathbf{k}} \to U_{\mathbf{k}} $$

in $\mathcal{HC}_{\mathbf{k}}$. Upon localising, we obtain an isomorphism

$$ \left({}_{\mathbf{k}}P_{\mathbf{k}'} \otimes_{U_{\mathbf{k}'}} {}_{\mathbf{k}'}P_{\mathbf{k}}\right) \xrightarrow{\sim} \mathcal{D}(\mathfrak{h}^{reg}/\Gamma) $$

by the multiplication map and thus we must have an isomorphism

$$ {}_{\mathbf{k}}P_{\mathbf{k}'} \otimes_{U_{\mathbf{k}'}} {}_{\mathbf{k}'}P_{\mathbf{k}} \xrightarrow{\sim} U_{\mathbf{k}} $$

in $\mathcal{HC}_{\mathbf{k}}$. The exact same arguments will show that

$$ {}_{\mathbf{k}'}P_{\mathbf{k}} \otimes_{U_{\mathbf{k}}} {}_{\mathbf{k}}P_{\mathbf{k}'} \xrightarrow{\sim} U_{\mathbf{k}'}. $$

So ${}_{\mathbf{k}'}P_{\mathbf{k}}$ is an invertible bimodule and hence $P_{\mathbf{k}\to\mathbf{k}'}$ and $P^{\sharp}_{\mathbf{k}\to\mathbf{k}'}$ are indeed equivalences. □

Corollary 4.2.11. *Let $\mathbf{k}, \mathbf{k}', \mathbf{l}, \mathbf{l}'$ be parameters for Γ with $\mathbf{k}' - \mathbf{k}, \mathbf{l}' - \mathbf{l} \in \Lambda(\Gamma)$ and $\mathbf{k}$ or $\mathbf{k}'$ regular and $\mathbf{l}$ or $\mathbf{l}'$ regular. Then ${}_{\mathbf{l}'}\mathcal{HC}_{\mathbf{k}'}$ and ${}_{\mathbf{l}}\mathcal{HC}_{\mathbf{k}}$ are equivalent as Abelian categories.*

Proof. The proof of Theorem 4.2.10 shows that under these hypotheses the bimodules ${}_{\mathbf{k}'}P_{\mathbf{k}}$ and ${}_{\mathbf{l}'}P_{\mathbf{l}}$ are invertible. □

4.3 Integral Parameter Values

To illustrate the great usefulness of these shift functors introduced in the previous section, we will investigate the category $\mathcal{HC}_{\mathbf{k}}(\Gamma)$ in the case when $\mathbf{k}$ is integral. This was first done for real reflection groups in [BEG03b] and in essence extended to arbitrary complex reflection groups in [BC09]. The method involved differential operators on quasi-invariants to construct all Harish-Chandra bimodules and the notion of quasi-invariants was later extended to all complex reflection groups by Berest and Chalykh. We start with a short lemma:

Lemma 4.3.1. *Suppose Γ is a complex reflection group and $\mathbf{k} \in \mathrm{Reg}(\Gamma)$ is such that up to isomorphism $\mathcal{HC}_{\mathbf{k}}$ has $\#\mathrm{Irr}(\Gamma)$ simple objects. Then there is an equivalence*

$$ (\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}}) \cong (\mathrm{rep}_{\mathbb{C}}(\Gamma), \otimes)\,. $$

Proof. By Corollary 3.4.21 we have an equivalence

$$ (\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}}) \cong (\mathrm{rep}_{\mathbb{C}}(\Gamma/N_{\mathbf{k}}), \otimes) $$

for some normal $N_{\mathbf{k}} \trianglelefteq \Gamma$. By hypothesis we must have

$$ \#\mathrm{Irr}\,(\Gamma/N_{\mathbf{k}}) = \#\mathrm{Irr}(\Gamma) $$

and since we have an injection of sets

$$\mathrm{Irr}\left(\Gamma/N_{\mathbf{k}}\right) \hookrightarrow \mathrm{Irr}(\Gamma)$$

this must mean that every irreducible Γ-representation factors through $\Gamma/N_{\mathbf{k}}$. If $N_{\mathbf{k}}$ were not the trivial group, then we could in particular deduce that Γ has no faithful irreducible representations. This is a contradiction, as can be seen from Theorem 2.1.12. Alternatively note that our definition of $H_{\mathbf{k}}(\Gamma)$ restricts us to complex reflection groups Γ for which $\mathfrak{h}$ is an irreducible, faithful representation. Thus $N_{\mathbf{k}} = \{1\}$ and we have $(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}}) \cong (\mathrm{rep}_{\mathbb{C}}(\Gamma), \otimes)$ as claimed. □

Theorem 4.3.2. *(Theorem 8.5 and Proposition 8.11 in [BEG03b] and essentially also Proposition 4.3 with Theorem 8.2 in [BC09]) Let* $\mathbf{k} \in \Lambda$*. Then there is an equivalence of categories*

$$(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}}) \cong (\mathrm{rep}_{\mathbb{C}}\Gamma, \otimes)\,.$$

Proof. By Theorem 4.2.10, we have an equivalence of categories

$$(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}}) \cong (\mathcal{HC}_{\mathbf{0}}, \otimes_{U_{\mathbf{0}}})$$

and from Theorem 2.3.9 it follows that the Dunkl representation of $U_{\mathbf{0}}$ gives an isomorphism

$$U_{\mathbf{0}} \cong \mathcal{D}(\mathfrak{h})^{\Gamma} \subset \mathcal{D}(\mathfrak{h}^{reg})^{\Gamma}.$$

This identifies $\mathbb{C}[\mathfrak{h}]^{\Gamma}\mathbf{e} \subset U_{\mathbf{0}}$ with $\mathbb{C}[\mathfrak{h}]^{\Gamma} \subset \mathcal{D}(\mathfrak{h})$ and if we denote by R the subring of $\mathcal{D}(\mathfrak{h})$ generated by derivations, then $\mathbb{C}[\mathfrak{h}^*]^{\Gamma}$ is identified with R^{Γ}. Any element of $\mathbb{C}[\mathfrak{h}]$ or R acts locally ad-nilpotently on $\mathcal{D}(\mathfrak{h})$ and thus $\mathbb{C}[\mathfrak{h}]^{\Gamma}$ and R^{Γ} act locally ad-nilpotently on $\mathcal{D}(\mathfrak{h})$. Hence, $\mathcal{D}(\mathfrak{h})$ is a $U_{\mathbf{0}} \cong \mathcal{D}(\mathfrak{h})^{\Gamma}$ Harish-Chandra bimodule. For any $\lambda \in \mathrm{rep}_{\mathbb{C}}(\Gamma)$ we set

$$\mathcal{D}_{\lambda} := \left(\lambda^{\vee} \otimes_{\mathbb{C}} \mathcal{D}(\mathfrak{h})\right)^{\Gamma}$$

where $\lambda^{\vee}$ denotes the dual representation of λ. Note that $\mathcal{D}_{\lambda}$ is a $\mathcal{D}(\mathfrak{h})^{\Gamma}$-bimodule and in particular each $\mathcal{D}_{\lambda}$ gives rise to a $U_{\mathbf{0}}$-Harish-Chandra bimodule (because if $\lambda \in \mathrm{Irr}(\Gamma)$, then $\mathcal{D}_{\lambda}$ isomorphic to a sub-bimodule of the Harish-Chandra bimodule $\mathcal{D}(\mathfrak{h})$). The Dunkl representation identifies $\Delta_{\mathbf{0}}(\mathrm{triv})$ with the $\mathcal{D}(\mathfrak{h})^{\Gamma}$-module $\mathcal{O}(\mathfrak{h})^{\Gamma}$ and more generally for $\lambda \in \mathrm{Irr}(\Gamma)$

$$\Delta_{\mathbf{0}}(\lambda) \mapsto \mathcal{O}_{\lambda} = \left(\lambda^{\vee} \otimes_{\mathbb{C}} \mathcal{O}(\mathfrak{h})\right)^{\Gamma}.$$

We therefore obtain an evaluation map

$$\mathcal{D}_{\lambda} \otimes_{\mathcal{D}^{\Gamma}} \mathcal{O}^{\Gamma} \to \mathcal{O}_{\lambda}$$

and it is easy to check that this is an isomorphism for it factors as follows:

$$\begin{aligned}\mathcal{D}_{\lambda} \otimes_{\mathcal{D}^{\Gamma}} \mathcal{O}^{\Gamma} &\cong \left(\lambda^{\vee} \otimes_{\mathbb{C}} \mathcal{D}\right)^{\Gamma} \otimes_{\mathcal{D}^{\Gamma}} \left(\mathrm{triv} \otimes_{\mathbb{C}} \mathcal{O}\right)^{\Gamma} \\ &\cong \left(\lambda^{\vee} \otimes \mathrm{triv} \otimes_{\mathbb{C}} \mathcal{D} \otimes_{\mathcal{D}} \mathcal{O}\right)^{\Gamma} \\ &\cong \left(\lambda^{\vee} \otimes_{\mathbb{C}} \mathcal{O}\right)^{\Gamma}.\end{aligned}$$

Thus we have an isomorphism

$$\mathcal{D}_\lambda \xrightarrow{\sim} \mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda))$$

and therefore the $\mathcal{D}_\lambda$ for $\lambda \in \mathrm{Irr}(\Gamma)$ are simple, non-isomorphic Harish-Chandra bimodules and by Corollary 3.4.10 they are precisely all simple Harish-Chandra bimodules up to isomorphism. Thus $\mathcal{HC}_{\mathbf{0}}$ has $\#\mathrm{Irr}(\Gamma)$ simple objects up to isomorphism and the result then follows from Lemma 4.3.1. □

Chapter 5

Cyclic Groups

In this chapter we will describe precisely for which regular parameter values non-trivial Harish-Chandra bimodules exist and present a way of constructing these. We will also be able to determine how many distinct (up to isomorphism) simple Harish-Chandra bimodules exist for a regular parameter. The methods used in this chapter are computational and largely independent of the work done in Chapter 3. We will use similar ideas and arguments in Chapter 7.

5.1 Rational Cherednik algebras of cyclic groups

For the purpose of this chapter, we will use W to denote a finite cyclic group of order m. In this section, we shall gather together some basic definitions and computations regarding the rational Cherednik algebra of W as well as the spherical subalgebra.

Definition 5.1.1. *We choose a generator $s \in W$ and let it act on $\mathfrak{h} = \mathbb{C}$ via multiplication by ρ^{-1}, where ρ is a primitive m-th root of unity. We then denote by E_p the one-dimensional representation of W on which s acts via ρ^p, so that $\mathfrak{h} = E_{m-1}$. For each E_p choose a basis vector $v_p \in E_p$.*

Although the above choices seem slightly odd, computationally they will be advantageous.

Recall from Example 2.1.5 that W is a complex reflection group with reflection representation $\mathfrak{h}$ and reflection hyperplane $\{0\} \subset \mathfrak{h}$ and $W_{\{0\}} = W$. Thus it has an associated rational Cherednik algebra. We let y be a basis vector of $\mathfrak{h}$ and x its dual in $\mathfrak{h}^*$. We may choose $\alpha_{\{0\}} = x$ and $v_{\{0\}} = y$ in the terminology of Definition 2.2.3. We denote by $\mathbf{e}_j$ the group idempotent associated to the character $\det|_{\mathfrak{h}}^{-j}$ with $\mathfrak{h} = E_{m-1}$ as in Definition 5.1.1. Let us write out $\mathbf{e}_j$ in full detail:

$$\mathbf{e}_j = \frac{1}{m} \sum_{i=0}^{m-1} \rho^{-ij} s^i.$$

Then the rational Cherednik algebra $H_{\mathbf{k}}(W)$ is generated by $y \in \mathfrak{h}, x \in \mathfrak{h}^*$ and $s \in W$ subject to the relations

$$sx = \rho x s, \quad sy = \rho^{-1} y s$$
$$[y, y] = 0 \quad [x, x] = 0$$
$$[y, x] = 1 + m \sum_{j=0}^{m-1} (k_j - k_{j+1}) \mathbf{e}_j$$

Lemma 5.1.2. *The choice of parameters* $\mathbf{k} = (k_0 = 0, k_1, \ldots, k_{m-1})$ *is in* $\mathrm{Reg}(W)$ *if and only if*

$$k_a - k_b - \frac{a-b}{m} \notin \mathbb{Z}$$

for all $a \neq b \in \{0, \ldots, m-1\}$.

Proof. Recall from Example 2.4.16 (or directly from Definition 2.4.14) that the Hecke algebra $\mathcal{H}_{\mathbf{q}}(W)$ is given by

$$\mathcal{H}_{\mathbf{q}}(W) = \frac{\mathbb{C}[T]}{(T-1)\prod_{j=1}^{m-1}(T - e^{-\frac{2\pi i j}{m}} e^{2\pi i k_j})} = \frac{\mathbb{C}[T]}{\prod_{j=0}^{m-1}(T - e^{-\frac{2\pi i j}{m}} e^{2\pi i k_j})}.$$

From Theorem 2.4.21 we can deduce that $\mathrm{Reg}(W) = \{\mathbf{k} \mid \mathcal{H}_{\mathbf{q}}(W) \text{ is semisimple}\}$ and $\mathcal{H}_{\mathbf{q}}(W)$ is semisimple if and only if it has no repeated roots. So $\mathbf{k} \in \mathrm{Reg}(W)$ if and only if for any $a \neq b \in \{0, 1 \ldots, m-1\}$ we have

$$\begin{aligned} e^{-\frac{2\pi i a}{m}} e^{2\pi i k_a} &\neq e^{-\frac{2\pi i b}{m}} e^{2\pi i k_b} \\ \iff e^{2\pi i\left(\frac{(a-b)}{m} + (k_b - k_a)\right)} &\neq 1 \\ \iff \frac{(a-b)}{m} + (k_b - k_a) &\notin \mathbb{Z} \\ \iff k_a - k_b - \frac{a-b}{m} &\notin \mathbb{Z} \end{aligned}$$

□

We will be able to give an alternative proof of Lemma 5.1.2 when determining the set of aspherical parameter values for W. Now recall that for each $E_p \in \mathrm{Irr}(W)$ we have the standard modules $\Delta(E_p)$ and we will next describe the action of the generators of $H_{\mathbf{k}}$ on the basis $\{x^n \otimes v_p \mid n \in \mathbb{N}_0\}$ of $\Delta(E_p)$: Note that

$$[y, x^n] = x^{n-1}(n + m\sum_{j=0}^{m-1}(k_j - k_{j+n})\mathbf{e}_j)$$

and so

$$\begin{aligned} y(x^n \otimes v_p) &= [y, x^n] \otimes v_p \\ &= x^{n-1}(n + m\sum_{j=0}^{m-1}(k_j - k_{j+n})\mathbf{e}_j)(1 \otimes v_p) \\ &= x^{n-1}(n + m(k_p - k_{p+n}))(1 \otimes v_p) \\ &= (n - m(k_{p+n} - k_p))(x^{n-1} \otimes v_p) \end{aligned}$$

The grading element $\underline{\mathrm{h}}$ is given by

$$\underline{\mathrm{h}} = -xy - \frac{1}{2} - \sum_{j}^{m-1} mk_j\mathbf{e}_j + \sum_{j=0}^{m-1} k_j.$$

acts on $(1 \otimes v_p)$ as follows:

$$\underline{\mathrm{h}}(1 \otimes v_p) = (-\frac{1}{2} - mk_p + \sum_{j=0}^{m-1} k_j)(1 \otimes v_p).$$

Definition 5.1.3. *Let W be a cyclic group of order m. For $p \in \{0, \ldots, m-1\}$ we set*

$$a_p := \begin{cases} 0 \text{ if } p = 0 \\ m - p \text{ else} \end{cases}$$
$$\text{i.e. } a_p = (1 - \delta_{0,p})(m - p).$$

Note further that we clearly have

$$\mathbb{C}[\mathfrak{h}]^W = \mathbb{C}[x^m] \text{ and } \mathbb{C}[\mathfrak{h}^*]^W = \mathbb{C}[y^m].$$

The $U_{\mathbf{k}}$-module $\mathbf{e}\Delta(E_p)$ then has basis given by $\mathbf{e}x^{rm}x^{a_p} \otimes v_p$ and we can give the action of $\mathbf{e}y^m\mathbf{e}$ on this basis by direct calculation: We have

$$y^l(x^n \otimes v_p) = \left(\prod_{r=0}^{l-1} (n - r - m(k_{p+n-r} - k_p)) \right) (x^{n-l} \otimes v_p)$$

and thus

$$\mathbf{e}y^{lm}(\mathbf{e}x^{a_p+nm} \otimes v_p) = \left(\prod_{r=0}^{lm-1} \left(a_p + nm - r - mk_{a_p+p+nm-r} + mk_p\right) \right) (x^{a_p+(n-l)m} \otimes v_p)$$

and the action of the grading element $\mathbf{e}\underline{\mathrm{h}}\mathbf{e}$ on $\mathbf{e}x^{a_p} \otimes v_p$ is

$$\mathbf{e}\underline{\mathrm{h}}\mathbf{e}(\mathbf{e}x^{a_p} \otimes v_p) = (-a_p - \frac{1}{2} - mk_p + \sum_{j=0}^{m-1} k_j)(\mathbf{e}x^{a_p} \otimes v_p).$$

By Theorem 2.4.2 the spherical subalgebra is generated by $\mathbf{e}x^m\mathbf{e}$ and $\mathbf{e}y^m\mathbf{e}$ if $\mathbf{k}$ is regular. It is also known that for cyclic groups the spherical subalgebra is a special case of a type of algebra first studied by Smith [Smi90] and Hodges [Hod93]. We give a complete description of $U_{\mathbf{k}}$ in terms of generators and relations in the next result but need to give an important definition first:

Definition 5.1.4. *Let $\mathbf{k} = (k_0 = 0, k_1, \ldots, k_{m-1})$ be any choice of parameters. Recall that we have set*

$$a_j = a_j(m) = (1 - \delta_{0,j})(m - j) \text{ for } 0 \leq j \leq m - 1.$$

We now define

$$\chi_{\mathbf{k}}(t) := \chi_{\mathbf{k}}^W(t) := \prod_{j=0}^{m-1} \left(t - \frac{a_j}{m} - k_j\right) \in \mathbb{C}[t]$$

Proposition 5.1.5. *For any choice of parameter $\mathbf{k}$ the algebra $U_{\mathbf{k}}$ is isomorphic to the algebra*

$R_{\chi_{\mathbf{k}}}$ with generators X, Y, Z and relations

$$XY = m^m \chi_{\mathbf{k}}(Z) \text{ and } YX = m^m \chi_{\mathbf{k}}(Z+1)$$
$$[Z, X] = X \text{ and } [Z, Y] = -Y$$

The isomorphism is given by

$$X \mapsto \mathbf{e}x^m\mathbf{e}$$
$$Y \mapsto \mathbf{e}y^m\mathbf{e}$$
$$Z \mapsto \frac{\mathbf{e}xy\mathbf{e}}{m}.$$

In particular if $\chi_{\mathbf{k}}(t) = \chi_{\mathbf{k}'}(t)$ then $U_{\mathbf{k}} \cong U'_{\mathbf{k}}$, this isomorphism preserves $\mathbb{C}[\mathfrak{h}]^W\mathbf{e}$ and $\mathbb{C}[\mathfrak{h}^*]^W\mathbf{e}$ and hence gives equivalences $\mathcal{O}_{\mathbf{k}} \cong \mathcal{O}_{\mathbf{k}'}$ as well as $\mathcal{HC}_{\mathbf{k}} \cong \mathcal{HC}_{\mathbf{k}'}$.

Proof. We need to verify first that $\mathbf{e}x^m\mathbf{e}, \mathbf{e}y^m\mathbf{e}$ and $\mathbf{e}xy\mathbf{e}$ fulfil the relations given. The last are easy to check, using the fact that $\mathbf{e}\underline{\mathrm{h}}\mathbf{e} = -\mathbf{e}xy\mathbf{e} - (\frac{1}{2} + k_1)\mathbf{e}$ we find

$$\left[\frac{\mathbf{e}xy\mathbf{e}}{m}, \mathbf{e}x^m\mathbf{e}\right] = \frac{-1}{m}[\mathbf{e}\underline{\mathrm{h}}\mathbf{e}, \mathbf{e}x^m\mathbf{e}] = \frac{-1}{m}\mathbf{e}[\underline{\mathrm{h}}, x^m]\mathbf{e} = \mathbf{e}x^m\mathbf{e}$$

and similarly for $[\frac{\mathbf{e}xy\mathbf{e}}{m}, \mathbf{e}y^m\mathbf{e}]$. For any polynomial $f(\mathbf{e}xy\mathbf{e})$ this then implies that

$$f(\mathbf{e}xy\mathbf{e})\mathbf{e}x^m\mathbf{e} = \mathbf{e}x^m\mathbf{e}f(\mathbf{e}xy\mathbf{e} + m) \text{ and } f(\mathbf{e}xy\mathbf{e})\mathbf{e}y^m\mathbf{e} = \mathbf{e}y^m\mathbf{e}f(\mathbf{e}xy\mathbf{e} - m).$$

We will recursively define two families of polynomials $p_s^W(t), q_s^W(t) \in \mathbb{C}$ by

$$p_0^W(t) = 1 \text{ and } p_{s+1}^W(t) = (t - (s + mk_{m-s}))\, p_s^W(t)$$

and

$$q_0^W(t) = 1 \text{ and } q_{s+1}^W(t) = (t + s - mk_{s+1} + mk_1))\, q_s^W(t)$$

Direct computation then shows that

$$\mathbf{e}x^{s+1}y^{s+1}\mathbf{e} = (\mathbf{e}xy\mathbf{e} - s - mk_{m-s})\, \mathbf{e}x^s y^s\mathbf{e}$$
$$\mathbf{e}y^{s+1}x^{s+1}\mathbf{e} = (\mathbf{e}yx\mathbf{e} + s - mk_{s+1} + mk_1)\, \mathbf{e}y^s x^s\mathbf{e}$$

and so we can conclude that

$$\mathbf{e}x^s y^s\mathbf{e} = p_s^W(\mathbf{e}xy\mathbf{e}) \text{ and } \mathbf{e}y^s x^s\mathbf{e} = q_s^W(\mathbf{e}yx\mathbf{e}).$$

Using the explicit formulae

$$p_{s+1}^W(t) = \prod_{l=0}^{s}(t - l - mk_{m-l}) \text{ and } q_{s+1}^W(t) = \prod_{l=0}^{s}(t + l - mk_{l+1} + mk_1)$$

we can deduce that

$$\mathbf{e}x^m y^m\mathbf{e} = \prod_{l=0}^{m-1}(\mathbf{e}xy\mathbf{e} - l - mk_{m-l}) = \prod_{l=0}^{m-1}(\mathbf{e}xy\mathbf{e} - a_{m-l} - mk_{m-l}) = m^m\chi_{\mathbf{k}}\left(\frac{\mathbf{e}xy\mathbf{e}}{m}\right)$$

and similarly

$$\mathbf{e}y^m x^m\mathbf{e} = \prod_{l=0}^{m-1}(\mathbf{e}yx\mathbf{e} + l - mk_{l+1} + mk_1) = m^m\chi_{\mathbf{k}}\left(\frac{\mathbf{e}yx\mathbf{e}}{m} + \frac{a_1}{m} + k_1\right) = m^m\chi_{\mathbf{k}}\left(\frac{\mathbf{e}xy\mathbf{e}}{m} + 1\right).$$

Hence the given map certainly defines a surjection $R_{\chi_{\mathbf{k}}} \twoheadrightarrow U_{\mathbf{k}}$. Since R has a grading induced by $\mathrm{ad}(Z)$ and $U_{\mathbf{k}}$ the grading induced by $\mathrm{ad}(\mathbf{e}xy\mathbf{e}) = -\mathrm{ad}(\mathbf{e}\underline{\mathbf{h}}\mathbf{e})$, we see that this map gives an isomorphism of finite-dimensional graded components as vector spaces, hence must be an isomorphism of algebras. □

Corollary 5.1.6. *Let W be cyclic of order m and $\mathbf{k}$ and $\mathbf{k}'$ parameters. If*

$$\chi_{\mathbf{k}}(t) = \chi_{\mathbf{k}'}(t)$$

the map

$$\begin{aligned} \phi : U_{\mathbf{k}} &\mapsto U_{\mathbf{k}'} \\ \mathbf{e}x^m\mathbf{e} &\mapsto \mathbf{e}x^m\mathbf{e} \\ \mathbf{e}y^m\mathbf{e} &\mapsto \mathbf{e}y^m\mathbf{e} \\ \mathbf{e}xy\mathbf{e} &\mapsto \mathbf{e}xy\mathbf{e} \end{aligned}$$

extends to an isomorphism $U_{\mathbf{k}} \cong U_{\mathbf{k}'}$ and restricts to the identity on $\mathbb{C}[\mathfrak{h}]^W\mathbf{e}$ and $\mathbb{C}[\mathfrak{h}^*]^W\mathbf{e}$.

Proof. Obvious from Proposition 5.1.5. □

5.2 Harish-Chandra bimodules for cyclic groups

Definition 5.2.1. *Let Γ be a complex reflection group. For a choice of parameters $\mathbf{k}$ we define $\Xi_{\mathbf{k}} = \Xi_{\mathbf{k}}(\Gamma)$ to be the set of isomorphism classes of simple Harish-Chandra bimodules.*

Proposition 5.2.2. *Suppose that W is cyclic and $\mathbf{k} \in \mathrm{Reg}(W)$, then $\Xi_{\mathbf{k}}$ is a cyclic group under $\otimes_{U_{\mathbf{k}}}$ with order*

$$\#\Xi_{\mathbf{k}} \cong \#\left(W/N_{\mathbf{k}}\right)$$

with $N_{\mathbf{k}}$ as in Corollary 3.4.21.

Proof. For any cyclic group G the set $\mathrm{Irr}(G)$ is a group under $\otimes$ and we have $\#G = \#\mathrm{Irr}(G)$. Since W is cyclic, so is any quotient $W/N_{\mathbf{k}}$ and as the equivalence $(\mathrm{rep}_{\mathbb{C}}(W/N_{\mathbf{k}}), \otimes) \cong (\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}})$ will take simple objects to simple objects, we can deduce that $\Xi_{\mathbf{k}}$ is a group under $\otimes_{U_{\mathbf{k}}}$ and that both $\Xi_{\mathbf{k}}$ and $W/N_{\mathbf{k}}$ must have the same order. □

Our aim for this chapter will be to determine $\Xi_{\mathbf{k}}$ depending on regular parameters $\mathbf{k}$, i.e. we need to determine the number of isomorphism classes of simple objects in $\mathcal{HC}_{\mathbf{k}}$. We will do this completely.

We know that $\#\Xi_{\mathbf{k}}$ has to be a divisor of $m = \#W$ by Proposition 5.2.2. We will show that any divisor of m occurs as the cardinality of $\Xi_{\mathbf{k}}$. We will give a combinatorial criterion to determine for which $\mathbf{k}$ each divisor is realised and describe the equivalence of categories

$$(\mathcal{HC}_{\mathbf{k}}(W), \otimes_{U_{\mathbf{k}}}) \cong (W/N_{\mathbf{k}}, \otimes)$$

in more detail, presenting in particular an explicit construction of the simple Harish-Chandra bimodules. One of our main computational tools will be the polynomial $\chi_{\mathbf{k}}(t)$ from Definition 5.1.4 as well as certain induction and restriction functors $\mathrm{Ind}_{W^d}^W$ and $\mathrm{Res}_{W^d}^W$ to be defined later in Definition 5.2.23.

Our main result in this chapter is the following:

Theorem 5.2.3. *Let W be a cyclic group of order $m = dr$, $\mathbf{k} \in \mathrm{Reg}(W)$ and assume that there exist $\beta_1, \ldots, \beta_{r-1} \in \mathbb{C}$ and $\underline{\lambda} = (\lambda_{i,j})_{\substack{0 \leq i \leq r-1 \\ 0 \leq j \leq d-1}} \in \Lambda(W)$ with $\lambda_{0,0} = 0$, satisfying the assumptions in Theorem 5.2.31. Then there is an equivalence of Abelian tensor categories*

$$(\mathcal{HC}_{\mathbf{k}}(W), \otimes_{U_{\mathbf{k}}}) \cong \left(\mathrm{rep}_{\mathbb{C}}\left(W/W^d\right), \otimes\right)$$

where $W^d = \{w^d \mid w \in W\}$, so that W^d is the unique subgroup of W of order r.

As usual it is computationally advantageous to work over the spherical subalgebra $U_{\mathbf{k}}$ instead of the rational Cherednik algebra $H_{\mathbf{k}}$. We first develop a purely combinatorial criterion for the existence of non-trivial Harish-Chandra bimodules in terms of the vanishing of the polynomial $\chi_{\mathbf{k}}$ under a certain operator which we will introduce now.

Definition 5.2.4. *Let R be a commutative ring; for $d \in \mathbb{N}_0$ and $\alpha \in R$ we define an operator*

$$\begin{aligned} \Omega_\alpha^d &: R[t] \to R[t] \\ f(t) &\mapsto \sum_{l=0}^{d} (-1)^l \binom{d}{l} \prod_{n=l}^{d-1} f(t-n) \prod_{n=0}^{l-1} f(t-\alpha-n) \end{aligned}$$

Lemma 5.2.5. *Let $f(t) \in R[t]$ and set $N = \deg f$.*

1. $\Omega_\alpha^{d+1}(f)(t) = f(t-d)\Omega_\alpha^d(f)(t) - f(t-\alpha)\Omega_\alpha^d(f)(t-1)$
2. *We have* $\deg \Omega_\alpha^d(f)(t) \leq d(N-1)$.

Proof. We can prove 1) by direct calculation

$$\begin{aligned}
\Omega_\alpha^{d+1}(f)(t) &= \sum_{l=0}^{d+1}(-1)^l\binom{d+1}{l}\prod_{n=l}^{d} f(t-n)\prod_{n=0}^{l-1} f(t-\alpha-n) \\
&= \sum_{l=0}^{d+1}(-1)^l\left(\binom{d}{l}+\binom{d}{l-1}\right)\prod_{n=l}^{d} f(t-n)\prod_{n=0}^{l-1} f(t-\alpha-n) \\
&= f(t-d)\sum_{l=0}^{d}(-1)^l\binom{d}{l}\prod_{n=l}^{d-1} f(t-n)\prod_{n=0}^{l-1} f(t-\alpha-n) \\
&+ f(t-\alpha)\sum_{l=1}^{d+1}(-1)^l\binom{d}{l-1}\prod_{n=l}^{d} f(t-1-(n-1))\prod_{n=1}^{l-1} f(t-1-\alpha-(n-1)) \\
&= f(t-\alpha)\sum_{l=0}^{d}(-1)^{l+1}\binom{d}{l}\prod_{n=l}^{d} f(t-1-n)\prod_{n=0}^{l-1} f(t-1-\alpha-n) \\
&+ f(t-d)\Omega_\alpha^d(f)(t) \\
&= f(t-d)\Omega_\alpha^d(f)(t) - f(t-\alpha)\Omega_\alpha^d(f)(t-1)
\end{aligned}$$

For 2), we will induct on d. Let us fix $f(t) \in R[t]$, then note that we have

$$\Omega_\alpha^0(f)(t) = 1 \text{ and } \Omega_\alpha^1(f)(t) = f(t) - f(t-\alpha)$$

so that both claims of 2) hold true for these values of d. Now suppose that both are true for some value of d. Using the recursive identity

$$\Omega_\alpha^{d+1}(f)(t) = f(t-d)\Omega_\alpha^d(f)(t) - f(t-\alpha)\Omega_\alpha^d(f)(t-1)$$

we see immediately that the degree of $\Omega_\alpha^{d+1}(f)(t)$ is bounded above by $N + \deg \Omega_\alpha^d(f)$ and the coefficients of $t^{N+\deg \Omega_\alpha^d(f)}$ in both summands cancel. Hence

$$\deg \Omega_\alpha^{d+1}(f) \leq N + \deg \Omega_\alpha^d(f) - 1.$$

Using the inductive hypothesis, we can conclude that indeed

$$\deg \Omega_\alpha^{d+1}(f) \leq (d+1)(N-1).$$

□

We can show the following vanishing criterion:

Proposition 5.2.6. *Let $f \in \mathbb{C}[t]$ and suppose that no two distinct zeros of f differ by an integer (but higher multiplicities are permitted) and that for some $n_0 \in \mathbb{N}$ and $\alpha \in \mathbb{C}$*

$$f(t-n_0) \mid \prod_{s=0}^{n_0} f(t-\alpha-s).$$

Then for any $d > n_0 \cdot \deg f$ we have

$$\Omega_\alpha^d(f) = 0.$$

Proof. Shifting the indeterminate t from t to $t-n+n_0$ in the relation $f(t-n_0) \mid \prod_{s=0}^{n_0} f(t-\alpha-s)$

we can deduce that

$$f(t-k) \mid \prod_{s=0}^{n_0} f(t-\alpha-s-k+n_0) = \prod_{s=k-n_0}^{n} f(t-\alpha-s)$$

and so in particular

$$f(t-k) \mid \prod_{s=0}^{k} f(t-\alpha-s) \text{ for } k \geq n_0$$

Setting $N = \deg f$ we can estimate the number of zeros of Ω_α^d for $d > n_0 \cdot N$:

$$\begin{aligned}
\Omega_\alpha^d(f)(t) &= \sum_{l=0}^{d}(-1)^l \binom{d}{l} \prod_{s=l}^{d-1} f(t-s) \prod_{s=0}^{l-1} f(t-\alpha-s) \\
&= \sum_{l=k+1}^{d} (-1)^l \binom{d}{l} \prod_{s=l}^{d-1} f(t-s) \prod_{s=0}^{l-1} f(t-\alpha-s) \\
&+ \sum_{l=0}^{n}(-1)^l \binom{d}{l} \prod_{s=l}^{d-1} f(t-s) \prod_{s=0}^{l-1} f(t-\alpha-s) \\
&= \left(\prod_{s=0}^{n} f(t-\alpha-s)\right) \left(\sum_{l=k+1}^{d} (-1)^l \binom{d}{l} \prod_{s=l}^{d-1} f(t-s) \prod_{s=k+1}^{l-1} f(t-\alpha-s)\right) \\
&+ f(t-k) \left(\sum_{l=0}^{k}(-1)^l \binom{d}{l} \prod_{\substack{s=l \\ s\neq k}}^{d-1} f(t-s) \prod_{s=0}^{l-1} f(t-\alpha-s)\right)
\end{aligned}$$

and thus $f(t-k) \mid \Omega_\alpha^d(f)(t)$ for any $n_0 \leq k \leq d-1$. Since no two distinct zeros of f differ by an integer, we can conclude that $f(t)$ and $f(t-k)$ have no common factors. Else they would have a common zero and thus we would have found two zeros of f differing by an integer, namely k. Thus the product $\prod_{k=n_0}^{d-1} f(t-k)$ divides $\Omega_\alpha^d(f)(t)$. The product has degree $N(d-n_0)$ whilst the degree of $\Omega_\alpha^d(f)(t)$ is bounded above by $d(N-1)$. It can easily be seen that for $d > n_0 \cdot N$ we have $Nd - Nn_0 > dN - d$ and thus

$$\Omega_\alpha^d(f)(t) = 0.$$

□

Definition 5.2.7. *Let* $\mathbf{k} = (k_0 = 0, k_1, \ldots, k_{m-1})$ *be any choice of parameters. Recall that we have set*

$$a_j = a_j(m) = (1-\delta_{0,j})(m-j) \text{ for } 0 \leq j \leq m-1.$$

We now set

$$\alpha^{q,p}(\nu) = \nu + \frac{a_p - a_q}{m} + k_p - k_q \text{ for } \nu \in \mathbb{N}_0.$$

If $q = 0$ *we will often drop it from our notation, and likewise for* ν. *So we will have* $\alpha^p(\nu) = \alpha^{0,p}(\nu)$ *and* $\alpha^p = \alpha^{0,p}(0)$.

Definition 5.2.8. *Let* V *be a Harish-Chandra bimodule and* $v \in V$. *We set*

$$d_{\mathbf{e}x^m\mathbf{e}}(v) = \min\{d \in \mathbb{N} \mid \mathrm{ad}(\mathbf{e}x^m\mathbf{e})^d(v) = 0\}$$

and

$$d_{\mathbf{e}y^m\mathbf{e}}(v) = \min\{d \in \mathbb{N} \mid \mathrm{ad}(\mathbf{e}y^m\mathbf{e})^d(v) = 0\}.$$

Lemma 5.2.9. *Up to scalar multiples, the* $\mathrm{ad}(\mathbf{e}\underline{\mathrm{h}}\mathbf{e})$*-eigenfunctions* $f \in \mathcal{L}(\Delta(E_q), \Delta(E_p))$ *with* $d_{\mathbf{e}x^m\mathbf{e}}(f) = 1$ *and* $d_{\mathbf{e}y^m\mathbf{e}}(f) \le d$ *are in bijection with the set of* $\nu \in \mathbb{N}_0$ *such that* $\Omega^d_{\alpha^{q,p}(\nu)}(\chi_{\mathbf{k}})(t) = 0$.
It follows that $\mathcal{L}(\Delta(E_q), \Delta(E_p)) \neq 0$ *if and only if* $\exists d \in \mathbb{N}, \nu \in \mathbb{N}_0$ *such that* $\Omega^d_{\alpha^{q,p}(\nu)}(\chi_{\mathbf{k}}) = 0$. *In particular if* $\mathbf{k}$ *is regular, non-trivial Harish-Chandra bimodules exist if and only if* $\exists d \in \mathbb{N}, \nu \in \mathbb{N}_0$ *and a non-trivial representation* $E_p \in \mathrm{Irr}(W)$ *such that* $\Omega^d_{\alpha^p(\nu)}(\chi_{\mathbf{k}}) = 0$.

Proof. Let us define

$$\chi^p_{\mathbf{k}}(t) := \prod_{r=0}^{m-1}\left(t + \nu + \frac{a_p - a_r}{m} - (k_r - k_p)\right) = \chi_{\mathbf{k}}\left(t + \frac{a_p}{m} + k_p\right).$$

Suppose $f \in \mathcal{L}(\Delta(E_q), \Delta(E_p))$ is an $\mathrm{ad}(\mathbf{e}\underline{\mathrm{h}}\mathbf{e})$-eigenfunction with $d_{\mathbf{e}x^m\mathbf{e}}(f) = 1$ and $d_{\mathbf{e}y^m\mathbf{e}}(f) \le d \in \mathbb{N}$. As f is an $\mathrm{ad}(\mathbf{e}\underline{\mathrm{h}}\mathbf{e})$-eigenfunction, we have $\alpha \in \mathbb{C}$ such that

$$\mathrm{ad}(\mathbf{e}\underline{\mathrm{h}}\mathbf{e})(f)(v) = \mathbf{e}\underline{\mathrm{h}}\mathbf{e}f(v) - f(\mathbf{e}\underline{\mathrm{h}}\mathbf{e}v) = \alpha f(v)$$

for any $v \in \Delta(E_q)$. Let us apply this to the basis $\{\mathbf{e}x^{a_q+nm} \otimes v_q \mid n \in \mathbb{N}_0\}$ of $\Delta(E_q)$. We set

$$f(\mathbf{e}x^{a_q+nm} \otimes v_q) = \sum_k \lambda_k \mathbf{e}x^{a_p+km} \otimes v_p$$

and then deduce

$$\begin{aligned}
\mathbf{e}\underline{\mathrm{h}}\mathbf{e}f(\mathbf{e}x^{a_q+nm} \otimes v_q) &= f(\mathbf{e}\underline{\mathrm{h}}\mathbf{e}x^{a_q+nm} \otimes v_q) + \alpha f(\mathbf{e}x^{a_q+nm} \otimes v_q) \\
&\iff \sum_k \lambda_k(-a_p - km - \frac{1}{2} - \sum_{p,l}(1 - m\delta_{p,l})k_l)\mathbf{e}x^{a_p+km} \\
&= \sum_k \lambda_k(-a_q - nm - \frac{1}{2} + \sum_{q,l}(1 - m\delta_{q,l})k_l + \alpha)\mathbf{e}x^{a_p+km} \otimes v_p \\
&\iff -a_p - km - \frac{1}{2} - \sum_{p,l}(1 - m\delta_{p,l})k_l = -a_q - nm - \frac{1}{2} + \sum_{q,l}(1 - m\delta_{q,l})k_l + \alpha
\end{aligned}$$

for all $k \in \mathbb{N}_0$. But note that this allows us to uniquely determine k depending on n and thus we set that f maps the basis $\{\mathbf{e}x^{a_q+nm} \otimes v_q \mid n \in \mathbb{N}_0\}$ of $\Delta(E_q)$ to the basis $\{\mathbf{e}x^{a_p+nm} \otimes v_p \mid n \in \mathbb{N}_0\}$ of $\Delta(E_p)$. Hence after taking a suitable scalar multiple of f we may assume that $f(\mathbf{e}x^{a_q} \otimes v_q) = \mathbf{e}x^{a_p+\nu m} \otimes v_p$ for some $\nu \in \mathbb{N}_0$. As $d_{\mathbf{e}x^m\mathbf{e}}(f) = 1$ -recall that this means nothing but

$$\mathrm{ad}(\mathbf{e}x^m\mathbf{e})(f) = 0 \iff f(\mathbf{e}x^m\bullet) = \mathbf{e}x^m f(\bullet)$$

- the value of ν uniquely determines the map f (up to scalar multiple). Moreover we have $\mathrm{ad}(\mathbf{e}y^m\mathbf{e})^d(f) = 0$ and thus

$$\sum_{l=0}^{d}(-1)^l\binom{d}{l}\mathbf{e}y^{m(d-l)}\mathbf{e}f(\mathbf{e}y^{lm}x^{a_q+nm} \otimes v_q) = 0$$

for all $n \in \mathbb{N}_0$. As

$$\mathbf{e}y^{lm}x^{a_q+nm} \otimes v_q = \mathbf{e}\prod_{r=0}^{lm-1}(a_q + mn - r - m(k_{m-r} - k_q))\, x^{a_q+(n-l)m} \otimes v_q$$

we have

$$\begin{aligned}
&m^{-dm}\sum_{l=0}^{d}(-1)^l\binom{d}{l}\mathbf{e}y^{m(d-l)}\mathbf{e}f(\mathbf{e}y^{lm}x^{a_q+nm} \otimes v_q)\\
&= m^{-dm}\sum_{l=0}^{d}(-1)^l\binom{d}{l}\left(\prod_{r=0}^{lm-1}(a_q + mn - r - m(k_{m-r} - k_q))\right)\left(\mathbf{e}y^{m(d-l)}\mathbf{e}x^{a_p+(\nu+n-l)m} \otimes v_p\right)\\
&= \sum_{l=0}^{d}(-1)^l\binom{d}{l}\left(\prod_{r=0}^{lm-1}(\frac{a_q - r}{m} + n - k_{m-r} + k_q)\right)\\
&\times\left(\prod_{r=0}^{(d-l)m-1}(\frac{a_p - r}{m} + \nu + n - l - k_{m-r} + k_p)\right)\left(x^{a_p+m(\nu+n-d)} \otimes v_p)\right)\\
&= \left(\sum_{l=0}^{d}(-1)^l\binom{d}{l}\prod_{t=l}^{d-1}\chi^p(n-t)\prod_{t=0}^{l-1}\chi^p(n-t-\alpha^{q,p}(\nu))\right)\left(x^{a_p+m(\nu+n-d)} \otimes v_p)\right)
\end{aligned}$$

i.e.

$$\Omega^d_{\alpha^{q,p}(\nu)}(\chi^p)(n) = 0 \ \forall n \in \mathbb{N}_0$$

which shows

$$\Omega^d_{\alpha^{q,p}(\nu)}(\chi^p)(t) = 0.$$

Noting that Ω^d_α annihilates a polynomial if and only if it annihilates all shifts of that polynomial, we see that a map f as described really does uniquely determine $\nu \in \mathbb{N}_0$ with $\Omega^d_{\alpha^{q,p}(\nu)}(\chi_{\mathbf{k}}) = 0$.

To show the reverse, fix $d \in \mathbb{N}$ and assume we have $\nu \in \mathbb{N}_0$ such that $\Omega^d_{\alpha^{q,p}(\nu)}(\chi_{\mathbf{k}}) = 0$. The above calculation read backwards will then show that the $\mathbb{C}[\mathfrak{h}]^W$-morphism $f : \Delta(E_q) \to \Delta(E_p)$ defined by $\mathbf{e}x^{a_q} \otimes v_q \mapsto \mathbf{e}x^{a_p+\nu m} \otimes v_p$ is $\mathrm{ad}(\mathbf{e}y^m\mathbf{e})$-nilpotent of degree $\leq d$ and it clearly is an $\mathrm{ad}(\mathbf{e}\underline{h}\mathbf{e})$-eigenfunction.

We further know from Theorem 3.4.8 and Lemma 3.4.12 that for regular $\mathbf{k}$ non-trivial Harish-Chandra bimodules exist if and only if $\mathcal{L}(\Delta(E_0), \Delta(E_p)) \neq 0$ for some non-trivial $E_p \in \mathrm{Irr}(W)$. We have just shown that this is the case if and only if $\Omega^d_{\alpha^p(\nu)}(\chi_{\mathbf{k}}) = 0$ for some $d \in \mathbb{N}, \nu \in \mathbb{N}_0$. □

We have already seen how $\chi^W_{\mathbf{k}}(t)$ can be a useful computational tool. Another reason why it is important is that it completely describes the structure of the spherical subalgebra $U_{\mathbf{k}}$, I. Losev has constructed a group action on the parameter spaces of cyclotomic rational Cherednik algebras - see for example Section 4 in [GL11] for a nice exposition or Section 6 in [Los11b] - of which the rational Cherednik algebras of cyclic groups are a special example. From this result, we can obtain in our case an action of S_{m-1} on the set of parameters for $H_{\mathbf{k}}(W)$

Definition 5.2.10. *We denote by $\cdot$ the action of S_{m-1} on $0 \times \mathbb{C}^{m-1}$ defined as follows: For* *each simple reflection $s = (j, j+1)$ and an element $(k_0 = 0, k_1, \dots, k_{m-1})$ we have*

$$s_{j,j+1} : (0, k_1, \dots, k_j, k_{j+1}, \dots, k_{m-1}) \mapsto \left(0, k_1, \dots, k_{j+1} - \frac{1}{m}, k_j + \frac{1}{m}, \dots, k_{m-1}\right)$$

and we extend this to the action $\cdot$. We will refer to this action as the "Losev action."

It follows from Losev's result that $U_{\mathbf{k}} \cong U_{w\cdot\mathbf{k}}$ for any $w \in S_{m-1}$ and that this isomorphism preserves $\mathbb{C}[\mathfrak{h}]^W, \mathbb{C}[\mathfrak{h}^*]^W$ and xy. In our special case, we have already shown this:

Corollary 5.2.11. *We have an isomorphism $U_{\mathbf{k}} \cong U_{w\cdot\mathbf{k}}$ for any $w \in S_{m-1}$ given by $\mathbf{e}x^m\mathbf{e} \mapsto \mathbf{e}x^m\mathbf{e}$, $\mathbf{e}y^m\mathbf{e} \mapsto \mathbf{e}y^m\mathbf{e}$ and $\mathbf{e}xy\mathbf{e} \mapsto \mathbf{e}xy\mathbf{e}$. Hence this isomorphism induces equivalences $\mathcal{O}(U_{\mathbf{k}}) \cong \mathcal{O}(U_{w\cdot\mathbf{k}})$ and $\mathcal{HC}(U_{\mathbf{k}}) \cong \mathcal{HC}(U_{w\cdot\mathbf{k}})$.*

Proof. Everything will follow from Corollary 5.1.6 once we have checked that $\chi_{\mathbf{k}}(t) = \chi_{w\cdot\mathbf{k}}(t)$. This is an easy computation if $w = (r, r+1)$ is a simple reflection and will follow from this for the general case:

$$\begin{aligned}
\chi_{w\cdot\mathbf{k}}(t) &= \prod_{j=0}^{m-1}\left(t - \frac{a_j}{m} - (w\cdot k_j)\right) \\
&= (t - \frac{a_r}{m} - k_{r+1} + \frac{1}{m})(t - \frac{a_{r+1}}{m} - k_r - \frac{1}{m}) \prod_{\substack{j=0 \\ j\neq r,r+1}}^{m-1}\left(t - \frac{a_j}{m} - k_j\right) \\
&= (t - \frac{m-r-1}{m} - k_{r+1})(t - \frac{m-r-1+1}{m} - k_r) \prod_{\substack{j=0 \\ j\neq r,r+1}}^{m-1}\left(t - \frac{a_j}{m} - k_j\right) \\
&= (t - \frac{a_{r+1}}{m} - k_{r+1})(t - \frac{a_r}{m} - k_r) \prod_{\substack{j=0 \\ j\neq r,r+1}}^{m-1}\left(t - \frac{a_j}{m} - k_j\right) \\
&= \prod_{j=0}^{m-1}\left(t - \frac{a_j}{m} - k_j\right) \\
&= \chi_{\mathbf{k}}(t)
\end{aligned}$$

□

Corollary 5.2.12. *The action of S_{m-1} preserves* $\mathrm{Reg}(W)$.

Proof. By Corollary 5.2.11 we have equivalences $\mathcal{O}_{\mathbf{k}} \cong \mathcal{O}_{w(\mathbf{k})}$ for all $w \in S_{m-1}$. Thus if $\mathcal{O}_{\mathbf{k}}$ is semisimple so are all $\mathcal{O}_{w(\mathbf{k})}$ and hence if $\mathbf{k} \in \mathrm{Reg}(W)$ then so is $w(\mathbf{k}) \in \mathrm{Reg}(W)$. □

Definition 5.2.13. *Let $m = dr$ with $d, r \in \mathbb{N}$. For $\beta_0 = 0$ and $\beta_1, \dots, \beta_{r-1} \in \mathbb{C}$ we define a parameter $\mathbf{l}(\beta_1, \dots, \beta_{r-1})$ by setting*

$$\mathbf{l}(\beta_1, \dots, \beta_{r-1})_j = l_j = \begin{cases} j\frac{(d-m)}{md} \text{ for } 0 \le j \le d-1 \\ \beta_1 + (j-d)\frac{(d-m)}{md} \text{ for } d \le j \le 2d-1 \\ \beta_2 + (j-2d)\frac{d-m}{md} \text{ for } 2d \le j \le 3d-1 \\ \vdots \\ \beta_{r-1} + (j-(r-1)d)\frac{d-m}{md} \text{ for } (r-1)d \le j \le m-1. \end{cases}$$

Definition 5.2.14. *For any $d \in \mathbb{N}_0$ we define a polynomial $\chi^{[d]}(t) \in \mathbb{C}[t]$ by*

$$\chi^{[d]}(t) = \prod_{j=0}^{d-1} \left(t - \frac{j}{d}\right).$$

Note that this is identical to $\chi_{\underline{0}}^{\mathbb{Z}_d}(t)$, the χ-polynomial of a cyclic group of order d at parameter value $\underline{0}$.

Recall that $\Lambda = \Lambda(W)$ denotes the set of integral parameters

$$\Lambda = \{\mathbf{k} \mid k_j \in \mathbb{Z} \forall\ 1 \leq j \leq m-1\}$$

as introduced in Definition 4.2.1.

Proposition 5.2.15. *Let $\mathbf{k}$ be a regular choice of parameters. Then the following are equivalent:*

1. *There exists a partition of $\{k_0, k_1, \dots, k_{m-1}\}$ into sets $P_0 = \{k_{p_0(0)}, k_{p_0(1)}, \dots, k_{p_0(d-1)}\}, P_1 = \{k_{p_1(0)}, \dots, k_{p_1(d-1)}\}, \dots,$ $P_{r-1} = \{k_{p_{r-1}(0)}, \dots, k_{p_{r-1}(d-1)}\}$ with the property that the sets*
$$\left\{p_s(j) - mk_{p_s(j)} - (p_s(0) - mk_{p_s(0)}) + m\mathbb{Z} \mid 0 \leq j \leq d-1\right\}$$
form a subgroup of $\mathbb{Z}/m\mathbb{Z}$ of order d for each $s \in \{0, \dots, r-1\}$.
2. *$\mathbf{k} \in S_{m-1} \cdot_L (\mathbf{l}(\beta_1, \dots, \beta_{r-1}) + \Lambda)$ for some regular $\mathbf{l}(\beta_1, \dots, \beta_{r-1})$.*
3. *$\Xi_{\mathbf{k}}$ has order divisible by d.*

Proof. We will begin by introducing more notation: We have a bijection

$$\begin{aligned} \widetilde{\ }: 0 \times \mathbb{C}^{m-1} &\to 0 \times \mathbb{C}^{m-1} \\ \mathbf{k} = (0, k_1, \dots, k_{m-1}) &\mapsto (0, 1 - mk_1, \dots, m-1-mk_{m-1}) = \widetilde{\mathbf{k}} \end{aligned}$$

We have two actions of S_{m-1} on $0 \times \mathbb{C}^{m-1}$, one is the action $\cdot$ from Definition 5.2.10 and the standard permutation action $\circ$. We claim that the map$\widetilde{\ }$ intertwines these actions: More specifically we will show that for any $\mathbf{k} \in 0 \times \mathbb{C}^{m-1}$ we have

$$\widetilde{w \cdot \mathbf{k}} = w \circ \widetilde{\mathbf{k}}.$$

Again we will prove this for simple reflections $s = (j, j+1)$ and the general case will follow from this. We compute

$$\begin{aligned} &s\cdot (0, 1 - mk_1, 2 - mk_2, \dots, m-1-mk_{m-1}) \\ &= \left(0, 1 - mk_1, 2 - mk_2, \dots, j - m(k_{j+1} - \frac{1}{m}), j - m(k_j + \frac{1}{m}), \dots, m-1-mk_{m-1}\right) \\ &= (0, 1 - mk_1, 2 - mk_2, \dots, j+1-mk_{j+1}, j - mk_j, \dots, m-1-mk_{m-1}) \end{aligned}$$

The intertwining relation follows. As S_{m-1} acts via isomorphisms through $\cdot$, we can conclude that if $\widetilde{\mathbf{k}'} \in S_{m-1} \circ \widetilde{\mathbf{k}}$, we have $U_{\mathbf{k}'} \cong U_{\mathbf{k}}$.
To show that (1) $\implies$ (2), fix sets P_s and elements $k_{p_s(j)} \in P_s$ as described above. As a subgroup of $\mathbb{Z}/m\mathbb{Z}$ the cyclic group of order d consists of the elements $m\mathbb{Z}, \frac{m}{d} + m\mathbb{Z}, 2\frac{m}{d} +$

$m\mathbb{Z}, \dots, (d-1)\frac{m}{d} + m\mathbb{Z}$. It follows that we have equalities of sets

$$\{p_s(j) - mk_{p_s(j)} - (p_s(0) - mk_{p_s(0)}) + m\mathbb{Z} \mid 0 \leq j \leq d-1\} = \{j\frac{m}{d} + m\mathbb{Z} \mid 0 \leq j \leq d-1\}$$

and we may in fact choose our enumeration of the elements of P_s such that and for each $j \in \{0, \dots, d-1\}$

$$p_s(j) - mk_{p_s(j)} - (p_s(0) - mk_{p_s(0)}) + m\mathbb{Z} = j\frac{m}{d} + m\mathbb{Z}.$$

Choose $w \in S_{m-1}$ so that the $sd+j$-th entry in $\widetilde{\mathbf{k}}^* = w \circ \widetilde{\mathbf{k}}$ is $p_s(j) - mk_{p_s(j)}$. We wish to determine the corresponding parameters $\mathbf{k}^*$. By definition, $\widetilde{\mathbf{k}}^*$ has the property that

$$\begin{aligned} sd + j - mk^*_{sd+j} - (sd - mk^*_{sd}) + m\mathbb{Z} &= j\frac{m}{d} + m\mathbb{Z} \\ \implies mk^*_{sd+j} \in sd + j - \frac{jm}{d} - sd + mk^*_{sd} + m\mathbb{Z} & \\ \implies k^*_{sd+j} \in k^*_{sd} + \frac{jd - jm}{md} + \mathbb{Z} & \\ \implies k^*_{sd+j} \in k^*_{sd} + j\frac{(d-m)}{md} + \mathbb{Z} & \end{aligned}$$

thus with $\beta_s = k^*_{sd}$ we have shown that

$$\mathbf{k}^* = w(\mathbf{k}) \in \mathbf{l}(\beta_1, \dots, \beta_{\frac{m}{d}-1}) + \Lambda.$$

For (2) $\implies$ (3), we note that the Losev action of the symmetric group and the integer translation functors from Chapter 4, Definition 4.2.6 more precisely, give an equivalence of categories $\mathcal{H}_{\mathbf{k}} \cong \mathcal{H}_{\mathbf{l}(\beta_1, \dots, \beta_{\frac{m}{d}-1})}$ and thus it will suffice to show that $\Xi_{\mathbf{l}(\beta_1, \dots, \beta_{\frac{m}{d}-1})}$ has order divisible by d. From now on we set $\mathbf{l} = \mathbf{l}(\beta_1, \dots, \beta_{\frac{m}{d}-1})$ for brevity. The χ-polynomial associated to $\mathbf{l}$ is

$$\begin{aligned} \chi_{\mathbf{l}}(t) &= \prod_{s=0}^{m-1} \left(t - (\frac{a_s}{m} + k_s)\right) \\ &= \left(t \prod_{j=1}^{d-1} \left(t - \frac{d-j}{d}\right)\right) \cdot \left(\prod_{s=1}^{\frac{m}{d}-1} \left(\prod_{j=0}^{d-1} \left(t - \frac{(m-j-sd)}{m} - \beta_s - j\frac{d-m}{md}\right)\right)\right). \end{aligned}$$

We can rewrite this as

$$\chi_{\mathbf{l}}(t) = \chi^{[d]}(t) \cdot \prod_{s=1}^{\frac{m}{d}-1} \chi^{[d]}\left(t - \beta_s - \frac{m - sd^2}{dm}\right)$$

since

$$\begin{aligned}(t-\beta_s-\frac{m-sd^2}{dm})\prod_{j=1}^{d-1}\left(t-\frac{d-j}{d}-\beta_s-\frac{m-sd^2}{dm}\right) &= \prod_{j=1}^{d}\left(t-\frac{d-j}{d}-\beta_s-\frac{m-sd^2}{dm}\right)\\ &= \prod_{j=1}^{d}\left(t-\frac{md-jm+m-sd^2}{dm}-\beta_s\right)\\ &= \prod_{j=0}^{d-1}\left(t-\frac{md-(j+1)m+m-sd^2}{dm}-\beta_s\right)\\ &= \prod_{j=0}^{d-1}\left(t-\frac{md-jm-sd^2}{dm}-\beta_s\right)\\ &= \prod_{j=0}^{d-1}\left(t-\frac{(m-j-sd)}{m}-\beta_s-j\frac{d-m}{md}\right).\end{aligned}$$

We have already noted that $\chi^{[d]}(t)$ is identical to the χ-polynomial for the cyclic group of order d at parameter values $\mathbf{0}$ and furthermore for $p=0,\dots,d-1$ we have

$$\alpha^p_{\mathbf{l}}(\nu)=\alpha^p_{\underline{0}}(\nu)$$

where the right-hand parameter is taken for the cyclic group of order d at $\underline{0}$. From this we can deduce that

$$\chi^{[d]}(t-1)\mid\chi^{[d]}(t-\alpha^p)\chi^{[d]}(t-\alpha^p-1)$$

as this relation holds for the χ-polynomial of $\mathbb{Z}/d\mathbb{Z}$ at $\underline{0}$. Hence we have

$$\chi_{\mathbf{l}}(t-1)\mid\chi_{\mathbf{l}}(t-\alpha^p)\chi_{\mathbf{l}}(t-\alpha^p-1)$$

for any $\nu\in\mathbb{N}_0$. Since $\mathbf{l}$ is regular, no two zeros of $\chi_{\mathbf{l}}(t)$ differ by an integer and therefore for any $n\geq m+1$ we have

$$\Omega^n_{\alpha^p}(\chi_{\mathbf{l}})(t)=0$$

for any $\nu\in\mathbb{N}_0$. In particular for any $p=0,\dots,d-1$ we have

$$V_p:=\mathcal{L}_{\mathbf{l}}(\Delta(E_0),\Delta(E_p))\neq 0$$

and these are all distinct by Corollary 3.4.11. Hence we have shown that

$$\#\Xi_{\mathbf{l}}\geq d.$$

We will now show that $V_p=V_1^{\otimes p}$. Clearly $V_1^{\otimes p}$ is a non-zero simple Harish-Chandra bimodule and must hence equal $\mathcal{L}_{\mathbf{l}}(\Delta(E_0),\Delta(E_j))$ for some $j\in\{0,\dots,m-1\}$. We know further from Corollary 3.4.11 that E_j is uniquely determined and weight considerations using regularity of $\mathbf{l}$ imply that we must have $j=p$: The weights of any weight vector in $V_1^{\otimes p}$ must lie in $p(-a_1+\kappa_{\mathbf{l}}(E_1)-\kappa_{\mathbf{l}}(E_0))+m\mathbb{Z}$ and by hypothesis it must also lie in $-a_j+\kappa_{\mathbf{l}}(E_j)-\kappa_{\mathbf{l}}(E_0)+m\mathbb{Z}$. Thus as $\kappa_{\mathbf{l}}(E_q)-\kappa_{\mathbf{l}}(E_0)=-ml_q$ for any q and $a_q=-q$ modulo m we have an equality

$$p-mpl_1+m\mathbb{Z}=j-ml_j+m\mathbb{Z}.$$

By construction of $\mathbf{l}$, $p - mpl_1 + m\mathbb{Z} = p - ml_p + m\mathbb{Z}$ and so we have $p - j + m(l_p - l_j) \in m\mathbb{Z}$. As $\mathbf{l}$ is regular, we must conclude $j = p$ by by Lemma 5.1.2. Thus

$$V_1^{\otimes p} = V_p$$

and further

$$V_1^{\otimes d} = U_{\mathbf{k}},$$

again by weight considerations. So $\{V_0, V_1, \ldots, V_{d-1}\}$ forms a subgroup of $\Xi_{\mathbf{l}}$ of order d and so d must divide $\#\Xi_{\mathbf{k}}$.

Finally let us prove (3) $\implies$ (1). As $\Xi_{\mathbf{k}}$ is a cyclic group under $\otimes_{U_{\mathbf{k}}}$ by Proposition 5.2.2 and by hypothesis it has order divisible by d it must contain an element of order d, say V. Define the sets P_s inductively as follows: We set

$$P_0 = \left\{\Delta(E_0), V \otimes_{U_{\mathbf{k}}} \Delta(E_0), \ldots, V^{\otimes d-1} \otimes_{U_{\mathbf{k}}} \Delta(E_0)\right\}$$

and associate to that a set of parameters as follows: By Lemma 3.4.23 all objects $V^{\otimes n} \otimes_{U_{\mathbf{k}}} \Delta(E_0)$ have to be simple again and thus equal some standard module, say

$$\Delta(E_{p_n}) = V^{\otimes n} \otimes_{U_{\mathbf{k}}} \Delta(E_0)$$

To each standard module $\Delta(E_q)$ we then associate the parameter k_q so that we have identified our set P_0 with

$$P_0 = \left\{\Delta(E_0), \Delta(E_{p_1}), \ldots, \Delta(E_{p_{d-1}})\right\} = \left\{k_0, k_{p_1}, \ldots, k_{p_{d-1}}\right\}.$$

It remains to check that this set fulfils the conditions of the proposition. Recall our definition of weights of Harish-Chandra bimodules and objects in $\mathcal{O}_{\mathbf{k}}$. Then by construction

$$V = \mathcal{L}(\Delta(E_0), \Delta(E_{p_1}))$$

and so

$$\mathfrak{w}(V) = -a_{p_1} - mk_{p_1} + m\mathbb{Z} = p_1 - mk_{p_1} + m\mathbb{Z}.$$

Then we must have

$$\mathfrak{w}(\Delta(E_{p_n})) = np_1 - mnk_{p_1} + m\mathbb{Z} = p_n - mk_{p_n} + m\mathbb{Z}$$

and as

$$\mathfrak{w}(V^{\otimes d}) = \mathfrak{w}(U_{\mathbf{k}}) = m\mathbb{Z}$$

we can also conclude that

$$dp_1 - mdk_{p_1} \in m\mathbb{Z}$$

and since $p_0 = 0, k_0 = 0$ we conclude that $\{p_n - mk_{p_n} + m\mathbb{Z}\} = \{p_n - mk_{p_n} - (p_0 - nmk_{p_0}) + m\mathbb{Z}\}$ really forms an additive group of order d.

Assuming we already have $P_0, \ldots, P_{s-1}$, we can define P_s by choosing $\Delta(\lambda)$ such that k_λ is not in any of the sets $P_0, \dots, P_{s-1}$ and then we set

$$P_s = \left\{\Delta(\lambda), V \otimes_U \Delta(\lambda), \dots, V^{\otimes d-1} \otimes_U \Delta(\lambda)\right\}.$$

The proof that this again fulfils the criteria of the Proposition is analogous. □

It might be interesting to note that we can rephrase the above criteria in terms of the parameters of associated Hecke algebra $\mathcal{H}_{\mathbf{q}}(W)$: Namely if $V \otimes_{U_{\mathbf{k}}} \Delta(E_a) = \Delta(E_b)$ then $\mathfrak{w}(V) = b - mk_b - (a - mk_a) + m\mathbb{Z}$ and so

$$\begin{aligned} e^{\frac{2\pi}{m} i\mathfrak{w}(V)} &= e^{\frac{2\pi i(b-a)}{m}} e^{-2\pi i(k_b - k_a)} \\ &= (e^{\frac{-2\pi ib}{m}} e^{2\pi ik_b})^{-1} (e^{\frac{-2\pi ia}{m}} e^{2\pi ik_a}) \end{aligned}$$

and thus we have shown

Corollary 5.2.16. *Let $\mathbf{k} = (k_0 = 0, k_1, \dots, k_{m-1})$ be a choice of parameters for W a cyclic group of order m and $\mathbf{q} = (q_0 = 1, q_1 = e^{2\pi ik_1}, \dots, q_{m-1} = e^{2\pi ik_{m-1}})$ the parameters of the associated Hecke algebra. Set $\tilde{q}_j = e^{\frac{-2\pi ij}{m}} q_j$. Then $\Xi_{\mathbf{k}}$ has order divisible by d if and only if there exists a partition of $\{\tilde{q}_j\}$ into m/d disjoint subsets*

$$Q_s = \{\tilde{q}_{s,0}, \dots, \tilde{q}_{s,d-1}\}, s = 0, \dots, \frac{m}{d} - 1$$

such that the sets

$$\{1, \tilde{q}_{s,0}^{-1}\tilde{q}_{s,1}, \dots, \tilde{q}_{s,0}^{-1}\tilde{q}_{s,d-1}\}$$

consists of all d-th roots of unity.

Corollary 5.2.16 proves Conjectures 1.
We can also give conditions on the parameters $\mathbf{k}$ to determine $\#\Xi_{\mathbf{k}}$.

Corollary 5.2.17. *The following statements are equivalent for regular $\mathbf{k}$ and $m = rd$:*

1. *There exists a partition of $\{k_0, k_1, \dots, k_{m-1}\}$ into sets $P_0 = \{k_{p_0(0)}, k_{p_0(1)}, \dots, k_{p_0(d-1)}\}, P_1 = \{k_{p_1(0)}, \dots, k_{p_1(d-1)}\}, \dots,$ $P_{r-1} = \{k_{p_{r-1}(0)}, \dots, k_{p_{r-1}(d-1)}\}$ with the property that the sets*

$$\left\{p_s(j) - mk_{p_s(j)} - (p_s(0) - mk_{p_s(0)}) + m\mathbb{Z} \mid 0 \le j \le d-1\right\}$$

   form a subgroup of $\mathbb{Z}/m\mathbb{Z}$ of order d and d is maximal.
2. *$\mathbf{k} \in S_{m-1} \cdot_L (\mathbf{l}(\beta_1, \dots, \beta_{r-1}) + \Lambda)$ for some regular $\mathbf{l}(\beta_1, \dots, \beta_{r-1})$ and d is maximal.*
3. *$\Xi_{\mathbf{k}}$ has order d.*

Proof. Immediate from the preceding proposition: Maximality of d in (1) implies maximality of d in (2), for if (2) were to hold for some $d' > d$ then we would have $\#\Xi_{\mathbf{k}} \ge d'$ and hence we could find a partition as described in (1) for $d' > d$ contradicting maximality. The rest follows similarly. □

This implies that the case of integral parameters is "essentially" the only regular case in which we attain the maximum number of simples.

Corollary 5.2.18. *Let $\mathbf{k}$ be a regular choice of parameters. The following statements are equivalent:*

1. $\mathbf{k} \in S_{m-1} \cdot_L \Lambda$.
2. *The parameter* $\mathbf{k}$ *is regular and* $\mathcal{HC}_{\mathbf{k}}$ *has exactly* m *isomorphism classes of simple objects.*
3. *There is an equality of sets* $\{p - mk_p + m\mathbb{Z}\} = \mathbb{Z}/m\mathbb{Z}$ *and this extends to an isomorphism of Abelian groups under addition. In particular, for any two parameters* k_a, k_b *we can find* k_c *such that* $a + b - m(k_a + k_b) = c - mk_c \bmod m$.

Proof. Follows from setting $d = m$ in the previous corollary. □

We will now turn to describing these non-trivial Harish-Chandra modules more closely, and in particular show how we can find them using an induction process.

Proposition 5.2.19. *Let* W *be a cyclic group of order* $m = rd$ *and* $W^r \subseteq W$ *the cyclic subgroup of order* d *generated by all* r*-th powers of elements of* W*. Consider* $U_{\mathbf{l}}(W)$ *and* $U_{\mathbf{k}}(W^r)$ *with* $\mathbf{l} = (l_0 = 0, l_1, \cdots, l_{m-1}) \in 0 \times \mathbb{C}^{m-1}$ *and* $\mathbf{k} = (k_0 = 0, k_1, \cdots, k_{d-1}) \in 0 \times \mathbb{C}^{d-1}$*. There exists an embedding* $\phi : U_{\mathbf{l}}(W) \hookrightarrow U_{\mathbf{k}}(W^r)$ *given by*

$$\begin{aligned} \mathbf{e}x^m\mathbf{e} &\mapsto (\mathbf{e}x^d\mathbf{e})^r \\ \mathbf{e}y^m\mathbf{e} &\mapsto (\mathbf{e}y^d\mathbf{e})^r \\ \mathbf{e}xy\mathbf{e} &\mapsto \mathbf{e}xy\mathbf{e}. \end{aligned}$$

if and only if

$$m^m \chi_{\mathbf{l}}^W \left(\frac{t}{m}\right) = d^m \prod_{s=0}^{r-1} \chi_{\mathbf{k}}^{W^r} \left(\frac{t}{d} - s\right).$$

Proof. Let us first prove the existence of such an embedding given the identity of polynomials:

By direct computation one can show that inside $U_{\mathbf{k}}$ we have

$$(\mathbf{e}x^d\mathbf{e})^r(\mathbf{e}y^d\mathbf{e})^r = d^{rd} \prod_{s=0}^{r-1} \chi_{\mathbf{k}}^{W^r} \left(\frac{\mathbf{e}xy\mathbf{e}}{d} - s\right) = d^m \prod_{s=0}^{r-1} \chi_{\mathbf{k}}^{W^r} \left(\frac{\mathbf{e}xy\mathbf{e}}{d} - s\right)$$

and

$$(\mathbf{e}y^d\mathbf{e})^r(\mathbf{e}x^d\mathbf{e})^r = d^m \prod_{s=0}^{r-1} \chi_{\mathbf{k}}^{W^r} \left(\frac{\mathbf{e}xy\mathbf{e}}{d} + s + 1\right) = d^m \prod_{s=1}^{r} \chi_{\mathbf{k}}^{W^r} \left(\frac{\mathbf{e}xy\mathbf{e}}{d} + s\right).$$

By hypothesis $m^m \chi_{\mathbf{l}}^W(\frac{t}{m}) = d^m \prod_{s=0}^{r-1} \chi_{\mathbf{k}}^{W^r}\left(\frac{t}{d} - s\right)$ and also $m^m \chi_{\mathbf{l}}^W(\frac{t}{m} + 1) = d^m \prod_{s=0}^{r-1} \chi_{\mathbf{k}}^{W^r}\left(\frac{t+m}{d} - s\right) = d^m \prod_{s=1}^{r} \chi_{\mathbf{k}}^{W^r}\left(\frac{t}{d} + s\right)$. So the subalgebra of $U_{\mathbf{k}}$ generated by $(\mathbf{e}x^d\mathbf{e})^r, (\mathbf{e}y^d\mathbf{e})^r$ and $\mathbf{e}xy\mathbf{e}$ will certainly be a quotient of $R_{\chi_{\mathbf{l}}} \cong U_{\mathbf{l}}$ via ϕ. To conclude that ϕ is an isomorphism we can again use the argument that it induces an isomorphism between graded components.

Now let us suppose that the map ϕ as defined in the proposition is indeed an isomorphism. Then it is easy to check that on the one hand we have

$$\mathbf{e}x^m\mathbf{e}\mathbf{e}y^m\mathbf{e} = m^m \chi_{\mathbf{l}}^W(\frac{\mathbf{e}xy\mathbf{e}}{m}) \in U_{\mathbf{l}}(W)$$

and on the other we have

$$(\mathbf{e}x^d\mathbf{e})^r(\mathbf{e}y^d\mathbf{e})^r = d^{dr} \prod_{s=0}^{r-1} \chi_{\mathbf{k}}^{W^r} \left(\frac{\mathbf{e}xy\mathbf{e}}{d} - s\right) \in U_{\mathbf{k}}(W^r).$$

These get identified by the isomorphism ϕ and thus we conclude

$$m^m \chi_{\mathbf{l}}^W \left(\frac{t}{m}\right) = d^m \prod_{s=0}^{r-1} \chi_{\mathbf{k}}^{W^r} \left(\frac{t}{d} - s\right).$$

□

We can rephrase this Proposition somewhat:

Definition 5.2.20. *Let W be a cyclic group of order $m = rd$ and $W^r \subseteq W$ the subgroup generated by all r-th powers of elements of W. There is an action of the quotient W/W^r on $U_{\mathbf{k}}(W^r)$ given by*

$$sW^r \cdot \mathbf{e}x^d\mathbf{e} = \rho^d \mathbf{e}x^d\mathbf{e},\ sW^r \cdot \mathbf{e}y^d\mathbf{e} = \rho^{-d}\mathbf{e}y^d\mathbf{e},\ sW^r \cdot \mathbf{e}xy\mathbf{e} = \mathbf{e}xy\mathbf{e}.$$

We will refer to this as a standard action of W/W^r on $U_{\mathbf{k}}(W^r)$.

Proposition 5.2.21. *Let us consider a standard action of W/W^r on $U_{\mathbf{k}}(W^r)$. Then there is an isomorphism $U_{\mathbf{k}}(W^r)^{W/W^r} \cong U_{\mathbf{l}}(W)$ given by*

$$\begin{aligned} \mathbf{e}x^m\mathbf{e} &\mapsto (\mathbf{e}x^d\mathbf{e})^r \\ \mathbf{e}y^m\mathbf{e} &\mapsto (\mathbf{e}y^d\mathbf{e})^r \\ \mathbf{e}xy\mathbf{e} &\mapsto \mathbf{e}xy\mathbf{e}. \end{aligned}$$

if and only if $\mathbf{l}$ is a parameter choice such that

$$m^m \chi_{\mathbf{l}}^W \left(\frac{t}{m}\right) = d^m \prod_{s=0}^{r-1} \chi_{\mathbf{k}}^{W^r} \left(\frac{t}{d} - s\right).$$

In particular, any such isomorphism is independent of the choice of standard action.

Proof. The invariants $U_{\mathbf{k}}(W^r)^{W/W^r}$ are the set

$$\left\{\mathbf{e}x^a y^b\mathbf{e} \mid a, b \in \mathbb{N}_0 \text{ such that } a - b = 0 \mod m\right\}$$

and as an algebra it is generated by $(\mathbf{e}x^d\mathbf{e})^r, (\mathbf{e}y^d\mathbf{e})^r, \mathbf{e}xy\mathbf{e}$ (as a bimodule in fact it is generated just by $\mathbf{e}xy\mathbf{e}$). A similar calculation to the one done in the previous proposition using the definition of $\mathbf{l}$ then completes the result. □

Definition 5.2.22. *Let W be a cyclic group of order $m = rd$ and W^r the subgroup of order d generated by r-th powers. An embedding $U_{\mathbf{l}}(W) \cong U_{\mathbf{k}}(W^r)^{W/W^r} \subseteq U_{\mathbf{k}}(W^r)$ obtained via a standard action will be called a standard embedding. As noted before, this is independent of the choice of standard action.*

In this setup we can now introduce functors relating $\mathcal{O}_{\mathbf{k}}$ and $\mathcal{O}_{\mathbf{l}}$ as well as $\mathcal{HC}_{\mathbf{k}}$ and $\mathcal{HC}_{\mathbf{l}}$:

Definition 5.2.23. *Given a a standard embedding $U_{\mathbf{l}}(W) \cong U_{\mathbf{k}}(W^r)^{(W/W^r)} \subseteq U_{\mathbf{k}}(W^r)$ we can regard $U_{\mathbf{k}}(W^r)$ as a $U_{\mathbf{l}}(W)$-$U_{\mathbf{k}}(W^r)$-bimodule (and vice versa) and thus we have functors*

$$\begin{aligned} \operatorname{Ind}_{W^r}^W &: \mathcal{O}_{\mathbf{k}}(W^r) \to \mathcal{O}_{\mathbf{l}}(W),\ M \mapsto {}_{U_{\mathbf{k}}^{W/W^r}} U_{\mathbf{k}} \otimes_{U_{\mathbf{k}}} M \\ \operatorname{Res}_{W^r}^W &: \mathcal{O}_{\mathbf{l}}(W) \to \mathcal{O}_{\mathbf{k}}(W^r),\ N \mapsto {}_{U_{\mathbf{k}}} U_{\mathbf{k}} \otimes_{U_{\mathbf{k}}{}^{W/W^r}} N \end{aligned}$$

as well as

$$\mathrm{Ind}_{W^r}^W\colon \mathcal{HC}_{\mathbf{k}}(W^r) \to \mathcal{HC}_{\mathbf{l}}(W),\ V \mapsto {}_{U_{\mathbf{k}}{}^{W/W^r}}U_{\mathbf{k}} \otimes_{U_{\mathbf{k}}} V \otimes_{U_{\mathbf{k}}} U_{\mathbf{k}}{}_{U_{\mathbf{k}}{}^{W/W^r}}$$
$$\mathrm{Res}_{W^r}^W\colon \mathcal{HC}_{\mathbf{l}}(W) \to \mathcal{HC}_{\mathbf{k}}(W^r),\ V \mapsto {}_{U_{\mathbf{k}}}U_{\mathbf{k}} \otimes_{U_{\mathbf{k}}{}^{W/W^r}} V \otimes_{U_{\mathbf{k}}^{W/W^r}} U_{\mathbf{k}}{}_{U_{\mathbf{k}}}.$$

We will often not notationally distinguish between these functors as it is clear from the context which version we will be referring to. Note that in both cases the functor $\mathrm{Ind}_{W^r}^W$ *is given by restricting the action of* $U_{\mathbf{k}}(W^r)$ *to* $U_{\mathbf{l}}(W)$.

We can record some basic and easily seen properties of these functors

Lemma 5.2.24. *The functors* $\mathrm{Ind}_{W^r}^W\colon \mathcal{HC}_{\mathbf{k}}(W^r) \to \mathcal{HC}_{\mathbf{l}}(W)$ *and* $\mathrm{Res}_{W^r}^W\colon \mathcal{HC}_{\mathbf{l}}(W) \to \mathcal{HC}_{\mathbf{k}}(W^r)$ *have the following properties:*

1. *The functor* $\mathrm{Ind}_{W^r}^W$ *is right adjoint to* $\mathrm{Res}_{W^r}^W$ *and* $\mathrm{Ind}_{W^r}^W$ *is exact.*
2. $\mathrm{Ind}_{W^r}^W$ *and* $\mathrm{Res}_{W^r}^W$ *preserve finite-dimensionality.*
3. *There is a natural surjection of functors* $\mathrm{Res}_{W^r}^W \circ \mathrm{Ind}_{W^r}^W \twoheadrightarrow Id_{\mathbf{k}}$ *and a natural injection of functors* $Id_{\mathbf{l}} \hookrightarrow \mathrm{Ind}_{W^r}^W \circ \mathrm{Res}_{W^r}^W$
4. *Any Harish-Chandra bimodule in* $\mathcal{HC}_{\mathbf{k}}$ *is a quotient of a module restricted from* $\mathcal{HC}_{\mathbf{l}}$. *Any Harish-Chandra bimodule in* $\mathcal{HC}_{\mathbf{l}}$ *is a submodule of a module induced from* $\mathcal{HC}_{\mathbf{k}}$.
5. *If* $\mathrm{Ind}_{W^r}^W(M) = 0$ *then* $M = 0$ *and if* $\mathrm{Res}_{W^r}^W(M) = 0$ *then* $M = 0$.

Proof.
1. Follows from $\otimes - \mathrm{Hom}$ adjointness and as $\mathrm{Ind}_{W^r}^W$ simply restricts the action of $U_{\mathbf{k}}$ to the subalgebra $U_{\mathbf{l}}$, exactness is clear as well.
2. The statement is clear for $\mathrm{Ind}_{W^r}^W$ so let us consider $\mathrm{Res}_{W^r}^W$. Suppose $F \in \mathcal{HC}_{\mathbf{l}}$ is finite-dimensional. As $U_{\mathbf{k}}$ is finitely generated as a right module over $U_{\mathbf{l}}$, we have a surjection $U_{\mathbf{l}}^{\oplus n} \twoheadrightarrow U_{\mathbf{k}}$ for some $n \in \mathbb{N}$. Then we obtain a surjection
$$U_{\mathbf{l}}^{\oplus n} \otimes_{U_{\mathbf{l}}} F \twoheadrightarrow U_{\mathbf{k}} \otimes_{U_{\mathbf{l}}} F$$
and as $U_{\mathbf{l}} \otimes_{U_{\mathbf{l}}} F$ is finite-dimensional, so is $U_{\mathbf{k}} \otimes_{U_{\mathbf{l}}} F$. A similar argument now shows that $U_{\mathbf{k}} \otimes_{U_{\mathbf{k}}(W/W^r)} V \otimes_{U_{\mathbf{k}}^{(W/W^r)}} U_{\mathbf{k}}$ is finite-dimensional too.
3. We have a natural map of bimodules $U_{\mathbf{k}} \otimes_{U_{\mathbf{l}}} U_{\mathbf{k}} \to U_{\mathbf{k}}$ induced from multiplication and this is evidently surjective. Now let us consider ${}_{U_{\mathbf{l}}}U_{\mathbf{k}} \otimes_{U_{\mathbf{k}}} U_{\mathbf{k}}{}_{U_{\mathbf{l}}} \cong {}_{U_{\mathbf{l}}}U_{\mathbf{k}}{}_{U_{\mathbf{l}}}$. We can decompose this into W/W^r-isotypic components and the summand corresponding to the trivial representation will regain $U_{\mathbf{l}}$. Hence $U_{\mathbf{l}}$ embeds into ${}_{U_{\mathbf{l}}}U_{\mathbf{k}} \otimes_{U_{\mathbf{k}}} U_{\mathbf{k}}{}_{U_{\mathbf{l}}}$ and this gives the injection of functors.
4. This follows immediately from the natural transformations of functors in 3).
5. Follows as M is a quotient (bi)module of $\mathrm{Res}_{W^r}^W(\mathrm{Ind}_{W^r}^W(M))$ or a sub-(bi)module of $\mathrm{Ind}_{W^r}^W(\mathrm{Res}_{W^r}^W(M))$.

$\square$

Lemma 5.2.25. *The functors* $\mathrm{Ind}_{W^r}^W\colon \mathcal{O}_{\mathbf{k}}(W^r) \to \mathcal{O}_{\mathbf{l}}(W)$ *and* $\mathrm{Res}_{W^r}^W\colon \mathcal{O}_{\mathbf{l}}(W) \to \mathcal{O}_{\mathbf{k}}(W^r)$ *have the following properties:*

1. *The functor* $\mathrm{Ind}_{W^r}^W$ *is right adjoint to* $\mathrm{Res}_{W^r}^W$ *and* $\mathrm{Ind}_{W^r}^W$ *is exact.*
2. $\mathrm{Ind}_{W^r}^W$ *and* $\mathrm{Res}_{W^r}^W$ *preserve finite-dimensionality.*
3. *There is a natural surjection of functors* $\mathrm{Res}_{W^r}^W \circ \mathrm{Ind}_{W^r}^W \twoheadrightarrow Id_{\mathbf{k}}$ *and a natural injection of functors* $Id_{\mathbf{l}} \hookrightarrow \mathrm{Ind}_{W^r}^W \circ \mathrm{Res}_{W^r}^W$

4. *Any module in $\mathcal{O}_{\mathbf{k}}$ is a quotient of a module restricted from $\mathcal{O}_{\mathbf{l}}$. Any module in $\mathcal{O}_{\mathbf{l}}$ is a submodule of a module induced from $\mathcal{O}_{\mathbf{k}}$.*
5. *If $\mathrm{Ind}_{W^r}^{W}(M) = 0$ then $M = 0$ and if $\mathrm{Res}_{W^r}^{W}(M) = 0$ then $M = 0$.*

Proof. Same as in Lemma 5.2.24. □

Corollary 5.2.26. *Suppose that we have a standard embedding $U_{\mathbf{l}}(W) \hookrightarrow U_{\mathbf{k}}(W^r)$. Then the following hold:*

1. *$U_{\mathbf{l}}$ is simple if and only if $U_{\mathbf{k}}$ is.*
2. *If $\mathbf{k}$ is regular, $\mathbf{l}$ is either regular or aspherical.*
3. *If $\mathbf{l}$ is regular, then $\mathbf{k}$ is regular.*

Proof.

1. $U_{\mathbf{l}}$ is non-simple if and only if it has non-zero finite-dimensional modules: The annihilator of any non-zero finite-dimensional module will be a proper 2-sided ideal of $U_{\mathbf{l}}$ and so $U_{\mathbf{l}}$ is not simple if it has finite-dimensional modules. Conversely, if $U_{\mathbf{l}}$ is not simple, it has a proper 2-sided ideal $\mathfrak{a}$. The ideal $\mathfrak{a}$ is then cofinite in $U_{\mathbf{l}}$ and so $U_{\mathbf{l}}/\mathfrak{a}$ is a finite-dimensional $U_{\mathbf{l}}$-module. By Lemma 5.2.24, $U_{\mathbf{l}}$ has finite-dimensional modules if and only if $U_{\mathbf{k}}$ has finite-dimensional modules which is the case if and only if $U_{\mathbf{k}}$ is non-simple.
2. We have just shown that if $U_{\mathbf{k}}$ is regular then $U_{\mathbf{l}}$ has no finite-dimensional modules. Hence $U_{\mathbf{l}}$ is simple by (1) and if $\mathbf{l}$ is not regular, then $H_{\mathbf{l}}$ and $U_{\mathbf{l}}$ cannot be Morita equivalent and so $\mathbf{l}$ must be aspherical.
3. Suppose that $\mathbf{l}$ is regular, then arguing as above we see that we only need to rule out that $\mathbf{k}$ is aspherical. The parameter $\mathbf{k}$ will be aspherical if and only if $\chi_{\mathbf{k}}^{W^r}(t)$ has repeated zeros as can be seen from the definition of $\chi_{\mathbf{k}}(t)$ in Definition 5.1.4 and the description of aspherical parameter values in Proposition 6.2.2. But by Proposition 5.2.21 we have the relation
$$m^m \chi_{\mathbf{l}}^{W}\left(\frac{t}{m}\right) = d^m \prod_{s=0}^{r-1} \chi_{\mathbf{k}}^{W^r}\left(\frac{t}{d} - s\right)$$
and in particular $\chi_{\mathbf{k}}^{W^r}\left(\frac{t}{d}\right)$ divides $\chi_{\mathbf{l}}^{W}\left(\frac{t}{m}\right)$. As $\mathbf{l}$ is regular, $\chi_{\mathbf{l}}^{W}(t)$ has no repeated zeros and thus neither does $\chi_{\mathbf{l}}^{W}\left(\frac{t}{m}\right)$. Hence none of its divisors can have repeated zeros and so $\chi_{\mathbf{k}}^{W^r}\left(\frac{t}{d}\right)$ has no repeated zeros, implying that $\chi_{\mathbf{k}}^{W^r}(t)$ has no repeated zeros so that $\mathbf{k}$ is not aspherical.

□

Definition 5.2.27. *A regular parameter $\mathbf{k}$ is called rigid if $\Xi_{\mathbf{k}}$ is trivial, i.e. if up to isomorphism the only simple Harish-Chandra bimodule is $U_{\mathbf{k}}$.*

Example 5.2.28.

1. *Any parameter $k_1 \notin \frac{1}{2}\mathbb{Z}$ is rigid for $\mathbb{Z}_2$. These are all regular (the set of non-regular parameters for $\mathbb{Z}_2$ is precisely $\frac{1}{2}+\mathbb{Z}$) and the only regular, non-rigid parameter values are those for which we have precisely two simple Harish-Chandra bimodules. For these, $(0, k_1)$ must lie in $(0,0) + \Lambda(\mathbb{Z}_2)$.*
2. *Let $\mathbf{k} = (k_0 = 0, k_1, \ldots, k_{m-1})$ be a parameter such that all k_j have pairwise distinct imaginary parts. Then $\mathbf{k}$ is regular by the result of Lemma 5.1.2 and that it must be rigid can be seen as follows: Take V a generator of the group $\Xi_{\mathbf{k}}$ (recall this is a group by Proposition 5.2.2), by the remarks following Theorem 3.4.25 we can assign a weight class $\mathfrak{w}(V)$ to V which is additive on tensor products and has the form $\mathfrak{w}(V) = -a_p - mk_p + m\mathbb{Z}$ for some $p \in \{0, \ldots, m-1\}$. As $\Xi_{\mathbf{k}}$ is isomorphic to a quotient of W we must have*

$V^{\otimes m} \cong U_{\mathbf{k}}$ and so $-ma_p - m^2 k_p \in m\mathbb{Z}$. By our choice of parameters, this implies $p = 0$ and hence $V \cong \mathcal{L}(\Delta(\mathrm{triv}), \Delta(\mathrm{triv})) \cong U_{\mathbf{k}}$.

Lemma 5.2.29. *Let W be a cyclic group of order $m = rd$ and $\mathbf{k} \in \mathrm{Reg}(W)$. Suppose that $\mathbf{k} \in S_{m-1} \cdot_L \mathbf{l}$ for some $\mathbf{l} = \mathbf{l}(\beta_1, \ldots, \beta_{r-1})$ and that r is minimal with respect to this property. Then there exists a rigid choice of parameters $\mathbf{d}$ for $W^d \subseteq W$ the cyclic group of order r, such that we have a standard embedding $U_{\mathbf{k}}(W) \hookrightarrow U_{\mathbf{d}}(W^d)$.*

Proof. We set $\mathbf{d} = (0, \gamma_1, \ldots, \gamma_{r-1})$ with

$$\gamma_j = d\beta_j + j\left(\frac{1-d}{r}\right).$$

Now we employ the criterion of Proposition 5.2.21: We have

$$\begin{aligned}
r^m \prod_{i=0}^{d-1} \chi_{\mathbf{d}}\left(\frac{t}{r} - i\right) &= \prod_{i=0}^{d-1}\prod_{j=0}^{r-1} (t - a_j(r) - r\gamma_j - ri) \\
&= \prod_{i=0}^{d-1}\prod_{j=0}^{r-1} \left(t - a_j(r) - r\left(d\beta_j + j\frac{1-d}{r}\right) - ri\right) \\
&= \prod_{i=0}^{d-1}\prod_{j=0}^{r-1} (t - a_j(r) - m\beta_j + j(d-1) - ri) \\
&= \left(\prod_{i=0}^{d-1}(t - ri)\right)\left(\prod_{i=0}^{d-1}\prod_{j=1}^{r-1} (t - m\beta_j - r(i+1) + jd)\right)
\end{aligned}$$

and it remains to evaluate

$$\begin{aligned}
m^m \chi_{\mathbf{l}}\left(\frac{t}{m}\right) &= m^m \chi^{[d]}\left(\frac{t}{m}\right) \cdot \prod_{j=1}^{r-1} \chi^{[d]}\left(\frac{t}{m} - \beta_j - \frac{m - jd^2}{dm}\right) \\
&= m^m \prod_{i=0}^{d-1}\left(\frac{t}{m} - \frac{i}{d}\right) \cdot \prod_{j=1}^{r-1}\prod_{i=0}^{d-1}\left(\frac{t}{m} - \beta_j - \frac{i}{d} - \frac{m - jd^2}{dm}\right) \\
&= \prod_{i=0}^{d-1}\left(\left(t - \frac{mi}{d}\right) \cdot \prod_{j=1}^{r-1}\left(t - m\beta_j - \frac{mi}{d} - \frac{m - jd^2}{d}\right)\right) \\
&= \prod_{i=0}^{d-1}\left((t - ri) \cdot \prod_{j=1}^{r-1}(t - m\beta_j - ri - (r - jd))\right) \\
&= \prod_{i=0}^{d-1}(t - ri) \cdot \prod_{i=0}^{d-1}\left(\prod_{j=1}^{r-1}(t - m\beta_j - r(i + 1 + jd))\right)
\end{aligned}$$

Hence

$$r^m \prod_{i=0}^{d-1} \chi_{\mathbf{d}}\left(\frac{t}{r} - i\right) = m^m \chi_{\mathbf{l}}\left(\frac{t}{m}\right)$$

so that by Proposition 5.2.21 we have a standard embedding $U_{\mathbf{k}}(W) \hookrightarrow U_{\mathbf{d}}(W^d)$. By hypothesis $\mathbf{k}$ was regular and so by Corollary 5.2.26 the parameter $\mathbf{d}$ is regular. It remains to prove rigidity.

Suppose $\mathbf{d}$ were not rigid, so that $\Xi_{\mathbf{b}}(W^d) = u > 1$. The induced bimodule $\mathrm{Ind}_{W^d}^{W}(U_{\mathbf{d}})$ is isomorphic to ${}_{U_{\mathbf{k}}}U_{\mathbf{d}}{}_{U_{\mathbf{k}}}$ and splits into d non-isomorphic summands by taking W/W^d - isotypic components $U_{\mathbf{d}} = \oplus_{\varpi \in \mathrm{Irr}(W/W^d)} U_{\mathbf{d}}^{\varpi}$. Thus we obtain at least d non-isomorphic simple objects by considering composition factors of the $U_{\mathbf{d}}^{\varpi}$. By hypothesis, $\Xi_{\mathbf{k}}(W)$ has d objects (minimality of r implies maximality of d and we can apply Corollary 5.2.17) and thus these must be all simple Harish-Chandra bimodules. Note that taking W/W^d-isotypic components is the same as decomposing $\mathrm{Ind}_{W^d}^{W}(V)$ into certain sums of $\mathrm{ad}(\mathbf{e}\underline{h}\mathbf{e})$-eigenspaces: We set $U_{\mathbf{d}}[\alpha] = \{u \in U_{\mathbf{d}} \mid [\mathbf{e}\underline{h}\mathbf{e}, u] = \alpha u\}$, then $U_{\mathbf{d}} = \oplus_{s\in\mathbb{Z}} U_{\mathbf{d}}[sr]$. As $U_{\mathbf{k}}(W)$-bimodules we then have

$$U_{\mathbf{d}} = \bigoplus_{i=0}^{r-1} \left(\bigoplus_{s\in\mathbb{Z}} U_{\mathbf{d}}[ir+sm] \right)$$

and we can identify

$$U_{\mathbf{d}}^{\varpi} = \bigoplus_{s\in\mathbb{Z}} U_{\mathbf{k}}[i(\varpi)r+sm]$$

where $i(\varpi)$ is chosen such that $\det s\mid_{\varpi} = \rho^{i(\varpi)d}$ i.e. $U_{\mathbf{d}}^{\varpi} = \oplus_{\substack{s\in\mathbb{Z} \\ \rho^s=\det s|_{\varpi}}} U_{\mathbf{d}}[s]$.

Now that we have already found all simple objects of $\mathcal{HC}_{\mathbf{k}}$, choose $V \in \Xi_{\mathbf{d}}(W^d)$ non-isomorphic to $U_{\mathbf{d}}(W^d)$ and consider $\mathrm{Ind}_{W^d}^{W}(V)$. This is a Harish-Chandra bimodule in $\mathcal{HC}_{\mathbf{k}}$ and by 5.2.24 it is non-zero. The weight class of any composition factor of $\mathrm{Ind}_{W^d}^{W}(V)$ is not equal to the weight class of any summand of $\mathrm{Ind}_{W^d}^{W}(U_{\mathbf{d}})$: As $U_{\mathbf{d}}$ and V are non-isomorphic their weights are not congruent modulo r and hence the weight classes of composition factors cannot be congruent modulo m since m is a multiple of r. So any composition factor of $\mathrm{Ind}_{W^d}^{W}(U_{\mathbf{d}})$ must be a simple object of $\mathcal{HC}_{\mathbf{k}}$ non-isomorphic to any of the d simples we had already identified. So $\Xi_{\mathbf{k}}(W)$ must contain at least $d+1$ objects, contradicting our assumptions and thus $\mathbf{d}$ must be rigid. □

Proposition 5.2.30. *Let A, B be rings and X an invertible A-B-bimodule (i.e. there exists an B-A-bimodule Y with $X \otimes_B Y \cong A$ and $Y \otimes_A X \cong B$). Then the following holds:*

1. *X is finitely generated as a left A- and as a right B-module.*
2. *X is a generator in the categories $A-\mathrm{mod}$ and $\mathrm{mod}-B$.*
3. *X is projective as a left A- and as a right B-module.*
4. *$X \otimes_B -$ is an equivalence between $B-\mathrm{mod} \xrightarrow{\sim} A-\mathrm{mod}$ with inverse $Y \otimes_A -$.*
5. *If $A = B$ and if A is Artinian, the bimodule length of X is equal to the bimodule length of A.*

Proof. The first four statements are well-known standard results and can be found in many books on ring theory or homological algebra (see e.g. [Bas68], Chapter 2 Corollary 2.4 and Theorem 3.2) The fifth statement clearly follows as $X \otimes_A -$ maps A to X. □

Now we can turn to the proof of our main result:

Theorem 5.2.31. *Let W be a cyclic group of order $m = dr$, $\mathbf{k} \in \mathrm{Reg}(W)$ and assume that there exist $\beta_1, \ldots, \beta_{r-1} \in \mathbb{C}$ and $\underline{\lambda} = (\lambda_{i,j})_{\substack{0\le i\le r-1 \\ 0\le j\le d-1}} \in \Lambda(W)$ with $\lambda_{0,0} = 0$, such that we may factorise $\chi_{\mathbf{k}}(t)$ as*

$$\chi_{\mathbf{k}}^{W}(t) = \left(\prod_{j=0}^{d-1} \left(t - \frac{j}{d} - \lambda_{0,j} \right) \right) \cdot \left(\prod_{i=1}^{r-1} \prod_{j=0}^{d-1} \left(t - \frac{j}{d} - \beta_i - \frac{m-id^2}{md} - \lambda_{i,j} \right) \right).$$

Let us finally assume that r is maximal with this property. Then there is an equivalence of

Abelian tensor categories

$$(\mathcal{HC}_{\mathbf{k}}(W), \otimes_{U_{\mathbf{k}}}) \cong \left(\mathrm{rep}_{\mathbb{C}}\left(W/W^d\right), \otimes\right).$$

Moreover under this equivalence, $U_{\mathbf{k}}$ is sent to triv *and $\mathrm{Ind}_{W^d}^{W}(U_{\mathbf{d}})$ is sent to $\mathbb{C}\left(W/W^d\right)$.*

Proof. of Theorem 5.2.31. The factorisation of $\chi_{\mathbf{k}}^{W}(t)$ together with regularity of $\mathbf{k}$ and maximality of r implies that $\#\Xi_{\mathbf{k}}(W) = d$ with $d = \frac{m}{r}$ by Corollary 5.2.17. Set $\mathbf{l} := \mathbf{k} - \underline{\lambda}$ and note that $\mathcal{HC}_{\mathbf{k}}$ is equivalent as an Abelian tensor category to $\mathcal{HC}_{\mathbf{l}}$ by Theorem 4.2.10 as $\underline{\lambda}$ is integral. So it will suffice to prove an equivalence $(\mathcal{HC}_{\mathbf{l}}(W), \otimes_{U_{\mathbf{l}}}) \cong \left(\mathrm{rep}_{\mathbb{C}}\left(W/W^d\right), \otimes\right)$. We can apply Lemma 5.2.29 to $U_{\mathbf{l}}$ to find a rigid parameter value $\mathbf{d}$ for W^d and a standard embedding $U_{\mathbf{l}}(W) \hookrightarrow U_{\mathbf{d}}(W^d)$. Then by Lemma 5.2.24, any Harish-Chandra bimodule is a submodule of $\mathrm{Ind}_{W^d}^{W}(U_{\mathbf{d}})$. Recall further that we have a decomposition

$$\mathrm{Ind}_{W^d}^{W}(U_{\mathbf{d}}) = \bigoplus_{\varpi \in \mathrm{Irr}(W/W^d)} U_{\mathbf{d}}^{\varpi}$$

and we claim that the assignment

$$F : \varpi \to U_{\mathbf{d}}^{\varpi}$$

extends to the desired equivalence

$$\mathrm{rep}_{\mathbb{C}}(W/W^d) \to \mathcal{H}_{\mathbf{l}}(W).$$

Step 1: $U_{\mathbf{d}}^{\mathrm{triv}} \cong U_{\mathbf{l}}$. This follows immediately from the description of the $U_{\mathbf{d}}^{\varpi}$ in the proof of Lemma 5.2.29.

Step 2: $U_{\mathbf{d}}^{\varpi}$ is invertible with inverse $U_{\mathbf{d}}^{\epsilon}$ where $\epsilon \in \mathrm{Irr}(W/W^d)$ such that $\varpi \cdot \epsilon = \mathrm{triv}$ in the representation ring of W/W^d. We have a natural morphism $f : U_{\mathbf{d}}^{\varpi} \otimes_{U_{\mathbf{l}}} U_{\mathbf{d}}^{\epsilon} \to U_{\mathbf{d}}^{\mathrm{triv}}$ induced from multiplication in $U_{\mathbf{d}}$. Since $U_{\mathbf{d}}^{\mathrm{triv}} \cong U_{\mathbf{l}}$ is simple by regularity of $\mathbf{l}$, we can deduce that f is surjective since it is non-zero. To show injectivity, suppose that we have $u_j \in U_{\mathbf{d}}^{\varpi}$ and $v_j \in U_{\mathbf{d}}^{\epsilon}$ such that $\sum_j u_j \otimes v_j \overset{f}{\mapsto} 0$. Pick $u_i' \in U_{\mathbf{l}}^{\varpi}, v_i' \in U_{\mathbf{l}}^{\varepsilon}$ such that

$$f\left(\sum_i u_i' \otimes_{U_{\mathbf{l}}} v_i'\right) = \sum_i u_i' v_i' = \mathbf{e}$$

(the multiplicative unit in $U_{\mathbf{l}}$). Then

$$\begin{aligned}
\left(\sum_j u_j \otimes v_j\right)\left(\sum_i u_i' v_i'\right) &= \sum_{i,j} u_j \otimes (v_j u_i' v_i') \\
&= \sum_{i,j} (u_j v_j u_i') \otimes v_i' \\
&= \left(\sum_j u_j v_j\right)\left(\sum_i u_i' \otimes v_i'\right) \\
&= f\left(\sum_j u_j \otimes v_j\right)\left(\sum_i u_i' \otimes v_i'\right) = 0
\end{aligned}$$

so that f is injective and hence an isomorphism

$$f : U_{\mathbf{d}}^{\varpi} \otimes_{U_{\mathbf{l}}} U_{\mathbf{d}}^{\epsilon} \xrightarrow{\sim} U_{\mathbf{d}}^{\mathrm{triv}} \cong U_{\mathbf{l}}.$$

An identical calculation proves that also $U_{\mathbf{d}}^{\epsilon} \otimes_{U_{\mathbf{l}}} U^{\varpi}\mathbf{d} \cong U_{\mathbf{l}}$. This proves invertibility.
Step 3: For any $\varpi \in \mathrm{Irr}(W/W^d)$ the bimodule $U_{\mathbf{d}}^{\varpi}$ is simple. To show this, suppose that we have a surjection of bimodules $U_{\mathbf{d}}^{\varpi} \twoheadrightarrow L$ for some Harish-Chandra L. Then we obtain an exact sequence $U_{\mathbf{d}}^{\epsilon} \otimes_{U_{\mathbf{l}}} U_{\mathbf{d}}^{\varpi} \to U_{\mathbf{d}}^{\epsilon} \otimes_{U_{\mathbf{l}}} L \to 0$ and hence $U_{\mathbf{d}}^{\epsilon} \otimes_{U_{\mathbf{l}}} L$ is a quotient of $U_{\mathbf{l}}$. As this is simple, we deduce that either $L = 0$ or $L = U_{\mathbf{d}}^{\varpi}$ using again the invertibility of $U_{\mathbf{d}}^{\varpi}$. Thus $U_{\mathbf{d}}^{\varpi}$ has no non-trivial quotients and is thus simple.
Step 4: The category $\mathcal{HC}_{\mathbf{l}}(W)$ is semisimple. Recall that any object of $\mathcal{HC}_{\mathbf{l}}$ is a subobject of a finite direct sum of copies of $\mathrm{Ind}_{W^d}^{W}(U_{\mathbf{d}})$. We have shown in steps 1-3 that $\mathrm{Ind}_{W^d}^{W}(U_{\mathbf{d}})$ is semisimple. Hence any object of $\mathcal{HC}_{\mathbf{l}}$ is semisimple and thus so is $\mathcal{HC}_{\mathbf{l}}$. Moreover, a similar argument shows that the simple objects of $\mathcal{HC}_{\mathbf{l}}$ are precisely the $U_{\mathbf{d}}^{\varpi}$ up to isomorphism.
Step 5: For any $\varpi, \epsilon \in \mathrm{Irr}(W/W^d)$ we let $\varpi \cdot \epsilon$ denote their product in the representation ring of W/W^d. We have a morphism $U_{\mathbf{d}}^{\varpi} \otimes_{U_{\mathbf{l}}} U_{\mathbf{d}}^{\epsilon} \to U_{\mathbf{d}}^{\varpi\cdot\epsilon}$ again induced by multiplication in $U_{\mathbf{d}}(W/W^d)$ and by simplicity of $U_{\mathbf{d}}^{\varpi\cdot\epsilon}$ this is surjective. We again aim to show that this is an isomorphism. We know that $U_{\mathbf{d}}^{\varpi} \otimes_{U_{\mathbf{l}}} U_{\mathbf{d}}^{\epsilon}$ is again invertible with inverse $U_{\mathbf{d}}^{\epsilon^{-1}} \otimes_{U_{\mathbf{l}}} U_{\mathbf{d}}^{\varpi^{-1}}$ and it hence is simple too, by Proposition 5.2.30.
Steps 1-5 now imply that F indeeds extends to an equivalence of Abelian tensor categories as claimed.
It remains to show that $F^{-1}(U_{\mathbf{l}}) = \mathrm{triv}$ and $F^{-1}(\mathrm{Ind}_{W^d}^{W}(U_{\mathbf{d}})) = \mathbb{C}(W/W^d)$. The latter statement follows from the decomposition $\mathrm{Ind}_{W^d}^{W}(U_{\mathbf{d}}) = \oplus_{\varpi\in\mathrm{Irr}(W/W^d)} U_{\mathbf{d}}^{\varpi}$ and the definition of F, and the first statement follows as $U_{\mathbf{l}} \cong U_{\mathbf{d}}^{\mathrm{triv}}$. □

Corollary 5.2.32. *Let W be a cyclic group and $\mathbf{k}$ a regular choice of parameters for $U_{\mathbf{k}}(W)$. Then the tensor functor associated to any simple Harish-Chandra bimodule in $\mathcal{HC}_{\mathbf{k}}$ is an autoequivalence of the category of left or right $U_{\mathbf{k}}$-modules and descends to an autoequivalence of $\mathcal{O}_{\mathbf{k}}$.*

Proof. The previous Theorem 5.2.31 implies that every simple object of $\mathcal{HC}_{\mathbf{k}}$ is invertible as any irreducible representation of a cyclic group is. The rest follows from Proposition 5.2.30. □

Chapter 6

Finite-dimensional bimodules of rational Cherednik algebras of cyclic groups

We have constructed the Harish-Chandra bimodules for rational Cherednik algebras of cyclic groups at regular parameter values in the last chapter and have investigated Harish-Chandra bimodules at regular parameter values for rational Cherednik algebras of other complex reflection groups in Chapter 3. In this chapter, we will now consider a subcategory $\mathrm{fd}(\mathcal{HC}_{\mathbf{k}})$ of $\mathcal{HC}_{\mathbf{k}}(\mathbb{Z}_m)$ for non-regular parameter values of rational Cherednik algebras of cyclic groups, namely that of finite-dimensional Harish-Chandra bimodules. The category $\mathrm{fd}(\mathcal{HC}_{\mathbf{k}})$ is computationally easily approachable as it coincides precisely with the category of finite-dimensional $H_{\mathbf{k}}$-bimodules (and this fact is true for all complex reflection groups, not just cyclic groups). This enables us to use techniques and results relating to $\mathcal{O}_{\mathbf{k}}$ to compute the quiver with relations of the category of finite-dimensional Harish-Chandra bimodules from the quiver with relations of blocks of $\mathcal{O}_{\mathbf{k}}$. From this we can deduce a wildness criterion for $\mathrm{fd}(\mathcal{HC}_{\mathbf{k}})$ and thus also $\mathcal{HC}_{\mathbf{k}}$ which is expressed as a condition on $\mathcal{O}_{\mathbf{k}}$.

We first record some numerical descriptions of the structure of category $\mathcal{O}_{\mathbf{k}}$. For this, we will make us of the so-called *naive duality* (see Section 4.2 in [GGOR03]) which will help us to relate the category of left finite-dimensional $H_{\mathbf{k}}$-modules and right $H_{\mathbf{k}}$-modules.

6.1 Naive Duality

The naive duality functor $(\bullet)^\vee$ allows us to compare the category $\mathcal{O}_{\mathbf{k}}^{op}$ with another category $\mathcal{O}$ and so in a sense allows us to derive "dual statements for free". This will be a useful tool for further calculations. It is derived from an isomorphism between a rational Cherednik algebra and the opposite of another rational Cherednik algebra.

Lemma 6.1.1. *Let W be a cyclic group of order m and $\mathbf{k} = (k_0 = 0, \dots, k_{m-1})$ a choice of parameters for W, not necessarily in $\mathrm{Reg}(W)$. There is an isomorphism*

$$\phi : H_{\mathbf{k}}(\mathfrak{h}) \xrightarrow{\sim} H_{\mathbf{k}}^{op}(\mathfrak{h}^*)$$

with $(\bullet)^\vee$ *denoting the dual of a representation:* $\lambda^\vee = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h},\mathbb{C})$*. The isomorphism* ϕ *is given by*

$$x \mapsto x,\ \ y \mapsto y,\ \ s \mapsto s^{-1}.$$

Proof. We shall identify $\mathfrak{h}^{**}$ and $\mathfrak{h}$ in this proof. We need to check that the parameters $\mathbf{k}^\dagger$ of $H_{\mathbf{k}^\dagger}(\mathfrak{h}^*)$ are as claimed. For clarity, we denote the multiplication in $H_{\mathbf{k}^\dagger}(\mathfrak{h}^*)$ by $\circ$ and we denote the commutator $[\bullet,\bullet]$ in $H_{\mathbf{k}^\dagger}(\mathfrak{h}^*)$ by $[\bullet,\bullet]^{op}$. Let us calculate:

$$\begin{aligned}\phi([y,x]) &= [\phi(y),\phi(x)]^{op}\\ &= y\circ x - x\circ y\\ &= xy-yx\end{aligned}$$

Note that as we are considering the rational Cherednik algebra of $H_{\mathbf{k}^\dagger}(\mathfrak{h}^*)$ we have $x\in\mathfrak{h}^*$ and $y\in\mathfrak{h}=\mathfrak{h}^{**}$ so that in order to read the relations in Definition 2.2.3 correctly, we need to consider the idempotents $\mathbf{e}_j^\dagger$ associated to $\det|_{\mathfrak{h}^*}$ so that

$$\mathbf{e}_j^\dagger = \mathbf{e}_{m-j}$$

and the appropriate version of Definition 2.2.3 $(*)$ will "swap x and y". Thus in $H_{\mathbf{k}^\dagger}(\mathfrak{h}^*)$ we have

$$xy-yx = 1+m\sum_{j=0}^{m-1}(k_j^\dagger - k_{j+1}^\dagger)\mathbf{e}_j^\dagger$$

and hence

$$\phi([y,x]) = 1+m\sum_{j=0}^{m-1}(k_j^\dagger - k_{j+1}^\dagger)\mathbf{e}_j^\dagger.$$

On the other hand,

$$\phi\left([y,x]) = \phi(1+m\sum_{j=0}^{m-1}(k_j-k_{j+1})\mathbf{e}_j\right) = 1+m\sum_{j=0}^{m-1}(k_j-k_{j+1})\mathbf{e}_{m-j}$$

so that we conclude

$$\sum_{j=0}^{m-1}(k_j-k_{j+1})\mathbf{e}_{m-j} = \sum_{j=0}^{m-1}(k_j^\dagger - k_{j+1}^\dagger)\mathbf{e}_j^\dagger = \sum_{j=0}^{m-1}(k_j^\dagger - k_{j+1}^\dagger)\mathbf{e}m-jj$$

and from this we easily deduce that

$$\mathbf{k} = \mathbf{k}^\dagger$$

as claimed. □

It is important to point out again that throughout this chapter we will be dealing exclusively with the cyclic case and as such will be retaining the notation from Chapter 5. For the remainder of this chapter, we will keep identifying $\mathfrak{h}^{**}$ and $\mathfrak{h}$. We see that $\mathbb{C}[\mathfrak{h}^{**}]_+ = \mathbb{C}[\mathfrak{h}]_+$ acts locally nilpotently on any $M\in\mathcal{O}_{\mathbf{k}}(\mathfrak{h}^*)$. Now let $M\in\mathcal{O}_{\mathbf{k}}$ and consider $\mathrm{Hom}_{\mathbb{C}}(M,\mathbb{C})$. This is a right

$H_{\mathbf{k}}$-module and so we can define the space

$$M^{\vee} := \{f \in \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C}) \mid f \cdot \mathbb{C}[\mathfrak{h}]_{+}^{N} = 0 \ N >> 0\}.$$

This is a right $H_{\mathbf{k}}$-module and we claim that as such it is finitely generated: As $(\bullet)^{\vee}$ is left exact and contravariant, it will suffice to prove this for simple modules. As we are in the cyclic case, any simple module is either finite-dimensional or isomorphic to a standard module. The statement is clear for finite-dimensional modules and so it remains to show that $M^{\vee}$ is finitely generated for a simple standard object M. For any fixed $N \in \mathbb{N}_0$ we set

$$(M^{\vee})_N = \{f \in \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C}) \mid f \cdot \mathbb{C}[\mathfrak{h}]_{+}{}^{N} = 0\}$$

and note that $M^{\vee} = \cup_{N \geq 0}(M^{\vee})_N$. Further, it is easy to see that $\dim_{\mathbb{C}}(M^{\vee})_N = N$. Identifying M with $\mathbb{C}[\mathfrak{h}]$ as a vector space, we see that $f \in (M^{\vee})_N$ if and only if $f(x^{N+i}) = 0$ for any $i \geq 0$. Denote by $\partial^r|_0$ the linear map $p(x) \mapsto (\partial^r p)(0)$ with ∂ the usual derivative on $\mathbb{C}[\mathfrak{h}]$. A basis of $(M^{\vee})_N$ is then given by $\partial^1|_0, \dots, \partial^{N-1}|_0$. We claim that $\partial^1|_0 =: \partial_0$ generates $M^{\vee}$ as a right module over $H_{\mathbf{k}}$. By simplicity of M, yx^r is a non-zero multiple of x^{r-1} for any $r \neq 0$ and thus we see that the composition $\partial_0 \circ y^r$ must be a non-zero multiple of $\partial^r|_0$. Therefore the submodule generated by ∂_0 contains a basis of $M^{\vee}$ and so $M^{\vee} = (\partial_0)H_{\mathbf{k}}$.

Using the isomorphism from Lemma 6.1.1 we may consider $M^{\vee}$ as a left $H_{\mathbf{k}}(\mathfrak{h}^*)$-module via ϕ (note that any ring isomorphism $R \xrightarrow{\sim} S$ induces an isomorphism $R^{op} \xrightarrow{\sim} S^{op}$ which is what we are implicitly using here). We have just shown that if $M \in \mathcal{O}_{\mathbf{k}}$, then $M^{\vee} \in \mathcal{O}_{\mathbf{k}}(\mathfrak{h}^*)$, so that the next definition does indeed makes sense.

Definition 6.1.2. *([GGOR03], Subsection 4.2.1) For any $H_{\mathbf{k}}$-module M in $\mathcal{O}_{\mathbf{k}}$ we set*

$$M^{\vee} = \{f \in \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C}) \mid f \cdot \mathbb{C}[\mathfrak{h}]_{+}{}^{N} f = 0 \ N >> 0\}$$

and regard this as a left $H_{\mathbf{k}}(\mathfrak{h}^*)$-module. This extends to a contravariant functor

$$(\bullet)^{\vee} : \mathcal{O}_{\mathbf{k}}(\mathfrak{h}) \to \mathcal{O}_{\mathbf{k}}(\mathfrak{h}^*).$$

We refer to this as the naive duality functor.

Recall the notation from Theorem 2.3.11, for any $\lambda \in \mathrm{Irr}(\Gamma)$ we denote by $L(\lambda)$ the associated simple module, by $P(\lambda)$ the indecomposable projective cover $L(\lambda)$ and by $I(\lambda)$ the indecomposable injective envelope of $L(\lambda)$. In any highest weight category, there exist not only standard objects but also so-called costandard objects. We will introduce these next.

Definition 6.1.3. *Let Γ be a complex reflection group and $\lambda \in \mathrm{Irr}(\Gamma)$. We set*

$$\nabla(\lambda) = \{f \in \mathrm{Hom}_{\mathbb{C}[\mathfrak{h}] \rtimes W}(H_{\mathbf{k}}, \lambda) \mid \mathbb{C}[\mathfrak{h}^*]_{+}{}^{N} f = 0 \ N >> 0\}.$$

As $H_{\mathbf{k}}$ is a $H_{\mathbf{k}}$-bimodule, $\nabla(\lambda)$ is a left $H_{\mathbf{k}}$-module.

To show that $\nabla(\lambda)$ is in $\mathcal{O}_{\mathbf{k}}$, we need only argue that it is finitely generated. We can proceed similarly to the proof that $M^{\vee}$ is finitely generated. By the PBW theorem, Theorem 2.2.4, any $f \in \nabla(\lambda)$ is uniquely determined by the values it takes on the subalgebra $\mathbb{C}[\mathfrak{h}^*]$ and using the nilpotence of the action of $\mathbb{C}[\mathfrak{h}^*]_+$ on f (note that this action is induced from the right multiplication in $H_{\mathbf{k}}$ we can see that $\nabla(\lambda)$ is generated by the map $f_0 : y \mapsto v_\lambda$ where v_λ is any non-zero vector in λ and hence $\nabla(\lambda)$ is in $\mathcal{O}_{\mathbf{k}}$.

Lemma 6.1.4. *([GGOR03], Subsection 2.3.2) Let $\lambda, \mu \in \operatorname{Irr}(\Gamma)$, then*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{O}_{\mathbf{k}}}(\Delta(\mu), \nabla(\lambda)) = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{else} \end{cases}$$

and $L(\mu)$ is the unique simple submodule of $\nabla(\mu)$.

The following results are again from [GGOR03] and describe the naive duality functor in greater detail:

Theorem 6.1.5. *([GGOR03], Proposition 4.6 and Subsection 4.2.1) For $\lambda \in \operatorname{Irr}(W)$ let $\lambda^\vee$ denote the dual representation. The following holds:*

1. *$(\bullet)^\vee$ defines a "contravariant equivalence" $\mathcal{O}_{\mathbf{k}}(\mathfrak{h}) \to \mathcal{O}_{\mathbf{k}}(\mathfrak{h}^*)$*
2. *$L_{\mathbf{k}}(\lambda, \mathfrak{h})^\vee = L_{\mathbf{k}}(\lambda^\vee, \mathfrak{h}^*)$*
3. *$\Delta_{\mathbf{k}}(\lambda, \mathfrak{h})^\vee = \nabla_{\mathbf{k}}(\lambda^\vee, \mathfrak{h}^*)$ and $\nabla_{\mathbf{k}}(\lambda, \mathfrak{h})^\vee = \Delta_{\mathbf{k}}(\lambda^\vee, \mathfrak{h}^*)$*
4. *$P_{\mathbf{k}}(\lambda, \mathfrak{h})^\vee = I_{\mathbf{k}}(\lambda^\vee, \mathfrak{h}^*)$ and $I_{\mathbf{k}}(\lambda, \mathfrak{h})^\vee = P_{\mathbf{k}}(\lambda^\vee, \mathfrak{h}^*)$*

Proposition 6.1.6. *(Proposition 3.3 in [GGOR03]) For any $\lambda \in \operatorname{Irr}(W)$ there are equalities*

$$[\Delta(\lambda)] = [\nabla(\lambda)] \text{ and } [P(\lambda)] = [I(\lambda)]$$

in the Grothendieck group of $\mathcal{O}_{\mathbf{k}}$, so that $\Delta(\lambda)$ and $\nabla(\lambda)$ have the same composition factors as have $P(\lambda)$ and $I(\lambda)$.

Theorem 6.1.5 and Proposition 6.1.6 are shown in [GGOR03] for general complex reflection groups rather than just cyclic groups.

We denote by $\kappa_{\mathbf{k}}(\lambda, \mathfrak{h})$ the eigenvalue of the grading element $\underline{\mathrm{h}} \in H_{\mathbf{k}}(\mathfrak{h})$ on $1 \otimes \lambda \subset \Delta_{\mathbf{k}}(\lambda, \mathfrak{h})$ and by $\kappa_{\mathbf{k}}(\lambda, \mathfrak{h}^*)$ the eigenvalue of the grading element $\underline{\mathrm{h}} \in H_{\mathbf{k}}(\mathfrak{h}^*)$ on $1 \otimes \lambda \subset \Delta_{\mathbf{k}}(\lambda, \mathfrak{h}^*)$.
If $\alpha \in \operatorname{Irr}(W)$ and $a \in \mathbb{N}$ are such that W acts on α via $\det|_{\mathfrak{h}}{}^{-a}$ then we find by direct calculation that

$$\kappa_{\mathbf{k}}(\mathfrak{h}, \alpha) = -\frac{1}{2} - mk_a + \sum_{j=0}^{m-1} k_j.$$

Thus as $(\det|_{\mathfrak{h}})^{-1} = \det|_{\mathfrak{h}^*}$ we have

$$\kappa_{\mathbf{k}}(\mathfrak{h}^*, \alpha^\vee) = -\frac{1}{2} - mk_a + \sum_{j=0}^{m-1} k_j$$

and so

$$\kappa_{\mathbf{k}}(\mathfrak{h}^*, \alpha^\vee) = \kappa_{\mathbf{k}}(\mathfrak{h}, \alpha) \; \forall \alpha \in \operatorname{Irr}(W).$$

$$\kappa_{\mathbf{k}}(\lambda, \mathfrak{h}) - \kappa_{\mathbf{k}}(\mu, \mathfrak{h}) = -(\kappa_{\mathbf{k}}(\mu^\vee, \mathfrak{h}^*) - \kappa_{\mathbf{k}}(\lambda^\vee, \mathfrak{h}^*))$$

Proposition 6.1.7. *We have*

$$\operatorname{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(L(\lambda), L(\mu)) \neq 0$$

if and only if there exists an exact sequence

$$0 \to \Delta(\mu) \to \Delta(\lambda) \to L(\lambda) \to 0$$

or

$$0 \to L(\mu) \to \nabla(\mu) \to \nabla(\lambda) \to 0$$

If $\mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(L(\lambda), L(\mu)) \neq 0$ *we have*

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(L(\lambda), L(\mu)) = 1.$$

Proof. By Proposition 1.12 of [BEG03a] the claim holds if $\lambda = \mu$ (as $\mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(L(\lambda), L(\lambda)) = 0$), so let us suppose from now on that $\mu \neq \lambda$. Some of the arguments regarding long exact sequences are standard arguments in any highest weight category, see for example Chapter 3 in [Hum08]. As any submodule of a standard module is isomorphic to a standard module or 0 (since W is cyclic), we have an exact sequence of the form

$$0 \to \Delta(\alpha) \to \Delta(\lambda) \to L(\lambda) \to 0$$

(with $\alpha \in \mathrm{Irr}(W)$ or $\Delta(\alpha) = 0$ in case $\Delta(\lambda)$ is simple). The associated long exact sequence of Ext-groups takes the form:

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(L(\lambda), L(\mu)) \to \mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(\Delta(\lambda), L(\mu)) \to \mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(\Delta(\alpha), L(\mu)) \to \\ \to \mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(L(\lambda), L(\mu)) \to \mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(\Delta(\lambda), L(\mu)) \to \mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(\Delta(\alpha), L(\mu)) \to \\ \to \dots \end{aligned}$$

We will use this to prove our statement. First, note that as $\lambda \neq \mu$ we have

$$\mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(L(\lambda), L(\mu)) = \mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(\Delta(\lambda), L(\mu)) = 0$$

and thus we can simplify the sequence to

$$0 \to \mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(\Delta(\alpha), L(\mu)) \to \mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(L(\lambda), L(\mu)) \to \mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(\Delta(\lambda), L(\mu)) \to \mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(\Delta(\alpha), L(\mu)) \to \dots$$

Now, we need to make some case distinctions:

Case 1: Suppose that $\kappa_{\mathbf{k}}(\mu) - \kappa_{\mathbf{k}}(\lambda) \notin \mathbb{N}$. We claim that $\mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(\Delta(\lambda), L(\mu)) = 0$. Whilst this holds by Lemma 2.9 in [GGOR03], it might be instructive to give a proof. For suppose we have an exact sequence

$$0 \to L(\mu) \to E \to \Delta(\lambda) \to 0.$$

Let $v \in E$ be a vector which maps to the highest weight vector $\bar{v}$ in $\Delta(\lambda)$. We can decompose v into a direct sum $v_1 + \ldots + v_k$ of generalised $\underline{\mathrm{h}}$-eigenvectors with distinct eigenvalues:

$$\begin{aligned} v_1 + \ldots + v_k &= v \\ (\underline{\mathrm{h}} - \kappa_i)^{n_i} v_i &= 0 \end{aligned}$$

with $n_i \in \mathbb{N}, \kappa_i \in \mathbb{C}$ and $\kappa_i \neq \kappa_j$ if $i \neq j$. Then

$$\left(\prod_{i=1}^{k}(\underline{\mathrm{h}} - \kappa_i)^{n_i}\right) v = 0$$

and under the projection map this gets mapped to

$$\left(\prod_{i=1}^{k}(\kappa_{\mathbf{k}}(\lambda) - \kappa_i)^{n_i}\right) \bar{v} = 0$$

and therefore we must conclude that $\kappa_j = \kappa_{\mathbf{k}}(\lambda)$ for some $j \in \{1, \ldots, k\}$. Without loss of generality let us suppose that $\kappa_1 = \kappa_{\mathbf{k}}(\lambda)$. Then replacing v by

$$\left(\prod_{i=2}^{k}(\kappa_{\mathbf{k}}(\lambda) - \kappa_i)^{n_i}\right) v = 0$$

we may assume that v is a generalised $\underline{\mathrm{h}}$-eigenvector with eigenvalue $\kappa_{\mathbf{k}}(\lambda)$. By hypothesis, $yv \in L(\mu)$ as the image of yv under the projection $E \to \Delta(\lambda)$ is zero. Suppose that $yv \neq 0$, then yv is a generalised $\underline{\mathrm{h}}$-eigenvector with eigenvalue $\kappa_{\mathbf{k}}(\lambda) + 1$.

From this we can deduce that

$$\kappa_{\mathbf{k}}(\mu) - \kappa_{\mathbf{k}}(\lambda) - 1 \in \mathbb{N}_0.$$

This contradicts the case hypothesis $\kappa_{\mathbf{k}}(\mu) - \kappa_{\mathbf{k}}(\lambda) \notin \mathbb{N}$ and so we must have

$$yv = 0.$$

Thus by the universal property of $\Delta(\lambda)$ we obtain a map

$$\Delta(\lambda) \to E$$

which by construction splits the sequence above, so that

$$E \cong L(\mu) \oplus \Delta(\lambda) \implies \mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(\Delta(\lambda), L(\mu)) = 0.$$

Therefore we can deduce exactness of

$$0 \to \mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(\Delta(\alpha), L(\mu)) \to \mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(L(\lambda), L(\mu)) \to 0$$

and so

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(L(\lambda), L(\mu)) \cong \mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(\Delta(\alpha), L(\mu)).$$

Hence we have

$$\begin{aligned} &\mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(L(\lambda), L(\mu)) \neq 0 \\ \iff &\mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(\Delta(\alpha), L(\mu)) \neq 0 \\ &\iff \alpha = \mu \end{aligned}$$

And by definition, the latter is the case if and only if

$$0 \to \Delta(\alpha) \to \Delta(\lambda) \to L(\lambda) \to 0$$

is exact. This proves the statement in the case $\kappa_{\mathbf{k}}(\mu) - \kappa_{\mathbf{k}}(\lambda) \notin \mathbb{N}$.
Case 2: Thanks to case 1, we may now assume that $\kappa_{\mathbf{k}}(\mu) - \kappa_{\mathbf{k}}(\lambda) \in \mathbb{N}$. Note that the naive duality functor induces an isomorphism

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}(\mathfrak{h})}(L(\lambda, \mathfrak{h}), L(\mu, \mathfrak{h})) \cong \mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}(\mathfrak{h}^*)}(L(\mu^\vee, \mathfrak{h}^*), L(\lambda^\vee, \mathfrak{h}^*))$$

and since $\kappa_{\mathbf{k}}(\lambda, \mathfrak{h}) - \kappa_{\mathbf{k}}(\mu, \mathfrak{h}) = \kappa_{\mathbf{k}}(\lambda^\vee, \mathfrak{h}^*) - \kappa_{\mathbf{k}}(\mu^\vee, \mathfrak{h}^*)$ we see that in this case

$$\kappa_{\mathbf{k}}(\lambda^\vee, \mathfrak{h}^*) - \kappa_{\mathbf{k}}(\mu^\vee, \mathfrak{h}^*) \notin \mathbb{N}.$$

Thus as in case 1, we deduce

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}(\mathfrak{h})}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}(\mathfrak{h}^*)}(L(\mu^\vee, \mathfrak{h}^*), L(\lambda^\vee, \mathfrak{h}^*)) \cong \mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}(\mathfrak{h}^*)}(\Delta(\beta^\vee, \mathfrak{h}^*), L(\lambda^\vee, \mathfrak{h}^*))$$

where we define $\Delta(\beta^\vee, \mathfrak{h}^*)$ by exactness of

$$0 \to \Delta(\beta^\vee, \mathfrak{h}^*) \to \Delta(\mu^\vee, \mathfrak{h}^*) \to L(\mu^\vee, \mathfrak{h}^*) \to 0$$

(so in particular we again allow $\Delta(\beta^\vee, \mathfrak{h}^*) = 0$). Hence we have

$$\begin{aligned} &\mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}(\mathfrak{h})}(L(\lambda), L(\mu)) \neq 0 \\ \iff &\mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}(\mathfrak{h}^*)}(L(\mu^\vee, \mathfrak{h}^*), L(\lambda^\vee, \mathfrak{h}^*)) \neq 0 \\ \iff &\mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}(\mathfrak{h}^*)}(\Delta(\beta^\vee, \mathfrak{h}^*), L(\lambda^\vee, \mathfrak{h}^*) \neq 0 \end{aligned}$$

As before this is the case if and only if $\beta^\vee = \lambda^\vee$, which is the case if and only if

$$0 \to \Delta(\lambda^\vee, \mathfrak{h}^*) \to \Delta(\mu^\vee, \mathfrak{h}^*) \to L(\mu^\vee, \mathfrak{h}^*) \to 0$$

is exact in $\mathcal{O}_{\mathbf{k}}(\mathfrak{h}^*)$.
Applying the naive duality functor again, we see that this sequence is exact if and only if

$$0 \to L(\mu) \to \nabla(\mu) \to \nabla(\lambda) \to 0$$

is exact in $\mathcal{O}_{\mathbf{k}}(\mathfrak{h})$. This proves our statement in the case $\kappa_{\mathbf{k}}(\mu) - \kappa_{\mathbf{k}}(\lambda) \in \mathbb{N}$ □

Corollary 6.1.8. *For any $\lambda, \mu \in \mathrm{Irr}(W)$ we have*

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(L(\lambda), L(\mu)) \in \{0, 1\}.$$

Proof. In the proof of Proposition 6.1.7 we have shown that

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(L(\lambda), L(\mu)) \cong \mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(\Delta(\alpha), L(\mu))$$

or

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(L(\lambda), L(\mu)) \cong \mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(L(\lambda), \nabla(\lambda)).$$

□

Corollary 6.1.9. *For any $\lambda, \mu \in \mathrm{Irr}(W)$ we have an isomorphism of vector spaces*

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(L(\mu), L(\lambda)).$$

Proof. We need only consider the case that one of $\mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(L(\lambda), L(\mu))$ or $\mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(L(\mu), L(\lambda))$ is non-zero. So without loss of generality let us assume that $\mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(L(\lambda), L(\mu)) \neq 0$. Then one of

$$(A)\ 0 \to \Delta(\mu) \to \Delta(\lambda) \to L(\lambda) \to 0$$

or

$$(B)\ 0 \to L(\mu) \to \nabla(\mu) \to \nabla(\lambda) \to 0$$

is exact. We need to show that one of

$$(A')\ 0 \to \Delta(\lambda) \to \Delta(\mu) \to L(\mu) \to 0$$

or

$$(B')\ 0 \to L(\lambda) \to \nabla(\lambda) \to \nabla(\mu) \to 0$$

We will show that $(A) \implies (B')$ and $(B) \implies (A')$.

So suppose that (A) is exact. Every non-zero proper submodule of $\Delta(\lambda)$ is of the form $\Delta(\alpha)$ for some $\alpha \in \mathrm{Irr}(W)$ with $\alpha \leq \mu$. Proposition 6.1.6 together with maximality of μ amongst all α with $\Delta(\alpha)$ a proper non-zero submodule of $\Delta(\lambda)$ implies (B') as every non-zero proper quotient of a costandard module is again a costandard associated to a representation lower in the $\mathbf{k}$-ordering.

The proof that $(B) \implies (A')$ is similar. □

6.2 Block Structure of $\mathcal{O}_{\mathbf{k}}$

Proposition 6.2.1. *Let $\mathbf{k} = (k_0 = 0, k_1, \ldots, k_{m-1})$ be an arbitrary choice of parameters.*

1. *We have an injection $\Delta(E_p) \hookrightarrow \Delta(E_q)$ if and only if there exists $n \in \mathbb{N}_0$ such that*

$$p \equiv n + q \bmod m$$
$$\text{and } n = m(k_p - k_q)$$

2. *We have a surjection $\nabla(E_p) \twoheadrightarrow \nabla(E_q)$ if and only if there exists $n \in \mathbb{N}_0$ such that*

$$p \equiv -n + q \bmod m$$
$$\text{and } -n = m(k_p - k_q)$$

Proof. **Case 1**:
$(\implies)$: Suppose we have an injection $f : \Delta(E_p) \hookrightarrow \Delta(E_q)$. Then the image of $1 \otimes v_p$ under f must be of the form $x^\ell \otimes v_q$ for some $\ell \in \mathbb{N}_0$ as it has to be an $\underline{\mathrm{h}}$-eigenvector and all eigenspaces

in $\Delta(E_q)$ are one-dimensional. Then

$$s(1 \otimes v_p) = \rho^p(1 \otimes v_p) \text{ and } s(x^\ell \otimes v_q) = \rho^{\ell+q}(x^\ell \otimes v_q)$$

and so

$$\rho^p = \rho^{\ell+q} \implies p \equiv \ell + q \bmod m.$$

Next we must have

$$0 = y(x^\ell \otimes v_q) = (\ell - m(k_{q+\ell} - k_q))(x^{\ell-1} \otimes v_q).$$

As $0 \equiv \ell + q \bmod m$, we conclude that we must have

$$0 = \ell - m(k_p - k_q).$$

Thus taking $n = \ell$ satisfies our statement.
$(\Leftarrow)$: Now suppose that for some $n \in \mathbb{N}_0$ we have

$$\begin{aligned} p &\equiv n + q \bmod m \\ \text{and } n &= m(k_p - k_q). \end{aligned}$$

We claim we can define an injection $\Delta(E_p) \hookrightarrow \Delta(E_q)$ by mapping $1 \otimes v_p \mapsto x^n \otimes v_q$. In fact, all we need to do is to verify that $x^n \otimes v_q$ is a highest weight vector of matching weights and appeal to the universal property of $\Delta(E_p)$. The chosen generator $s \in W$ acts on $x^n \otimes v_q$ by $\rho^{n+q} = \rho^p$ as $p \equiv n + q \bmod m$ and so the W-representations spanned by $1 \otimes v_p$ and $x^n \otimes v_q$ agree. Further we find

$$\begin{aligned} y(x^n \otimes v_q) &= (n - m(k_{q+n} - k_q))(x^{n-1} \otimes v_q) \\ &= (n - m(k_p - k_q))(x^{n-1} \otimes v_q) \\ &= 0 \end{aligned}$$

and so we indeed have a map $\Delta(E_p) \to \Delta(E_q)$ given by $1 \otimes v_p \mapsto x^n \otimes v_q$ as claimed. This has to be an injection, as $\Delta(E_q)$ has no non-zero finite-dimensional submodules.

Case 2: We can reduce this to case 1, since the sequence

$$\nabla(E_p) \to \nabla(E_q) \to 0$$

is exact in $\mathcal{O}_{\mathbf{k}}(\mathfrak{h})$ if and only if

$$0 \to \Delta(E_q^\vee, \mathfrak{h}^*) \to \Delta(E_p^\vee, \mathfrak{h}^*)$$

is exact in $\mathcal{O}_{\mathbf{k}}(\mathfrak{h}^*)$. And by case 1, this is the case if and only if there is $\ell \in \mathbb{N}_0$ such that

$$\begin{aligned} q &\equiv \ell + p \bmod m \\ \ell &= m(k_q - k_p) \end{aligned}$$

and thus on setting $n = -\ell$ we have shown our claim. $\square$

We can now come to the complete description of $\mathrm{Asp}(W)$ which was announced in Chapter 2. The aspherical parameter values of the groups $G(m,p,n)$ were determined by Dunkl and Griffeth in [DG] and our result is just a special case of theirs with $p = n = 1$.

Proposition 6.2.2. *The choice of parameters* $\mathbf{k} = (k_0 = 0, k_1, \ldots, k_{m-1})$ *is aspherical* $\iff$ *there exist* $p \neq q \in \{0, \ldots, m-1\}$ *such that*

$$\delta_{p,0} - \delta_{q,0} + \frac{p-q}{m} - (k_p - k_q) = 0$$

which is equivalent to demanding

$$\frac{a_q}{m} + k_q = \frac{a_p}{m} + k_p.$$

Proof. $(\Rightarrow)$: Suppose that $\mathbf{k}$ is aspherical. Then there exists a simple module $L(E_p)$ in $\mathcal{O}_{\mathbf{k}}(H_{\mathbf{k}})$ such that $\mathbf{e}L(E_p) = 0$. Such a simple module has to be finite-dimensional as non finite-dimensional simple modules in $\mathcal{O}_{\mathbf{k}}(W)$ are standard modules. So we have a $q \neq p$ such that

$$0 \to \Delta(E_q) \to \Delta(E_p) \to L(E_p) \to 0$$

is exact in $\mathcal{O}_{\mathbf{k}}(H_{\mathbf{k}})$ and thus

$$0 \to \mathbf{e}\Delta(E_q) \to \mathbf{e}\Delta(E_p) \to 0$$

is exact as a sequence of $U_{\mathbf{k}}$-modules. Both are still highest weight modules and thus their highest weights must be the same. Therefore

$$\begin{aligned} -a_p + \kappa_{\mathbf{k}}(E_p) &= -a_q + \kappa_{\mathbf{k}}(E_q) \\ \implies a_q - a_p &= \kappa_{\mathbf{k}}(E_q) - \kappa_{\mathbf{k}}(E_p) \end{aligned}$$

and upon substituting $a_r = (1 - \delta_{0,r})(m - r)$ we deduce

$$\begin{aligned} &(m-q)(1-\delta_{q,0}) - (m-p)(1-\delta_{p,0}) = mk_p - mk_q \\ \implies &m(\delta_{p,0} - \delta_{q,0}) + p - q = m(k_p - k_q) \\ \implies &\delta_{p,0} - \delta_{q,0} + \frac{p-q}{m} = k_p - k_q \end{aligned}$$

which proves the claim.

$(\Leftarrow)$: Now suppose that for distinct p, q we have

$$\delta_{p,0} - \delta_{q,0} + \frac{p-q}{m} - (k_p - k_q) = 0.$$

We consider

$$n := p - q + m(\delta_{p,0} - \delta_{q,0})$$

and note that $m(k_p - k_q) = n$ as well as $p - q \equiv n(m)$. Thus we deduce from Proposition 6.2.1 if $n > 0$ we have an injection

$$\Delta(E_p) \hookrightarrow \Delta(E_q)$$

whereas if $n \leq 0$ then

$$\Delta(E_q) \hookrightarrow \Delta(E_p),$$

So from now on let us suppose without loss of generality that $n \leq 0$ and thus that $\Delta(E_q) \hookrightarrow \Delta(E_p)$. Moreover from the proof of Proposition 6.2.1 we can see that this embedding is given by $1 \otimes v_q \mapsto x^{-n} \otimes v_p$.

We can now calculate the dimension of $\mathbf{e}\left(\Delta(E_p)/\Delta(E_q)\right) \cong \mathbf{e}\Delta(E_p)/\mathbf{e}\Delta(E_q)$: The highest weight vector of $\mathbf{e}\Delta(E_p)$ is $\mathbf{e}x^{a_p} \otimes v_p$ and that of $\mathbf{e}\Delta(E_q)$ is $\mathbf{e}x^{a_q} \otimes v_q$ and the map $\mathbf{e}\Delta(E_q) \hookrightarrow \mathbf{e}\Delta(E_p)$ is given by $\mathbf{e}x^{a_q} \otimes v_q \mapsto \mathbf{e}x^{a_q-n} \otimes v_p$. Thus

$$\begin{aligned}
\dim_{\mathbb{C}}\left(\mathbf{e}\Delta(E_p)/\mathbf{e}\Delta(E_q)\right) &= \frac{1}{m}(a_q - n - a_p) \\
&= \frac{1}{m}\left((1-\delta_{q,0})(m-q) - (p - q + m(\delta_{p,0} - \delta_{q,0})) - (1-\delta_{p,0})(m-p)\right) \\
&= \frac{1}{m}\left(m - q - m\delta_{q,0} + q\delta_{q,0} - m + p + m\delta_{p,0} - p\delta_{p,0}\right) \\
&- \frac{1}{m}\left(p - q + m(\delta_{p,0} - \delta_{q,0})\right) \\
&= \frac{1}{m}\left(-q - m\delta_{q,0} + p + m\delta_{p,0}\right) - \frac{1}{m}\left(p - q + m(\delta_{p,0} - \delta_{q,0})\right) \\
&= 0
\end{aligned}$$

So $\mathbf{k}$ is indeed aspherical as any simple module in a Jordan-Hölder series of $\mathbf{e}\Delta(E_p)/\mathbf{e}\Delta(E_q)$ will be annihilated by $\mathbf{e}$. □

Definition 6.2.3. *1. Let $\mathcal{C}$ be an Abelian category. For simple objects, L and S we write $L \sim S$ if $\operatorname{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(L,S) \neq 0$ or $\operatorname{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(S,L) \neq 0$ or $L = S$. We extend this to an equivalence relation on the set of (equivalence classes of) simple objects and refer to an equivalence class as a block of $\mathcal{C}$. Now suppose $\mathcal{B}$ is a block of $\mathcal{C}$ and $\mathcal{C}$ is such that every object has finite length and that the Jordan-Hölder Theorem holds. Then we will also refer to the full subcategory of all modules whose composition factors lie in $\mathcal{B}$ as a "block" of $\mathcal{C}$ and denote this subcategory again by $\mathcal{B}$.*

Proposition 6.2.4. *Let $\mathcal{B}$ denote a block of $\mathcal{O}_{\mathbf{k}}$ and suppose that $E_p \in \mathcal{B}$ then*

$$\begin{aligned}
\mathcal{B} &= \{E_q \mid \exists n \in \mathbb{Z} \text{ s.t. } p - q \equiv n(m) \text{ and } m(k_p - k_q) = n\} \\
&= \{E_q \mid \frac{p-q}{m} - (k_p - k_q) \in \mathbb{Z}\}
\end{aligned}$$

Proof. We shall first show that the sets

$$\begin{aligned}
\mathcal{B}' &:= \{E_q \mid \exists n \in \mathbb{Z} \text{ s.t. } p - q \equiv n(m) \text{ and } m(k_p - k_q) = n\} \\
&\text{and} \\
\mathcal{B}'' &:= \{E_q \mid \frac{p-q}{m} - (k_p - k_q) \in \mathbb{Z}\}
\end{aligned}$$

agree and then that they coincide with the block of $\mathcal{O}_{\mathbf{k}}$ containing E_p.

So suppose that $E_q \in \mathcal{B}'$ and consider $z = p - q - m(k_p - k_q)$. By definition of $\mathcal{B}'$ we see $z \equiv 0(m)$ and thus $\frac{z}{m} \in \mathbb{Z}$, hence $\frac{p-q}{m} - (k_p - k_q) \in \mathbb{Z}$ and $E_q \in \mathcal{B}''$, so $\mathcal{B}' \subseteq \mathcal{B}''$.

To show the reverse inclusion, take $E_q \in \mathcal{B}''$ and set $n = m(k_p - k_q)$. Then n must be integral as $n \in p - q + m\mathbb{Z}$ and by construction we have both $m(k_p - k_q) = n$ as well as $p - q \equiv n(m)$ so $E_q \in \mathcal{B}'$ and $\mathcal{B}' \subseteq \mathcal{B}''$. Hence these sets agree.

It remains to show that say $\mathcal{B}' = \mathcal{B}$. Clearly if $E_q \in \mathcal{B}'$ then either $\Delta(E_q) \hookrightarrow \Delta(E_p)$ or $\Delta(E_p) \hookrightarrow \Delta(E_q)$ and therefore $E_q \in \mathcal{B}$ and $\mathcal{B}' \subseteq \mathcal{B}$. Conversely suppose that $E_q \in \mathcal{B}$. By the definition of blocks and using Corollary 6.1.9 we can therefore assume that we have a sequence

$$E_q = E_{a_0}, E_{a_1}, \ldots, E_{a_\ell} = E_p$$

such that

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(L(E_{a_i}), L(E_{a_{i+1}})) \neq 0.$$

From Proposition 6.1.7 and Proposition 6.2.1 it then follows that there exist $n_0, \ldots, n_{\ell-1}$ such that $a_i \equiv n_i + a_{i+1}$ modulo m and $m(k_{a_i} - k_{a_{i+1}}) = n_i$. Then we deduce from repeated substitution that

$$a_0 \equiv n_0 + n_1 + \ldots + n_{\ell-1} + a_\ell \text{ mod } m$$

and

$$\begin{aligned} m(k_{a_0} - k_{a_\ell}) &= m(k_{a_0} - k_{a_1}) + m((k_{a_1} - k_{a_2}) + \ldots + m(k_{a_{\ell-1}} - k_{a_\ell}) \\ &= m(k_{a_0} - k_{a_\ell}). \end{aligned}$$

Thus setting $n = n_0 + \ldots + n_{\ell-1}$ and recalling $p = a_0$ and $q = a_\ell$ we have shown $p \equiv n + q$ and $m(k_p - k_q) = n$. So $E_q \in \mathcal{B}'$ and we indeed have the equality

$$\mathcal{B} = \mathcal{B}' = \mathcal{B}''.$$

□

Corollary 6.2.5. *Let $\mathcal{B}$ be a block of $\mathcal{O}_{\mathbf{k}}$, then there exists an enumeration $\{\lambda_1, \ldots, \lambda_\ell\}$ of its elements such that*

$$\Delta(\lambda_i) \hookrightarrow \Delta(\lambda_{i+1}) \text{ for } i = 1, \ldots \ell - 1.$$

Proof. Fix a $E_p \in \mathcal{B}$, for any $E_q \in \mathcal{B}$ there exists $n_q \in \mathbb{Z}$ such that $q - p \equiv n_q(m)$ and $m(k_q - k_p) = n_q$ by Proposition 6.2.4. Note that if $n_q = n_r$ we have $q - p \equiv r - p(m)$ and therefore $q = r$ as $r, q \in \{0, \ldots, m-1\}$. So the n_q are pairwise distinct and thus there exists a unique enumeration $\{\mu_1, \ldots, \mu_\ell\}$ such that $n_i < n_j$ if and only if $i < j$. Then applying Proposition 6.2.1 we see that $\Delta(\mu_j) \hookrightarrow \Delta(\mu_i)$ for $i < j$. So we may choose $\{\lambda_1, \ldots, \lambda_\ell\}$ to be the reverse enumeration of $\{\mu_1, \ldots, \mu_\ell\}$. □

We have seen in Theorem 4.1.14 that a block $\mathcal{B}$ of $\mathcal{O}_{\mathbf{k}}$ when viewed as a category will be equivalent to $\mathrm{End}(P_{\mathcal{B}})^{op} - \mathrm{mod}$ for any progenerator $P_{\mathcal{B}}$ of $\mathcal{B}$. As $\mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(M, N)$ is finite-dimensional for any two objects in $\mathcal{O}_{\mathbf{k}}$ we thus have an equivalence to a module category over a finite-dimensional algebra which makes performing computations easy. There is an efficient language to compute $\mathrm{End}(P_{\mathcal{B}})^{op}$, namely that of quivers with relations. These are standard techniques, so we will only give a brief sketch.

Lemma 6.2.6. *A finite-dimensional complex algebra has only finitely many simple modules up to isomorphism.*

Proof. As A is finite-dimensional, it has a composition series and by the Jordan-Hölder theorem, A has only finitely many composition factors up to isomorphism. Any simple module S is a quotient of A and thus equal to a composition factor of A. □

Definition 6.2.7. *1. Let A be a finite-dimensional algebra, $S_1, \dots, S_n$ an enumeration of the pairwise non-isomorphic simple modules of A. The* Ext*-quiver of A is the quiver with vertices $S_1, \dots, S_n$ and $\dim_{\mathbb{C}} \operatorname{Ext}^1_A(S_i, S_j)$ the number of arrows $S_i \to S_j$.*

2. Let $\mathcal{C}$ be an Abelian $\mathbb{C}$-linear category with finitely many isomorphism classes of simple objects. If P is a progenerator of $\mathcal{C}$ such that $\operatorname{End}_{\mathcal{C}}(P)$ is finite-dimensional. then we refer to the Ext*-quiver of $\operatorname{End}_{\mathcal{C}}(P)^{op}$ as the* Ext*-quiver of $\mathcal{C}$.*

Lemma 6.2.8. *1. If A and A' are Morita-equivalent finite-dimensional complex algebras the* Ext*-quivers of A and A' are isomorphic.*

2. If $\mathcal{C}$ and $\mathcal{C}'$ are equivalent Abelian $\mathbb{C}$-linear category with finitely many isomorphism classes of simple objects. Then the Ext*-quivers of $\mathcal{C}$ and $\mathcal{C}'$ are isomorphic.*

Proof. The second statement follows from the first and the first statement is clear as the definition of the Ext-quiver is categorical: If $F : A - \mathrm{Mod} \to A' - \mathrm{Mod}$ is an equivalence it will take simple objects to simple objects and induce isomorphisms of Ext-groups and hence give an isomorphism of directed graphs from the Ext-quiver of A to that of A'. □

Next theorem can be found in Section 4.3 of [Gab80].

Theorem 6.2.9. *Let A be a finite-dimensional complex algebra and Q its* Ext*-quiver. Then for some ideal $I \lhd \mathbb{C}Q$, the categories $\mathbb{C}Q/I - \mathrm{mod}$ and $A - \mathrm{mod}$ are equivalent.*

So in particular, if $\mathcal{C}$ is an Abelian $\mathbb{C}$-linear Noetherian category with progenerator P such that $\operatorname{End}_{\mathcal{C}}(P)^{op}$ is finite-dimensional, then by Theorem 4.1.14, $\mathcal{C}$ and $\operatorname{End}_{\mathcal{C}}(P)^{op} - \mathrm{mod}$ are equivalent. Thus $\mathcal{C}$ will be equivalent to $\mathbb{C}Q/I - \mathrm{mod}$ for Q the Ext-quiver of $\operatorname{End}_{\mathcal{C}}(P)^{op}$ and Q can be determined from the categorical data of $\mathcal{C}$, the number of simple objects and the dimensions of Ext-groups. So now we have a first approach to computing the structure of a block $\mathcal{B}$ of $\mathcal{O}_{\mathbf{k}}$, namely by identifying the simples in $\mathcal{B}$ and then computing the dimensions of their Ext-groups. We have already done this and therefore we see that the quiver of a block of $\mathcal{O}_{\mathbf{k}}$ has the form (without relations)

$$\begin{tikzcd} \lambda_1 \arrow[r, bend left, "a_1"] & \lambda_2 \arrow[l, bend left, "b_1"] \arrow[r, bend left, "a_2"] & \lambda_3 \arrow[l, bend left, "b_2"] \arrow[r, bend left, "a_3"] & \cdots \arrow[l, bend left, "b_3"] \arrow[r, bend left, "a_{n-1}"] & \lambda_n \arrow[l, bend left, "b_{n-1}"] \end{tikzcd}$$

where the representations λ_j form a string:

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n.$$

We need to explain the notation $\lambda_i \leq \lambda_j$

Definition 6.2.10. *Let Γ be a complex reflection group and $\mu, \lambda \in \operatorname{Irr}(\Gamma)$. For fixed parameters $\mathbf{k}$ we write $\lambda \leq \mu$ if there is a non-zero morphism $\Delta(\lambda) \to \Delta(\mu)$.*

This is stronger than the order $\prec_{\mathbf{k}}$ defined in Theorem 2.3.11. Recall that $\prec_{\mathbf{k}}$ is defined by $\lambda \prec_{\mathbf{k}} \mu$ if $\kappa_{\mathbf{k}}(\mu) - \kappa_{\mathbf{k}}(\lambda) \in \mathbb{N}$. Clearly if $\lambda \leq \mu$ then $\lambda \prec_{\mathbf{k}} \mu$ but the converse is not true, with counterexamples being found for integral parameter values.

Let us explain some of the background leading to Theorem 6.2.9. All of these are well-known results and we have either given a reference or a proof.

Definition 6.2.11. *A finite-dimensional complex algebra is called basic if all of its simple modules have dimension* 1.

Lemma 6.2.12. *Any finite-dimensional complex algebra A is Morita equivalent to a basic algebra A'.*

Proof. As A is finite-dimensional, we have an isomorphism

$$A = \oplus_{i=1}^{n} P_i^{\oplus r_i}$$

where each P_i is an indecomposable left A-module and the P_i are pairwise non-isomorphic. Then each P_i is projective and $P = P_1 \oplus P_2 \oplus \ldots \oplus P_n$ is a progenerator of $A - \text{Mod}$. Thus A and $\text{End}_A(P)^{op}$ are Morita equivalent by Theorem 4.1.12. We claim that $\text{End}_A(P)^{op}$ is basic. By Morita equivalence, the simple modules of $\text{End}_A(P)^{op}$ are the modules $\text{Hom}_A(P, S_i)$ where $S_1, \ldots, S_r$ is an enumeration of the simple modules of A. We need to show that this space is one-dimensional.
In a first step, let us show that $P_i/\text{rad}P_i$ is simple. We follow the arguments in the proof of Theorem 4.9 of [Val09]. Suppose $\pi_i : P_i \twoheadrightarrow S$ is surjective. Denote by Q_i a submodule of minimal dimension of P_i such that $\pi_i|_Q$ is surjective. Using the projective property of P_i there is a map $\theta : P_i \to Q$ such that

$$\begin{tikzcd}
 & P_i \arrow[dl, "\theta"'] \arrow[d, "\pi_i"] & \\
Q \arrow[r, "\pi_i"'] & S \arrow[r] \arrow[d] & 0 \\
 & 0 &
\end{tikzcd}$$

commutes. Then θ must be surjective by minimality of Q. Denote by $\imath : Q \hookrightarrow P_i$ the inclusion map. Then $\pi_i(\theta(\imath(Q))) = \pi_i(\imath(Q)) = S$ and so $\theta(\imath(Q))$ is a submodule of Q surjecting onto S under π_i. By minimality we must have $\theta\imath(Q) = Q$ and hence $P_i \cong Q \oplus \ker(\theta)$. By indecomposability we must have $\ker(\theta) = 0$ and thus $P_i/\text{rad}P_i$ must be simple as else Q would be a proper submodule of P. Hence the dimension $\dim_{\mathbb{C}} \text{Hom}_A(P, S)$ is equal to the number of P_i whose head is equal to the simple S. We claim this is the case for precisely one P_i.

Suppose S is a simple module with $P_i \twoheadrightarrow S$. We claim that if $P_j \twoheadrightarrow S$ then $P_i \cong P_j$. So suppose we have

$$\begin{tikzcd}
 & P_i \arrow[d, "\pi_i"] & \\
Q_i \arrow[r, "\pi_i"'] & S \arrow[r] \arrow[d] & 0 \\
 & 0 &
\end{tikzcd}$$

Then by the projective property of P_i, P_j we obtain morphisms θ, ϕ such that

$$\begin{tikzcd}
 & P_i \arrow[d, "\imath"] \arrow[dl, bend left, "\theta"] & \\
P_j \arrow[ur, bend left, "\phi"] \arrow[r, "\jmath"'] & S \arrow[r] \arrow[d] & 0 \\
 & 0 &
\end{tikzcd}$$

commutes. So we have $\imath\phi = \jmath$ and $\jmath\theta = \imath$. Thus we have $\imath = \imath\phi\theta$ and hence $\imath = \imath(\phi\theta)^k$ for any $k \in \mathbb{N}$. Thus $\phi\theta \in \mathrm{End}_A(P_i)$ is not nilpotent and by Fitting's Lemma it must be automorphism of P_i. Repeating the argument we obtain that $\theta\phi$ is an automorphism of P_j and so θ, ϕ are isomorphisms. So $P_i \cong P_j$ if $P_i \mathrm{rad} P_i \cong P_j \mathrm{rad} P_j$. So indeed

$$\dim_{\mathbb{C}} \mathrm{Hom}_A(P, S) = 1$$

and $\mathrm{End}_A(P)^{op}$ is basic. $\square$

So it will suffice to show Theorem 6.2.9 for basic algebras. In fact a stronger result holds in this situation. This can be found as Proposition 4.1.7 in [Ben10].

Theorem 6.2.13. *Let A be a basic algebra and Q its* Ext*-quiver. There is a surjection $\mathbb{C}Q \twoheadrightarrow A$.*

Clearly Theorem 6.2.13 and Lemma 6.2.12 imply Theorem 6.2.9. We shall quote some results in the theory of finite-dimensional algebras that we can use to sketch a proof of Theorem 6.2.13.

Theorem 6.2.14. *Let A be a finite-dimensional algebra over $\mathbb{C}$ with $S_1, \ldots, S_n$ an enumeration of the distinct simple objects of A. Let $J(A)$ denote the Jacobson radical of A. Then the following hold:*

1. *For each S_i there exists a unique (up to isomorphism) projective module P_i*
2. *$P_i/\mathrm{rad}P_i \cong S_i$.*
3. *P_i is indecomposable.*
4. *For each simple module S_i there exists an idempotent $f_i \in A$ such that $P_i = Af_i$ with P_i as in* (1)
5. *The idempotents $\{f_i \mid i = 1, \ldots, n\}$ form a complete set of orthogonal idempotents of A, so that $1 = f_1 + \ldots + f_n$, $f_i f_j = \delta_{i,j} f_j$ and n is maximal.*
6. *$A = P_1 \oplus \ldots \oplus P_n$ as left A-modules.*
7. $\dim_{\mathbb{C}} \left(f_j J(A) f_i / f_j J(A)^2 f_i\right) = \dim_{\mathbb{C}} \mathrm{Ext}^1_A(S_i, S_j)$

Proof. By Theorem 4.2 in [ARS10] every A-module has a projective cover which is unique up to isomorphism. Thus the P_i exist and are unique up to isomorphism. By Proposition 4.3 in [ARS10] we thus have an isomorphism $P_i/\mathrm{rad}P_i \xrightarrow{\sim} S_i/\mathrm{rad}S_i \cong S_i$. Clearly this also shows that each P_i is indecomposable.

Now let $f_1, \ldots, f_N$ be a complete set of orthogonal non-zero primitive idempotents of A, so that $1 = f_1 + \ldots + f_N, f_i f_j = \delta_{ij} f_i$ and no f_i can be written as the sum of two idempotent elements. By Proposition 4.8 (d) in [ARS10] such a set always exists and each Af_i is indecomposable. Any such Af_i is clearly surjective and $A \cong Af_1 \oplus \ldots \oplus Af_N$ as $1 = f_1 + \ldots + f_N$, the sum is direct by orthogonality. We claim that $N = n$.

For any simple S_j we have $\dim_{\mathbb{C}} \mathrm{Hom}_A(A, S_j) = \dim_{\mathbb{C}} S_j = 1$ and thus for some f_a we must

have $\mathrm{Hom}_A(Af_a, S_j) \neq 0$. We claim that if $\mathrm{Hom}(Af_a, S_j) \neq 0$ and $\mathrm{Hom}_A(Af_b, S_j) \neq 0$ then $a = b$. As vector spaces we have an isomorphism $\mathrm{Hom}(Af_a, S_j) \cong \mathbb{C} f_a S_j$ taking a morphism ϕ to $\phi(f_a) = f_a \phi(f_a)$. Thus we have an isomorphism of vector spaces $f_a S_j = f_b S_j$. As S_j is one-dimensional we must in fact have an equality $f_a S_j = f_b S_j = S_j$ and so $S_j = f_a S_j = f_a f_b S_j = 0$ if $a \neq b$ which is absurd. Thus we have $a = b$ meaning that the condition $\mathrm{Hom}_A(Af_a, S_j) \neq 0$ uniquely determines f_a and we can deduce $N \geq n$.
If $N > n$ we must be able to choose f_c such that $\mathrm{Hom}_A(Af_c, S_i) = 0$ for all simple modules S_i. In particular $\mathrm{Hom}_A(Af_c, Afc/\mathrm{rad}(Af_c)) = 0$ which shows $Af_c = 0$. But $f_c \in Af_c$ and so $f_c = 0$ contrary to our hypothesis. Thus we must have $N = n$.
So after renumbering we may assume that $\mathrm{Hom}_A(Af_j, S_i) \neq \delta_{ij}$ for $i, j = 1, \ldots, n$ and this implies $Af_j/\mathrm{rad}(Af_j) \cong S_j$. By Theorem 4.4 (a) this implies that Af_j is a projectove cover of S_j and so by uniqueness $Af_j \cong P_j$.
The final equality of dimensions is Proposition 2.4.3 in [Ben10]. □

We can now give a proof of Theorem 6.2.13, retaining the notation of Theorem 6.2.14 and following the proof as laid out in Section 4.3 in [Gab80] or Proposition 4.1.7 in [Ben10]:

Proof. (of Theorem 6.2.13) We define a map $\Phi : \mathbb{C}Q \to A$ as follows: The vertex idempotent e_i of the vertex S_i is mapped to the idempotent f_i. To define Φ on the arrows we proceed as follows: For any $i, j \in \{1, \ldots, n\}$ we choose a vector space decomposition $f_j J(A) f_i \cong f_j J(A)^2 f_i \oplus V_{i,j}$ and let $v_{i,j}^1, \ldots, v_{i,j}^{d_{ij}}$ be a basis of $V_{i,j}$ with $d_{i,j} = \dim_{\mathbb{C}} \mathrm{Ext}_A^1(S_i, S_j)$. If $a_1, \ldots, a_{d_{i,j}}$ is any enumeration of the arrows from S_i to S_j we then define $\Phi(a_k) = v_{i,j}^k$. It can be checked that Φ is a well-defined algebra morphism and it remains to show surjectivity. But Φ is surjective as a morphism $\mathbb{C}Q \to A/J(A)^2$ and by Proposition 1.2.8 in [Ben10] this implies surjectivity of $\Phi : \mathbb{C}Q \to A$. □

We will now turn our attention again to determine the relations on the quivers of blocks of $\mathcal{O}_{\mathbf{k}}$ for a cyclic group. The underlying quiver as already stated is

$$\begin{tikzcd} \lambda_1 \arrow[r, bend left, "a_1"] & \lambda_2 \arrow[l, bend left, "b_1"] \arrow[r, bend left, "a_2"] & \lambda_3 \arrow[l, bend left, "b_2"] \arrow[r, bend left, "a_3"] & \cdots \arrow[l, bend left, "b_3"] \arrow[r, bend left, "a_{n-1}"] & \lambda_n \arrow[l, bend left, "b_{n-1}"] \end{tikzcd}$$

with $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. Recall that the notation $\lambda \leq \mu$ means that we have an embedding $\Delta(\lambda) \hookrightarrow \Delta(\mu)$.

Lemma 6.2.15. *For the block structure as described above we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(P(\lambda_s), P(\lambda_r)) = 1 + n - \max\{r, s\}.$$

Proof. This is an easy calculation that follows nearly immediately from BGG reciprocity. We have

$$\begin{aligned} \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(P_s, P_r) &= [P(\lambda_r) : L(\lambda_s)] \\ &= \sum_{\gamma \in \mathrm{Irr}(W)} [P(\lambda_r) : \Delta(\gamma)][\Delta(\gamma) : L(\lambda_s)] \\ &= \sum_{t=1}^{n} [P(\lambda_r) : \Delta(\lambda_t)][\Delta(\lambda_t) : L(\lambda_s)] \\ &= \sum_{t=1}^{n} [\Delta(\lambda_t) : L(\lambda_r)][\Delta(\lambda_t) : L(\lambda_s)] \end{aligned}$$

We have

$$[\Delta(\lambda_t) : L(\lambda_u)] = \begin{cases} 0 \text{ if } t < u \\ 1 \text{ if } t \geq u \end{cases}$$

so that

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{O}_{\mathbf{k}}}(P_s, P_r) = 1 + n - \max\{r, s\}.$$

□

Definition 6.2.16. *Let Γ be a complex reflection group and $\lambda \in \operatorname{Irr}(\Gamma)$. For $n \in \mathbb{N}$ we define the generalised standard module of order n to be*

$$\Delta^{(n)}(\lambda) := H_{\mathbf{k}} \otimes_{\mathbb{C}[\mathfrak{h}^*]\#\Gamma} \left(\mathbb{C}[\mathfrak{h}^*]/\mathbb{C}[\mathfrak{h}^*]_+^n \otimes_{\mathbb{C}} \lambda\right).$$

We let the group Γ act diagonally on $\mathbb{C}[\mathfrak{h}^*]/\mathbb{C}[\mathfrak{h}^*]_+^n \otimes_{\mathbb{C}} \lambda$ for the purposes of forming the first tensor product.

Lemma 6.2.17. *Suppose that W is cyclic, then $\Delta^{(n)}(E_p)$ has basis $\{x^a \otimes y^b \otimes v_p\}$ for $a \in \mathbb{N}_0$ and $0 \leq b \leq n-1$.*

Proof. We have $\mathbb{C}[\mathfrak{h}^*]_+ = \mathbb{C}[y]y$ and now apply the PBW theorem, Theorem 2.2.4. □

Example 6.2.18. *1. $\Delta^{(1)}(\lambda) = \Delta(\lambda)$.*

2. Let $W = \mathbb{Z}_2$, then if $\mathbf{k} = (k_0 = 0, k_1)$ is regular, $\Delta^{(2)}(E_p) \cong \Delta(E_0) \oplus \Delta(E_1)$ for any $p \in \{0, 1\}$.

The generalised standard modules were used in [Gua03] and [GGOR03] to show the existence of enough projectives in $\mathcal{O}_{\mathbf{k}}$: Projective covers turn out to be summands of suitably large generalised standard modules.

Proposition 6.2.19. *(Remarks following Proposition 4.4 in [Gua03], Corollary 2.7 in [GGOR03]) Let Γ be a complex reflection group, $\mathbf{k}$ a choice of parameters and $\lambda \in \operatorname{Irr}(\Gamma)$. The projective cover $P(\lambda)$ is a homomorphic image of $\Delta^{(n)}(\lambda)$ for some suitable $n \in \mathbb{N}$ depending on $\mathbf{k}$.*

Corollary 6.2.20. *Let Γ be a complex reflection group and $\mathbf{k}$ a choice of parameters. For any $\lambda \in \operatorname{Irr}(\Gamma)$, the projective cover $P(\lambda)$ is cyclic and generated by generalised $\underline{\mathrm{h}}$-eigenvector v_λ^∞ of eigenvalue $\kappa_{\mathbf{k}}(\lambda)$ such that $\mathbb{C}v_\lambda^\infty \cong \lambda$ as Γ-representations.*

Proof. Follows from Proposition 6.2.19 as the statement is true for $\Delta^{(n)}(\lambda)$ and we can set $v_\lambda^\infty = f(1 \otimes 1 \otimes \lambda)$ under the map $f : \Delta^{(n)}(\lambda) \twoheadrightarrow P(\lambda)$. □

This makes the next result slightly unsurprising: For the rest of this section, we shall now make the following **standing hypothesis** that any finite-dimensional simple object in $\mathcal{O}_{\mathbf{k}}$ has dimension 1:

$$(H1) : \dim_{\mathbb{C}} L(\lambda) < \infty \implies \dim_{\mathbb{C}} L(\lambda) = 1.$$

The validity of this hypothesis up to Morita equivalence will be shown later in this chapter, see Corollary 6.3.19.

Theorem 6.2.21. *Let* $\mathbf{k}$ *be a choice of parameters such that* $(H1)$ *holds and*

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$$

be a block of $\mathcal{O}_{\mathbf{k}}$ *with* $\Delta(\lambda_r) \hookrightarrow \Delta(\lambda_{r+1})$ *for* $r = 1, \ldots, n-1$. *Then the projective cover* $P(\lambda_r)$ *of* $L(\lambda_r)$ *is*

$$P(\lambda_r) = \Delta^{(1+n-r)}(\lambda_r).$$

Proof. For ease of notation, we identify the representations λ_r with their basis vectors. As $\dim_{\mathbb{C}} L(\lambda_{r+1}) = 1$ the embeddings $\Delta(\lambda_r) \hookrightarrow \Delta(\lambda_{r+1})$ must be given by

$$1 \otimes \lambda_r \mapsto x \otimes \lambda_{r+1}$$

and since both must transform in the same way under W we have

$$\lambda_r = E_1 \otimes \lambda_{r+1}, \lambda_{r+1} = E_{m-1} \otimes \lambda_r.$$

We will first show that $\Delta^{(1+n-r)}(\lambda_r)$ has a filtration with subquotients $\Delta(\lambda_r), \Delta(\lambda_{r+1}), \ldots, \Delta(\lambda_n)$ and the prove the same statement for $P(\lambda_r)$. Then we will show the existence of a surjective morphism $\Delta^{(1+n-r)}(\lambda_r) \twoheadrightarrow P(\lambda_r)$ which must then be an isomorphism as both modules have the same lengths (since their standard filtrations have identical subquotients).
We will begin with the first assertion and construct such a filtration on $\Delta^{(1+n-r)}(\lambda_r)$. The module $\Delta^{(1+n-r)}(\lambda_r)$ has basis $\{x^a \otimes y^b \otimes \lambda_r \mid a \in \mathbb{N}_0, 0 \leq b \leq n-r\}$; we define

$$F_j = H_{\mathbf{k}}(1 \otimes y^j \otimes \lambda_r), \text{ for } j = 0, \ldots, 1+n-r.$$

Then F_j/F_{j+1} is free of rank 1 as a $\mathbb{C}[\mathfrak{h}]$-module and is generated by $1 \otimes y^j \otimes \lambda_r + F_{j+1}$ which is singular vector transforming as λ_{r+j} under W and having weight $\kappa_{\mathbf{k}}(\lambda_{r+j})$. Thus by the universal property of standard modules we have a map $\Delta(\lambda_{r+j}) \to F_j/F_{j+1}$ taking $1 \otimes \lambda_{r+j}$ to $1 \otimes y^j \otimes \lambda_r + F_{j+1}$ and this map must be an isomorphism as it is an isomorphism of $\mathbb{C}[\mathfrak{h}]$-modules. So $\Delta^{(1+n-r)}(\lambda_i)$ has a standard filtration as claimed.

To show that $P(\lambda_r)$ has a filtration with the same subquotients we need only appeal to BGG reciprocity, which states that the multiplicity of $\Delta(\mu)$ in any standard filtration of $P(\lambda)$ equals the multiplicity of $L(\lambda)$ in a Jordan-Hölder series of $\Delta(\mu)$:

$$[P(\lambda) : \Delta(\mu)] = [\Delta(\mu) : L(\lambda)].$$

Thus from the choice of enumeration of the λ_r we have

$$[P(\lambda_r) : \Delta(\lambda_j)] = [\Delta(\lambda_j) : L(\lambda_r)] = \begin{cases} 1 \text{ if } j \geq r \\ 0 \text{ else} \end{cases}.$$

So both $P(\lambda_r)$ and $\Delta^{(1+n-r)}(\lambda_r)$ have standard filtrations with the same subquotients.

By Corollary 6.2.20 $P(\lambda_r)$ is generated by $v^{\infty}_{\lambda_r}$ and as $P(\lambda_r) \in \mathcal{O}_{\mathbf{k}}$, $v^{\infty}_{\lambda_r}$ is annihilated by some power of $y \in \mathbb{C}[\mathfrak{h}^*]$. Because $P(\lambda_r)$ has a standard filtration with $1+n-r$ terms we see that

$y^{1+n-r}v^{\infty}_{\lambda_r} = 0$ and by the universal property of $\Delta^{(1+n-r)}(\lambda_r)$ we obtain a map

$$\Delta^{(1+n-r)}(\lambda_r) \twoheadrightarrow P(\lambda_r)$$

which is surjective as its image contains a generator of $P(\lambda_r)$. As both have the same length, this map is an isomorphism. $\square$

Our following calculations will underscore the importance of $(H1)$ when making explicit calculations. We keep the notational convention in the proof of Theorem 6.2.21 and add some new ones: First, we will write the parameter k_p as k_{E_p}, the group idempotent $\mathbf{e}_j$ as $\mathbf{e}_{E_j}$ (and still identify the representations λ_r with their basis vectors).
We can now make an easy calculation in $\Delta^{(1+n-r)}(\lambda_r)$:

$$\begin{aligned}
[y,x]x^n \otimes y^c \otimes \lambda_r &= \left(1 + m \sum_{\mu \in \mathrm{Irr}(W)} (k_\mu - k_{E_{m-1}\otimes\mu})\mathbf{e}_\mu\right) x^n \otimes y^c \otimes \lambda_r \\
&= x^n \otimes \left(1 + m \sum_{\mu \in \mathrm{Irr}(W)} (k_\mu - k_{E_{m-1}\otimes\mu})\mathbf{e}_{E^{-n}_{m-1}\otimes\mu}\right) y^c \otimes \lambda_r \\
&= \left(1 - m(k_{E^{1+n-c}_{m-1}\otimes\lambda_r} - k_{E^{n-c}_{m-1}\otimes\lambda_r})\right) x^n \otimes y^c \otimes \lambda_r
\end{aligned}$$

This gives us all the information we need to determine the relations for the paths in blocks of $\mathcal{O}_{\mathbf{k}}$ (assuming $(H1)$).

Proposition 6.2.22. *Let $\mathbf{k}$ be a choice of parameters such that $(H1)$ holds. Recall that the quiver $Q_{\mathcal{B}}$ of a block $\mathcal{B}$ has the form*

$$\lambda_1 \underset{b_1}{\overset{a_1}{\rightleftarrows}} \lambda_2 \underset{b_2}{\overset{a_2}{\rightleftarrows}} \lambda_3 \underset{b_3}{\overset{a_3}{\rightleftarrows}} \cdots \underset{b_{n-1}}{\overset{a_{n-1}}{\rightleftarrows}} \lambda_n\ .$$

The following relations hold:

1. *For $1 \leq r \leq n-2$ we have $a_r b_r = b_{r+1}a_{r+1}$.*
2. *We have $a_{n-1}b_{n-1} = 0$.*

and these are all relations.

Proof. Suppose r between 1 and $n-1$ is fixed. Both $a_r b_r$ and $b_{r+1}a_{r+1}$ induce an endomorphism of $P(\lambda_{r+1})$ and we can check if they are identical by showing that the image of the generating vector $1 \otimes 1 \otimes \lambda_{r+1}$ is the same under both. The endomorphism induced by $a_r b_r$ maps

$$1 \otimes 1 \otimes \lambda_{r+1} \mapsto xy(1 \otimes 1 \otimes \lambda_{r+1})$$

and the endomorphism induced by $b_{r+1}a_{r+1}$ maps

$$1 \otimes 1 \otimes \lambda_{r+1} \mapsto yx(1 \otimes 1 \otimes \lambda_{r+1}).$$

Now we have

$$\begin{aligned} yx(1\otimes 1\otimes \lambda_{r+1}) &= xy(1\otimes 1\otimes \lambda_{r+1}) + [y,x](1\otimes 1\otimes \lambda_{r+1}) \\ &= xy(1\otimes 1\otimes \lambda_{r+1}) \\ &+ \left(1 + m\sum_{\mu\in \mathrm{Irr}(W)}(k_\mu - k_{E_{m-1}\otimes\mu})\mathbf{e}_\mu\right)(1\otimes 1\otimes \lambda_{r+1}) \\ &= xy(1\otimes 1\otimes \lambda_{r+1}) + \big(1 + m(k_{\lambda_{r+1}} - k_{E_{m-1}\otimes\lambda_{r+1}})\big)(1\otimes 1\otimes \lambda_{r+1}) \\ &= xy(1\otimes 1\otimes \lambda_{r+1}) + \big(1 - m(k_{\lambda_r} - k_{\lambda_{r+1}})\big)(1\otimes 1\otimes \lambda_{r+1}) \end{aligned}$$

Now

$$\begin{aligned} 1 &= \dim_{\mathbb{C}} L(\lambda_{r+1}) \\ &= \kappa_{\mathbf{k}}(\lambda_{r+1}) - \kappa_{\mathbf{k}}(\lambda_r) \\ &= \sum_\mu (1 - m\delta_{\lambda_{r+1},\mu})k_\mu - \sum_\mu (1 - m\delta_{\lambda_r,\mu})k_\mu \\ &= m(k_{\lambda_r} - k_{\lambda_{r+1}}) \end{aligned}$$

and thus

$$yx(1\otimes 1\otimes \lambda_{r+1}) = xy(1\otimes 1\otimes \lambda_{r+1})$$

which proves that indeed

$$\forall\, r\in\{1,\dots,n-1\}:\ a_rb_r = b_{r+1}a_{r+1}.$$

Now to check the second relation $a_{n-1}b_{n-1} = 0$. The element of $\mathrm{End}_{\mathcal{O}_{\mathbf{k}}}(P(\lambda_n))$ induced from $a_{n-1}b_{n-1}$ is given by

$$\begin{aligned} 1\otimes 1\otimes \lambda_n \mapsto\ & yx(1\otimes 1\otimes \lambda_n) \\ &= xy(1\otimes 1\otimes \lambda_n) + [y,x](1\otimes 1\otimes \lambda_n) \\ &= (1 + m(k_{\lambda_n} - k_{\sigma\otimes\lambda_n}))(1\otimes 1\otimes \lambda_n) \\ &= \big(1 - m(k_{\lambda_{n-1}} - k_{\lambda_n})\big)(1\otimes 1\otimes \lambda_n) \\ &= 0. \end{aligned}$$

Thus we have verified that the relations (1) and (2) hold.

To show that these are all relations, we need to count the number of distinct paths between any two nodes λ_s, λ_t subject to the relations in the statement and show that this number equals $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(P(\lambda_t), P(\lambda_s)) = 1 + n - \max\{t,s\}$. As a first step, we shall prove that we have

$$(a_rb_r)^{n-r} = 0.$$

We have already shown the case $r = n-1$. So assume the relation holds for $r+1, r+2, \dots, n-1$

and we attempt to prove it for r. We have

$$\begin{aligned}(a_r b_r)^{n-r} &= (b_{r+1}a_{r+1})^{n-r}\\ &= b_{r+1}(a_{r+1}b_{r+1})^{n-r-1}b_{r+1}\\ &= a_r(a_{r+1}b_{r+1})^{n-(r+1)}b_r\\ &= 0.\end{aligned}$$

Let us first consider the case that $t \geq s$, any path from λ_t to λ_s (giving an element of $\mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(P_s, P_t)$) will be equivalent to a path from λ_t to λ_s that only contains cycles of the form $a_t b_t$ using the relations $b_r a_r = a_{r+1}b_{r+1}$. As $(a_r b_r)^{n-r} = 0$, a non-zero path can only contain cycles of the form $(a_t b_t)^j$ with powers $j \in \{0, 1, \ldots, n-t\}$ and thus the number of possible isomorphism classes is equal to $1 + n - t$ and therefore identical to $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(P_s, P_t)$. Similarly, if $s \geq t$ we can see that it suffices to consider paths $\lambda_t \to \lambda_s$ containing loops of the form $a_{s-1}b_{s-1}$ and again the only possible powers of $a_{s-1}b_{s-1}$ are $0, 1, \ldots, n-(s-1)-1$ and thus the number of possible isomorphism classes of paths is $1+n-(s-1)-1 = 1+n-s$ again identical to $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(P_s, P_t)$. □

Definition 6.2.23. *Let $\mathcal{A}(\mathcal{B})$ denote the basic finite-dimensional algebra with $\mathcal{A}(\mathcal{B})-\mathrm{mod} \cong \mathcal{B}$. That is $\mathcal{A}(\mathcal{B})$ is the opposite of the endomorphism algebra of a minimal projective generator of $\mathcal{B}$. Let $\mathbb{C}Q_{\mathcal{B}}$ be the path algebra of the quiver $Q_{\mathcal{B}}$. We let $I_{\mathcal{B}}$ be the two-sided ideal of $\mathbb{C}Q_{\mathcal{B}}$ generated by*

$$\{a_{n-1}b_{n-1}, a_r b_r - b_{r+1}a_{r+1} \mid 1 \leq r \leq n-2\}$$

Corollary 6.2.24. *With notation as in Definition 6.2.23, we have*

$$\mathbb{C}Q_{\mathcal{B}}/I_{\mathcal{B}} \cong \mathcal{A}(\mathcal{B}).$$

Proof. We have just shown this in Proposition 6.2.22: We have an epimorphism $\mathbb{C}Q_{\mathcal{B}}/I_{\mathcal{B}} \twoheadrightarrow \mathcal{A}(\mathcal{B})$ and since the equality $\dim_{\mathbb{C}} \mathbf{e}_{\lambda_s} (\mathbb{C}Q_{\mathcal{B}}/I_{\mathcal{B}}) \mathbf{e}_{\lambda_t} = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{O}}(P_s, P_t)$ holds for all $s, t \in \{1, \ldots, n\}$ we can conclude that it is an isomorphism. □

6.3 The Quiver Picture

Crawley-Boevey and Holland have used quiver-theoretic methods to investigate families of algebras that appear as noncommutative deformations of the coordinate rings of Kleinian singularities. For Kleinian singularities of type A, these are strongly connected to rational Cherednik algebras of cyclic groups and their spherical subalgebras and provide an alternative way to approach these objects. The key reference is [CBH98].

Denote by Γ a non-trivial finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ Then Γ naturally acts on $\mathbb{C}^2$ and the quotient variety $\mathbb{C}^2/\Gamma$ is called a Kleinian or Du-Val-singularity.

There is a canonical way to associate a graph to such a group.

Definition 6.3.1. *Let $\Gamma \leq \mathrm{SL}_2(\mathbb{C})$ a finite subgroup. Let $S_0, S_1, \ldots, S_n$ be the simple representations of Γ (up to isomorphism) with S_0 the trivial representation. Let us denote* *by V the natural representation of Γ obtained by restricting the action of $\mathrm{SL}_2(\mathbb{C})$. We then define the McKay graph $Q(\Gamma)$ of Γ as follows:*

1. *The vertices of $Q(\Gamma)$ are $\{S_0, S_1, \dots, S_n\}$*
2. *The number of edges from S_i to S_j is the multiplicity of S_i in the representation $V \otimes_{\mathbb{C}} S_j$*

It can be checked that this is well-defined in that the multiplicity of S_i in $V \otimes_{\mathbb{C}} S_j$ is equal to the multiplicity of S_j in $V \otimes_{\mathbb{C}} S_i$.

Example 6.3.2. *Let ρ be a primitive m-th root of unity in $\mathbb{C}$ and we let Γ be the group generated by*

$$\gamma := \begin{pmatrix} \rho^{-1} & 0 \\ 0 & \rho \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

so that Γ is cyclic of order m. Mimicking our previous notation, we let S_r be the one-dimensional representation of Γ on which γ acts via ρ^r. Giving V the basis a_1, a_2 we see that V is given by

$$\gamma \cdot a_1 = \epsilon^{-1} a_1 \text{ and } \gamma \cdot a_2 = \epsilon a_2.$$

Then it is clear that

$$S_j \otimes V \cong (S_j \otimes \mathbb{C}a_1) \oplus (S_j \otimes \mathbb{C}a_2) \cong S_{j-1} \oplus S_{j+1}$$

with indices read modulo m. Thus the McKay graph of Γ is

$$\begin{tikzcd}[arrows=dash]
 & & S_0 \arrow[dll] \arrow[drr] & & \\
S_{m-1} \arrow[r] & S_{m-2} \arrow[r] & \cdots \arrow[r] & S_2 \arrow[r] & S_1
\end{tikzcd}$$

This is affine Dynkin of type $\tilde{A}_{m-1}$ with S_0 as extending vertex.

In fact it is true that any McKay graph for $\Gamma \leq \mathrm{SL}_2(\mathbb{C})$ is affine Dynkin with S_0 as extending vertex and there is a well-known result which classifies the possible groups Γ up to isomorphism and the resulting McKay graphs. The following can be easily deduced from the remarks at the start of Section 4.2 of [Nak99], in fact it is almost a verbatim quote:

Lemma 6.3.3. *([Nak99]) Let K, Γ be as above. Then we are in one of the following cases:*

1. *Γ is cyclic of order $m+1$ and the McKay graph is of type $\tilde{A}_m$.*
2. *Γ is binary dihedral of order $4(m-1)$, i.e. has a presentation $\Gamma = \langle a, b \mid a^{m-1} = b^2, b^4 = e, bab^{-1} = a^{-1} \rangle$. Its McKay graph is of type $\tilde{D}_m$.*
3. *Γ is a binary tetra-, octa- or icosahedral group and the McKay graph is of type $\tilde{E}_6, \tilde{E}_7$ or $\tilde{E}_8$ respectively.*

Definition 6.3.4. *Suppose that we have an arbitrary finite quiver $Q = (Q_0, Q_1)$. The double $\bar{Q}$ of Q is the quiver obtained from Q by keeping the vertex set unchanged and adding an additional arrow $a^* : j \to i$ for every arrow $a : i \to j$ in the original quiver Q.*

Definition 6.3.5. *The set of weights of Q is $\mathrm{Fun}(Q_0, \mathbb{C}) = \mathbb{C}^{Q_0}$, the set of functions from the vertices to $\mathbb{C}$. A given weight λ will usually be identified with an element of the path algebra*

$\mathbb{C}Q$ (or alternatively $\mathbb{C}\bar{Q}$) as follows:

$$\lambda \leftrightarrow \sum_{i \in Q_0} \lambda(i) e_i$$

where $e_i \in \mathbb{C}Q$ is the idempotent of the vertex i.
The commutator element of $\mathbb{C}\bar{Q}$ is the sum

$$c = \sum_{a \in Q_1} [a, a^*].$$

Crawley-Boevey and Holland have proved a close relation between deformations of the coordinate ring of the quotient singularity $\mathbb{C}^2/\Gamma$ and deformations of $\mathbb{C}\overline{Q}/c$. To introduce this, we need a further identification of the weights $\mathrm{Fun}(Q_0, \mathbb{C})$ with the centre $Z(\mathbb{C}\Gamma)$ of the group algebra.

Lemma 6.3.6. *There is an isomorphism of complex vector spaces $Z(\mathbb{C}\Gamma) \xrightarrow{\sim} \mathrm{Fun}(Q_0, \mathbb{C})$ given by characters: Recall that $S_0 = \mathrm{triv}, S_1, \ldots, S_n$ are the irreducible representations of Γ, then we map*

$$\begin{aligned} Z(\mathbb{C}\Gamma) &\to \mathrm{Fun}(Q_0, \mathbb{C}) \\ \lambda &\mapsto (S_i \mapsto \mathrm{Tr}|_{S_i}(\lambda)). \end{aligned}$$

Proof. To see that this is an isomorphism, note that $\dim_{\mathbb{C}} Z(\mathbb{C}\Gamma) = n = \# Q_0 = \dim_{\mathbb{C}} \mathrm{Fun}(Q_0, \mathbb{C})$, namely a $\mathbb{C}$-basis of $Z(\mathbb{C}\Gamma)$ is given by $\{\sum_{\gamma \in C} \mid C \text{ a conjugacy class of } \Gamma\}$. □

Definition 6.3.7. *([CBH98], Page 606) Let λ be a weight of the finite quiver Q. We define the associated deformed preprojective algebra to be the quotient*

$$\Pi^\lambda(Q) := \mathbb{C}\bar{Q}/(c - \lambda).$$

It is clear from the construction that $\mathbb{C}\bar{Q}$ is independent of the orientation of Q and it is in fact true that the same holds for $\Pi^\lambda(Q)$.

Lemma 6.3.8. *([CBH98], Lemma 2.2) If Q' is obtained from Q by reversing an arrow, a say, then an isomorphism $\Pi^\lambda(Q) \xrightarrow{\sim} \Pi^\lambda(Q')$ is given by sending $a \in \bar{Q}_1$ to $a^* \in \bar{Q}'_1$ and $a^* \in \bar{Q}'_1$ to $-a \in \bar{Q}_1$ and leaving all other arrows unchanged.*

From now on, let us assume that Γ is a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. We can endow the McKay graph of Γ with an arbitrary orientation and consider $\Pi^\lambda(Q(\Gamma))$. By Lemma 6.3.8 this is well-defined, independent of the orientation chosen and only depends on the isomorphism type of the group Γ. Thus we will also denote $\Pi^\lambda(Q(\Gamma))$ simply by $\Pi^\lambda(\Gamma)$.

Definition 6.3.9. *([CBH98], Page 608) Let Γ be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. For a weight $\lambda \in Z(\mathbb{C}\Gamma)$ we define an algebra*

$$\mathcal{S}^\lambda(\Gamma) := \frac{\mathbb{C}\langle x, y \rangle \# \Gamma}{(yx - xy - \lambda)}.$$

It follows easily from Theorem 3.4 in [CBH98] that there is an isomorphism between $H_{\mathbf{k}}(W)$ and $\Pi^\lambda(W)$ if W is cyclic. For this we need some preliminary work. Keeping our notation from

Chapter 5 and the beginning of this chapter, we may set

$$V = \mathfrak{h} \otimes \mathfrak{h}^*$$

and the action of W on V then identifies the chosen generator s with the matrix $\begin{pmatrix} \rho^{-1} & 0 \\ 0 & \rho \end{pmatrix}$. Thus we are in the situation of Example 6.3.2 and the associated McKay graph of W is $\tilde{A}_{m-1}$. We fix an orientation on the double graph and label the arrows as follows:

$$\begin{tikzcd}
 & & S_0 \arrow[dll, bend right, "a_{m-1}^*"'] \arrow[drr, "a_0"'] & & \\
S_{m-1} \arrow[urr, "a_{m-1}"] \arrow[r, bend right, "a_{m-2}^*"'] & S_{m-2} \arrow[l, "a_{m-2}"'] \arrow[r, bend right, "a_{m-3}^*"'] & \cdots\cdots \arrow[l, "a_{m-3}"'] \arrow[r, bend right, "a_2^*"'] & S_2 \arrow[l, "a_2"'] \arrow[r, bend right, "a_1^*"'] & S_1 \arrow[l, "a_1"'] \arrow[ull, bend right, "a_0^*"']
\end{tikzcd}$$

Proposition 6.3.10. *(Theorem 3.4 in [CBH98]) Let W be cyclic of order m. Let $\mathbf{k}$ be an arbitrary choice of parameters and define*

$$\begin{aligned}
\lambda(\mathbf{k}) &= \sum_{j=0}^{m-1} \lambda(\mathbf{k})_j \mathbf{e}_j \\
&= \sum_{j=0}^{m-1} (1 + m(k_j - k_{j+1}))\mathbf{e}_j \in Z(\mathbb{C}W) = \mathbb{C}W.
\end{aligned}$$

Then there is an isomorphism $\Pi^{\lambda(\mathbf{k})}(W) \cong H_{\mathbf{k}}(W)$ given by mapping

$$\begin{aligned}
a_i^* &\mapsto \mathbf{e}_i y \mathbf{e}_{i+1} \\
a_i &\mapsto \mathbf{e}_{i+1} x \mathbf{e}_i \\
e_j &\mapsto \mathbf{e}_j.
\end{aligned}$$

Here the $\mathbf{e}_j \in \mathbb{C}W$ are the group idempotents whereas the e_j are the vertex idempotents of the double quiver of W with e_j the vertex idempotent of the representation E_j

We can thus prove results on $H_{\mathbf{k}}(W)$ by applying quiver-theoretic methods.

Definition 6.3.11. *The Ringel form of a quiver Q is the $\mathbb{Z}$-bilinear form*

$$\begin{aligned}
\langle \bullet, \bullet \rangle : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} &\to \mathbb{Z} \\
(\alpha, \beta) &\mapsto \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{a \in Q_1} \alpha_{ta} \beta_{ha}.
\end{aligned}$$

The bilinear symmetric form on Q is

$$\begin{aligned}
(\bullet, \bullet) : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} &\to \mathbb{Z} \\
(\alpha, \beta) &\mapsto \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle.
\end{aligned}$$

Definition 6.3.12. *For each loop-free (!) vertex $i \in Q_0$ we define $\epsilon_i \in \mathbb{Z}^{Q_0}$ to be the vector with entry 1 in i-th position and zeros otherwise. The simple reflection s_i is the $\mathbb{Z}$-linear map*

$$s_i : \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_0}$$

given by

$$\alpha \mapsto s_i(\alpha) = \alpha - (\alpha, \epsilon_i)\epsilon_i.$$

The Weyl group $\mathcal{W} = \mathcal{W}(Q)$ of Q is

$$\mathcal{W} := \langle s_i \mid i \in Q_0 \rangle \leq \mathrm{GL}_{\#Q_0}(\mathbb{Z}).$$

To continue with our example of cyclic Γ of order m from above, let us compute the action of the corresponding Weyl group for the following orientation of the McKay graph of Γ:

$$\begin{tikzcd}
 & & S_0 \arrow[drr, "a_0"] & & \\
S_{m-1} \arrow[urr, "a_{m-1}"] & S_{m-2} \arrow[l, "a_{m-2}"'] & \cdots \arrow[l, "a_{m-3}"'] & S_2 \arrow[l, "a_2"'] & S_1 \arrow[l, "a_1"']
\end{tikzcd}$$

Every vertex is loop-free and the Weyl group will be generated by m simple reflections. For a vertex i the simple reflection s_i acts on $\alpha = (\alpha_j)_j$ as follows:

$$\begin{aligned}
s_i(\alpha) &= \alpha - (\alpha, \epsilon_i)\epsilon_i \\
&= \alpha - \left(\langle \alpha, \epsilon_i \rangle + \langle \epsilon_i, \alpha \rangle\right) \epsilon_i \\
&= \alpha - \left(\alpha_i - \alpha_{i-1} + \alpha_i - \alpha_{i+1}\right) \epsilon_i \\
&= \alpha - \left(2\alpha_i - (\alpha_{i-1} + \alpha_{i+1})\right) \epsilon_i
\end{aligned}$$

and so

$$s_i(\alpha)_j = \begin{cases} \alpha_j \text{ if } j \neq i \\ \alpha_{i+1} + \alpha_{i-1} - \alpha_i \text{ if } j = i \end{cases}.$$

Definition 6.3.13. *For a loop-free vertex $i \in Q_0$ we have the dual reflections on $\mathbb{C}^{Q_0}$ defined by*

$$\begin{aligned}
r_i : \mathbb{C}^{Q_0} &\to \mathbb{C}^{Q_0} \\
\lambda &\mapsto r_i(\lambda) = (\lambda_j - (\epsilon_i, \epsilon_j)\lambda_i)_{j \in Q_0}
\end{aligned}$$

The importance of the dual reflections for us lies in the following theorem :

Theorem 6.3.14. *(Theorem 5.1 and Corollary 5.2 in [CBH98])*

1. *Let $i \in Q_0$ be a loop-free vertex and λ a weight with $\lambda_i \neq 0$. Then there is an equivalence of categories*
$$E_i : \Pi^\lambda(Q) - \mathrm{Mod} \cong \Pi^{r_i(\lambda)}(Q) - \mathrm{Mod}.$$
2. *Let λ be a weight and $\mu \in \mathcal{W}(Q)\lambda$. hen there is an equivalence of categories*
$$E_i : \Pi^\lambda(Q) - \mathrm{Mod} \cong \Pi^\mu(Q) - \mathrm{Mod}.$$

Note that for $\lambda_i = 0$ both λ and $r_i\lambda$ are identical.

The functor E_i can be described as follows (see the proof of Theorem 5.1 in [CBH98]):

1. We fix an orientation of Q such that the vertex $i \in Q_0$ is a sink of Q, meaning that no arrow has i as its tail. Any $\Pi^\lambda(Q)$-module V corresponds to a representation of $\overline{Q}$. We denote the corresponding representation again by V and let V_j be the vector space at

vertex j and for any arrow $b \in \overline{Q}_1$ we denote the linear map $V_{tb} \to V_{hb}$ by V_b. For any vertex $j \in Q_0$ the relation

$$\sum_{\substack{a \in Q_1 \\ ha=j}} V_a V_{a^*} - \sum_{\substack{a \in Q_1 \\ ta=j}} V_{a^*} V_a - \lambda_j \mathrm{id}_{V_j} = 0$$

must be satisfied.

2. We denote by $V_\oplus$ the direct sum of all vector spaces V_j such that j is the tail of an arrow in Q with head i (counted with multiplicities), so we have

$$V_\oplus = \bigoplus_{\substack{a \in Q_1 \\ ha=i}} V_{ta}.$$

3. For any such arrow $a \in Q_1$ we have natural inclusion and projection maps $\imath_a, \pi_a$ respectively

$$\imath_a : V_{ta} \hookrightarrow V_\oplus$$
$$\pi_a : V_\oplus \twoheadrightarrow V_{ta}$$

4. We then define maps $\imath : V_i \to V_\oplus$ and $\pi : V_\oplus \to V_i$ by

$$\imath : V_i \to V_\oplus \qquad\qquad \pi : V_\oplus \to V_i$$
$$\imath := \sum_{\substack{a \in Q_1 \\ ha=i}} \imath_a V_{a^*} \qquad\qquad \pi := \frac{1}{\lambda_i} \sum_{\substack{a \in Q_1 \\ ha=i}} V_a \pi_a$$

5. We define a new representation $E_i V$ of $\overline{Q}$ by setting

$$(E_i V)_j = \begin{cases} V_j & \text{if } j \neq i \\ \mathrm{im}(1 - \imath\pi) & \text{if } j = i \end{cases}$$

for any vertex $j \in Q_0$. The equality $\mathrm{im}(1 - \imath\pi) = \ker \pi$ holds by construction. The maps of $E_i V$ are as follows

$$(E_i V)_a = \begin{cases} V_a & \text{if } ha \neq i \\ -\lambda_i (1 - \imath\pi)\imath_a & \text{if } ha = i \end{cases}$$

and

$$(E_i V)_{a^*} = \begin{cases} V_{a^*} & \text{if } ha \neq i \\ \pi_a|_{EV_i} & \text{if } ha = i \end{cases}$$

for any arrow $a \in Q_1$. Recall that a^* denotes the opposite arrow of a.

6. As $r_i(r_i\lambda) = \lambda_i$, a similar construction gives a functor $E_i^* : \Pi^{r_i\lambda}(Q)-\mathrm{Mod} \cong \Pi^\lambda(Q)-\mathrm{Mod}$ inverse to E_i.

For cyclic W we have an isomorphism $H_{\mathbf{k}} \cong \Pi^{\lambda(\mathbf{k})}(W)$ by Proposition 6.3.10 and therefore the above Theorem 6.3.14 will allow us to apply quiver -theoretic methods to the study of $H_{\mathbf{k}}$.
If $\Pi^\lambda(W) \cong H_{\mathbf{k}}(W)$ then $\Pi^{r_i\lambda}(W)$ is again isomorphic to a rational Cherednik algebra (at $t = 1$) and we denote this by $H_{r_i\mathbf{k}}(W)$. We can now prove:

Corollary 6.3.15. *1. Let $i \in \{0, 1 \ldots, m-1\}$ such that $1 + m(k_i + k_{i+1}) \neq 0$. Then there is an equivalence of categories $E_i : H_{\mathbf{k}} - \mathrm{Mod} \cong H_{r_i\mathbf{k}} - \mathrm{Mod}$ restricting to equivalences*

$$E_i : \mathcal{O}_{\mathbf{k}} \cong \mathcal{O}_{r_i\mathbf{k}}$$

and

$$E_i : H_{\mathbf{k}} - \mathrm{fin} \cong H_{\mathbf{k}} - \mathrm{fin}.$$

2. If $\mathbf{k}' \in \mathcal{W}(Q)\mathbf{k}$ then there is an equivalence of categories $E_i : H_{\mathbf{k}} - \mathrm{Mod} \cong H_{r_i\mathbf{k}} - \mathrm{Mod}$ restricting to equivalences $E_i : \mathcal{O}_{\mathbf{k}} \cong \mathcal{O}_{r_i\mathbf{k}}$ and $E_i : H_{\mathbf{k}} - \mathrm{fin} \cong H_{\mathbf{k}} - \mathrm{fin}$.

Proof. 1. We shall consider the double quiver

$$\begin{tikzcd}
& & S_i \arrow[dll, bend right, "a^*_{i-1}"'] & & \\
S_{i-1} \arrow[urr, "a_{i-1}"'] \arrow[r, shift right, "a^*_{i-2}"'] & S_{i-2} \arrow[l, shift right, "a_{i-2}"'] \arrow[r, shift right, "a^*_{i-3}"'] & \cdots\cdots \arrow[l, shift right, "a_{i-3}"'] \arrow[r, shift right, "a^*_{i+2}"'] & S_{i+2} \arrow[l, shift right, "a_{i+2}"'] \arrow[r, shift right, "a^*_{i+1}"'] & S_{i+1} \arrow[l, shift right, "a_{i+1}"'] \arrow[ull, bend right, "a_i"'] \arrow[ull, "a^*_i"]
\end{tikzcd}$$

By Lemma 6.3.8 and Proposition 6.3.10 we have an isomorphism $H_{\mathbf{k}}(W) \xrightarrow{\sim} \Pi^{\lambda}(W)$ with $\lambda = \lambda(\mathbf{k})$ given by

$$\mathbf{e}_j y \mathbf{e}_{j+1} \mapsto \begin{cases} a_j^* & \text{if } j \neq i \\ -a_i & \text{if } j = i \end{cases} \qquad \mathbf{e}_{j+1} x \mathbf{e}_j \mapsto \begin{cases} a_j & \text{if } j \neq i \\ a_i^* & \text{if } j = i \end{cases}.$$

Under this isomorphism, a module $M \in \mathcal{O}_{\mathbf{k}}$ is mapped to a $\overline{Q}$ representation V on which

$$\mathbf{e}_j y^m \mathbf{e}_j = a_j^* a_{j+1}^* \ldots a_{i-1}^* (-a_i) a_{i+1}^* \ldots a_{m-1}^* = -a_j^* \ldots a_{i-1}^* a_i a_{i+1}^* \ldots a_{m-1}^*$$

acts locally nilpotently for all $j = 0, \ldots, m-1$. We will show that $-E_i(a_j^* \ldots a_{i-1}^* a_i a_{i+1}^* \ldots a_{i-2}^*)$ again acts locally nilpotently on E_iV. So consider first the space $(E_iV)_j = V_j$ for $j \neq i$. Recall that the space E_iV is a subspace of $V_{i-1} \oplus V_{i+1}$. The action $E_iV_{a_j^*}$ in E_iV of the arrows a_j^* is as follows:

$$E_iV_{a_j^*} = \begin{cases} V_{a_j^*} = \mathbf{e}_j y \mathbf{e}_{j+1} & \text{if } j \neq i, i-1 \\ \pi_{a_j} & \text{if } j = i, i-1 \end{cases}.$$

The action $E_iV_{a_j}$ in E_iV of the arrows a_j is as follows:

$$E_iV_{a_j} = \begin{cases} V_{a_j} = \mathbf{e}_{j+1} x \mathbf{e}_j & \text{if } j \neq i, i-1 \\ -(1 + m(k_i - k_{i+1}))(1 - \imath\pi)\imath_{a_j} & \text{if } j = i, i-1 \end{cases}.$$

So we will need to describe the composition $a_{i-1}^* a_i$ in more detail. Pick

$$u \in (E_iV)_{i+1} = V_{i+1}),$$

which corresponds to the space $\mathbf{e}_{i+1}M$, and recall that $(E_iV)_i = V_{i-1} \oplus V_{i+1}$. Then

$$\begin{aligned}
&-\pi_{a_{i-1}}(1+m(k_i-k_{i+1}))(1-\imath\pi)\imath_{a_i}u \\
&= -(1+m(k_i-k_{i+1}))(1-\imath\pi)(0,u) \\
&= -(1+m(k_i-k_{i+1}))\pi_{a_{i-1}}\left((0,u) - \frac{1}{1+m(k_i-k_{i+1})}\imath V_{a_i}(u)\right) \\
&= -(1+m(k_i-k_{i+1}))\pi_{a_{i-1}}\left((0,u) - \frac{1}{1+m(k_i-k_{i+1})}\imath(-\mathbf{e}_i y \mathbf{e}_{i+1}u)\right) \\
&= -(1+m(k_i-k_{i+1}))\pi_{a_{i-1}}\left((0,u) + \frac{1}{1+m(k_i-k_{i+1})}(\mathbf{e}_{i-1}y^2\mathbf{e}_{i+1}u, \mathbf{e}_{i+1}x\mathbf{e}_i y\mathbf{e}_{i+1}u)\right) \\
&= \pi_{a_{i-1}}\left(\mathbf{e}_{i-1}y^2\mathbf{e}_{i+1}u, -(1+m(k_i-k_{i+1})u - \mathbf{e}_{i+1}xy\mathbf{e}_{i+1}u\right) \\
&= \mathbf{e}_{i-1}y^2\mathbf{e}_{i+1}u
\end{aligned}$$

Thus for $j \neq i$ the element $-E_i(a_j^* \dots a_{i-1}^* a_i a_{i+1}^* \dots a_{m-1}^*)$ will act locally nilpotently. To deduce that $-E_i(a_i a_{i+1}^* \dots a_{i-1}^*)$ acts locally nilpotently just note that $E_i(a_{-1}a_i a_{i+1}^* \dots a_{i-2}^*)^{k-1}$ occurs as a subword of $E_i(a_i a_{i+1}^* \dots a_{i-1}^*)^k$ and so localy nilpotence of $E_i(a_i a_{i+1}^* \dots a_{i-1}^*)$ follows.

Thus we have shown that $\mathbf{e}_j y^m \mathbf{e}_j \in H_{r_i\mathbf{k}}(W)$ acts locally nilpotently on E_iM for all j. Thus $y^m = \mathbf{e}_0 y^m \mathbf{e}_0 + \dots \mathbf{e}_{m-1}y^m\mathbf{e}_{m-1}$ acts locally nilpotently on E_iM and thus $E_iM \in \mathcal{O}_{r_i\mathbf{k}}$. Any Morita equivalence preserves finite-dimensional modules, this is Corollary 4.1.8.

2. This follows since the functor giving the equivalence $H_{\mathbf{k}}(W) - \mathrm{Mod} \cong H_{w(\mathbf{k})}(W) - \mathrm{Mod}$ is a composition of functors of the type E_i and identity functors.

□

A further note might be of interest: Calculating the action of the dual reflections on the space of parameters for a cyclic group we find:

$$\begin{aligned}
(r_i\lambda(\mathbf{k}))_j &= \lambda(\mathbf{k})_j - (\epsilon_i, \epsilon_j)\lambda(\mathbf{k})_i \\
&= \lambda(\mathbf{k})_j - (\langle\epsilon_i, \epsilon_j\rangle + \langle\epsilon_j, \epsilon_i\rangle)\,\lambda(\mathbf{k})_i \\
&= \lambda(\mathbf{k})_j + (-2\delta_{j,i} + \delta_{j,i+1} + \delta_{j,i-1})\,\lambda(\mathbf{k})_i \\
&= \begin{cases} \lambda(\mathbf{k})_j & \text{if } j \neq i-1, i, i+1 \\ \lambda(\mathbf{k})_{i-1} + \lambda(\mathbf{k})_i & \text{if } j = i-1 \\ -\lambda(\mathbf{k})_i & \text{if } j = i \\ \lambda(\mathbf{k})_{i+1} + \lambda(\mathbf{k})_i & \text{if } j = i+1 \end{cases}
\end{aligned}$$

Now since $\lambda(\mathbf{k})_i = 1 + m(k_i - k_{i+1})$ and $k_0 = 0$ we can determine $\mathbf{k}$ from $\lambda(\mathbf{k})$ via

$$\begin{aligned}
\lambda(\mathbf{k})_0 &= 1 + m(k_0 - k_1) \\
\lambda(\mathbf{k})_1 &= 1 + m(k_1 - k_2) \\
&\vdots
\end{aligned}$$

and so

$$\begin{aligned}
k_0 &= 0\\
k_1 &= \frac{1}{m}(1-\lambda(\mathbf{k})_0)\\
k_2 &= \frac{1}{m}(2-\lambda(\mathbf{k})_0-\lambda(\mathbf{k})_1)\\
&\vdots\\
k_j &= \frac{1}{m}(j-\lambda(\mathbf{k})_0-\lambda(\mathbf{k})_1-\ldots-\lambda(\mathbf{k})_{j-1})
\end{aligned}$$

Using this, we can determine the parameter $r_i\mathbf{k}$ corresponding to $r_i\lambda(\mathbf{k})$.
First, let us consider the case that $i \neq 0$. Then we compute

$$\begin{aligned}
(r_i\mathbf{k})_j &= \frac{1}{m}(j-\sum_{l=0}^{j-1}(r_i\lambda(\mathbf{k}))_l)\\
&= \frac{1}{m}\left(j-\sum_{l=0}^{j-1}(\lambda(\mathbf{k})_l+(-2\delta_{l,i}+\delta_{l,i+1}+\delta_{l,i-1})\,\lambda(\mathbf{k})_i)\right)\\
&= \begin{cases}
\frac{1}{m}\left(j-\sum_{l=0}^{j-1}\lambda(\mathbf{k})_l\right) & \text{if } j<i\\
\frac{1}{m}\left(i-\left(\sum_{l=0}^{i-1}\lambda(\mathbf{k})_l\right)-\lambda(\mathbf{k})_i\right) & \text{if } j=i\\
\frac{1}{m}\left(i+1-\left(\sum_{l=0}^{i}\lambda(\mathbf{k})_l\right)+\lambda(\mathbf{k})_i\right) & \text{if } j=i+1\\
\frac{1}{m}\left(j-\sum_{l=0}^{j-1}\lambda(\mathbf{k})_l\right) & \text{if } j>i+1
\end{cases}
\end{aligned}$$

so in other words for $i \neq 0, m-1$

$$(r_i\mathbf{k})_j = \begin{cases}
k_j \text{ if } j<i\\
k_{i+1}-\frac{1}{m} & \text{if } j=i\\
k_i+\frac{1}{m} & \text{if } j=i+1\\
k_j & \text{if } j>i+1
\end{cases}$$

If $i=0$ we have

$$(r_0\mathbf{k})_j = \begin{cases}
-k_1+\frac{2}{m} & \text{if } j=1\\
k_j-k_1+\frac{1}{m} & \text{if } j>1
\end{cases}$$

and if $i=m-1$ we have

$$(r_{m-1}\mathbf{k})_j = \begin{cases}
k_j-k_{m-1}-\frac{1}{m} & \text{if } j\neq m-1\\
-k_{m-1}-\frac{2}{m} & \text{if } j=m-1
\end{cases}$$

This agrees with the the action of the group $S_m = \langle r_1,\ldots,r_{m-1}\rangle$ that we would also obtain from Losev's construction, see [GL11], Section 4 and both should yield the same functors. In Chapter 5 we have only worked with the subgroup of $\mathcal{W}(Q)$ generated by $r_1,\ldots,r_{m-2}$, the existence of the two further equivalences induced by r_0 and r_{m-1} do not change our results from Chapter 5, they would merely enable us to move between distinct S_{m-1}-orbits on the parameter space keeping $\#\Xi_{\mathbf{k}}(W)$ constant.

Finally we note another categorical equivalence we will need. It stems from an isomorphism $\Pi^\lambda(W) \xrightarrow{\sim} \Pi^{\alpha\lambda}(W)$ for any $\alpha \in \mathbb{C}^*$

Lemma 6.3.16. *For any weight λ and any $\alpha \in \mathbb{C}^*$ there are isomorphisms*

$$\tau_\pm : \Pi^\lambda(W) \xrightarrow{\sim} \Pi^{\alpha\lambda}(W).$$

Proof. Choose $\alpha' \in \mathbb{C}^*$ such that $\alpha'^2 = \alpha$. Then the isomorphism we are looking for are given as follows:

$$\tau_+ : \Pi^\lambda(W) \xrightarrow{\sim} \Pi^{\alpha\lambda}(W)$$
$$a_j \mapsto \frac{1}{\alpha'} a_j, \;\; a_j^* \mapsto \frac{1}{\alpha'} a_j^*$$

and τ_- is defined with $-\alpha'$ instead. □

Definition 6.3.17. *For any complex number $\alpha \in \mathbb{C}$ we denote by $\Re\alpha$ the real part of α and by $\Im\alpha$ the imaginary part of α. A weight $\lambda = (\lambda_i)_i \in K^{Q_0}$ will be called dominant if it fulfils the following condition: For each $i \in Q_0$ either $\Re\lambda_i > 0$ or $\Re\lambda_i = 0$ and $\Im\lambda_i > 0$.*

The following is again taken again from [CBH98]:

Lemma 6.3.18. *([CBH98], see Lemma 7.2 and Lemma 7.3) Suppose Q is an extended Dynkin quiver derived from the McKay graph of W and set $\delta = (\dim_{\mathbb{C}} S_i)_{S_i \in \mathrm{Irr}(W)}$ with S_0 the trivial representation. Suppose λ is such that*

$$\lambda \cdot \delta = \sum_i \lambda_i \delta_i \neq 0.$$

Then $\lambda/\lambda \cdot \delta$ is conjugate under the Weyl group action to a unique dominant weight λ^+. If μ is a dominant weight then each finite-dimensional simple representation of the deformed preprojective algebra $\Pi^\mu(Q)$ is one-dimensional.

For us this means that we can now finally assert the validity of hypothesis $(H1)$ up to categorical equivalence:

Corollary 6.3.19. *For any choice of parameters $\mathbf{k}$ there exist parameters $\mathbf{k}^+$ such that $H_{\mathbf{k}^+}$ is Morita equivalent to $H_{\mathbf{k}}$, $\mathcal{O}_{\mathbf{k}^+}$ and $\mathcal{O}_{\mathbf{k}}$ are equivalent and each finite-dimensional simple module in $\mathcal{O}_{\mathbf{k}^+}$ is one-dimensional.*

Proof. Let Q denote the Dynkin quiver of W. By Proposition 6.3.10 we have an isomorphism

$H_{\mathbf{k}} \cong \Pi^{\lambda(\mathbf{k})}(W)$ for $\lambda(\mathbf{k})$ defined by $\lambda(\mathbf{k})_j = 1 + m(k_j - k_{j+1})$. We have

$$\begin{aligned}
\lambda(\mathbf{k}) \cdot \delta &= \sum_{j=0}^{m-1} \lambda(\mathbf{k})_j \\
&= \sum_{j=0}^{m-1} (1m(k_j - k_{j+1})) \\
&= m + m \sum_{j=0}^{m-1} (k_j - k_{j+1}) \\
&= m + m \left(\sum_{j=0}^{m-1} k_j - \sum_{j=1}^{m} k_j \right) \\
&= m \neq 0.
\end{aligned}$$

Thus applying Lemma 6.3.16 and Proposition 6.3.10 we have a series of isomorphisms

$$H_{\mathbf{k}}(W) \cong \Pi^{\lambda(\mathbf{k})}(W) \cong \Pi^{\lambda(\mathbf{k})/m}(W)$$

and these induce equivalences $H_{\mathbf{k}}(W) - \mathrm{Mod} \cong \Pi^{\lambda(\mathbf{k})}(W) - \mathrm{Mod} \cong \Pi^{\lambda(\mathbf{k})/m}(W) - \mathrm{Mod}$. By Lemma 6.3.18, $\lambda(\mathbf{k})/m$ is conjugate under $\mathcal{W}(Q)$to a dominant weight λ^+. By Theorem 6.3.14 we have a Morita equivalence between $\Pi^{\lambda\mathbf{k}/m}(W)$ and $\Pi^{\lambda^+}(W)$. Note that by Lemma 6.3.16 we have a Morita equivalence between $\Pi^{\lambda(\mathbf{k})}(W)$ and $\Pi^{\lambda\mathbf{k}/m}(W)$ (given by an isomorphism $\Pi^{\lambda(\mathbf{k})}(W) \cong \Pi^{\lambda\mathbf{k}/m}(W)$) so that in summary we have obtained a Morita equivalence between $H_{\mathbf{k}}$ and $\Pi^{\lambda^+}(W)$. Now $\Pi^{\lambda^+}(W) \cong \Pi^{m\lambda^+}(W)$ (again by Lemma 6.3.16) and we have

$$\begin{aligned}
\sum_i m\lambda_i^+ = m \sum_i \lambda_i^+ &= m \sum_i (\lambda_i/m) \\
&= \sum_i \lambda_i = m
\end{aligned}$$

as the action of $\mathcal{W}(Q)$ preserves the value $\sum_i \lambda_i$ for any weight λ. Thus we can define $\mathbf{k}^+$ by

$$\begin{aligned}
k_0^+ &= 0 \\
1 + m(k_i^+ - k_{i+1}^+) &= m\lambda_i^+
\end{aligned}$$

and so

$$H_{\mathbf{k}^+} \cong \Pi^{m\lambda^+}(W).$$

Thus we have finally obtained a Morita equivalence between $H_{\mathbf{k}}$ and $H_{\mathbf{k}}^+$ given by the compositions

$$\begin{aligned}
H_{\mathbf{k}} - \mathrm{Mod} &\cong \Pi^{\lambda}(W) - \mathrm{Mod} \cong \Pi^{\lambda/m}(W) - \mathrm{Mod} \\
&\cong \Pi^{\lambda^+}(W) - \mathrm{Mod} \cong \Pi^{m\lambda^+}(W) - \mathrm{Mod} \\
&\cong H_{\mathbf{k}^+} - \mathrm{Mod}.
\end{aligned}$$

This restricts to an equivalence $\mathcal{O}_{\mathbf{k}} \cong \mathcal{O}_{\mathbf{k}^+}$ as the equivalences $\Pi^{\lambda}(W) - \mathrm{Mod} \cong \Pi^{\lambda/m}(W) - \mathrm{Mod}$ and $\Pi^{\lambda^+}(W) - \mathrm{Mod} \cong \Pi^{m\lambda^+}(W) - \mathrm{Mod}$ are induced from isomorphisms which rescale arrows and hence preserve nilpotent actions and by applying Corollary 6.3.15. By Lemma 6.3.18, every simple finite-dimensional object in $\mathcal{O}_{\mathbf{k}^+}$ has dimension 1. $\square$

Thus we deduce from Proposition 6.2.22

Corollary 6.3.20. *Let* $\mathbf{k}$ *be a choice of parameters and let* $\mathcal{B}$ *be a block of* $\mathcal{O}_{\mathbf{k}}$ *length* n*. Then* $Q_{\mathcal{B}}$ *has the form* $\mathcal{B}$ *is equivalent to* $\mathcal{A}(\mathcal{B})$-mod *where* $\mathcal{A}(\mathcal{B})$ *is the algebra of the quiver* $Q_{\mathcal{B}}$*. The quiver* $Q_{\mathcal{B}}$ *has the form*

$$\begin{tikzcd} \lambda_1 \arrow[r, bend left, "a_1"] & \lambda_2 \arrow[l, bend left, "b_1"] \arrow[r, bend left, "a_2"] & \lambda_3 \arrow[l, bend left, "b_2"] \arrow[r, bend left, "a_3"] & \cdots \arrow[l, bend left, "b_3"] \arrow[r, bend left, "a_{n-1}"] & \lambda_n \arrow[l, bend left, "b_{n-1}"] \end{tikzcd}$$

of $\mathcal{B}$ *subject to the relations*

$$\begin{aligned} a_{n-1}b_{n-1} &= 0 \text{ and} \\ a_r b_r &= b_{r+1}a_{r+1} \text{ for } 1 \leq r \leq n-2. \end{aligned}$$

6.4 Finite-dimensional modules of $H_{\mathbf{k}}(W)$.

Definition 6.4.1. *For any category* $\mathcal{C}$ *of modules or bimodules over* $H_{\mathbf{k}}$ *we will use* $\mathrm{fd}(\mathcal{C})$*to denote the full subcategory of all finite-dimensional modules or bimodules in* $\mathcal{C}$*.*

Example 6.4.2. *We have* $H_{\mathbf{k}}-\mathrm{fin} = \mathrm{fd}(\mathcal{O}_{\mathbf{k}})$ *and* $\mathrm{fd}(\mathcal{HC}_{\mathbf{k}})$ *denotes the full subcategory of all finite-dimensional Harish-Chandra* $H_{\mathbf{k}}$*-bimodules.*

Definition 6.4.3. *For given parameter values* $\mathbf{k}$ *we set*

$$\mathrm{Irr_{\mathbf{k}}}^{fin}(W) := \{\lambda \in \mathrm{Irr}(W) \mid \dim_{\mathbb{C}} L_{\mathbf{k}}(\lambda) < \infty\}$$

Lemma 6.4.4. *Let* $\mathcal{B}$ *be a block of* $\mathcal{O}_{\mathbf{k}}$*, viewed as an equivalence class of simple objects in* $\mathcal{O}_{\mathbf{k}}$*. Then* $\mathcal{B} \cap \mathrm{Irr_{\mathbf{k}}}^{fin}(W)$ *is a block of* $\mathrm{fd}(\mathcal{O}_{\mathbf{k}})$ *and viewing* $\mathcal{B}$ *as a category we have that* $\mathrm{fd}(\mathcal{B})$ *is the block of* $\mathrm{fd}(\mathcal{O}_{\mathbf{k}})$ *associated to the simple objects in* $\mathcal{B} \cap \mathrm{Irr_{\mathbf{k}}}^{fin}(W)$*.*

Proof. Viewing $\mathcal{B}$ as a subcategory of $\mathcal{O}_{\mathbf{k}}$, we know that it is given by the quiver

$$\begin{tikzcd} \lambda_1 \arrow[r, bend left, "a_1"] & \lambda_2 \arrow[l, bend left, "b_1"] \arrow[r, bend left, "a_2"] & \lambda_3 \arrow[l, bend left, "b_2"] \arrow[r, bend left, "a_3"] & \cdots \arrow[l, bend left, "b_3"] \arrow[r, bend left, "a_{n-1}"] & \lambda_n \arrow[l, bend left, "b_{n-1}"] \end{tikzcd}$$

with $L(\lambda_1)$ being the only infinite-dimensional simple object in $\mathcal{B}$. Now we have that

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(L(\lambda_{i+1}), L(\lambda_i)) \neq 0$$

since $\Delta(\lambda_i) \hookrightarrow \Delta(\lambda_{i+1})$. Furthermore, $\mathrm{fd}(\mathcal{O}_{\mathbf{k}})$ is a Serre subcategory of $\mathcal{O}_{\mathbf{k}}$ since it is closed under taking extensions and we have an isomorphism of groups $\mathrm{Ext}^1_{\mathcal{O}_{\mathbf{k}}}(L(\lambda_{i+1}), L(\lambda_i)) \cong \mathrm{Ext}^1_{\mathrm{fd}(\mathcal{O}_{\mathbf{k}})}(L(\lambda_{i+1}), L(\lambda_i)))$. This implies that $L(\lambda_2), L(\lambda_3), \ldots, L(\lambda_n)$ all are in the same block of $\mathrm{fd}(\mathcal{O}_{\mathbf{k}})$. Note also that each block of $\mathrm{fd}(\mathcal{O}_{\mathbf{k}})$ is a subset of a block of $\mathcal{O}_{\mathbf{k}}$ as the finite-dimensional subcategory is Serre. Thus we can conclude that

$$\{L(\lambda_2), \ldots, L(\lambda_n)\} = \{L(\lambda_1), \ldots, L(\lambda_n)\} \cap \mathrm{Irr_{\mathbf{k}}}^{fin}(W)$$

is a block of $\mathrm{fd}(\mathcal{O}_{\mathbf{k}})$. The second assertion simply reformulates the first. $\square$

Now we know the underlying graph of the category $\mathrm{fd}(\mathcal{O}_{\mathbf{k}})$ and we still need to determine the relations on the arrows. This will be easy once we have a description of the projective indecomposables in $\mathrm{fd}(\mathcal{O}_{\mathbf{k}})$ and in providing this we will in fact show that projective indecomposables exist in $\mathrm{fd}(\mathcal{O}_{\mathbf{k}})$.

Definition 6.4.5. *For a module $M \in \mathcal{O}_{\mathbf{k}}$ we set*

$$\Sigma_{\mathbf{k}}(M) := \{\alpha \in \mathbb{C} \mid (\underline{\mathrm{h}} - \alpha)^N m = 0 \text{ some } N \in \mathbb{N} \text{ and } m \in M, m \neq 0\}$$

and refer to this as the spectrum of M. In words, the spectrum of a module in $\mathcal{O}_{\mathbf{k}}$ is the set of all generalised $\underline{\mathrm{h}}$-eigenvalues of elements of M. We further set

$$\Sigma_{\mathbf{k}} = \cup_{M \in \mathcal{O}_{\mathbf{k}}} \Sigma_{\mathbf{k}}(M)$$

and also

$$\Sigma_{\mathbf{k}}^{fin} := \cup_{M \in \mathrm{fd}(\mathcal{O}_{\mathbf{k}})} \Sigma_{\mathbf{k}}(M).$$

Lemma 6.4.6. *We have*

$$\Sigma_{\mathbf{k}} = \cup_{\lambda \in \mathrm{Irr}(W)} \Sigma_{\mathbf{k}}(L(\lambda))$$

and

$$\Sigma_{\mathbf{k}}^{fin} := \cup_{\lambda \in \mathrm{Irr}_{\mathbf{k}}{}^{fin}(W)} \Sigma_{\mathbf{k}}(L(\lambda)).$$

Proof. Let M be a module in $\mathcal{O}_{\mathbf{k}}$ and m a generalised $\underline{\mathrm{h}}$-eigenvector with eigenvalue α. Now M has a Jordan-Hölder series, say

$$M = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_n \supseteq M_{n+1} = 0$$

with $M_i/M_{i+1} \cong L(\lambda_i)$ for some $\lambda_i \in \mathrm{Irr}(W)$. Now pick i maximal such that $m \in M_i$ and consider its image $\bar{m} \in L(\lambda_i) = M_i/M_{i+1}$. By hypothesis this is non-zero and also a generalised $\underline{\mathrm{h}}$-eigenvalue, so that $(\underline{\mathrm{h}} - \beta)^{N_\beta} \bar{m} = 0$ for some $\beta \in \Sigma_{\mathbf{k}}(L(\lambda_i))$ and $N_\beta \in \mathbb{N}$. Now clearly also $(\underline{\mathrm{h}} - \alpha)^N \bar{m} = 0$ and by uniqueness of eigenvalues we deduce that

$$\alpha = \beta \in \Sigma_{\mathbf{k}}(L(\lambda_i)).$$

Similar reasoning applies to $\Sigma_{\mathbf{k}}^{fin}$ as $\mathrm{fd}(\mathcal{O}_{\mathbf{k}})$ is also a finite-length category. □

Proposition 6.4.7. *There exists a non-zero, two-sided, cofinite ideal $I \trianglelefteq H_{\mathbf{k}}$ such that $IM = 0$ for any finite-dimensional $H_{\mathbf{k}}$-module M. Moreover there exists a unique ideal $J_{\mathbf{k}}$ maximal with respect to these properties.*

Proof. From Lemma 6.4.6 we deduce that $\Sigma_{\mathbf{k}}^{fin}$ is finite and hence bounded, so contained in a disc in $\mathbb{C}$ of finite radius. Therefore we may choose $N >> 0$ such that $\alpha \pm N \notin \Sigma_{\mathbf{k}}^{fin}$ for any $\alpha \in \Sigma_{\mathbf{k}}^{fin}$. Now let $m \in M$ be any generalised $\underline{\mathrm{h}}$-eigenvector with eigenvalue $\alpha \in \Sigma_{\mathbf{k}}^{fin}$. Then if $x^N m \in M$ is nonzero, it is again a generalised $\underline{\mathrm{h}}$-eigenvector with eigenvalue $\alpha - N$. We have observed that this cannot hold by choice of N and thus $x^N m = 0$. Similarly $y^N m = 0$. Thus the two-sided ideal

$$I = H_{\mathbf{k}} x^N H_{\mathbf{k}} + H_{\mathbf{k}} y^N H_{\mathbf{k}} \subseteq H_{\mathbf{k}}$$

is such that $IM = 0$ for any finite-dimensional module M.

We may now choose

$$J_{\mathbf{k}} = \{a \in H_{\mathbf{k}} \mid aM = 0 \ \forall M \text{ s.t. } \dim_{\mathbb{C}} M < \infty\}.$$

By definition, $J_{\mathbf{k}}$ is certainly a two-sided ideal and we have just shown that $J_{\mathbf{k}}$ is non-empty since $I \subseteq J_{\mathbf{k}}$, which obviously also shows that it is cofinite. If I' is any two-sided ideal of $H_{\mathbf{k}}$ annihilating all finite-dimensional modules of $H_{\mathbf{k}}$ then again by definition, $I' \subseteq J_{\mathbf{k}}$ and thus $J_{\mathbf{k}}$ is maximal among such ideals. □

Definition 6.4.8. *We will denote by $J_{\mathbf{k}}$ the ideal of $H_{\mathbf{k}}$ introduced in Proposition 6.4.7, that is the ideal of $H_{\mathbf{k}}$ which is maximal among those ideals annihilating all finite-dimensional modules.*

Corollary 6.4.9. *Every module M of $\mathcal{O}_{\mathbf{k}}$ has a well-defined, maximal, finite-dimensional quotient module $\bar{M}$ (maximal in the sense that every other finite-dimensional quotient of M is a quotient of $\bar{M}$).*

Proof. We claim first that if M is in $\mathcal{O}_{\mathbf{k}}$, then $J_{\mathbf{k}}M \neq 0$ if and only if M is infinite-dimensional. If $J_{\mathbf{k}}M \neq 0$ then M has to be infinite-dimensional by definition of $J_{\mathbf{k}}$.If M is infinte-dimensional, consider a Jordan-Hölder series of M and note that an infinite-dimensional simple module must occur as a subfactor. But if L is infinite-dimensional simple, then L is free of rank one as a $\mathbb{C}[\mathfrak{h}]$-module and so $x^n L \neq 0$ for any $n \in \mathbb{N}_0$ and so $J_{\mathbf{k}}L \neq 0$.
Now consider

$$\bar{M} = M/J_{\mathbf{k}}M.$$

Then $\bar{M}$ is again in $\mathcal{O}_{\mathbf{k}}$ and as $J_{\mathbf{k}}\bar{M} = 0$ we can deduce that it is finite-dimensional by our first claim. If $M' = M/N$ is another finite-dimensional quotient of M, then we again have $J_{\mathbf{k}}M' = 0$ and thus $J_{\mathbf{k}}M \subseteq N$ and so M' is a quotient of $\bar{M} = M/J_{\mathbf{k}}M$. □

Definition 6.4.10. *Let W be cyclic of order m and $\mathbf{k}$ a choice of parameters. For any module $M \in \mathcal{O}_{\mathbf{k}}$ we denote by $\bar{M}$ the maximal finite-dimensional quotient of M from Corollary 6.4.9.*

Proposition 6.4.11. *Up to isomorphism, the projective indecomposable modules in* $\mathrm{fd}(\mathcal{O}_{\mathbf{k}})$ *are given by*

$$\bar{P}(\lambda) \text{ for } \lambda \in {\mathrm{Irr}_{\mathbf{k}}}^{fin}(W).$$

For any $\lambda \in {\mathrm{Irr}_{\mathbf{k}}}^{fin}(W)$ we have a surjection $\bar{P}(\lambda) \twoheadrightarrow L(\lambda)$. In particular, $\mathrm{fd}(\mathcal{O}_{\mathbf{k}})$ *has enough indecomposable projective modules.*

Proof. Let $P(\lambda)$ be an indecomposable projective object in $\mathcal{O}_{\mathbf{k}}$. Then by definition $\bar{P}(\lambda)$ is an object of $\mathrm{fd}(\mathcal{O}_{\mathbf{k}})$ and it is non-zero if and only if $\lambda \in {\mathrm{Irr}_{\mathbf{k}}}^{fin}(W)$ since the head of $P(\lambda)$ is finite-dimensional if and only if $\lambda \in {\mathrm{Irr}_{\mathbf{k}}}^{fin}(W)$. Since $P(\lambda)$ has simple head, so has $\bar{P}(\lambda)$ and therefore $\bar{P}(\lambda)$ remains indecomposable.
We thus need to show that $\bar{P}(\lambda)$ is projective in $\mathrm{fd}(\mathcal{O}_{\mathbf{k}})$ and that any indecomposable projective of $\mathrm{fd}(\mathcal{O}_{\mathbf{k}})$ has this form. As $P(\lambda)$ is projective in $\mathcal{O}_{\mathbf{k}}$, the functor

$$\mathrm{Hom}_{H_{\mathbf{k}}}(P(\lambda), -) : \mathrm{fd}(\mathcal{O}_{\mathbf{k}}) \to \mathrm{Vect}_{\mathbb{C}}$$

is certainly exact. Now if $F \in \mathrm{fd}(\mathcal{O}_{\mathbf{k}})$ and $f \in \mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(P(\lambda), F)$, then $\mathrm{im}(f) \subseteq F$ is a finite-dimensional quotient of $P(\lambda)$, and hence there exists a unique, well-defined morphism

$$\bar{f} : \bar{P}(\lambda) \twoheadrightarrow \mathrm{im}(f) \hookrightarrow F$$

whose lift from $\bar{P}(\lambda)$ to $P(\lambda)$ agrees with f. Thus we have an isomorphism of vector spaces

$$\mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(P(\lambda), F) \cong \mathrm{Hom}_{\mathrm{fd}(\mathcal{O}_{\mathbf{k}})}(\bar{P}(\lambda), F)$$

and hence the functor

$$\mathrm{Hom}_{\mathrm{fd}(\mathcal{O}_{\mathbf{k}})}(\bar{P}(\lambda), -) : \mathrm{fd}(\mathcal{O}_{\mathbf{k}}) \to \mathrm{Vect}_{\mathbb{C}}$$

is exact too. Thus $\bar{P}(\lambda)$ is projective in $\mathrm{fd}(\mathcal{O}_{\mathbf{k}})$.
To show that $\bar{P}(\lambda)$ covers $L(\lambda)$ for $\lambda \in {\mathrm{Irr}_{\mathbf{k}}}^{fin}(W)$, consider the map $\bar{\varphi} : \bar{P}(\lambda) \to L(\lambda)$ induced from the covering map $\varphi : P(\lambda) \to L(\lambda)$. We want to show that its kernel is superfluous, i.e. that if we have a submodule $M \subseteq \bar{P}(\lambda)$ such that $M + \ker \bar{\varphi} = \bar{P}(\lambda)$, then $M = \bar{P}(\lambda)$. So suppose such a submodule $M \subseteq \bar{P}(\lambda)$ is given. Then we can lift this to a submodule N of $P(\lambda)$ such that $N + \ker \varphi = P(\lambda)$. As $\ker \varphi$ is superfluous, we can deduce that $N = P(\lambda)$ and so $M = \bar{P}(\lambda)$. This establishes that $\ker \bar{\varphi}$ is indeed superfluous.
Note that for $\lambda, \mu \in {\mathrm{Irr}_{\mathbf{k}}}^{fin}(W)$ we have $\bar{P}(\lambda) \neq \bar{P}(\mu)$ if $\lambda \neq \mu$ as their heads are non-isomorphic. Thus we have found an indecomposable, projective module with simple head for each simple object of $\mathrm{fd}(\mathcal{O}_{\mathbf{k}})$. Since $\mathrm{fd}(\mathcal{O}_{\mathbf{k}})$ is the category of finitely generated modules of the finite-dimensional algebra $H_{\mathbf{k}}/J_{\mathbf{k}}$, we see that these are indeed up to isomorphism all indecomposable projectives in $\mathrm{fd}(\mathcal{O}_{\mathbf{k}})$. □

We now need to identify what maps between projectives the arrows of the quiver correspond to and what relations hold between these.

Lemma 6.4.12. *Let M be a module in $\mathcal{O}_{\mathbf{k}}$. Then $J_{\mathbf{k}}M$ does not have any finite-dimensional quotients. In particular, if M belongs to a block of $\mathcal{O}_{\mathbf{k}}$, then the head of $J_{\mathbf{k}}M$ is a direct sum of copies of the unique infinite-dimensional simple in that block.*

Proof. If $J_{\mathbf{k}}M$ had a finite-dimensional quotient, then $M/J_{\mathbf{k}}M$ could not be the maximal finite-dimensional quotient of M. In particular the head of $J_{\mathbf{k}}M$ can only contain infinite-dimensional simples. □

Corollary 6.4.13. *Let M be a module in $\mathcal{O}_{\mathbf{k}}$, then $J_{\mathbf{k}}M$ is the submodule of M generated by $\bigcup_{\lambda \notin {\mathrm{Irr}_{\mathbf{k}}}^{fin}(W)} \bigcup_{f \in \mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}} f(P(\lambda)) \subseteq M$.*

Proof. Follows as the head of $J_{\mathbf{k}}M$ contains only simples of the form $L(\lambda)$ with $\lambda \notin {\mathrm{Irr}_{\mathbf{k}}}^{fin}(W)$. □

Lemma 6.4.14. *Let $\mathbf{k}$ be such that all finite-dimensional modules have dimension 1. Let $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a block of category $\mathcal{O}_{\mathbf{k}}$, enumerated as before so $\{\lambda_2, \ldots, \lambda_n\} \subseteq {\mathrm{Irr}_{\mathbf{k}}}^{fin}(W)$. Then*

$$\dim_{\mathbb{C}} \bar{P}(\lambda_r) = (r-1)(1+n-r)$$

and a basis is given by

$$\{x^p \otimes y^c \otimes \lambda_r \mid 0 \leq p \leq r-2, 0 \leq c \leq n-r\}.$$

Moreover this module is cyclic generated by the image of $1 \otimes 1 \otimes \lambda_r$.

Proof. That $\bar{P}(\lambda_r)$ is cyclic and generated by $1 \otimes 1 \otimes \lambda_r$ is trivial as the statement holds for $P(\lambda_r)$.

Let us consider the vector

$$x^{r-1} \otimes 1 \otimes \lambda_r \in P(\lambda_r).$$

This generates a subquotient isomorphic to $L(\lambda_1)$, the infinite-dimensional simple in this block:

$$y(x^{r-1} \otimes 1 \otimes \lambda_r) = x^{r-1} \otimes y \otimes \lambda_r + (r - 1 + m(k_{\lambda_r} - k_{\lambda_1}))\,(x^{r-2} \otimes 1 \otimes \lambda_r).$$

Since

$$\begin{aligned} m(k_{\lambda_r} - k_{\lambda_1}) &= m\left((k_{\lambda_r} - k_{\lambda_{r-1}}) + (k_{\lambda_{r-1}} - k_{\lambda_{r-2}}) + \ldots + (k_{\lambda_2} - k_{\lambda_1})\right) \\ &= -(1 + 1 + \ldots + 1) \\ &= -(r-1) \end{aligned}$$

this simplifies to

$$y(x^{r-1} \otimes 1 \otimes \lambda_r) = x^{r-1} \otimes y \otimes \lambda_r.$$

Thus the module $H_{\mathbf{k}}(x^{r-1} \otimes 1 \otimes \lambda_r)$ has the quotient $\frac{H_{\mathbf{k}}(x^{r-1}\otimes 1\otimes\lambda_r)}{H_{\mathbf{k}}(x^{r-1}\otimes y\otimes\lambda_r)}$. This is generated by a singular vector on which W acts as λ_1 and hence it is a quotient of $\Delta(\lambda_1)$. This is simple and so

$$\frac{H_{\mathbf{k}}(x^{r-1} \otimes 1 \otimes \lambda_r)}{H_{\mathbf{k}}(x^{r-1} \otimes y \otimes \lambda_r)} \cong L(\lambda_1)$$

which is infinite-dimensional. Thus $J_{\mathbf{k}}P(\lambda_r)$ must contain $x^{r-1} \otimes 1 \otimes \lambda_r$ and by a similar calculation as before we see that

$$\{x^{r-1} \otimes y \otimes \lambda_r, x^{r-1} \otimes y^2 \otimes \lambda_r, \ldots, x^{r-1} \otimes y^{n-r} \otimes \lambda_r\} \subseteq H_{\mathbf{k}}(x^{r-1} \otimes 1 \otimes \lambda_r).$$

As the quotient module $\frac{P(\lambda_r)}{H_{\mathbf{k}}(x^{r-1}\otimes 1\otimes\lambda_r)}$ is already finite-dimensional, it must be the maximal finite-dimensional quotient of $P(\lambda_r)$.
It can be checked that in fact $H_{\mathbf{k}}(x^{r-1} \otimes 1 \otimes \lambda_r)$ is generated by $\{x^{r-1} \otimes 1 \otimes \lambda_r, x^{r-1} \otimes y^1 \otimes \lambda_r, \ldots, x^{r-1} \otimes y^{n-r} \otimes \lambda_r\}$ as a $\mathbb{C}[\mathfrak{h}]$-module and thus the quotient $\bar{P}(\lambda_r)$ really has basis

$$\{x^p \otimes y^c \otimes \lambda_r \mid 0 \le p \le r-2, 0 \le c \le n-r\}$$

as claimed and the statement about the dimension follows. □

Corollary 6.4.15. *The map*

$$\begin{aligned} \mathrm{Hom}_{\mathrm{fd}(\mathcal{O}_{\mathbf{k}})}(\bar{P}(\lambda_l), \bar{P}(\lambda_r)) &\to \bar{P}(\lambda_r) \\ f &\mapsto f(1 \otimes 1 \otimes \lambda_l) \end{aligned}$$

is and isomorphism onto its image. The image is the span

$$\mathbb{C}\{x^p \otimes y^c \otimes \lambda_r \mid 0 \le p \le r-2, 0 \le c \le n-r, r+c-p = l\}.$$

In particular $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{fd}(\mathcal{O}_{\mathbf{k}})}(\bar{P}(\lambda_l), \bar{P}(\lambda_r)) = \min(n-r+1, l-1) - \max(0, l-r)$.

Proof. Note that the set $S = \{x^p \otimes y^c \otimes \lambda_r \mid 0 \le p \le r-2, 0 \le c \le n-r, r+c-p = l\}$ contains precisely $\min(n-r+1, l-1) - \max(0, l-r))$ elements. So the formula for the dimension of the Hom-space follows once the isomorphism is shown and we have proved that S is a set of linearly independent vectors. But linear independence of S follows from Lemma 6.4.14.

By Lemma 6.4.14, $\bar{P}(\lambda_l)$ is cyclic with generating vector $1 \otimes 1 \otimes \lambda_l$. Any morphism $f : \bar{P}(\lambda_l) \to \bar{P}(\lambda_r)$ is therefore uniquely determined by the image of the generating vector of $\bar{P}(\lambda_l)$. It follows that $f(1 \otimes 1 \otimes \lambda_l)$ must again span λ_l as a $\mathbb{C}W$-module and thus the space of possible images of $1 \otimes 1 \otimes \lambda_l$ must be contained in the λ_l-isotypic component of $\bar{P}(\lambda_r)$. Considering the basis of $\bar{P}(\lambda_l)$ from Lemma 6.4.14, we see that this must be spanned by

$$\{x^p \otimes y^c \otimes \lambda_r \mid 0 \leq p \leq r-2, 0 \leq c \leq n-r, r+c-p = l\}$$

as claimed. Hence we have an injection

$$\mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(\bar{P}(\lambda_l), \bar{P}(\lambda_r)) \hookrightarrow \mathbb{C}\{x^p \otimes y^c \otimes \lambda_r \mid 0 \leq p \leq r-2, 0 \leq c \leq n-r, r+c-p = l\}$$

and it remains to show surjectivity. This will follows from the projective property of $P(\lambda_l)$. Choose any $x^p \otimes y^c \otimes \lambda_r \in S \subseteq \bar{P}(\lambda_r)$ and consider the submodule $H_{\mathbf{k}}(x^p \otimes y^c \otimes \lambda_r) \leq \bar{P}(\lambda_r)$ generated by $x^p \otimes y^c \otimes \lambda_r$ as well as its quotient

$$A = \frac{H_{\mathbf{k}}(x^p \otimes y^c \otimes \lambda_r)}{H_{\mathbf{k}}(yx^p \otimes y^c \otimes \lambda_r)}.$$

If $A \neq 0$ it is generated by the image of $x^p \otimes y^c \otimes \lambda_r$ which transforms under W as λ_l and is annihilated by y. Hence there is a surjection $\Delta(\lambda_l) \twoheadrightarrow A$ and thus a surjection $P(\lambda) \twoheadrightarrow A$ giving a surjection $\bar{P}(\lambda_l) \twoheadrightarrow A$ which lifts to a map from $\bar{P}(\lambda_l)$ onto the submodule of $\bar{P}(\lambda_r)$ generated by $x^p \otimes y^c \otimes \lambda_r$. Thus we have found an element of $\mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(\bar{P}(\lambda_l), \bar{P}(\lambda_r))$ for any element $x^p \otimes y^c \otimes \lambda_r$ in S such that $\frac{H_{\mathbf{k}}(x^p \otimes y^c \otimes \lambda_r)}{H_{\mathbf{k}}(yx^p \otimes y^c \otimes \lambda_r)} \neq 0$. Repeating the calculation in the proof of Lemma 6.4.14 we see that

$$y(x^p \otimes y^c \otimes \lambda_r) = x^{p-1} \otimes y^{c+1} \otimes \lambda_r$$

so that indeed $\frac{H_{\mathbf{k}}(x^p \otimes y^c \otimes \lambda_r)}{H_{\mathbf{k}}(yx^p \otimes y^c \otimes \lambda_r)} \neq 0$ for any element of S. □

We now have enough information to work out the quiver of a block of $\mathrm{fd}(\mathcal{O}_{\mathbf{k}})$. Recall that from a block $\mathcal{B}$

$$\begin{tikzcd} \lambda_1 \arrow[r, bend left, "a_1"] & \lambda_2 \arrow[l, bend left, "b_1"] \arrow[r, bend left, "a_2"] & \lambda_3 \arrow[l, bend left, "b_2"] \arrow[r, bend left, "a_3"] & \cdots \arrow[l, bend left, "b_3"] \arrow[r, bend left, "a_{n-1}"] & \lambda_n \arrow[l, bend left, "b_{n-1}"] \end{tikzcd}$$

of category $\mathcal{O}_{\mathbf{k}}$ we obtain a block $\mathrm{fd}(\mathcal{B})$

$$\begin{tikzcd} \lambda_2 \arrow[r, bend left, "\bar{a}_2"] & \lambda_3 \arrow[l, bend left, "\bar{b}_2"] \arrow[r, bend left, "\bar{a}_3"] & \cdots \arrow[l, bend left, "\bar{b}_3"] \arrow[r, bend left, "\bar{a}_{n-1}"] & \lambda_n \arrow[l, bend left, "\bar{b}_{n-1}"] \end{tikzcd}$$

of $\mathrm{fd}(\mathcal{O}_{\mathbf{k}})$. As before we can now work out that the arrow

$$\bar{a}_r : \lambda_r \to \lambda_{r+1}$$

corresponds to the morphism

$$\bar{P}(\lambda_{r+1}) \to \bar{P}(\lambda_r),\ 1 \otimes 1 \otimes \lambda_{r+1} \mapsto 1 \otimes y \otimes \lambda_r$$

and the arrow

$$\bar{b}_r : \lambda_{r+1} \to \lambda_r$$

corresponds to the morphism

$$\bar{P}(\lambda_r) \to \bar{P}(\lambda_{r+1}),\ 1 \otimes 1 \otimes \lambda_r \mapsto x \otimes 1 \otimes \lambda_{r+1}.$$

Thus the relations

$$(1)\ \bar{a}_{n-1}\bar{b}_{n-1} = 0$$

and

$$(2)\ \bar{a}_r\bar{b}_r = \bar{b}_{r+1}\bar{a}_{r+1}$$

also hold in $\mathrm{fd}(\mathcal{B})$, induced from those in $\mathcal{B}$. However we also obtain the further relation

$$(3)\ \bar{b}_2\bar{a}_2 = 0$$

from direct calculation. We again claim that the relations $(1) - (3)$ are all relations in $\mathrm{fd}(\mathcal{B})$. As before we start with a simple lemma:

Lemma 6.4.16. *The following relations hold in* $\mathrm{fd}(\mathcal{B})$*:*

1. $\bar{a}_{n-1}\bar{b}_{n-1} = 0$
2. $\bar{b}_2\bar{a}_2 = 0$
3. $\bar{a}_r\bar{b}_r = \bar{b}_{r+1}\bar{a}_{r+1}$
4. *For* $2 \leq r \leq n-1$ *we have* $(\bar{a}_r\bar{b}_r)^{n-r} = 0$.
5. *For* $2 \leq r \leq n-1$ *we have* $(\bar{a}_r\bar{b}_r)^{r} = 0$.
6. *For* $2 \leq r \leq n-1$ *we have* $(\bar{a}_r\bar{b}_r)^{\min(n-r,r)} = 0$.
7. *For* $2 \leq r \leq n-1$ *we have* $(\bar{b}_r\bar{a}_r)^{r-1} = 0$
8. *For* $2 \leq r \leq n-1$ *we have* $(\bar{b}_r\bar{a}_r)^{n-r+1} = 0$
9. *For* $2 \leq r \leq n-1$ *we have* $(\bar{b}_r\bar{a}_r)^{\min(r-1,n-r+1)} = 0$

Proof. The first four statements have already been proven or follow easily form the corresponding statements for $\mathcal{B}$. The sixth follows from the fourth and the fifth and similarly the final statement follows from the seventh and eighth, so it suffices to prove numbers 5, 7, 8. For this we will use induction on r:
Statement 5): For $r = 2$, we have

$$(\bar{a}_2\bar{b}_2)^2 = \bar{a}_2\bar{b}_2\bar{a}_2\bar{b}_2 = \bar{a}_2(\bar{b}_2\bar{a}_2)\bar{b}_2 = 0.$$

Now if the statement has been established for r, let us consider $r+1$. We find

$$\begin{aligned}(\bar{a}_{r+1}\bar{b}_{r+1})^{r+1} &= \bar{a}_{r+1}(\bar{b}_{r+1}\bar{a}_{r+1})^r\bar{b}_{r+1}\\ &= \bar{a}_{r+1}(\bar{a}_r\bar{b}_r)^r\bar{b}_{r+1}\\ &= 0.\end{aligned}$$

Statement 7): For $r = 2$, we have

$$\bar{b}_2\bar{a}_2 = 0.$$

Let us consider the case of $r \to r+1$. Then

$$\begin{aligned}(\bar{b}_{r+1}\bar{a}_{r+1})^r &= (\bar{a}_r\bar{b}_r)^r \\ &= \bar{a}_r(\bar{b}_r\bar{a}_r)^{r-1}\bar{b}_r = 0\end{aligned}$$

Statement 8): For $r = n-1$, we have

$$(\bar{b}_{n-1}\bar{a}_{n-1})^2 = \bar{b}_{n-1}(\bar{a}_{n-1}\bar{b}_{n-1})\bar{a}_{n-1} = 0$$

and assuming the statement holds for the case $r+1$ we can consider the case of r:

$$\begin{aligned}(\bar{b}_r\bar{a}_r)^{n+r-1} &= \bar{b}_r(\bar{a}_r\bar{b}_r)^{n-r}\bar{a}_r \\ &= 0\end{aligned}$$

this proves the 8th statement. □

Proposition 6.4.17. *For a block $\mathcal{B}$ of $\mathcal{O}_{\mathbf{k}}$ as before, the algebra* $\mathrm{End}_{\mathrm{fd}(\mathcal{O}_{\mathbf{k}})}(\oplus_{\lambda\in\mathrm{fd}(\mathcal{B})}\bar{P}(\lambda))^{op}$ *is isomorphic to* $\mathbb{C}Q^{fd}_{\mathcal{B}}/I^{fd}_{\mathcal{B}} =: \mathcal{A}^{fin}(\mathcal{B})$ *with* $Q^{fd}_{\mathcal{B}}$ *the quiver*

$$\begin{tikzcd} \lambda_2 \arrow[r, bend left, "\bar{a}_2"] & \lambda_3 \arrow[l, bend left, "\bar{b}_2"] \arrow[r, bend left, "\bar{a}_3"] & \cdots \arrow[l, bend left, "\bar{b}_3"] \arrow[r, bend left, "\bar{a}_{n-1}"] & \lambda_n \arrow[l, bend left, "\bar{b}_{n-1}"] \end{tikzcd}$$

and $I^{fd}_{\mathcal{B}}$ *the two-sided ideal in* $\mathbb{C}Q^{fin}$ *generated by*

$$\{\bar{a}_{n-1}\bar{b}_{n-1}, \bar{b}_2\bar{a}_2, \bar{a}_r\bar{b}_r = \bar{b}_{r+1}\bar{a}_{r+1} \mid 2 \leq r \leq n-2\}.$$

Moreover, the category $\mathrm{fd}(\mathcal{B})$ *is equivalent to* $\mathcal{A}^{fd}(\mathcal{B}) - \mathrm{mod}$.

Proof. We only need to show that these are all the relations, and by Morita equivalence we may assume that we are in the case that each finite-dimensional simple module has dimension one. We need to compare the dimension of the space of possible paths $\lambda_r \to \lambda_l$ with

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{fd}(\mathcal{O}_{\mathbf{k}})}(\bar{P}(\lambda_l), \bar{P}(\lambda_r)) = \min(n-r+1, l-1) - \max(0, l-r).$$

As before we will make case distinctions and compute an upper bound on the dimension of the space of paths, which we will see will coincide with the dimension of the morphism space.
Case 1: $r \geq l$. As before, the relation $\bar{a}_i\bar{b}_i = \bar{b}_{i+1}\bar{a}_{i+1}$ enables us to move loops. So we may assume that the only loops in this path are loops of the form $a_{r-1}b_{r-1}$, giving us maximal number of $\min(n-r+1, r-1)$ loops. Assuming the only loops occurring are of the form $\bar{b}_l\bar{a}_l$ gives an upper bound of $\min(n-l+1, l-1)$ on the number of paths and thus the dimension of the path space is bounded above by

$$\min(\min(n-r+1, r-1), \min(n-l+1, l-1)) = \min(n-r+1, r-1, n-l+1, l-1).$$

Since $r \geq l$, this is reduced to

$$\min(n-r+1, l-1) = \min(n-r+1, l-1) - \max(0, l-r)$$

and we have an equality as we wanted.
Case 2: $r < l$. We can compute upper bounds by assuming all loops are of the form $\bar{a}_{r-1}\bar{b}_{r-1}$,

giving us an upper bound of $\min(n-r+1, r-1)$. Conversely, assuming all loops appearing are of the form $\bar{b}_l\bar{a}_l$ giving an upper bound of $\min(n-l+1, l-1)$. Thus we arrive again at the upper bound of

$$\min(n-r+1, r-1, n-l+1, l-1).$$

Since $r < l$, this is $\min(r-1, n-l+1)$. We need to compare the possible values of this with our target of $\min(n-r+1, l-1) - \max(0, l-r) = \min(n-r+1, l-1) - l + r$..
If $\min(r-1, n-l+1) = r-1$ then $\min(n-r+1, l-1) = l-1$ and so $\min(n-r+1, l-1)-l+r = l-1-l+r = r-1$ which is what we wanted.
If $\min(r-1, n-l+1) = n-l+1$ then $\min(n-r+1, l-1) = n-r+1$ and so $\min(n-r+1, l-1) - l + r = n-r+1-l+r = n-l+1$ which again fits.
Thus in all cases the dimension of the space of paths is bounded above by $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{fd}(\mathcal{O}_{\mathbf{k}})}(\bar{P}(\lambda_l), \bar{P}(\lambda_r))$ and thus they must be equal, meaning that we have indeed found all relations. $\square$

6.5 Finite-Dimensional Harish-Chandra Bimodules

Recall the notation from Proposition 6.4.17.

Definition 6.5.1. *For a choice of parameters* $\mathbf{k}$ *we set*

$$\mathcal{A}_{\mathbf{k}}^{fd} := \bigoplus_{\mathcal{B}} \mathcal{A}^{fd}(\mathcal{B}) = \bigoplus_{\mathcal{B}} \left(\mathbb{C}Q_{\mathcal{B}}^{fd}/I_{\mathcal{B}}^{fd}\right)$$

where the sum runs over all blocks of $\mathcal{O}_{\mathbf{k}}$*. For the sake of completeness, let us remark that* $\mathcal{A}_{\mathcal{B}}^{fd} = 0$ *if the set* $\mathrm{fd}(\mathcal{B})$ *is empty.*

Definition 6.5.2. *For a block* $\mathcal{B}$ *of* $\mathcal{O}_{\mathbf{k}}$*, we denote by* $\mathrm{length}(\mathcal{B})$ *the number of isomorphism classes of simple objects in* $\mathcal{B}$*. We refer to this as the length of* $\mathcal{B}$*.*

Corollary 6.5.3. $H_{\mathbf{k}} - \mathrm{fin}$ *is equivalent to* $\mathcal{A}_{\mathbf{k}}^{fd} - \mathrm{mod}$*. The category* $\mathrm{fin} - H_{\mathbf{k}}$ *of finite-dimensional right* $H_{\mathbf{k}}$ *modules is also equivalent to* $\mathcal{A}_{\mathbf{k}}^{fd} - \mathrm{mod}$*.*

Proof. For any block $\mathcal{B}$ we have $\mathcal{A}^{fd}(\mathcal{B}) \cong \mathrm{End}_{\mathcal{O}_{\mathbf{k}}}(P)^{op}$, where $P = \oplus_r P(\lambda_r)$, by Lemma 6.4.17 and applying Theorem 4.1.14 proves the first statement. So we only need to show the equivalence

$$\mathrm{fin} - H_{\mathbf{k}} \cong \mathcal{A}_{\mathbf{k}}^{fd} - \mathrm{mod}.$$

We have equivalences

$$\mathrm{mod} - H_{\mathbf{k}}(\mathfrak{h}) \cong H_{\mathbf{k}}^{op} - \mathrm{mod} \cong H_{\mathbf{k}}(\mathfrak{h}^*) - \mathrm{mod}$$

using the isomorphism $H_{\mathbf{k}}^{op} \xrightarrow{\sim} H_{\mathbf{k}}(\mathfrak{h}^*)$ from Lemma 6.1.1 So we have an equivalence

$$\mathrm{fin} - H_{\mathbf{k}} \cong H_{\mathbf{k}}(\mathfrak{h}^*) - \mathrm{fin} \cong \mathcal{A}_{\mathbf{k}}^{fd}(\mathfrak{h}^*) - \mathrm{mod}$$

and by Theorem 6.1.5 also an equivalence $\mathcal{O}_{\mathbf{k}}^{op}(\mathfrak{h}) \cong \mathcal{O}_{\mathbf{k}}(\mathfrak{h}^*)$. This in particular shows that for each block in $\mathcal{O}_{\mathbf{k}}$ we can find a block of equal length in $\mathcal{O}_{\mathbf{k}}(\mathfrak{h}^*)$. For any block $\mathcal{B}$ the presentation of $\mathcal{A}_{\mathbf{k}}^{fd}(\mathcal{B})$ depends only on the length of $\mathcal{B}$ and thus we see that $\mathcal{A}_{\mathbf{k}}^{fd}(\mathfrak{h})$ and $\mathcal{A}_{\mathbf{k}}^{fd}(\mathfrak{h}^*)$ have identical presentations and are thus isomorphic. $\square$

Lemma 6.5.4. *The category of finite-dimensional Harish-Chandra $H_{\mathbf{k}} - H_{\mathbf{k}'}$-bimodules is equivalent to $\mathcal{A}_{\mathbf{k}}^{fd} \otimes_{\mathbb{C}} \mathcal{A}_{\mathbf{k}'}^{fd} - \mathrm{mod}$.*

Proof. We identify $H_{\mathbf{k}} - H_{\mathbf{k}'}$-bimodules with left $H_{\mathbf{k}} \otimes_{\mathbb{C}} H_{\mathbf{k}'}^{op}$-modules and then with $H_{\mathbf{k}}(\mathfrak{h}) \otimes_{\mathbb{C}} H_{\mathbf{k}'}(\mathfrak{h}^*)$-modules. Let $P_{\mathbf{k}}(\mathfrak{h})$ be a progenerator of $H_{\mathbf{k}}(\mathfrak{h}) - \mathrm{fin}$ with $\mathrm{End}_{\mathcal{O}_{\mathbf{k}}(\mathfrak{h})}(P_{\mathbf{k}}(\mathfrak{h}))^{op} \cong \mathcal{A}_{\mathbf{k}}$ and $P_{\mathbf{k}'}(\mathfrak{h}^*)$ a progenerator of $H_{\mathbf{k}}(\mathfrak{h}^*) - \mathrm{fin}$ with $\mathrm{End}_{\mathcal{O}_{\mathbf{k}'}(\mathfrak{h}^*)}(P_{\mathbf{k}'}(\mathfrak{h}^*))^{op} \cong \mathcal{A}_{\mathbf{k}}$. Then $P_{\mathbf{k}}(\mathfrak{h}) \otimes_{\mathbb{C}} P_{\mathbf{k}'}(\mathfrak{h}^*)$ is a progenerator of the category of finite-dimensional $H_{\mathbf{k}}(\mathfrak{h}) \otimes_{\mathbb{C}} H_{\mathbf{k}'}(\mathfrak{h}^*)$-modules and we have

$$\mathrm{End}_{H_{\mathbf{k}}(\mathfrak{h}) \otimes_{\mathbb{C}} H_{\mathbf{k}'}(\mathfrak{h}^*) - \mathrm{mod}}(P_{\mathbf{k}}(\mathfrak{h}) \otimes_{\mathbb{C}} P_{\mathbf{k}'}(\mathfrak{h}^*)) \cong \mathcal{A}_{\mathbf{k}}^{fd} \otimes_{\mathbb{C}} \mathcal{A}_{\mathbf{k}'}^{fd}$$

and the result follows from Theorem 4.1.14. $\square$

Definition 6.5.5. *Let Q^1, Q^2 be finite quivers with vertices Q_0^i and arrows Q_1^i and $i \in \{1, 2\}$. We define a quiver $Q^1 \otimes Q^2$ by setting:*

$$\begin{aligned} (Q^1 \otimes Q^2)_0 &:= Q_0^1 \times Q_0^2 \text{ and} \\ (Q^1 \otimes Q^2)_1 &:= Q_1^1 \times Q_0^2 \cup Q_0^1 \times Q_1^2. \end{aligned}$$

Further we define the head and tail maps h and t respectively as follows:

$$\begin{aligned} h : (Q^1 \otimes Q^2)_1 &\to (Q^1 \otimes Q^2)_0 \\ (a, e_2) &\mapsto (ha, e_2) \\ (e_1, b) &\mapsto (e_1, hb) \\ &\text{and} \\ t : (Q^1 \otimes Q^2)_1 &\to (Q^1 \otimes Q^2)_0 \\ (a, e_2) &\mapsto (ta, e_2) \\ (e_1, b) &\mapsto (e_1, tb) \end{aligned}$$

Finally, let us introduce a two-sided ideal S of $Q^1 \otimes Q^2$:

$$S := \langle \{ (ha, b)(a, tb) - (a, hb)(ta, b) \mid a \in Q_1^1, b \in Q_1^2 \} \rangle$$

and we will refer to this as the "commuting squares relations".

Example 6.5.6. *For the type A quiver*

$$Q := \circ \longrightarrow \circ \longrightarrow \circ$$

the quiver $Q \otimes Q$ is given by

$$\begin{tikzcd} \circ \arrow[r] \arrow[d] & \circ \arrow[r] \arrow[d] & \circ \arrow[d] . \\ \circ \arrow[r] \arrow[d] & \circ \arrow[r] \arrow[d] & \circ \arrow[d] \\ \circ \arrow[r] & \circ \arrow[r] & \circ \end{tikzcd}$$

The relations S then just mean that each square commutes.

The following are well-known results regarding tensor products of quiver algebras and a less verbose exposition can for example be found in [Les94] whose approach we follow:

Proposition 6.5.7. *Let Q^1, Q^2 be finite quivers. Then the map*

$$\begin{aligned} \phi : \mathbb{C}(Q^1 \otimes Q^2) &\to \mathbb{C}Q^1 \otimes_{\mathbb{C}} \mathbb{C}Q^2 \\ (e_1, e_2) &\mapsto e_1 \otimes e_2 \\ (a, e_2) &\mapsto a \otimes e_2 \\ (e_1, b) &\mapsto e_1 \otimes b \\ a \in Q_1^1, b \in &Q_1^2, e_i \in Q_0^i \end{aligned}$$

is surjective with kernel S. In particular it descends to an isomorphism

$$\mathbb{C}(Q^1 \otimes Q^2)/S \cong \mathbb{C}Q^1 \otimes_{\mathbb{C}} \mathbb{C}Q^2.$$

Proof. Step 1: ϕ is well-defined. Suppose that p and q are non-zero paths in $Q^1 \otimes Q^2$ such that $pq = 0$. We need to prove that $\phi(pq) = 0$. Let $(e_1, e_2) \in Q_0^1 \times Q_0^2$ be the head of q and $(f_1, f_2) \in Q_0^1 \times Q_0^2$ the tail of p. Since we have not imposed any relations yet, we see that $pq = 0 \implies (e_1, e_2) \neq (f_1, f_2)$. Now $\phi(q) = q_1 \otimes q_2$ and $\phi(p) = p_1 \otimes p_2$ where q_i, p_i are paths in Q^i with heads e_i and tails f_i respectively. As $(e_1, e_2) \neq (f_1, f_2)$ one of $f_1 e_1$ or $f_2 e_2$ will equal zero and thus $\phi(pq) = p_1 q_1 \otimes p_2 q_2 = 0$. Now suppose that p_r and q_r are non-zero paths in $Q^1 \otimes Q^2$ such that $\sum_r \lambda_r p_r q_r = 0$ with the λ_r being scalars. Again we need to prove that $\sum_r \lambda_r \phi(p_r q_r) = 0$. By multiplying with vertex idempotents, we may assume that each p_r has the same head and each q_r has the same tail, so that all compositions $p_r q_r$ are paths between the same vertices. Since we have no relations, different paths between identical vertices are linearly independent, and thus $\sum_r \lambda_r p_r q_r = 0$ implies that $p_r q_r = 0$ whenever $\lambda_r \neq 0$ for all r. We have already shown that $p_r q_r = 0 \implies \phi(p_r q_r) = 0$ and so $\sum_r \lambda_r \phi(p_r q_r) = 0$.

Step 2: ϕ is surjective. A $\mathbb{C}$-basis of $\mathbb{C}Q^1 \otimes_{\mathbb{C}} \mathbb{C}Q^2$ is given by vectors of the form $p \otimes q$ with p a path in Q^1 and q a path in Q^2. Writing these in terms of arrows we have

$$p = u_n u_{n-1} \ldots u_2 u_1 \text{ and } q = v_k v_{k-1} \ldots v_2 v_1.$$

Without loss of generality, let us assume that $n \geq k$. Now consider the element

$$\alpha := (u_n, hv_k) \ldots (u_{k+1}, hv_k)(u_k, hv_k)(tu_k, v_k) \ldots (u_2, hv_2)(tu_2, v_2)(u_1, hv_1)(tu_1, v_1).$$

We find that

$$\phi(\alpha) = (u_n u_{n-1} \ldots u_2 u_1) \otimes (v_k v_{k-1} \ldots v_2 v_1) = p \otimes q$$

and hence ϕ is surjective.

Step 3: $\ker \phi = S$. It is easy to check that $S \subseteq \ker \phi$, so we need to show the reverse inclusion. This will require some more work. Let us call a path $p = u_n u_{n-1} \ldots u_2 u_1$ in $Q^1 \otimes Q^2$ *nicely ordered* if there is $i \in \{1, \ldots, n\}$ such that all arrows $u_i, u_{i-1}, \ldots, u_1$ are of the form $(a, e) \in Q_1^1 \otimes Q_0^2$ and all arrows $u_n, u_{n-1}, \ldots, u_{i+1}$ are of the form $(f, b) \in Q_0^1 \otimes Q_1^2$. Now let us suppose that we know the following:

1. For each path p there exists a unique nicely ordered path p^o such that $p - p^o \in S$.

2. $\phi(p) = \phi(p^o)$.
3. $\phi(p) = \phi(q)$ for two paths p, q if and only if $p^o = q^o$.

Given these facts, we can argue as follows: Let $\sum_r \lambda_r p_r \in \ker\phi$ be given. By pre- and postmultiplying with suitable vertex idempotents we may suppose that any two paths p_r have the same tails and heads. Passing to ordered paths we obtain $\sum_r \lambda_r p_r - \sum_r \lambda_r p_r^o \in S$ and $\sum_r \lambda_r p_r^o \in \ker\phi$. Let q_l denote the distinct ordered paths that appear among the p_r^o, so that $\sum_r \lambda_r p_r^o = \sum_l \mu_l q_l$ with $\mu_l = \sum_{p_r^o = q_l} \lambda_r$. As all q_l are distinct, their images under ϕ are distinct as well and of the form (path) $\otimes$ (path) and hence linearly independent. Thus $\mu_l = 0$ for all l. Thus $0 = \sum_l \mu_l q_l = \sum_r \lambda_r p_r^o$ and hence $\sum_r \lambda_r p_r \in S$. Thus $\ker\phi = S$.

So the proof of injectivity is reduced to a proof of the statements (1)-(3). Statement (2) is clear as $S \subseteq \ker\phi$ and so we must prove the first and third only.
To prove the first statement, for a path p we define

$$i(p) := \begin{cases} \max\{i \mid u_i \in Q_0^1 \otimes Q_1^2 \text{ and } u_{i+1} \in Q_1^1 \otimes Q_0^2\} \text{ if such an } i \text{ exists} \\ 0 \text{ else} \end{cases}$$

i.e. $i(p)$ denotes the "last arrow" in p which is "out of order". In particular, a path is nicely ordered if and only if $i(p) = 0$. To prove the existence of p^o, we will induct on $i(p)$.

As noted before, the statement is vacuously true for $i(p) = 0$ so let us assume that $i(p) = 1$. Then if $p = u_n u_{n-1} \dots u_2 u_1$ we have $u_1 = (e, b)$ and there is $2 \le k \le n$ with $u_2, u_3, \dots, u_k \in Q_1^1 \otimes Q_0^2$ and $u_{k+1}, \dots, u_n \in Q_0^1 \otimes Q_1^2$. Let us consider the sub-path

$$p_- = u_k \dots u_2 u_1$$

and set

$$u_2 = (a_2, f), u_3 = (a_3, f), \dots, u_k = (a_k, f).$$

We then have

$$\begin{aligned}
&(a_k, f)(a_{k-1}, f) \dots (a_2, f)(e, b) - (ha_k, b)(a_k, tb)(a_{k-1}, tb) \dots (a_2, tb) \\
&= (a_k, f)(a_{k-1}, f) \dots (a_2, f)(e, b) - (a_k, f)(a_{k-1}, f) \dots (ha_2, b)(a_2, tb) \\
&+ (a_k, f)(a_{k-1}, f) \dots (a_3, f)(ha_2, b)(a_2, tb) - (a_k, f)(a_{k-1}, f) \dots (ha_3, b)(a_3, tb)(a_2, tb) \\
&\dots \\
&+ (ha_k, b)(a_k, tb)(a_{k-1}, tb) \dots (a_2, tb) - (ha_k, b)(a_k, tb)(a_{k-1}, tb) \dots (a_2, tb) \\
&\in S
\end{aligned}$$

and thus taking $p_-^o = (ha_k, b)(a_k, tb)(a_{k-1}, tb) \dots (a_2, tb)$ we deduce that

$$p^o = u_n \dots u_{k+1} p_-^o$$

is a nicely ordered path such that $p - p^o \in S$ (as S is a two-sided ideal). Now we need to turn to the case that $i(p) > 1$.

Suppose that $i(p) = k > 1$ and that $p = u_n u_{n-1} \dots u_2 u_1$. We can consider the two sub-paths

$$p_+ = u_n u_{n-1} \dots u_{k+1} u_k \text{ and } p_- = u_{k-1} \dots u_1.$$

Then $i(p_+) = 1$ and $i(p_-) \leq k-2$ by construction. So by the inductive hypothesis, we have nicely ordered paths p_-^o and p_+^o such that $p_- - p_-^o \in S$ and $p_+ - p_+^o \in S$ and thus $p - p_+^o p_-^o \in S$. So let us consider the path $p_+^o p_-^o$. By construction,

$$i(p_+^o p_-^o) \leq k-1$$

and so we may apply the inductive hypothesis again to obtain a path p^o such that $p_+^o p_-^o - p^o \in S$. Clearly the also $p - p^o$ and p^o is nicely ordered.

Hence we have shown the existence of a nicely ordered path as claimed. We will return to uniqueness a bit later. To show the third statement, note that the implication $p^o = q^o \implies \phi(p) = \phi(q)$ holds true due to the second statement. So we need to show that if $\phi(p) = \phi(q)$, then $p^o = q^o$. Now supposing that $p = u_n u_{n-1} \dots u_2 u_1$ is a nicely ordered path with

$$\phi(p) = (a_i a_{i-1} \dots a_1) \otimes (b_n \dots b_{i+1})$$

we see that we must have $u_i = (a_i, e), u_{i-1} = (a_{i-1}, e), \dots, u_1 = (a_1, e) \in Q_1^1 \times Q_0^2$ and $u_n = (f, b_n), \dots, u_{i+1} = (f, b_{i+1}) \in Q_0^2 \times Q_1^2$ and thus

$$p = (f, b_n) \dots (f, b_{i+1})(a_i, e) \dots (a_1, e).$$

Thus a nicely ordered path is uniquely determined by its image under ϕ and the third statement follows.

Finally we need to show uniqueness of p^o for a given path p. This follows from the third statement though, as p uniquely determines $\phi(p)$. □

Lemma 6.5.8. *Let U, V be complex vector spaces and $N \leq U$ a subspace. Then the kernel of the natural surjection*

$$U \otimes_{\mathbb{C}} V \twoheadrightarrow (U/N) \otimes_{\mathbb{C}} V$$

is $N \otimes_{\mathbb{C}} V \subseteq U \otimes_{\mathbb{C}} V$.

Proof. We consider the short exact sequence $0 \to N \to U \to U/N \to 0$ and using the exactness of $- \otimes_{\mathbb{C}} V$ we see that

$$0 \to N \otimes_{\mathbb{C}} V \to U \otimes_{\mathbb{C}} V \to (U/N) \otimes_{\mathbb{C}} V \to 0$$

is exact too. □

Proposition 6.5.9. *Let Q^1, Q^2 be finite quivers and let $I^i \in \mathbb{C}Q^i$ be admissible ideals of relations, that is an ideal generated by paths of length at least 2. We set*

$$I^{(1,2)} := \langle (I^1 \times Q_1^2) \cup (Q_1^1 \times I^2) \cup S \rangle$$

the two-sided ideal in $\mathbb{C}(Q^1 \otimes Q^2)$ generated by $(I^1 \times Q_1^2) \cup (Q_1^1 \times I^2)$ and the commuting squares relations. Then

$$\mathbb{C}Q^1/I^1 \otimes_{\mathbb{C}} \mathbb{C}Q^2/I^2 \cong \mathbb{C}(Q^1 \otimes Q^2)/I^{(1,2)}.$$

Proof. To start, let us note that we have already dealt with the case of $I^1 = I^2 = 0$ in the previous Proposition 6.5.7. We shall again consider the map

$$\phi : \mathbb{C}(Q^1 \otimes Q^2) \to \mathbb{C}Q^1 \otimes_{\mathbb{C}} \mathbb{C}Q^2$$

introduced there.

First, let us only suppose that $I^2 = 0$. From Lemma 6.5.8 we know that the kernel of the surjection

$$\mathbb{C}Q^1 \otimes_{\mathbb{C}} \mathbb{C}Q^2 \twoheadrightarrow \mathbb{C}(Q^1/I^1) \otimes_{\mathbb{C}} \mathbb{C}Q^2$$

is just $I^1 \otimes_C \mathbb{C}Q^2$. Under the isomorphism $\mathbb{C}(Q^1 \otimes Q^2)/S \cong \mathbb{C}Q^1 \otimes_{\mathbb{C}} \mathbb{C}Q^2$ induced by ϕ, the space $I^1 \otimes_{\mathbb{C}} \mathbb{C}Q^2$ is mapped to the ideal $(I^1 \times Q_1^2 \cup S)/S \trianglelefteq \mathbb{C}(Q^1 \otimes Q^2)/S$ and hence we obtain an isomorphism

$$\mathbb{C}(Q^1 \otimes Q^2)/(I^1 \cup S) \cong (\mathbb{C}Q^1/I^1) \otimes_C \mathbb{C}Q^2$$

Now to deal with the general case, note that the surjection

$$\mathbb{C}(Q^1 \otimes Q^2)/S \twoheadrightarrow \mathbb{C}(Q^1/I^1) \otimes_{\mathbb{C}} \mathbb{C}(Q^2/I^2)$$

factors as

$$\mathbb{C}(Q^1 \otimes Q^2)/S \twoheadrightarrow \mathbb{C}(Q^1/I^1) \otimes_{\mathbb{C}} \mathbb{C}Q^2 \twoheadrightarrow \mathbb{C}(Q^1/I^1) \otimes_{\mathbb{C}} \mathbb{C}(Q^2/I^2)$$

and arguing as before and aplying applying Lemma 6.5.8 to this composition we obtain our desired result. $\square$

Hence we can now compute a presentation of $\mathcal{A}_{\mathbf{k}}^{fd} \otimes_{\mathbb{C}} \mathcal{A}_{\mathbf{k}'}^{fd}$ as a quiver algebra with relations for presentations of $\mathcal{A}_{\mathbf{k}}^{fd}$ and $\mathcal{A}_{\mathbf{k}'}^{fd}$. Specialising for the time being to the case of $\mathcal{A}^{fd}(\mathcal{B})_{\mathbf{k}} \otimes_{\mathbb{C}} \mathcal{A}^{fd}(\mathcal{B})_{\mathbf{k}}$, we see that for $\mathcal{B}$ of length n, this is given by the quiver algebra of the square quiver

$$\begin{tikzcd}
\lambda_2 \arrow[r, bend left, "\bar a_2"] \arrow[d, bend left] & \lambda_3 \arrow[l, bend left, "\bar b_2"] \arrow[r, bend left, "\bar a_3"] \arrow[d, bend left] & {} \arrow[l, bend left, "\bar b_3"] \arrow[d, bend left] & \cdots & {} \arrow[r, bend left, "\bar a_{n-2}"] \arrow[d, bend left] & \lambda_{n-1} \arrow[l, bend left, "\bar b_{n-2}"] \arrow[r, bend left, "\bar a_{n-1}"] \arrow[d, bend left] & \lambda_n \arrow[l, bend left, "\bar b_{n-1}"] \arrow[d, bend left] \\
\lambda_2 \arrow[r, bend left, "\bar a_2"] \arrow[d, bend left] \arrow[u, bend left] & \lambda_3 \arrow[l, bend left, "\bar b_2"] \arrow[r, bend left, "\bar a_3"] \arrow[d, bend left] \arrow[u, bend left] & {} \arrow[l, bend left, "\bar b_3"] \arrow[d, bend left] \arrow[u, bend left] & \cdots & {} \arrow[r, bend left, "\bar a_{n-2}"] \arrow[d, bend left] \arrow[u, bend left] & \lambda_{n-1} \arrow[l, bend left, "\bar b_{n-2}"] \arrow[r, bend left, "\bar a_{n-1}"] \arrow[d, bend left] \arrow[u, bend left] & \lambda_n \arrow[l, bend left, "\bar b_{n-1}"] \arrow[d, bend left] \arrow[u, bend left] \\
\vdots \arrow[r, bend left, "\bar a_2"] \arrow[d, bend left] \arrow[u, bend left] & \vdots \arrow[l, bend left, "\bar b_2"] \arrow[r, bend left, "\bar a_3"] \arrow[d, bend left] \arrow[u, bend left] & {} \arrow[l, bend left, "\bar b_3"] \arrow[d, bend left] \arrow[u, bend left] & \cdots & {} \arrow[r, bend left, "\bar a_{n-2}"] \arrow[d, bend left] \arrow[u, bend left] & \vdots \arrow[l, bend left, "\bar b_{n-2}"] \arrow[r, bend left, "\bar a_{n-1}"] \arrow[d, bend left] \arrow[u, bend left] & \vdots \arrow[l, bend left, "\bar b_{n-1}"] \arrow[d, bend left] \arrow[u, bend left] \\
\lambda_2 \arrow[r, bend left, "\bar a_2"] \arrow[d, bend left] \arrow[u, bend left] & \lambda_3 \arrow[l, bend left, "\bar b_2"] \arrow[r, bend left, "\bar a_3"] \arrow[d, bend left] \arrow[u, bend left] & {} \arrow[l, bend left, "\bar b_3"] \arrow[d, bend left] \arrow[u, bend left] & \cdots & {} \arrow[r, bend left, "\bar a_{n-2}"] \arrow[d, bend left] \arrow[u, bend left] & \lambda_{n-1} \arrow[l, bend left, "\bar b_{n-2}"] \arrow[r, bend left, "\bar a_{n-1}"] \arrow[d, bend left] \arrow[u, bend left] & \lambda_n \arrow[l, bend left, "\bar b_{n-1}"] \arrow[d, bend left] \arrow[u, bend left] \\
\lambda_2 \arrow[r, bend left, "\bar a_2"] \arrow[u, bend left] & \lambda_3 \arrow[l, bend left, "\bar b_2"] \arrow[r, bend left, "\bar a_3"] \arrow[u, bend left] & {} \arrow[l, bend left, "\bar b_3"] \arrow[u, bend left] & \cdots & {} \arrow[r, bend left, "\bar a_{n-2}"] \arrow[u, bend left] & \lambda_{n-1} \arrow[l, bend left, "\bar b_{n-2}"] \arrow[r, bend left, "\bar a_{n-1}"] \arrow[u, bend left] & \lambda_n \arrow[l, bend left, "\bar b_{n-1}"] \arrow[u, bend left]
\end{tikzcd}$$

Definition 6.5.10. *We define the finite block length of a choice of parameters* $\mathbf{k}$ *and the* *corresponding category* $\mathcal{O}_{\mathbf{k}}$ *as follows:*

$$\mathrm{fbl}(\mathbf{k}) := \mathrm{fbl}(\mathcal{O}_{\mathbf{k}}) := \max_{\mathcal{B} \text{ a block of } \mathcal{O}_{\mathbf{k}}} \{\mathrm{length}(\mathrm{fd}(\mathcal{B}))\}.$$

In words, it is the maximal length of a block of $\mathrm{fd}(\mathcal{O}_{\mathbf{k}})$.

Example 6.5.11. *The parameter* $\mathbf{k}$ *is regular if and only if* $\mathrm{fbl}(\mathbf{k}) = 0$.

Example 6.5.12. *If* $W = \mathbb{Z}/2\mathbb{Z}$ *then*

$$\mathrm{fbl}((0, k_1)) = \begin{cases} 0 \text{ if } k_1 \notin \frac{1}{2} + \mathbb{Z} \\ 1 \text{ if } k_1 \in \frac{1}{2} + \mathbb{Z} \end{cases}$$

Lemma 6.5.13. *Let* $\mathbf{k}$ *be a choice of parameters, we have an equality*

$$\mathrm{fbl}(\mathcal{O}_{\mathbf{k}}(\mathfrak{h})) = \mathrm{fbl}(\mathcal{O}_{\mathbf{k}}(\mathfrak{h}^*)).$$

Proof. We have an equivalence of categories

$$^\vee : \mathcal{O}_{\mathbf{k}}(\mathfrak{h}) \xrightarrow{\sim} \mathcal{O}_{\mathbf{k}}^{op}(\mathfrak{h}^*)$$

and since $1+\mathrm{fbl}(\mathcal{O}_{\mathbf{k}}(\mathfrak{h}))$ is equal to the maximal length of a block of $\mathcal{O}_{\mathbf{k}}(\mathfrak{h})$ the result follows. □

6.6 Tame and Wild

Definition 6.6.1. *A finite-dimensional* $\mathbb{F}$*-algebra (*$\mathbb{F}$ *algebraically closed) A is called*

1. *representation-finite or of finite type, if there is only a finite number of isomorphism classes of indecomposable A-modules;*
2. *representation-infinite or of infinite type if it is not of finite type;*
3. *tame, if for each integer* $d > 0$ *there are finitely many* $A - \mathbb{F}[u]$*-bimodules* $B_1^{(d)}, \ldots, B_{k(d)}^{(d)}$ *that are free when viewed as right* $\mathbb{F}[u]$*-modules and all but finitely many indecomposable A-modules of dimension d are of the form* $B_i \otimes S$ *where S is a simple* $\mathbb{F}[u]$*-module;*
4. *wild, if there is a finitely generated* $A - \mathbb{F}\langle u, v\rangle$*-bimodule Q, again free as a right* $\mathbb{F}\langle u, v\rangle$*-module, such that the associated tensor functor* $\mathbb{F}\langle u, v\rangle - \mathrm{mod} \to A - \mathrm{mod}$ *preserves isomorphism classes and indecomposability.*

So the isomorphism classes of tame algebras can be classified using at most one continuous parameter, whereas classifying the isomorphism classes of indecomposable modules of a wild algebra requires at least two continuous parameters (and is in a sense "impossible", see Proposition 6.6.3).

Note that many authors also require a tame algebra not to be of finite representation type, whereas our definition of tame includes the finite case.

The following theorem of Drozd underscores the importance of the concepts of tame and wild in the representation theory of finite-dimensional algebras:

Theorem 6.6.2. *([Dro80]) A finite-dimensional algebra over an algebraically closed field is either tame or wild and cannot be both.*

The rather flippant remark that classifying indecomposables of a wild algebra is impossible can be illustrated with the following well-known proposition:

Proposition 6.6.3. *Let A be a finite-dimensional wild algebra over an algebraically closed field and B any finite-dimensional algebra. Then there exists a functor $F : B - \mathrm{mod} \to A - \mathrm{mod}$ such that F preserves indecomposability and isomorphism classes. Hence F embeds the indecomposable B-modules in the category $A - \mathrm{mod}$*

Proof. This is part (a) of Corollary 1.6 of [SS10]. □

Thus, classifying indecomposables of a wild algebra means classifying indecomposables for arbitrary finite-dimensional algebras.
We will need the following general and well-known properties of tame and wild algebras:

Lemma 6.6.4. *Let A be a finite-dimensional $\mathbb{F}$-algebra.*

1. *If B is a homomorphic image of A and A is tame, then so is B.*
2. *If B is a homomorphic image of A and B is wild, then so is A.*
3. *If $e \in A$ is an idempotent and $B = eAe$ and B is wild, then so is A.*

Proof. The first two statements are equivalent and the second statement is $I.4.7(a)$ in [Erd90]. The second statement is $I.4.7(b)$ in [Erd90]. □

Also notable is the following well-known fact

Proposition 6.6.5. *Let A be a finite-dimensional local algebra over an algebraically closed field. If the minimal number of generators of A is at least 3, then A is wild.*

Proof. This follows for example from Ringel's list of tame and wild local algebras in [Rin75] or can be found in [Erd90], see $I.10.10(a)$. □

Using our knowledge of $\mathcal{A}_{\mathbf{k}}^{fd}$ we can prove the following:

Lemma 6.6.6. *Let $\mathbf{k}$ be a choice of parameters such that $\mathrm{fbl}(\mathbf{k}) \leq 2$. Then $\mathrm{fd}(\mathcal{HC}_{\mathbf{k}})$ is a tame category.*

Proof. Realising $\mathcal{A}_{\mathbf{k}}^{fd} \otimes_{\mathbb{C}} \mathcal{A}_{\mathbf{k}}^{fd}$ as a quiver algebra with relations as described above, our hypothesis implies that each of the finitely many connected component is either a single vertex or it is of the form

$$\begin{tikzcd} \bullet_1 \arrow[r, bend left, "u"] & \bullet_2 \arrow[l, bend left, "v"] \end{tikzcd}$$

with relations

$$uv = vu = 0$$

or of the form

$$\begin{tikzcd} \bullet_1 \arrow[r, bend left, "u_1"] \arrow[d, bend left, "v_4"] & \bullet_2 \arrow[l, bend left, "v_1"] \arrow[d, bend left, "u_2"] \\ \bullet_4 \arrow[u, bend left, "u_4"] \arrow[r, bend left, "v_3"] & \bullet_3 \arrow[l, bend left, "u_3"] \arrow[u, bend left, "v_2"] \end{tikzcd}$$

with relations

$$u_i v_i = v_i u_i = 0$$

and all possible squares commute, meaning

$$u_2 u_1 = v_3 v_4, u_3 u_2 = v_4 v_1, u_4 u_3 = v_1 v_2, u_1 u_4 = v_2 v_3.$$

It will suffice to show that each of these individually are tame. Clearly the case of a block consisting of a single vertex is trivial, so let us turn to the second case. Let us consider the cyclic group of order 2, $\mathbb{Z}_2 = \{1, s\}$, and the group idempotents $\mathbf{e}_1 = \frac{1}{2}(1-s), \mathbf{e}_2 = \frac{1}{2}(1-s)$. There is an action of $\mathbb{Z}_2$ on the polynomial algebra $\mathbb{C}[U]$ given by $s(U) = -U$.Thus we can form the smash product $\mathbb{C}[U,V]\#\mathbb{Z}_2$ and we have a surjection

$$\mathbb{C}[U]\#\mathbb{Z}_2 \twoheadrightarrow \mathbb{C}(\, \bullet \underset{v}{\overset{u}{\rightleftarrows}} \bullet \,)/(\text{relations})$$

given by

$$\begin{aligned} \mathbf{e}_2 U \mathbf{e}_1 &\mapsto u \\ \mathbf{e}_1 U \mathbf{e}_2 &\mapsto v \\ \mathbf{e}_i &\mapsto e_i. \end{aligned}$$

The relations of the quiver then imply that in fact U^2 lies in the kernel of this map and so we descend to an epimorphism

$$\frac{\mathbb{C}[U]}{(U^2)}\#\mathbb{Z}_2 \twoheadrightarrow \mathbb{C}(\, \bullet \underset{v}{\overset{u}{\rightleftarrows}} \bullet \,)/(\text{relations})$$

which is in fact an isomorphism. Since $\frac{\mathbb{C}[U]}{(U^2)}$ is tame and $\mathbb{C}\mathbb{Z}_2$ is semisimple, so is the image of the smash poduct.

So now let us suppose that we are in the third case. Again we will construct an epimorphism from a smash product onto the algebra of the quiver with relations. In this case, we will consider $\mathbb{C}\langle U, V\rangle\#\mathbb{Z}_4$. Here $\mathbb{Z}_4 = \{1, s, s^2, s^3\}$ is the cyclic group of order 4 and we denote its basis of idempotents by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$. An action of $\mathbb{Z}_4$ on $\mathbb{C}\langle U, V\rangle$ is given by $s(U) = iU$ and $s(V) = -iV$. The required surjection now is given by

$$\begin{aligned} \mathbf{e}_2 U \mathbf{e}_1 &\mapsto u_1, \quad \mathbf{e}_1 V \mathbf{e}_2 \mapsto v_1 \\ \mathbf{e}_3 U \mathbf{e}_2 &\mapsto u_2, \quad \mathbf{e}_2 V \mathbf{e}_3 \mapsto v_2 \\ \mathbf{e}_4 U \mathbf{e}_3 &\mapsto u_3, \quad \mathbf{e}_3 V \mathbf{e}_4 \mapsto v_3 \\ \mathbf{e}_1 U \mathbf{e}_4 &\mapsto u_4, \quad \mathbf{e}_4 V \mathbf{e}_1 \mapsto v_4 \\ \mathbf{e}_i &\mapsto e_i \end{aligned}$$

Under this map we see that

$$\begin{aligned} UV &= (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4)U(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4)V(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4) \\ &= (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4)U\,(\mathbf{e}_1 V \mathbf{e}_2 + \mathbf{e}_2 V \mathbf{e}_3 + \mathbf{e}_3 V \mathbf{e}_4 + \mathbf{e}_4 V \mathbf{e}_1) \\ &= \mathbf{e}_2 U \mathbf{e}_1 V \mathbf{e}_2 + \mathbf{e}_3 U \mathbf{e}_2 V \mathbf{e}_3 + \mathbf{e}_4 U \mathbf{e}_3 V \mathbf{e}_4 + \mathbf{e}_1 U \mathbf{e}_4 V \mathbf{e}_1 \\ &\mapsto u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4 = 0 \end{aligned}$$

and similarly $VU \mapsto 0$. Thus this descends to an epimorphism

$$\frac{\mathbb{C}[U,V]}{(UV)}\#\mathbb{Z}_4 \twoheadrightarrow \mathbb{C}(\text{quiver})/(\text{relations}).$$

Further, we can see that $U^2 - V^2$ lies in the kernel of this map:

$$\begin{aligned}
U^2 - V^2 &= \mathbf{e}_1 U^2 \mathbf{e}_3 + \mathbf{e}_2 U^2 \mathbf{e}_4 + \mathbf{e}_3 U^2 \mathbf{e}_1 + \mathbf{e}_4 U^2 \mathbf{e}_2 - (\mathbf{e}_1 V^2 \mathbf{e}_3 + \mathbf{e}_2 V^2 \mathbf{e}_4 + \mathbf{e}_3 V^2 \mathbf{e}_1 + \mathbf{e}_4 V^2 \mathbf{e}_2) \\
&\mapsto u_4u_3 + u_1u_4 + u_2u_1 + u_3u_2 - (v_1v_2 + v_2v_3 + v_3v_4 + v_4v_1) \\
&= (u_4u_3 - v_1v_2) + (u_1u_4 - v_2v_3) + (u_2u_1 - v_3v_4) + (u_3u_2 - v_4v_1) \\
&= 0
\end{aligned}$$

since all squares commute. Thus we obtain an epimorphism

$$\frac{\mathbb{C}[U,V]}{(UV, U^2 - V^2)} \# \mathbb{Z}_4 \twoheadrightarrow \mathbb{C}(\text{quiver})/(\text{relations}).$$

Now the algebra $\frac{\mathbb{C}[U,V]}{(UV,U^2-V^2)}$ is known to be tame by [Rin75] and thus again is the quiver algebra with relations. $\square$

Proposition 6.6.7. *Let* $\mathbf{k}$ *be a choice of parameters such that* $\mathrm{fbl}(\mathbf{k}) \geq 3$. *Then* $\mathrm{fd}(\mathcal{HC}_{\mathbf{k}})$ *is a wild category.*

Proof. We will need to analyse the quiver $Q^{fd}_{\mathcal{HC}_{\mathbf{k}}}$ of $\mathcal{A}^{fd}_{\mathbf{k}} \otimes \mathcal{A}^{fd}_{\mathbf{k}^\dagger}$. For a suitable $n \geq 3$, this quiver will have a connected component $\bar{Q}^{fd}_{\mathcal{HC}_{\mathbf{k}}}$ of the shape

$$\begin{tikzcd}
\lambda_2 \arrow[r, bend left, "a_2"] \arrow[d, bend left] & \lambda_3 \arrow[l, bend left, "b_2"] \arrow[r, bend left, "a_3"] \arrow[d, bend left] & {} \arrow[l, bend left, "b_3"] \arrow[d, bend left] & \cdots & \lambda_{n-1} \arrow[r, bend left, "a_{n-1}"] \arrow[d, bend left] & \lambda_n \arrow[l, bend left, "b_{n-1}"] \arrow[d, bend left] \\
\lambda_2 \arrow[u, bend left] \arrow[r, bend left, "a_2"] \arrow[d, bend left] & \lambda_3 \arrow[u, bend left] \arrow[l, bend left, "b_2"] \arrow[r, bend left, "a_3"] \arrow[d, bend left] & {} \arrow[u, bend left] \arrow[l, bend left, "b_3"] \arrow[d, bend left] & \cdots & \lambda_{n-1} \arrow[u, bend left] \arrow[r, bend left, "a_{n-1}"] \arrow[d, bend left] & \lambda_n \arrow[u, bend left] \arrow[l, bend left, "b_{n-1}"] \arrow[d, bend left] \\
\vdots \arrow[u, bend left] \arrow[r, bend left, "a_2"] \arrow[d, bend left] & \vdots \arrow[u, bend left] \arrow[l, bend left, "b_2"] \arrow[r, bend left, "a_3"] \arrow[d, bend left] & {} \arrow[u, bend left] \arrow[l, bend left, "b_3"] \arrow[d, bend left] & \cdots & \vdots \arrow[u, bend left] \arrow[r, bend left, "a_{n-1}"] \arrow[d, bend left] & \vdots \arrow[u, bend left] \arrow[l, bend left, "b_{n-1}"] \arrow[d, bend left] \\
\lambda_2 \arrow[u, bend left] \arrow[r, bend left, "a_2"] & \lambda_3 \arrow[u, bend left] \arrow[l, bend left, "b_2"] \arrow[r, bend left, "a_3"] & {} \arrow[u, bend left] \arrow[l, bend left, "b_3"] & \cdots & \lambda_{n-1} \arrow[u, bend left] \arrow[r, bend left, "a_{n-1}"] & \lambda_n \arrow[u, bend left] \arrow[l, bend left, "b_{n-1}"]
\end{tikzcd}$$

We will first deal with the case of $n = 4$ and our quiver being

$$\begin{tikzcd}
(2,2) \arrow[r, bend left, "{(a_2,e_2)}"] \arrow[d, bend left, "{(e_2,a_2)}"] & \boxed{(3,2)} \arrow[l, bend left, "{(b_2,e_2)}"] \arrow[r, bend left, "{(a_3,e_2)}"] \arrow[d, bend left, "{(e_3,a_2)}"] & (4,2) \arrow[l, bend left, "{(b_3,e_2)}"] \arrow[d, bend left, "{(e_4,a_2)}"] \\
\boxed{(2,3)} \arrow[u, bend left, "{(e_2,b_2)}"] \arrow[r, bend left, "{(a_2,e_3)}"] \arrow[d, bend left, "{(e_2,a_3)}"] & (3,3) \arrow[u, bend left, "{(e_3,b_2)}"] \arrow[l, bend left, "{(b_2,e_3)}"] \arrow[r, bend left, "{(a_3,e_3)}"] \arrow[d, bend left, "{(e_3,a_3)}"] & \boxed{(4,3)} \arrow[u, bend left, "{(e_4,b_2)}"] \arrow[l, bend left, "{(b_3,e_3)}"] \arrow[d, bend left, "{(e_4,a_3)}"] \\
(2,4) \arrow[u, bend left, "{(e_2,b_3)}"] \arrow[r, bend left, "{(a_2,e_4)}"] & \boxed{(3,4)} \arrow[u, bend left, "{(e_3,b_3)}"] \arrow[l, bend left, "{(b_2,e_4)}"] \arrow[r, bend left, "{(a_3,e_4)}"] & (4,3) \arrow[u, bend left, "{(e_4,b_3)}"] \arrow[l, bend left, "{(b_3,e_4)}"]
\end{tikzcd}$$

Setting

$$e_+ = e_{(3,2)} + e_{(2,3)} + e_{(3,4)} + e_{(4,3)}$$

we will show that

$$R_+ = e_+(\mathbb{C}\bar{Q}^{fd}_{\mathcal{HC}_{\mathbf{k}}})e_+$$

is wild. In other words, we will be investigating

$$\mathrm{End}_{\mathrm{fd}(HC_{\mathbf{k}})}\left((P(\lambda_2)\otimes_{\mathbb{C}} P(\lambda_3^\vee))\oplus(P(\lambda_3)\otimes_{\mathbb{C}} P(\lambda_2^\vee))\oplus(P(\lambda_4)\otimes_{\mathbb{C}} P(\lambda_3^\vee))\oplus(P(\lambda_3)\otimes_{\mathbb{C}} P(\lambda_4^\vee))\right).$$

Let us consider a graded basis for the modules

$$P(\lambda_2), P(\lambda_3), P(\lambda_4)$$

in terms of paths of the quiver in $\mathcal{O}_{\mathbf{k}}$ (this carries a natural grading by the length of paths).

We obtain the following:

1. for $P(\lambda_2)$ we have

$$\begin{array}{c} e_2 \\ a_2 \\ a_3a_2 \end{array}$$

2. for $P(\lambda_3)$ we have

$$\begin{array}{ccc} & e_3 & \\ b_2 & & a_3 \\ & a_2b_2 & \end{array}$$

3. for $P(\lambda_4)$ we have

$$\begin{array}{c} e_4 \\ b_3 \\ b_2b_3 \end{array}$$

and from these we can deduce descriptions of the Harish-Chandra bimodules involved as follows:

1. for $P(\lambda_3)\otimes_{\mathbb{C}} P(\lambda_2^\vee)$ a graded basis is given by

$$\begin{array}{llll} \boxed{3,2} & & & \\ 3,3 & 2,2 & 4,2 & \\ \boxed{3,4} & \boxed{2,3} & ,\boxed{4,3} & \boxed{3,2} \\ 2,4 & 4,4 & 3,3 & \\ \boxed{3,4} & & & \end{array}$$

2. for $P(\lambda_2)\otimes_{\mathbb{C}} P(\lambda_3^\vee))$ a graded basis is given by

$$\begin{array}{llll} \boxed{2,3} & & & \\ 2,2 & 2,4 & 3,3 & \\ \boxed{2,3} & \boxed{3,2} & \boxed{3,4} & \boxed{4,3} \\ 4,2 & 4,4 & 3,3 & \\ \boxed{4,3} & & & \end{array}$$

3. for $P(\lambda_4) \otimes_{\mathbb{C}} P(\lambda_3^\vee))$ a graded basis is given by

$$\begin{array}{llll}
\boxed{4,3} & & & \\
4,2 & 4,4 & 3,3 & \\
\boxed{4,3} & \boxed{3,2} & \boxed{3,4} & \boxed{2,3} \\
3,3 & 2,2, & 2,4 & \\
\boxed{2,3} & & &
\end{array}$$

4. for $P(\lambda_3) \otimes_{\mathbb{C}} P(\lambda_4^\vee))$ a graded basis is given by

$$\begin{array}{llll}
\boxed{3,4} & & & \\
3,3 & 2,4 & 4,4 & \\
\boxed{3,2} & \boxed{2,3} & \boxed{4,3} & \boxed{3,4} \\
2,2 & 4,2 & 3,3 & \\
\boxed{3,2} & & &
\end{array}$$

The framed vectors give rise to elements of R_+, i.e. a morphism between one of the projective modules, and they again form a basis of R_+. Now let us consider the quotient of R_+ by all paths of length ≥ 3. We consider the algebra

$$\mathbb{C}\langle z_0, z_1, z_2, z_3\rangle \# \mathbb{Z}_4$$

where the action of $\mathbb{Z}_4$ on $\mathbb{C}\langle z_0, z_1, z_2, z_3\rangle$ is defined by

$$s(z_j) = \rho^j z_j.$$

We define a map

$$\mathbb{C}\langle z_0, z_1, z_2, z_3\rangle \# \mathbb{Z}_4 \to R_+/(\text{paths length} \geq 3)$$

by

$$\begin{aligned}
\mathbf{e}_0 z_0 \mathbf{e}_0 &\mapsto (b_3 a_3, e_2) \\
\mathbf{e}_1 z_0 \mathbf{e}_1 &\mapsto (e_2, b_3 a_3) \\
\mathbf{e}_2 z_0 \mathbf{e}_2 &\mapsto (b_3 a_3, e_4) \\
\mathbf{e}_3 z_0 \mathbf{e}_3 &\mapsto (e_4, b_3 a_3)
\end{aligned}$$

as well as

$$\begin{aligned}
\mathbf{e}_0 z_1 \mathbf{e}_3 &\mapsto 4,3 \to 3,2 = (b_3, b_2)\\
\mathbf{e}_1 z_1 \mathbf{e}_0 &\mapsto 3,2 \to 2,3 = (b_2, a_2)\\
\mathbf{e}_2 z_1 \mathbf{e}_1 &\mapsto 2,3 \to 3,4 = (a_2, a_3)\\
\mathbf{e}_3 z_1 \mathbf{e}_2 &\mapsto 3,4 \to 4,3 = (a_3, b_3)\\
\mathbf{e}_0 z_2 \mathbf{e}_2 &\mapsto 3,4 \to 3,2 = (e_3, b_2 b_3)\\
\mathbf{e}_1 z_2 \mathbf{e}_3 &\mapsto 4,3 \to 2,3 = (b_2 b_3, e_3)\\
\mathbf{e}_2 z_2 \mathbf{e}_0 &\mapsto 3,2 \to 3,4 = (e_3, a_3 a_2)\\
\mathbf{e}_3 z_2 \mathbf{e}_1 &\mapsto 2,3 \to 4,3 = (a_3 a_2, e_3)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{e}_0 z_3 \mathbf{e}_1 &\mapsto 2,3 \to 3,2 = (a_2, b_2)\\
\mathbf{e}_1 z_3 \mathbf{e}_2 &\mapsto 3,4 \to 2,3 = (b_2, b_3)\\
\mathbf{e}_2 z_3 \mathbf{e}_3 &\mapsto 4,3 \to 3,4 = (b_3, a_3)\\
\mathbf{e}_3 z_3 \mathbf{e}_0 &\mapsto 3,2 \to 4,3 = (a_3, a_2).
\end{aligned}$$

We then obtain a surjection

$$\mathbb{C}\langle z_0, z_1, z_2, z_3\rangle \# \mathbb{Z}_4 \twoheadrightarrow \frac{R_+}{\langle \text{paths length} \geq 3\rangle}$$

whose kernel contains

$$z_j z_k \text{ for } i, k \in \{0, 1, 2, 3\}.$$

A dimension count then shows that both algebras have dimension 20 and so we have an isomorphism

$$\frac{\mathbb{C}\langle z_0, z_1, z_2, z_3\rangle \# \mathbb{Z}_4}{\langle z_j z_k \mid 0 \leq i, k \leq 3\rangle} \xrightarrow{\sim} \frac{R_+}{(\text{paths length} \geq 3)}.$$

Now $\frac{\mathbb{C}\langle z_0, z_1, z_2, z_3\rangle \# \mathbb{Z}_4}{\langle z_j z_k | 0 \leq i, k \leq 3\rangle}$ is a local algebra with maximal ideal generated by $\{z_0, z_1, z_2, z_3\}$ and minimal number of generators 4. By Proposition 6.6.5 this implies that R_+ is wild as it surjects onto a wild algebra, namely $\frac{\mathbb{C}\langle z_0, z_1, z_2, z_3\rangle \# \mathbb{Z}_4}{\langle z_j z_k | 0 \leq i, k \leq 3\rangle}$.

Now we can turn to the case of $n > 4$. Considering the generators and relations of the quivers in category $\mathrm{fd}(\mathcal{O}_{\mathbf{k}})$ for different block lengths, we find that the algebra of the quiver with relations

$$\begin{tikzcd} \lambda_2 \arrow[r, bend left, "\bar{a}_2"] & \lambda_3 \arrow[l, bend left, "\bar{b}_2"] \arrow[r, bend left, "\bar{a}_3"] & \cdots \arrow[l, bend left, "\bar{b}_3"] \arrow[r, bend left, "\bar{a}_{n-1}"] & \lambda_n \arrow[l, bend left, "\bar{b}_{n-1}"] \end{tikzcd}$$

is a quotient of the algebra of the quiver with relations

$$\begin{tikzcd} \lambda_2 \arrow[r, bend left, "\bar{a}_2"] & \lambda_3 \arrow[l, bend left, "\bar{b}_2"] \arrow[r, bend left, "\bar{a}_3"] & \cdots \arrow[l, bend left, "\bar{b}_3"] \arrow[r, bend left, "\bar{a}_{n-1}"] & \lambda_n \arrow[l, bend left, "\bar{b}_{n-1}"] \arrow[r, bend left, "\bar{a}_n"] & \lambda_{n+1} \arrow[l, bend left, "\bar{b}_n"] \end{tikzcd}$$

and this surjection is given by

$$\begin{aligned}\bar{a}_i &\mapsto \bar{a}_i \text{ for } 2 \leq i \leq n-1\\ \bar{b}_i &\mapsto \bar{b}_i \text{ for } 2 \leq i \leq n-1\\ \bar{a}_n &\mapsto 0\\ \bar{b}_n &\mapsto 0.\end{aligned}$$

this induces maps between the block algebras of $\mathrm{fd}(\mathcal{HC}_{\mathbf{k}})$ and hence the algebras of blocks of $\mathrm{fd}(\mathcal{HC}_{\mathbf{k}})$ map onto the algebras of blocks of smaller size. We have just established that the algebra of a block of length 9 (in $\mathrm{fd}(\mathcal{HC}_{\mathbf{k}})$) - i.e. corresponding to the case $n = 4$ - is wild. As any algebra of a block of greater square size - so corresponding to the case of $n > 4$ - will have this as a quotient they will be wild too. $\square$

Chapter 7

Finite Coxeter Groups

In this chapter we will investigate the category $\mathcal{HC}_{\mathbf{k}}(W)$ for regular parameters when W is a finite irreducible Coxeter group. This is an important class of reflection groups which coincides precisely with the real reflection groups. We will obtain a complete description of $\mathcal{HC}_{\mathbf{k}}(W)$ in this case.

7.1 Coxeter Groups

Coxeter groups form an important class of groups and occur frequently in Lie theory, for example all Weyl groups are Coxeter groups.

Definition 7.1.1. *A Coxeter system (W, S) consists of a group W and a finite set of generators $S \subset W$ subject only to relations are of the form*

$$(st)^{n(s,t)} = 1$$

for $s, t \in S$ and $n(s,t) \in \mathbb{N} \cup \{\infty\}$ such that $n(s,s) = 1$ and $n(s,t) = n(t,s) \geq 2$. We shall refer to $\#S$ as the "rank of the system (W, S).

We have already defined the rank of a complex reflection group in Definition 2.1.9. We will soon show in Theorem 7.1.6 that each finite Coxeter group is a complex reflection group and that the rank of W as a complex reflection group equals the rank of the Coxeter system (W, S) so that we may drop the distinction.
To each Coxeter system we can associate a graph, the so-called Coxeter graph, which we will introduce next.

Definition 7.1.2. *Let (W, S) be a Coxeter system. The Coxeter graph $\Gamma(W, S)$ is the undirected graph with vertex set S and edges drawn as follows:*
- *If $n(s,t) = 2$ then no edge is drawn between s and t.*
- *If $n(s,t) \geq 3$ an edge labelled with $n(s,t)$ is drawn between s and t*

It is often customary to leave edges with $n(s,t) = 3$ unlabelled.

Example 7.1.3. *We take $W = S_n$ and $S = \{(i, i+1) \in S_n \mid 1 \leq i \leq n-1\}$. This forms a Coxeter system with Coxeter graph*

$$\circ_1 \text{———} \circ_2 \text{———} \cdots \text{———} \circ_{n-1}$$

If we had taken $W = S_n$ *with* $S = \{(ij) \mid 1 \leq i < j \leq n\}$ *instead we would not have obtained a Coxeter system for* $n \geq 3$*: Taking* $s = (12)$*,* $t = (23)$ *and* $u = (13)$ *we would have had the additional relation* $st = us$ *which does not follow from the Coxeter relations.*

Example 7.1.4. *For another example, let us consider the group* W *of all permutations* f *of the set* $\{-n, -(n-1)\ldots, -1, 1, 2, \ldots n\}$ *such that* $f(-i) = -f(i)$*. We set*

$$\begin{aligned}
f_1 &= (-1,1) \text{ i.e. } f_0(\pm 1) = \mp 1, f(i) = i \text{ else}\\
f_2 &= ((-2,-1),(1,2)) \text{ i.e. } f(\pm 1) = \pm 2, f(\pm 2) = \pm 1\\
f_3 &= ((-3,-2),(2,3))\\
&\vdots\\
f_n &= ((-n,-(n-1)),(n-1,n))
\end{aligned}$$

then $(W, S = \{f_0, \ldots, f_{n-1}\})$ *is a Coxeter system with Coxeter graph*

$$\circ_0 \overset{4}{\text{——}} \circ_1 \text{———} \cdots \text{———} \circ_{n-1}$$

All Weyl groups are Coxeter groups (and indeed our previous examples have precisely been the Weyl groups of type A and BC), although we also have Coxeter groups that are not Weyl groups:

Example 7.1.5. *Consider the dihedral group* $I_2(n)$ *of order* $2n$*. This has presentation*

$$I_2(n) = \langle a, b \mid b^n = 1 = a^2, ab = b^{-1}a\rangle.$$

Setting

$$s = a, t = ab$$

we find that $s^2 = t^2 = 1$ *and* $(st)^n = (ts)^n = 1$ *and a simple counting argument shows that these are the relations. Thus dihedral groups are also Coxeter groups but not, in general, Weyl groups.*

We will often speak of "the Coxeter group W" or even worse "the Coxeter group (W,S)" when we should properly be speaking of the "Coxeter system (W,S)". The following result is well-known and can be found for example in [Hum92]

Theorem 7.1.6.
1. *Let* (W,S) *be a Coxeter system. Then there exists a real vector space* V_S *of dimension* $\#S$ *such that* W *acts faithfully on* V_S *by real reflections, i.e. each* $s \in S$ *pointwise fixes a hyperplane in* V_S *and sends a non-zero vector to its negative.*
2. *The finite real reflection groups are precisely the finite Coxeter groups.*

In particular the rank of W *as complex reflection group equals* $\#S$*.*

Proof. The construction of V_S and the action of W is presented in Section 5.3 of [Hum92], faithfulness is shown in Section 5.4 of the same book. Clearly if W is finite it is a finite real reflection group by this construction. Conversely, for any finite real reflection group a Coxeter presentation is constructed in Section 1.9 of [Hum92]. □

Definition 7.1.7. *The representation of* (W,S) *in the statement of Theorem 7.1.6 will be called the "geometric representation" of* (W,S)*, following [Hum92].*

Definition 7.1.8. *For any Coxeter group (W, S) we have two distinguished one-dimensional representations, the trivial representation* triv *and the sign representation* sgn *given by $s \mapsto -1$ for all $s \in S$. The kernel of the representation* sgn *will be denoted W^+ and is commonly called the even subgroup of W.*

The Coxeter groups have a natural notion of irreducibilty (which is not the same as being simple as a group) which we shall introduce next. Recall that we call a graph connected if there is a path connecting any two vertices with each other.

Definition 7.1.9. *Let (W, S) be a Coxeter system. We call such a system irreducible if the Coxeter graph $\Gamma(W, S)$ is connected.*

The reason for this nomenclature is as follows: Suppose that (W, S) is a Coxeter system such that $\Gamma(W, S)$ is not connected. Denote the connected components of the Coxeter graph by $\{\Gamma_i\}_i$ and denote the subset of S corresponding to the vertices of a Γ_i by S_i. Then the subgroup $W_i \leq W$ generated by the elements of S_i is again a Coxeter group with Coxeter graph $\Gamma(W_i, S_i) = \Gamma_i$ and we have an isomorphism

$$W \cong \times_i W_i,$$

see Section 6.1 in [Hum92].
So irreducible Coxeter systems are precisely those Coxeter systems which do not decompose into other Coxeter systems. That irreducibility really depends on the Coxeter system (W, S) rather than the isomorphism class of the group W is shown in the following illustrative exercise taken from Section 2.2 of [Hum92]:

Example 7.1.10. *Consider the group $I_2(6)$ (the dihedral group of order 12) with presentation $\langle s, t \mid s^2 = t^2 = 1, (st)^6 = (ts)^6 = 1\rangle$. The Coxeter graph is*

$$\circ \overset{6}{\text{———}} \circ$$

and thus $(I_2(6), S = \{s, t\})$ is irreducible. Now consider the set $S' = \{u, v, w\}$ with

$$u = s, v = (ts)^3, w = s(ts)^2.$$

This is a generating set of $I_2(6)$ as $s = u$ and $t = vw$ and it remains to check the relations. Clearly $u^2 = v^2 = w^2 = 1$ and we have $(uv)^2 = (vu)^2 = 1$, as well as $(uw)^3 = (wu)^3 = 1$ and $(vw)^2 = (wv)^2 = 1$. Thus u, v and w, v commute and we have an isomorphism $I_2(6) \cong \langle u, w\rangle \times \langle v\rangle$. By example 7.1.3 u and w fulfil the Coxeter relations of the generators $(12), (23)$ of S_3 and a counting argument now shows that we have all relations and have shown an isomorphism $I_2(6) \cong S_3 \times \mathbb{Z}_2$. The first presentation then corresponds to a representation of $I_2(6)$ in which both s, t act via reflections on a 2-dimensional space where the product $(st)^3 = v$ does not act as a reflection. The second presentation then corresponds to a 3-dimensional representation of $I_2(6)$ which is the sum of the irreducible reflection representation of S_3 as in Example 2.1.6 (2) and the sign representation of $\mathbb{Z}_2$. This is clearly not an irreducible representation of $I_2(6)$.

The following classification of finite Coxeter groups can be taken from e.g. Section 2.4 of [Hum92], the finite Coxeter groups are identified as those Coxeter groups with "positive definite graph" in Section 2.3 with the proof given in Theorem 6.4 of the same book :

Theorem 7.1.11. *Any finite irreducible Coxeter system (W, S) group is isomorphic to one of the groups in the 4 infinite families and 6 exceptional groups listed in Table 7.1*

Table 7.1: Table of Coxeter groups

Name	Order	Coxeter Graph
$A_n, n \geq 1,\ S_{n+1}$	$(n+1)!$	$\circ_1 \text{ --- } \cdots \text{ --- } \circ_n$
$BC_n, n \geq 3,\ \mathbb{Z}_2 \wr S_n$	$2^n n!$	$\circ_1 \overset{4}{\text{ --- }} \circ_2 \text{ --- } \cdots \text{ --- } \circ_n$
$D_n, n \geq 4$	$2^{n-1} n!$	$\begin{array}{ccccc} \circ_1 & & & & \\ & \diagdown & & & \\ & & \circ_3 \text{ --- } \cdots \text{ --- } \circ_n & & \\ & \diagup & & & \\ \circ_2 & & & & \end{array}$
$I_2(n)$, dihedral groups	$2n$	$\circ_1 \overset{n}{\text{ --- }} \circ_2$
E_6	51840	$\begin{array}{ccccccccc} \circ_1 & \text{---} & \circ_2 & \text{---} & \circ_3 & \text{---} & \circ_5 & \text{---} & \circ_6 \\ & & & & \vert & & & & \\ & & & & \circ_4 & & & & \end{array}$
E_7	2903040	$\begin{array}{ccccccccccc} \circ_1 & \text{---} & \circ_2 & \text{---} & \circ_3 & \text{---} & \circ_5 & \text{---} & \circ_6 & \text{---} & \circ_7 \\ & & & & \vert & & & & & & \\ & & & & \circ_4 & & & & & & \end{array}$
E_8	696729600	$\begin{array}{ccccccccccccc} \circ_1 & \text{---} & \circ_2 & \text{---} & \circ_3 & \text{---} & \circ_5 & \text{---} & \circ_6 & \text{---} & \circ_7 & \text{---} & \circ_8 \\ & & & & \vert & & & & & & & & \\ & & & & \circ_4 & & & & & & & & \end{array}$
F_4	1152	$\circ_1 \text{ --- } \circ_2 \overset{4}{\text{ --- }} \circ_3 \text{ --- } \circ_4$
H_3	120	$\circ_1 \overset{5}{\text{ --- }} \circ_2 \text{ --- } \circ_3$
H_4	14400	$\circ_1 \overset{5}{\text{ --- }} \circ_2 \text{ --- } \circ_3 \text{ --- } \circ_4$

Note that the group $I_2(6)$ is a Weyl group, usually denoted G_2 in this context. The rank of the Coxeter groups listed is equal to the subscript in their name, so that for example the symmetric group S_{n+1} of type A_n has rank n, the group F_4 has rank 4, the dihedral groups $I_2(m)$ (see Example 7.1.5) have rank 2 etc.

To classify all possible normal subgroups of those finite irreducible Coxeter groups, we will need to introduce some more terminology:

Definition 7.1.12. *Let (W, S) and (V, Σ) be two Coxeter systems and $f : W \to V$ a group homomorphism. Then f is called a graph homomorphism if $f(s) \in \Sigma$ or $f(s) = 1 \in V$ for any $s \in S$ and $\Sigma \subseteq f(S)$. In particular any graph homomorphism is surjective as a group* *homomorphism.*

Example 7.1.13. *1. The natural inclusion map $S_n \hookrightarrow S_{n+1}$ is not a graph homomorphism as it is not surjective.*

2. If W is a finite Coxeter group and W^+ its even subgroup, then identifying W/W^+ with S_2 the projection map $W \twoheadrightarrow W/W^+$ is a graph homomorphism.

Theorem 7.1.14. *(Theorem 0.1 in [Max98]) If (W, S) is an irreducible finite Coxeter group of rank at least 3 and H a normal subgroup of W, then either H is cyclic of order 2 or it is the kernel of some graph homomorphism. In particular if H is not cyclic of order 2 the quotient W/H is again Coxeter.*

An explicit list of all normal subgroups of finite irreducible Coxeter groups of rank at least 3 is given in Table 3 in [Max98], the cases for which normal subgroups of order 2 are listed in Proposition 7.1.20.

Theorem 7.1.15. *(Table 3 in [Max98]) Let W be a finite irreducible Coxeter group of rank at least 4. The normal subgroups of W other than W, W^+ and $\mathbb{Z}_2$ when appropriate are kernels of graph homomorphisms as in the following tables Table 7.2 and Table 7.3.*

Table 7.2: Table of normal subgroups of Coxeter groups

Group	Map	Image
B_2	$s_1 \mapsto 1$	A_1
	$s_2 \mapsto 1$	A_1
$B_n, n \geq 3$	$s_1 \mapsto \sigma_1 \quad s_i \mapsto 1, 2 \leq i \leq n$	A_1
	$s_n \mapsto 1 \quad s_i \mapsto s_1, 2 \leq i \leq n$	A_1
	$s_1 \mapsto \sigma_1 \quad s_i \mapsto s_1, 2 \leq i \leq n$	$A_1 \times A_1$
	$s_1 \mapsto 1$	A_{n-1}
	$s_1 \mapsto \sigma_1$	$A_1 \times A_{n-1}$
B_4	$s_1 \mapsto 1 \quad s_2 \mapsto s_1,$	A_2
	$s_2 \mapsto 1, s_1 \mapsto \sigma_1$	$A_1 \times A_2$
$D_n, n \neq 4$	$s_1 \mapsto s_1, s_i \mapsto s_{i-1}, 2 \leq i \leq n$	A_{n-1}
D_4	$s_1, s_2 \mapsto s_1$	A_2
	$s_1 \mapsto s_1, s_2 \mapsto s_3, s_3 \mapsto 2_2, s_4 \mapsto s_3$	$D_3 = A_3$
	$s_2 \mapsto s_1, s_3 \mapsto s_2, s_4 \mapsto s_3$	$D_3 = A_3$

In these tables, the entry under "Map" describes the graph homomorphism f by giving those values on elements of S for which $f(s_i) \neq s_i$ and σ_i denotes a generator of a group of type A. "Image" describes the quotient $W/\ker(f)$.

Table 7.3: Table of normal subgroups of Coxeter groups, ctd.

Group	Map	Image
F_4	$s_1, s_2 \mapsto 1, s_3, s_4 \mapsto \sigma_4$	A_1
	$s_1, s_2 \mapsto \sigma_1, s_3, s_4 \mapsto 1$	A_1
	$s_1, s_2 \mapsto \sigma_1, s_3, s_4 \mapsto \sigma_4$	$A_1 \times A_1$
	$s_1, s_2 \mapsto 1$	A_2
	$s_3, s_4 \mapsto 1$	A_2
	$s_1, s_2 \mapsto \sigma_1$	$A_1 \times A_2$
	$s_4 \mapsto \sigma_4$	$A_2 \times A_1$
	$s_i \mapsto \sigma_i, 1 \leq i \leq 4,$	$A_2 \times A_2$
$G_2 = I_2(6)$	$s_1 \mapsto 1$	A_1
	$s_2 \mapsto 1$	A_1
	$s_1 \mapsto \sigma_1, s_2 \mapsto \sigma_2$	$A_1 \times A_1$

So it only remains to check the normal subgroups of finite Coxeter groups of rank $1, 2$ or 3. For rank 1 we only obtain the group S_2 and the only normal subgroups are the trivial ones. For rank 3 the only irredcuible finite Coxeter group is S_4 with normal subgroups A_4 and the Klein four-group. The finite Coxeter groups of rank 2 are all dihedral, and their normal subgroups and quotients are well-known:

Proposition 7.1.16. *Let W be a Coxeter group of type $I_2(n)$ with*

$$W = \langle s, t \mid s^2 = t^2 = (st)^n = 1 \rangle.$$

Then the normal subgroups of W are as follows:

1. *If n is odd, the non-trivial normal subgroups of W are of the form $\langle (st)^d \rangle$ for some $d \mid n$. The quotient of W by $\langle (st)^d \rangle$ is isomorphic to the group $I_2(d)$ and in particular is again dihedral.*
2. *If n is even, then a non-trivial normal subgroup of W is either $\langle s, (st)^2 \rangle$ or $\langle t, (st)^2 \rangle$ or is of the form $\langle (st)^d \rangle$ for some $d \mid n$. Again the quotient of W by $\langle (st)^d \rangle$ is isomorphic to the group $I_2(d)$, the quotient of W by $\langle s, st \rangle$ or $\langle t, st \rangle$ is isomorphic to $\mathbb{Z}_2$, the cyclic group of order 2.*

Proof. The conjugacy classes in the dihedral groups $I_2(n)$ are known to be as follows:

1. If n is odd, the conjugacy classes of $I_2(n)$ are given by $\mathcal{S}$, the set of reflections, the set $\{1\}$ and the pairs $\{(st)^i, (st)^{-i}\}$. As every normal subgroup is a union of conjugacy classes and $\mathcal{S}$ is a set of generators of $I_2(n)$, we see that any non-trivial normal subgroup must be generated by a union of sets of pairs $\{(st)^i, (st)^{-i}\}$ and the claim on the form of normal

subgroups follows. If $N = \langle (st)^d \rangle$ for some $d \mid n$, we see that the quotient $I_2(n)/N$ must be a quotient of the group with presentation $\langle s, t \mid s^2 = t^2 = (st)^d = 1 \rangle$, i.e. $I_2(n)/N$ is a quotient of $I_2(d)$ and a counting argument shows this is an isomorphism.

2. If n is even, the conjugacy classes of $I_2(n)$ are given by the set $\{1\}$ and the pairs $\{(st)^i, (st)^{-i}\}$ as well as two sets $\mathcal{S}_s$ and $\mathcal{S}_t$ of reflections - the conjugates of s and t. As $S_s \cup S_t$ is again a set of generators of $I_2(n)$, the same arguments as in case 1 will show that a non-trivial normal subgroup of $I_2(n)$ will either be of the form $\langle (st)^d \rangle$ for some $d \mid n$ or be generated by a union of pairs $\{(st)^i, (st)^{-i}\}$ and one of S_s and S_t. But if N contains S_s then it contains $(st)^2 = (st)s(ts)s$ and if N contains S_t then $(ts)^2 = (ts)t(st)t \in N$. So let N be a normal subgroup generated by one of S_s or S_t and a union of pairs $\{(st)^i, (st)^{-i}\}$. If $(st)^i \in N$ for some odd i then $(st)^i$ generates the subgroup $\langle st \rangle$ and since either s or t is in N, N contains a set of generators of W and so is all of W. Thus if N is non-trivial, it can only contain even-indexed pairs $\{(st)^i, (st)^{-i}\}$ and as N contains $(st)^2 \in N$ we have that N is one of $\langle s, (st)^2 \rangle$ or $\langle t, (st)^2 \rangle$ (which are equal to $\langle S_s \rangle$ or $\langle S_t \rangle$ respectively).

□

Definition 7.1.17. *Let W be a finite Coxeter group. For any $w \in W$, we denote by $\ell(w) \in \mathbb{N}_0$ the minimal length of w as a word in the elements of S. The map*

$$\ell : W \to \mathbb{N}_0, \; w \mapsto \ell(w)$$

is the length function of (W, S).

If W is a finite Coxeter group, the length function $\ell : W \to \mathbb{N}_0$ must attain a maximal value for some $w_0 \in W$. It will turn out that such an element is unique and of order 2. This is again a well-known fact concerning finite Coxeter groups and is explained in Section 1.8 of [Hum92].

Proposition 7.1.18. *(Section 1.8 in [Hum92]) Let (W, S) be a Coxeter system with W finite. Then there exists a unique element $w_0 \in W$ such that $\ell(w_0)$ is maximal. Moreover, w_0 is of order 2.*

The proof relies on several properties of the length function which we have not introduced yet and whose discussion would take us too far afield.

Definition 7.1.19. *The element mentioned in Proposition 7.1.18 is called the longest element of (W, S).*

The following can be deduced from the case-by-case distinction following Definition 1.21 [Fra01] together with Table 3 in [Max98]:

Proposition 7.1.20. *If W is a finite irreducible Coxeter group then W has a normal subgroup of order 2 only in types B_n, D_n for n even, $I_2(n)$ for n even, E_7, E_8, F_4, H_3 and H_4. In these cases this subgroup is central and generated by the longest element.*

We will quote a proposition on conjugacy in Coxeter groups, which will help us to determine the number of distinct parameters of their rational Cherednik algebras.

Proposition 7.1.21. *Let (W, S) be a Coxeter system with W finite. Then the elements acting as reflections on the geometric representation are precisely the conjugates of elements in S and two elements $s, t \in S$ are conjugate if and only if there are elements $s_0, \ldots, s_n \in S$ such that $s = s_0, t = s_n$ and $s_i s_{i+1}$ has odd order for $i = 0, \ldots, n-1$.*

Proof. The claim about reflections of the geometric representation is shown as in Proposition 1.14 of [Hum92]. The claim about conjugacy of elements of S is Exercise 5.3 in the same book and we will present a proof following the hints given in the text:
First, let us suppose that $s, t \in S$ are such that st has odd order, say $2n+1$ for some $n \geq 1$ (note that by definition of a Coxeter group, if $s \neq t$ then the order of st is at least 2). Thus we have

$$(st)^{2n+1} = s(ts)^n t(st)^n = 1$$

and thus

$$(ts)^n t(st)^n = s$$

and s and t are indeed conjugate. Thus if there are elements $s_0, \ldots, s_n \in S$ such that $s = s_0, t = s_n$ and $s_i s_{i+1}$ has odd order for $i = 0, \ldots, n-1$, then $s_0, s_1, \ldots, s_n$ all are conjugate in W and thus $s = s_0$ and $t = s_n$ are conjugate as claimed.
Now let us suppose that we have $s, t \in S$ with s and t conjugate in W. We denote by S^* the set of all $s^* \in S$ such that there are elements $s_0, \ldots, s_n \in S$ with $s = s_0, s^* = s_n$ and $s_i s_{i+1}$ having odd order for $i = 0, \ldots, n-1$. We only need to show that s cannot be conjugate to any $u \in S \setminus S^*$. Define a map

$$f : S \to \mathbb{Z}_2, s \mapsto \begin{cases} 1 \text{ if } s \in S^* \\ -1 \text{ if } s \in S \setminus S^* \end{cases} .$$

We claim that f extends to a homomorphism $W \to \mathbb{Z}_2$ and to show this it will suffice to check that f respects the Coxeter relations. To verify this, it will be enough to show that if $s^* \in S^*$ and $u \in S \setminus S^*$, then $f((s^*u)^{n(s^*,u)}) = 1$ and so we need to show that $n(s^*, u)$ is even. But this is the case by definition of S^* and hence f indeed gives a morphism $W \to \mathbb{Z}_2$. Now $s \in \ker(f)$ and hence every conjugate of s is in $\ker(f)$ too, thus in particular if s and t are conjugate then $f(s) = f(t) = 1$. So every element of S that is conjugate to s must lie in S^* as claimed. □

Note that the orders of the elements $s_i s_{i+1}$ can be read off easily from the Coxeter graph.

7.2 Rational Cherednik algebras of finite Coxeter groups and their Harish-Chandra bimodules

From the Coxeter graphs given in Table 7.1 and Proposition 7.1.21 we can see that the rational Cherednik algebra of a finite Coxeter group (with the geometric representation) will have one independent parameter in types A_n, D_n, $I_2(n)$ for n odd, E_6, E_7, E_8, H_3 and H_4, and two independent parameters in types $BC_n, I_2(n)$ for n even, and F_4. Table 7.4 and Table 7.5 later in the chapter will give the Hecke algebras of the finite Coxeter groups in terms of generators and relations. That these tables are indeed accurate can be seen either directly from the definition of Hecke algebras, Definition 2.4.14 and Theorem 2.4.15 or we can appeal to Proposition 4.22 in [BMR98]. Note that the generators are in 1-1-correspondence with the vertices of the Coxeter diagram and generators T_i, T_j that do not have an explicit relation amongst each other commute (so e.g. T_i, T_j commute in type A_n if $|i-j| > 1$). Table 7.4 will be on the next page.

Definition 7.2.1. *Let (W, S) be a Coxeter system with W finite and Γ the associated Coxeter graph. Let Γ_{odd} be the graph obtained from Γ by deleting all edges with an even label.*

Table 7.4: Hecke algebras with 1 parameter

Type	Braid Relations	Hecke Relation
A_n	$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$	$(T_i-1)(T_i+q)$
D_n	$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, i \geq 2$ $T_1T_3T_1 = T_3T_1T_3$	$(T_i-1)(T_i+q)$
$I_2(n), n$ odd	$\underbrace{T_1T_2\ldots T_1}_{n \text{ factors}} = \underbrace{T_2T_1\ldots T_2}_{n \text{ factors}}$	$(T_i-1)(T_i+q)$
E_6	$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, i \neq 4$ $T_3T_5T_3 = T_5T_3T_5$	$(T_i-1)(T_i+q)$
E_7	$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, i \neq 4$ $T_3T_5T_3 = T_5T_3T_5$	$(T_i-1)(T_i+q)$
E_8	$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, i \neq 4$ $T_3T_5T_3 = T_5T_3T_5$	$(T_i-1)(T_i+q)$
H_3	$T_1T_2T_1T_2T_1 = T_2T_1T_2T_1T_2$ $T_2T_3T_2 = T_3T_2T_3$	$(T_i-1)(T_i+q)$
H_4	$T_1T_2T_1T_2T_1 = T_2T_1T_2T_1T_2$ $T_2T_3T_2 = T_3T_2T_3, T_3T_4T_3 = T_4T_3T_4$	$(T_i-1)(T_i+q)$

Example 7.2.2. *1. If W has one conjugacy class of reflections we have $\Gamma = \Gamma_{odd}$.*

2. *In type BC_n we have $\Gamma_{odd} = \circ_1 \qquad \circ_2 - \cdots - \circ_n$*
3. *In type $I_2(n)$ with n even we have $\Gamma_{odd} = \circ_1 \qquad \circ_2$*
4. *In type F_4 we have $\Gamma_{odd} = \circ_1 - \circ_2 \qquad \circ_3 - \circ_4$*

We can give some basic criteria to determine when a Hecke algebra $\mathcal{H}_{\mathbf{q}}(W)$ is semisimple.

Proposition 7.2.3. *Let W be a finite irreducible Coxeter group and $\mathcal{H}_{\mathbf{q}}(W)$ its Hecke algebra.*

1. *Suppose that $\mathcal{H}_{\mathbf{q}}(W)$ has only one independent parameter or that both of its parameters are equal $q = q_1 = q_2$. The algebra $\mathcal{H}_{\mathbf{q}}(W)$ is semisimple if and only if $\sum_{w\in W}(q)^{\ell(w)} \neq 0$.*
2. *The algebra $\mathcal{H}_{\mathbf{q}}(W)$ is not semisimple if for one of its parameters q_j we have $\sum_{w\in W_J}(q_j)^{\ell(w)} = 0$ where W_J is the parabolic subgroup of W associated to the component Γ_j of Γ_{odd}.*
3. *The algebra $\mathcal{H}_{\mathbf{q}}(W)$ is not semisimple if one of its parameters q_j equals -1.*

Proof. 1. This is the Main Theorem in [GU89].

2. Let J be the set of vertices of the component Γ_j of Γ_{odd} associated to q_j and denote by $\mathcal{H}_J$ the subalgebra of $\mathcal{H}_{\mathbf{q}}(W)$ generated by $\{T_i \mid i \in J\}$. Then $\mathcal{H}_J$ is either all of $\mathcal{H}$ in case $\mathcal{H}$ only has one independent parameter or $\mathcal{H}_J$ is a Hecke algebra of type A if $\mathcal{H}$ has two independent parameters. By (1), $\mathcal{H}_J$ is non-semisimple since $\sum_{w\in W_J}(q_j)^{\ell(w)} = 0$.

We now define

$$\psi : \mathcal{H} \to \mathcal{H}_J$$
$$T_i \mapsto \begin{cases} T_i \text{ if } i \in J \\ 1 \text{ else} \end{cases}$$

By inspection of the presentations of $\mathcal{H}$ and $\mathcal{H}_J$, we can see that ϕ is a well-defined surjective morphism of finite-dimensional algebras. As $\mathcal{H}_J$ is not semisimple, $\mathcal{H}$ cannot be semisimple either.

3. Suppose $q_j = -1$, we note that $\sum_{w \in W_J} (-1)^{\ell(w)}$ is the inner product of the trivial character of W_J with the character of the sign representation of W_J and thus $\sum_{w \in W_J} (-1)^{\ell(w)}=0$.

□

We shall now slightly simplify our notation concerning the parameters $\mathbf{k}$ of the rational Cherednik algebra $H_{\mathbf{k}}(W)$ of a finite Coxeter group and $\mathbf{q}$ of the associated Hecke algebras $\mathcal{H}_{\mathbf{q}}(W)$ by from now on only listing the independent parameters. So instead of writing

$$\mathbf{k} = (k_{H_1,0} = 0, k_{H_1,1} = k, k_{H_2,0} = 0, k_{H_2,1} = k, \ldots, k_{H_r,0} = 0, k_{H_r,1} = k)$$

we will write

$$\mathbf{k} = (k_1)$$

and similarly $\mathbf{q} = q$; if W has two independent parameters we will similarly write $\mathbf{k} = (k_1, k_2)$ and $\mathbf{q} = (q_1, q_2)$ and will specify when appropriate to what conjugacy class of reflections we associate the parameters k_i.

Theorem 7.2.4. *Let W be an irreducible finite Coxeter group with one conjugacy class of reflections (see Table 7.4) and let $\mathbf{k} = (k_1) \in \mathrm{Reg}(W)$. Then $\mathcal{HC}_{\mathbf{k}}(W)$ is non-trivial if and only if $k_1 \in \mathbb{Z}$ and so $(\mathcal{HC}_{\mathbf{k}}(W), \otimes_{U_{\mathbf{k}}}) \cong (W, \otimes)$.*

Proof. Suppose that $\mathbf{q}$ is such that $\mathcal{H}_{\mathbf{q}}(W)$ is semisimple with W an irreducible finite Coxeter group. Suppose that $\mathcal{HC}_{\mathbf{k}}$ is non-trivial, then by Theorem 3.4.17 this gives rise to a non-trivial finite-dimensional simple $\mathcal{H}_{\mathbf{q}}$-module M such that $M \otimes_{\mathbb{C}} M$ is again a $\mathcal{H}_{\mathbf{q}}$-module. Here $M \otimes_{\mathbb{C}} M$ is the tensor product of B_W-modules and thus the action of the generators T_j on $M \otimes_{\mathbb{C}} M$ is the diagonal one. Since M is non-trivial (and simple) at least one of the generators T_j cannot act as the identity on M, call this generator T_i. Since

$$(*)\ (T_i - 1)(T_i + q_1) = 0$$

this shows that T_i has an eigenvector $m \in M$ with eigenvalue $-q_1$. Then $m \otimes m \in M \otimes_{\mathbb{C}} M$ is a T_i-eigenvector with eigenvalue $q_1^2 = (-q_1)^2$ as

$$T_i(m \otimes m) = (T_i m) \otimes (T_i m) = q_1^2 (m \otimes m).$$

Thus by $(*)$ we must conclude that

$$q_1^2 = 1 \text{ or } q_1^2 = -q_1$$

or equivalently

$$q_1 \in \{-1, 0, 1\}.$$

By definition $q_1 \in \mathbb{C}^*$ and hence we must rule out $q_1 = 0$. Similarly, our hypothesis that $\mathcal{H}_{\mathbf{q}}(W)$ is semisimple rules out $q_1 = -1$ by Proposition 7.2.3. Hence we must conclude that $q_1 = 1$ and as $q_1 = e^{2\pi i k_1}$. Thus $k_1 \in \mathbb{Z}$ and the theorem is shown. □

This again proves Conjecture 1 for Coxeter groups with one conjugacy class of reflections. So it remains to deal with the irreducible finite Coxeter groups of types BC_n, $I_2(n)$ for n even and F_4. We shall again give their Hecke algebras in a table.

Table 7.5: Hecke algebras with 2 parameters

Type	Braid Relations	Hecke Relation
$BC_n, n \geq 3$	$T_1T_2T_1T_2 = T_2T_1T_2T_1$ $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, i \geq 2$	$(T_1 - 1)(T_1 + q_1)$ $(T_i - 1)(T_i + q_2), i \geq 2$
$I_2(n), n$ even	$\underbrace{T_1T_2 \ldots T_1T_2}_{n-\text{factors}} = \underbrace{T_2T_1 \ldots T_2T_1}_{n-\text{factors}}$	$(T_1 - 1)(T_1 + q_1)$ $(T_2 - 1)(T_2 + q)$
F_4	$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, i = 1, 3$ $T_2T_3T_2T_3 = T_3T_2T_3T_2$	$(T_i - 1)(T_i + q_1), i = 1, 2$ $(T_i - 1)(T_i + q_2), i = 3, 4$

Proposition 7.2.5. *Let W be an irreducible finite Coxeter group of type BC_n, F_4 or $I_2(n)$ with n even. Suppose that $\mathbf{k} \in \mathrm{Reg}(W)$ and $\mathcal{HC}_{\mathbf{k}} = \mathcal{HC}_{(k_1,k_2)}$ is non-trivial. Then k_1 or k_2 is integral.*

Proof. Note that in all these cases Γ_{odd} has precisely two components: $\Gamma_{odd} = \Gamma_1 \cup \Gamma_2$ where Γ_1 is the component of Γ_{odd} containing 1 and Γ_2 is the component containing n. Further, using Proposition 7.1.21 we can see that any two reflections s_i, s_j in the same component of Γ_{odd} are conjugate. The associated elements T_i, T_j of the Hecke algebra therefore fulfil the Hecke relations at the same parameter.

Assuming $\mathcal{HC}_{\mathbf{k}}$ is non-trivial as before implies the existence of a non-trivial, finite-dimensional simple $\mathcal{H}_{\mathbf{q}}(W)$-module M such that $M \otimes_{\mathbb{C}} M$ is again a $\mathcal{H}_{\mathbf{q}}(W)$ module under the diagonal action of the T_j. As M is non-trivial, for at least one component Γ_j of Γ_{odd} at least one of the associated generators -call it T_i - must have a non-empty $-q_j$-eigenspace. As before considering the action of T_i on $M \otimes_{\mathbb{C}} M$ and in particular the action on $m \otimes m$ for a $-q_j$ eigenvector $m \in M$, we conclude that $q_j^2 = 1$ or $q_j^2 = -q_j$. This again leaves us with $q_j = 1$ and thus $k_j \in \mathbb{Z}$. □

Let us first fix some more notation. We shall keep the notation of Table 7.1 and Table 7.5. Let W be a finite irreducible Coxeter group with two conjugacy classes of reflections and parameters k_1, k_2 associated to the parameters q_1, q_2 respectively of the associated Hecke algebra. We denote by W_i the subgroup of W generated by all reflections conjugate to reflections associated to the parameter k_i for $i = 1, 2$. Each W_i is a normal subgroup of W as its set of generators forms a conjugacy class in W. Recall that $\Lambda(W) \subseteq \operatorname{Reg}(W)$ denotes the set of integral parameter values and that a regular parameter is called rigid if $U_{\mathbf{k}}$ is the only non-zero simple Harish-Chandra bimodule up to isomorphism. The following proof was suggested to the author by Iain Gordon based on an earlier proof for the case of dihedral groups.

Theorem 7.2.6. *Let W be a finite irreducible Coxeter group with two conjugacy classes of reflections and parameters k_1, k_2 and let $\mathbf{k} \in \operatorname{Reg}(W)$.*

1. *We have $(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}}) \cong (\operatorname{rep}_{\mathbb{C}} W, \otimes)$ if and only if $\mathbf{k} \in \Lambda(W)$.*
2. *Let $\{a, b\} = \{1, 2\}$. If $k_a \in \mathbb{Z}$ and $k_b \notin \mathbb{Z}$ then $(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}}) \cong (\operatorname{rep}_{\mathbb{C}}(W/W_b), \otimes)$.*
3. *If $k_1, k_2 \notin \mathbb{Z}$ then $\mathbf{k}$ is rigid*

Proof. 1. That $(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}}) \cong (\operatorname{rep}_{\mathbb{C}} W, \otimes)$ if $\mathbf{k} \in \Lambda(W)$ follows from Theorem 4.3.2. So let us suppose that $(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}}) \cong (\operatorname{rep}_{\mathbb{C}} W, \otimes)$. Arguing as in Proposition 7.2.5 we see that $q_1 = q_2 = 1$ and so $k_1, k_2 \in \mathbb{Z}$ so that $\mathbf{k} \in \Lambda(W)$.

2. Suppose $k_a \in \mathbb{Z}$ and $k_b \notin \mathbb{Z}$. By Theorem 4.2.10 we may suppose that $k_a = 0$. We denote by $R_{\mathbf{k}}(W_b)$ the subalgebra of $H_{\mathbf{k}}(W)$ generated by $\mathfrak{h}, \mathfrak{h}^*$ and the elements of W_b. We denote by $\mathbf{e}_b$ the principal idempotent of W_b, $\mathbf{e}_b = \frac{1}{\#W_b} \sum_{w \in W_b} w$. Let us consider $\mathbf{e}_b R_{\mathbf{k}}(W_b) \mathbf{e}_b$.

We claim that W/W_b acts on $\mathbf{e}_b R_{\mathbf{k}}(W_b) \mathbf{e}_b$. Consider the action of W on $H_{\mathbf{k}}(W)$ by conjugation (and on $\mathfrak{h} \subseteq H_{\mathbf{k}}(W)$ this action coincides with the reflection action of W on $\mathfrak{h}$). If $w \in W_a$ then $w \mathbf{e}_b w^{-1} = \mathbf{e}_b$ as W_b is generated by a conjugacy class in W and if $w \in W_b$ then $w \mathbf{e}_b = \mathbf{e}_b w^{-1} = \mathbf{e}_b$ by construction so that the conjugation action of W_a preserves $\mathbf{e}_b R_{\mathbf{k}}(W_b) \mathbf{e}_b$ and the conjugation action of W_b leaves $\mathbf{e}_b R_{\mathbf{k}}(W_b) \mathbf{e}_b$ invariant. As $W_a \cup W_b$ is a generating set of W we may deduce that the conjugation action of W preserves $\mathbf{e}_b R_{\mathbf{k}}(W_b) \mathbf{e}_b$. and that the action of W on $\mathbf{e}_b R_{\mathbf{k}}(W_b) \mathbf{e}_b$ descends to an action of W/W_b as claimed.

Next, we aim to show that there is an isomorphism $(\mathbf{e}_b R_{\mathbf{k}}(W_b) \mathbf{e}_b)^{W/W_b} \cong U_{\mathbf{k}}(W)$ given by

$$\begin{aligned} \psi : (\mathbf{e}_b R_{\mathbf{k}}(W_b) \mathbf{e}_b)^{W/W_b} &\to U_{\mathbf{k}} \\ r &\mapsto \mathbf{e} r \mathbf{e}. \end{aligned}$$

We let $w_1, \ldots, w_k$ denote a set of representatives of the distinct cosets forming the quotient W/W_b and let us agree to choose $w_1 = 1$. Then $\mathbf{e} = \frac{\#W_b}{\#W} \sum_j w_j \mathbf{e}_b$ and as $r \in (\mathbf{e}_b R_{\mathbf{k}}(W_b) \mathbf{e}_b)^{W/W_b}$ we have $w_j r = r w_j$ for any $j = 1, \ldots, k$. Since we also trivially have $\mathbf{e}_b r = r \mathbf{e}_b$ we can deduce that for $r \in (\mathbf{e}_b R_{\mathbf{k}}(W_b) \mathbf{e}_b)^{W/W_b}$ we have $\mathbf{e} r = r \mathbf{e}$ and so ψ is indeed a morphism of $\mathbb{C}$ -algebras from $(\mathbf{e}_b R_{\mathbf{k}}(W_b) \mathbf{e}_b)^{W/W_b}$ to $U_{\mathbf{k}}$. Because $\mathbb{C}[\mathfrak{h}]^W, \mathbb{C}[\mathfrak{h}^*]^W \subset R_{\mathbf{k}}(W_b)$ we deduce that the image of ψ contains $\mathbb{C}[\mathfrak{h}]^W \mathbf{e}, \mathbb{C}[\mathfrak{h}^*]^W \mathbf{e}$. By Theorem 2.4.2 the set $\mathbb{C}[\mathfrak{h}]^W \mathbf{e} \cup \mathbb{C}[\mathfrak{h}^*]^W \mathbf{e}$ generates $U_{\mathbf{k}}$ as a $\mathbb{C}-$algebra for regular $\mathbf{k}$ and we conclude that ψ is surjective. It remains to show that ψ is injective. So let $r \in (\mathbf{e}_b R_{\mathbf{k}}(W_b) \mathbf{e}_b)^{W/W_b}$ be such that $\psi(r) = 0$. Using $\mathbf{e} r = r \mathbf{e}$ and $\mathbf{e} = \frac{\#W_b}{\#W} \sum_j \mathbf{e}_b w_j$ we find

$$\begin{aligned}\psi(r) &= \mathbf{e}r\mathbf{e} = r\mathbf{e} \\ &= \frac{\#W_b}{\#W}\sum_j r\mathbf{e}_b w_j \\ &= \frac{\#W_b}{\#W}\sum_j r w_j\end{aligned}$$

and hence $\frac{\#W_b}{\#W}\sum_j rw_j = 0$. But by the PBW Theorem, Theorem 2.2.4, the elements rw_j must all be linearly independent and thus we must have $r = 0$ (as we chose $w_1 = 1$). So ψ is indeed an isomorphism $(\mathbf{e}_b R_{\mathbf{k}}(W_b)\mathbf{e}_b)^{W/W_b} \xrightarrow{\sim} U_{\mathbf{k}}$ as claimed.
Thus $\mathbf{e}_b R_{\mathbf{k}}(W_b)\mathbf{e}_b$ is a $U_{\mathbf{k}}(W)$-bimodule and as $H_{\mathbf{k}}(W)$ is Harish-Chandra we can deduce that $\mathbf{e}_b R_{\mathbf{k}}(W_b)\mathbf{e}_b$ is in fact a $U_{\mathbf{k}}(W)$-Harish-Chandra bimodule with a $U_{\mathbf{k}}$-equivariant action of W/W_b. Each W/W_b-isotypic component of $\mathbf{e}_b R_{\mathbf{k}}(W_b)\mathbf{e}_b$ will again be a $U_{\mathbf{k}}(W)$-Harish-Chandra bimodule (and since $\mathbb{C}[\mathfrak{h}] \subset R_{\mathbf{k}}(W_b)$ these will all be non-zero). We claim that from this we can deduce the existence of at least $\mathrm{Irr}(W/W_b)$ distinct simple Harish-Chandra bimodules. We employ a localisation argument. As $k_a = 0$ the commutator $[y, x] \in \mathbb{C}W_b$ for any $y \in \mathfrak{h}, x \in \mathfrak{h}^*$ and hence using the Dunkl representation we see that upon localising to $\mathfrak{h}^{reg}$ we have an isomorphism

$$R_{\mathbf{k}}(W_b)|_{\mathfrak{h}^{reg}} \cong \mathcal{D}(\mathfrak{h}^{reg})\#W_b$$

and thus

$$\mathbf{e}_b R_{\mathbf{k}}(W_b)|_{\mathfrak{h}^{reg}}\mathbf{e}_b \cong \mathcal{D}(\mathfrak{h}^{reg})^{W_b} \cong \mathcal{D}(\mathfrak{h}^{reg}/W_b)$$

as the action of W_b is free on $\mathfrak{h}^{reg}$, see Proposition 2.3.8. For any $\alpha \in \mathrm{Irr}(W/W_b)$ we see that the α-isotypic component of $\mathbf{e}_b R_{\mathbf{k}}(W_b)\mathbf{e}_b$ localises to the α-isotypic component of $\mathcal{D}(\mathfrak{h}^{reg}/W_b)$. Thus by localising the tensor product of an α-isotypic component of $\mathbf{e}_b R_{\mathbf{k}}(W_b)\mathbf{e}_b$ with $\mathbf{e}\Delta(\mathrm{triv})$ we obtain the α-isotypic component of $\mathcal{O}(\mathfrak{h}^{reg})$. As this is also the localisation of $\mathbf{e}\Delta(\alpha)$. Using semisimplicity of $\mathcal{HC}_{\mathbf{k}}$ we can deduce that each α-isotypic component of $\mathbf{e}_b R_{\mathbf{k}}(W_b)\mathbf{e}_b$ is a simple Harish-Chandra bimodule. Hence we have obtained $\#\mathrm{Irr}(W/W_b)$ distinct simple Harish-Chandra bimodules.

We still need to establish the equivalence $(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}}) \cong (\mathrm{rep}_{\mathbb{C}}(W/W_b), \otimes)$. Following the proof of Theorem 3.4.21 we need to consider the associated Hecke algebra.As in proof of Theorem 3.4.21 we denote by $\mathrm{Irr}^{\mathcal{HC}}_{\mathbf{k}}(W)$ the set of irreducible W-representations λ with $\mathcal{L}(\Delta(\mathrm{triv}), \Delta(\lambda)) \neq 0$ and $M(\mu)$ denotes the $\mathcal{H}_{\mathbf{q}}(W)$-module $\mathrm{KZ}_{\mathbf{k}}(\Delta(\mu))$ for any $\mu \in \mathrm{Irr}(W)$. Let T be any generator of $\mathcal{H}_{\mathbf{q}}(W)$ associated to a reflection in W_b. Then if $M(\lambda)$ with $\lambda \in \mathrm{Irr}^{\mathcal{HC}}_{\mathbf{k}}(W)$ is a non-trivial $\mathcal{H}_{\mathbf{q}}(W)$-module, the eigenvalues of T on $M(\lambda)$ are 1 or $-q_b$. If T had an eigenvector with eigenvalue $-q_b$ then we could consider $M(\lambda) \otimes M(\lambda)$ to conclude as in the proof of Theorem 7.2.4 that $q_b = 1$ and so $k_b \in \mathbb{Z}$. But by hypothesis $k_b \notin \mathbb{Z}$ and so T must act as the identity on any $M(\lambda)$ with $\lambda \in \mathrm{Irr}^{\mathcal{HC}}_{\mathbf{k}}$ and thus any such $M(\lambda)$ is a module over the quotient $\frac{\mathcal{H}_{\mathbf{q}}(W)}{\langle T-1 | T \text{ assoc. to } W_b\rangle}$ that is the quotient by the two-sided ideal generated by the elements $T - 1$ for any generator T of $\mathcal{H}_{\mathbf{q}}(W)$ associated to a reflection in W_b.
By integrality of k_a, the Hecke relations for any T associated to a reflection in W/W_b,

equivalently a reflection in W_a, has the form $(T-1)(T+1)$ and so we have an isomorphism

$$\frac{\mathcal{H}_{\mathbf{q}}(W)}{\langle T-1 \mid T \text{ assoc. to } W_b\rangle} \cong \mathbb{C}\,(W/W_b)\,.$$

This together with the fact that $\mathcal{HC}_{\mathbf{k}}(W)$ has $\#\mathrm{Irr}(W/W_b)$ distinct simple objects gives the equivalence $(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}}) \cong (\mathrm{rep}_{\mathbb{C}}(W/W_b), \otimes)$ as claimed.

3. If neither k_a nor k_b are integral $\mathbf{k}$ is rigid by Proposition 7.2.5.

□

This now completes the proof of Conjecture 1 for all finite irreducible Coxeter groups.

Corollary 7.2.7. *Let W be of type BC_n and $\mathbf{k} \in \mathrm{Reg}(W)$.*

1. *If $k_1 \in \mathbb{Z}, k_2 \notin \mathbb{Z}$ then $(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}}) \cong (\mathrm{rep}_{\mathbb{C}}\mathbb{Z}_2, \otimes)$*
2. *If $k_1 \notin \mathbb{Z}, k_2 \in \mathbb{Z}$ then $(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}}) \cong (\mathrm{rep}_{\mathbb{C}}S_n, \otimes)$*

Proof. Follows from Theorem 7.2.6 and Tables 7.2 and 7.3. □

Corollary 7.2.8. *Let W be of type $I_2(n)$ with n even and $\mathbf{k} \in \mathrm{Reg}(W)$.*

1. *If $k_1 \in \mathbb{Z}, k_2 \notin \mathbb{Z}$ then $(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}}) \cong (\mathrm{rep}_{\mathbb{C}}\mathbb{Z}_2, \otimes)$*
2. *If $k_1 \notin \mathbb{Z}, k_2 \in \mathbb{Z}$ then $(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}}) \cong (\mathrm{rep}_{\mathbb{C}}\mathbb{Z}_2, \otimes)$*

Proof. Follows from Theorem 7.2.6 and Tables 7.2 and 7.3. □

Corollary 7.2.9. *Let W be of type F_4 and $\mathbf{k} \in \mathrm{Reg}(W)$.*

1. *If $k_1 \in \mathbb{Z}, k_2 \notin \mathbb{Z}$ then $(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}}) \cong (\mathrm{rep}_{\mathbb{C}}S_3, \otimes)$*
2. *If $k_1 \notin \mathbb{Z}, k_2 \in \mathbb{Z}$ then $(\mathcal{HC}_{\mathbf{k}}, \otimes_{U_{\mathbf{k}}}) \cong (\mathrm{rep}_{\mathbb{C}}S_3, \otimes)$*

Proof. Follows from Theorem 7.2.6 and Tables 7.2 and 7.3. □

Bibliography

[Ach07] P. Achar, *Introduction to Perverse Sheaves*, Online lecture notes at the author's webpage https://www.math.lsu.edu/∼pramod/tc/ps/psln.pdf, 2007, Last accessed April 2012.

[AM94] M. F. Atiyah and I.G. MacDonald, *Introduction to Commutative Algebra*, The Perseus Books Group, 1994, New Edition.

[Ari95] S. Ariki, *Representation Theory of a Hecke Algebra of $G(r,p,n)$*, J. Algebra **177** (1995), 164–185.

[Ari07] ______, *Finite dimensional Hecke algebras*, arXiv.org **0708.0446v4** (2007).

[ARS10] M. Auslander, I. Reiten, and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Studies in Advanced Mathematics, vol. 36, Cambridge University Press, Cambridge, 2010, eBook Edition.

[Bas68] H. Bass, *Algebraic K-Theory*, first ed., W. A. Benjamin Inc., 1968.

[BC09] Yu. Berest and O. Chalykh, *Quasi-invariants of complex reflection groups*, arXiv.org **0912.4518.v2** (2009).

[BE09] R. Bezrukavnikov and P. Etingof, *Parabolic induction and restriction functors for rational Cherednik algebras*, Selecta Math. (N.S.) **14** (2009), no. 3-4, 397–425.

[BEG03a] Y. Berest, P. Etingof, and V. Ginzburg, *Finite-dimensional representations of rational Cherednik algebras*, Int. Math. Res. Not. (2003), no. 19, 1053–1088. MR MR1961261 (2004h:16027)

[BEG03b] Yu. Berest, P. Etingof, and V. Ginzburg, *Cherednik algebras and differential operators on quasi-invariants*, Duke Math. J. **118** (2003), 279–337.

[Ben10] D. J. Benson, *Representations and Cohomology I: Basic representation theory of finite groups and associative algebras*, Cambridge Studies in Advanced Mathematics, vol. 30, Cambridge University Press, Cambridge, 2010.

[BG03] K. A. Brown and I. Gordon, *Poisson orders, symplectic reflection algebras and representation theory*, J. Reine Angew. Math. **559** (2003), 193–216.

[BM93] M. Broué and G. Malle, *Zyklotomische Heckealgebren*, Astérisque **212** (1993), 119–189, Représentations unipotentes génériques et blocs des groupes réductifs finis.

[BMR98] M. Broué, G. Malle, and R. Rouquier, *Complex reflection groups, braid groups, Hecke algebras*, J. Reine Angew. Math. **500** (1998), 127–190.

[Bro01] M. Broué, *Reflection Groups, Braid Groups, Hecke Algebras, Finite Reductive Groups*, Current Developments in Mathematics **2000** (2001), 1–107.

[Bro10] ______, *Introduction to Complex Reflection Groups and their Braid Groups*, Lecture Notes in Mathematics, vol. 1988, Springer, 2010.

[CBH98] W. Crawley-Boevey and M. P. Holland, *Noncommutative deformations of Kleinian singularities*, Duke Math. J. **92** (1998), no. 3, 605–635.

[Che55] C. Chevalley, *Invariants of finite groups generated by reflections*, Amer. J. Math. **77** (1955), 778–782.

[Chl09] M. Chlouveraki, *Blocks and Families for Cyclotomic Hecke Algebras*, Lecture Notes in Mathematics, vol. 1981, Springer, Berlin, 2009.

[CPS88] E. Cline, B. Parshall, and L. Scott, *Finite dimensional algebras and highest weight categories*, J. Reine Angew. Math. **391** (1988), 8599.

[DG] C. Dunkl and S. Griffeth, *Generalized Jack polynomials and the representation theory of rational Cherednik algebras*, Selecta Math. to appear.

[Dro80] Yu. Drozd, *Tame and wild matrix problems*, Representation Theory II, Lecture Notes in Mathematics, vol. 832, Springer, 1980, pp. 242–258.

[EG02] P. Etingof and V. Ginzburg, *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism*, Invent. Math. **147** (2002), no. 2, 243–348.

[Erd90] K. Erdmann, *Blocks of tame representation type*, Lecture Notes in Mathematics, no. 1428, Springer-Verlag, New York, 1990.

[Fra01] W. N. Franzsen, *Automorphisms of Coxeter Groups*, Ph.D. Thesis, University of Sydney, Australia, 2001.

[Gab80] P. Gabriel, *Auslander-Reiten sequences and representation-finite algebras* , Representation Theory I, Ottawa 1979, Lecture Notes in Mathematics, vol. 831, Springer, Berlin/New York, 1980, pp. 1–71.

[GGOR03] V. Ginzburg, N. Guay, E. Opdam, and R. Rouquier, *On the category $\mathcal{O}$ for rational Cherednik algebras*, Invent. Math. **154** (2003), 617–651.

[Gin] V. Ginzburg, *Lectures on D-Modules*, Online lecture notes, available at Sabin Cautis' webpage http://www.math.columbia.edu/∼scautis/dmodules/dmodules/ginzburg.pdf, With collaboration of Baranovsky, V. and Evens S.

[Gin03] ———, *On primitive ideals*, Selecta Mathematica, New Series **9** (2003), 379–407.

[GL11] I. Gordon and I. Losev, *On category $\mathcal{O}$ for cyclotomic rational Cherednik algebras*, arXiv.org **1109.2315** (2011).

[Gor08] I. G. Gordon, *Symplectic reflection algebras*, Trends in representation theory of algebras and related topics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, pp. 285–347.

[GU89] A. Gyoja and K. Uno, *On the semisimplicity of Hecke algebras*, J. Math. Soc. Japan **41** (1989), 75–79.

[Gua03] N. Guay, *Projective modules in the category $\mathcal{O}$ for the Cherednik algebra*, Journal of Pure and Applied Algebra **182** (2003), 209–221.

[GZ05] K. R. Goodearl and J. J. Zhang, *Homological properties of quantized coordinate rings of semisimple groups*, arXiv.org (2005).

[Hat] A. Hatcher, *Algebraic Topology*, Cambridge University Press.

[Hod93] Timothy J. Hodges, *Noncommutative deformations of type-A Kleinian singularities*, J. Algebra **161** (1993), no. 2, 271–290.

[HTT08] R. Hotta, K. Takeuchi, and T. Tanisaki, *D-modules, Perverse Sheaves, and Representation Theory*, Progress in Mathematics, vol. 236, Birkhäuser Boston Inc., Boston, MA, 2008, Translated from the 1995 Japanese edition by Takeuchi.

[Hum92] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, first ed., Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1992.

[Hum08] ———, *Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$*, Graduate Studies in Mathematics, vol. 94, American Mathematical Society, 2008.

[Jan79] J.C. Jantzen, *Einhüllende Algebren halbeinfacher Lie-Algebren*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 750, Springer, 1979.

[Jan83] ———, *Moduln mit einem höchsten Gewicht*, Lecture Notes in Mathematics, vol. 3, Springer, 1983.

[JL01] G. James and M. Liebeck, *Representations and Characters of Groups*, second ed., Cambridge University Press, 2001.

[Kas03] M. Kashiwara, *D-modules and microlocal calculus*, Translations of Mathematical Monographs, vol. 217, American Mathematical Society, Providence, RI, 2003, Translated from the 2000 Japanese original by Mutsumi Saito, Iwanami Series in Modern Mathematics.

[KL00] G. R. Krause and T. H. Lenagan, *Growth of Algebras and Gelfand-Kirillov dimension*, revised edition ed., Graduate Studies in Mathematics, vol. 22, American Mathematical Society, 2000.

[Lam98] T.Y. Lam, *Lectures on Modules and Rings*, Graduate Texts in Mathematics, no. 189, Springer, 1998.

[Les94] Z. Leszczyński, *On the representation type of tensor product algebras*, Fund. Math. **144** (1994), 143–161.

[Lev90] T. Levasseur, *Relèvements d'opérateurs différentiels*, Operator Algebras, Unitary Representations, Enveloping Algebras and Invariant Theory, Prog. Math., vol. 92, Birkhäuser, Boston, 1990.

[Los11a] I. Losev, *Completions of symplectic reflection algebras*, arXiv.org **1001.0239v4** (2011).

[Los11b] I. Losev, *Isomorphisms of quantizations via quantization of resolutions*, arXiv.org **1010.3182v3** (2011).

[Max98] G Maxwell, *The Normal subgroups of finite and affine Coxeter groups*, Proceedings of the London Mathematical Society **76** (1998), 359–382.

[Maz09] V. Mazorchuk, *Lectures on $\mathfrak{sl}_2\mathbb{C}$-modules*, Imperial College Press, 2009.

[MM10] V. Mazorchuk and V. Miemietz, *Serre functors for Lie algebras and superalgebras*, arXiv.org **1008.1166** (2010).

[MR01] J.C. McConnell and J.C. Robson, *Noncommutative Noetherian Rings*, Graduate Studies in Mathematics, vol. 30, American Mathematical Society, 2001.

[Nak99] H. Nakajima, *Lectures on Hilbert Schemes of Points on Surfaces*, University Lecture Series, vol. 18, American Mathematical Society, Providence, RI, 1999.

[Rin75] C.M. Ringel, *The representation type of local algebras*, ICRA 1974, Lecture Notes in Mathematics, vol. 488, Springer, 1975, pp. 282–305.

[Rot09] J. J. Rotman, *An Introduction to Homological Algebra*, second ed., Universitext, Springer, New York, 2009.

[Smi90] S. P. Smith, *A class of algebras similar to the enveloping algebra of* $\mathfrak{sl}(2)$, Trans. Amer. Math. Soc. **322** (1990), no. 1, 285–314.

[SS10] D. Simson and A. Skowronski, *Elements of the Representation Theory of Associative Algebras Volume 3, Representation-infinite Tilted Algebras* , London Mathematical Society Student Texts, vol. 72, Cambridge University Press, Cambridge, 2010.

[ST54] G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Canadian J. Math. **6** (1954), 274–304.

[Ste64] R. Steinberg, *Differential equations invariant under finite reflection groups*, Trans. Amer. Math. Soc. **112** (1964), 392–400.

[Val06] R. Vale, *On category $\mathcal{O}$ for the rational Cherednik algebra of the complex reflection group* $(\mathbb{Z}/\ell\mathbb{Z}) \wr S_n$, Ph. D. Thesis, University of Glasgow, UK, 2006.

[Val09] ———, *Topics in Finite-Dimensional Algebras*, Online lecture notes at the author's webpage http://www.math.cornell.edu/∼rvale/fdalgebras.pdf, 2009, Last accessed June 2012.

[Wat60] C.E. Watts, *Intrinsic Characterizations of Some Additive Functors*, Proceedings of the American Mathematical Society **11** (1960), 5–8.