Lecture 9

We're going to be a little more sketchy today in order to get to the end of deformations. The results we are aiming for are well illustrated by the following example.

<u>Example</u>: $G = GL_n(\mathbb{C})$, B = upper Δ^{ar} matrices, U = strictly upper Δ^{ar} matrices, T = diagonal matrices; $\mathfrak{b} = \mathrm{Lie}(B)$, $\mathfrak{h} = \mathrm{Lie}(T)$, $\mathfrak{g} = \mathrm{Lie}(G)$.
(NB $B = U \rtimes T$)

$$\begin{tikzcd}
G \curvearrowright G/B = \text{flag variety} = \{F^0 \subset F^1 \subset F^2 \subset \dots \subset F^n : \dim F_i = i\} \arrow[r, hook] & \mathbb{P}(V(\omega_1) \oplus \dots \oplus V(\omega_{n-1})) \\
G \curvearrowright G/U = \text{base affine space} \arrow[u, "/T"] \arrow[r, hook] & V(\omega_1) \oplus \dots \oplus V(\omega_{n-1}).
\end{tikzcd}$$

with $F^0 = 0$, $F^n = \mathbb{C}^n$; $g \mapsto g \cdot (\text{highest weights})$; $G/U \circlearrowleft T$.

$T^*(G/U) = G \times_U \mathfrak{b}^*$, symplectic with two moment maps

$$\begin{tikzcd}
T^*(G/U) \arrow[r] \arrow[d] & \mathfrak{h}^* \\
\mathfrak{g}^* &
\end{tikzcd}
\qquad
\begin{tikzcd}
{[g, \alpha]} \arrow[r, mapsto] \arrow[d, mapsto] & \alpha|_{\mathfrak{h}} \\
g \cdot \alpha &
\end{tikzcd}$$

Both are T-equivariant for right action of T
$[g, \alpha] \cdot t = [gt, t^{-1}\alpha]$

So we have induced
$$\begin{tikzcd}
T^*(G/U)/T = \tilde{\mathfrak{g}} = G \times_B \mathfrak{b}^* \arrow[r, "\nu"] \arrow[d, "\mu"] & \mathfrak{h}^* \arrow[d] \\
\mathfrak{g}^* \arrow[r, "\alpha"] & \mathfrak{h}^*/W
\end{tikzcd}$$

On identifying $\mathfrak{g} \simeq \mathfrak{g}^*$ via Tr:
$$\begin{tikzcd}
\tilde{\mathfrak{g}} = \{(F^\cdot, X) \in G/B \times \mathfrak{g} : X(F^i) \subseteq F^i \ \forall i\} \arrow[r, "\nu"] \arrow[d, "\mu"] & \mathfrak{h} \arrow[d, "\text{forget ordering}"] \\
\mathfrak{g} \arrow[r, "X \mapsto \mathrm{evals}(X)"] & \mathfrak{h}/W
\end{tikzcd}$$

$\nu: (F^\cdot, X) \mapsto (X|_{F^i/F^{i-1}})_{i=1..n}$, $\mu: (F^\cdot, X) \mapsto X$

This is the Grothendieck-Springer <u>semiuniversal</u> deformation. Note each fibre $\nu^{-1}(\lambda)$ is symplectic as it is the hamiltonian reduction of $T^*(G/U)$ w.r.t. T at λ. It is called a twisted cotangent bundle: $T^*_\lambda(G/B)$

<u>Theorem</u> (Namikawa) Let X be a conical symplectic singularity and let $\pi: Y \longrightarrow X$ be a partial resolution of X where Y is $\mathbb{Q}$-factorial and has terminal singularities such that K_Y, the canonical sheaf on Y, is trivial (*) (i.e. π crepant)

(i) $\exists\ \mathcal{Y}$ & $\mathcal{X}$ varieties over $\mathbb{A}^d$ s.t. at $0 \in \mathbb{A}^d$ $\mathcal{Y}$ and $\mathcal{X}$ are both the universal Poisson deformations of Y and X

(ii) $\exists$ commutative diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\hat{\pi}} & \mathcal{X} \\ \nu \downarrow & & \downarrow \alpha \\ \mathbb{A}^d & \xrightarrow{f} & \mathbb{A}^d \end{array}$$

s.t. a) $\hat{\pi}$ extends π at 0

b) f is a Galois covering

(iii) $\mathcal{Y}$ is a locally trivial deformation of Y

(iv) The morphism $\nu_t : \mathcal{Y}_t \longrightarrow \mathcal{X}_{f(t)}$ is an isomorphism at a general point $t \in \mathbb{A}^d$.

<u>BACK TO THE EXAMPLE</u>: $\pi : \underset{Y}{\underset{\parallel}{T^*(G/B)}} \xrightarrow{\pi} \underset{X}{\underset{\parallel}{\mathcal{N}}}$, the Springer resolution.

Then $\mathcal{Y} = \tilde{\mathfrak{g}}$, $\mathcal{X} = \mathfrak{g}$ and we have $\mathcal{Y} \xrightarrow{\nu} \mathbb{A}^d = \mathfrak{h}$, $\mathcal{X} \xrightarrow{\alpha} \mathbb{A}^d = \mathfrak{h}/W$

Moreover $T^*_\lambda(G/B) = \{[g, \lambda + u] : u \in Lie(U)\} \xrightarrow{\varphi_\lambda} \{[g,u]\} = T^*(G/B)$ is a diffeomorphism. If $\lambda = diag(\lambda_1 \ldots \lambda_n)$ with $\lambda_i \neq \lambda_j$ for all $i \neq j$, then $T^*_\lambda(G/B) \xrightarrow{\nu} \mathfrak{g}$ is an isomorphism onto its image since if $X \in \mathfrak{g}$ has the form $X = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ then the only flag that is stable under X with $X|_{F^i/F^{i-1}} = \lambda_i$ is $F^i = span\{e_1, \ldots, e_i\}$. □

<u>Corollary</u> (Namikawa) Let X be a conical symplectic singularity. Then X admits a symplectic projective resolution if and only if X has a smoothing by a Poisson deformation.

<u>Proof</u>: $\Rightarrow$ $\mathcal{Y}_t$ is smooth, so $\mathcal{X}_{f(t)}$ is too.

$\Leftarrow$ take $\pi: Y \to X$ as above. Then $\exists t$ s.t. $\mathcal{Y}_t = \mathcal{X}_{f(t)}$ is smooth since f is onto. Then $\mathcal{Y}_0 \underset{diff}{\simeq} \mathcal{Y}_t$ is smooth. □

(*) By advances in the Minimal Model Programme due to Birkar-Cascini-Hacon-McKernan, such a partial resolution exists.

Other examples of this picture include:

- semi-universal deformation of Kleinian singularities
- restriction of Grothendieck – Springer resolution to Slodowy slices at nilpotent x (Lehn - Namikawa - Sorger) provided $H^2(T^*G/B, \mathbb{C}) \longrightarrow H^2(\pi^{-1}(x), \mathbb{C})$ is an isomorphism.
- one might expect "generalized Calogero - Moser spaces" to be fibre products of universal deformations of orbit spaces $T^*\mathfrak{h}/G$ with $G \leq GL(\mathfrak{h})$.

(*) PERIOD MAP

Proof of theorem: 1) We still have to see PD_X is unobstructed.

a) Use lemma 4.5 to pass from PD_X to PD_U for $U = X_{reg} \cup$ symp. leaves of codim 2. $U \setminus X_{reg} =: \Sigma$.

b) Since U only has Kleinian singularities, we can resolve minimally (i.e. crepantly, i.e. symplectically) $\pi : \hat{U} \longrightarrow U$ and local cohomology calc^n (as before) will show $H^1(\hat{U}, \mathcal{O}_{\hat{U}}) = H^2(\hat{U}, \mathcal{O}_{\hat{U}}) = 0$.

c) From this cohomology result a theorem of Wahl $\Rightarrow \exists\ PD_{\hat{U}} \longrightarrow PD_U$ i.e. scheme deformation of $\hat{U}$ descends to U (Wahl) & then Poisson structure on $\pi^{-1}(U_{reg})$ pushes to Poisson structure on U.

d) Represent $PD_{\hat{U}}$ by $R_{\hat{U}}$, PD_U by R_U. Then $\tau : R_U \longrightarrow R_{\hat{U}}$.

e)
$$\begin{array}{ccccccc} 0 \to & PD_{lt, U}(\mathbb{C}[\varepsilon]) & \longrightarrow & PD_U(\mathbb{C}[\varepsilon]) & \longrightarrow & H^0(\Sigma, \mathcal{H}) & \\ & \uparrow{\scriptstyle 2} & & \uparrow & & & \\ 0 \longrightarrow & H^2(U, \pi_* \mathbb{C}) & \longrightarrow & H^2(\hat{U}, \mathbb{C}) & \longrightarrow & H^0(U, R^2\pi_* \mathbb{C}) & \longrightarrow 0 \quad \text{(Leray)} \\ & \| & & \| & & & \\ & \mathbb{C} & & PD_{\hat{U}}(\mathbb{C}[\varepsilon]) & & & \end{array}$$

$\exists$ interesting comparison of $\mathcal{H}$ & $R^2\pi_*\mathbb{C}$: first is Poisson deformations of U considered only as deformations; second is Poisson deformations of $\hat{U}$ considered only as deformations (and as a sheaf, pushed down to U). Can show that monodromy of the two are related, essentially by uniqueness of minimal resolution. In particular

$$\dim H^0(U, \mathcal{H}) = \dim H^0(U, R^2\pi_*\mathbb{C})$$

$\therefore \dim PD_U(\mathbb{C}[\varepsilon]) \leq \dim PD_{\hat{U}}(\mathbb{C}[\varepsilon])$

f) $\dim \tau^{-1}(0)$ is finite: i.e. show it has no curves passing through it.

This is then the study of deformations of $\hat{\mathcal{U}}$ over $\mathbb{C}[[t]]$ that are trivial on passing down to the associated deformation on $\mathcal{U}$. But around each point $p \in \Sigma$ the deformation on $\pi^{-1}(p) \subset \hat{\mathcal{U}}$ must be trivial since this is a Kleinian singularity calculation. Now glue together

g) $PD_{\hat{\mathcal{U}}}$ is unobstructed: because controlled by $H^2(\hat{\mathcal{U}}^{an}, \mathbb{C})$ and Gauss-Manin $\Rightarrow$ $H^2(\mathcal{U}^{an}, S) = H^2(\mathcal{U}^{an}, \mathbb{C}) \otimes S$ for any artinian S.

$\Rightarrow R_{\hat{\mathcal{U}}}$ is regular.

h) $\dim R_{\hat{\mathcal{U}}} \leq \dim R_{\mathcal{U}} + \dim \tau^{-1}(0) = \dim R_{\mathcal{U}} \leq \dim PD_{\mathcal{U}}(\mathbb{C}[\varepsilon]) \leq \dim PD_{\hat{\mathcal{U}}}(\mathbb{C}[\varepsilon])$

$$\dim PD_{\hat{\mathcal{U}}}(\mathbb{C}[\varepsilon]) = \dim R_{\hat{\mathcal{U}}}$$

$\Rightarrow R_{\mathcal{U}}$ regular, so unobstructed.

2) We now lift $\mathbb{C}^\times$-action on X to $\mathbb{C}^\times$-action on Y:

a) $\pi: Y \longrightarrow X$ includes inside it $\pi: \hat{\mathcal{U}} \longrightarrow \mathcal{U}$. So $\mathbb{C}^\times$-action lifts by uniqueness of min^l resolution to $\hat{\mathcal{U}}$. π is semi-small $\Rightarrow$ $\operatorname{codim}(Y \setminus \hat{\mathcal{U}}) \geq \frac{1}{2} \operatorname{codim}(X \setminus \mathcal{U}) \geq 2$.

b) Extend the action from $\hat{\mathcal{U}}$ to Y by using π-ample bundles and discreteness of appropriate Picard groups.

3) Use Grothendieck's algebraization theorem: the universal deformations of Y and X respectively have families $\{Y_n\}_n$ and $\{X_n\}$ over R_Y/m_Y^n & R_X/m_X^n and moreover $Y_n \longrightarrow X_n$ is projective (with ample line bundle a lift of a relatively ample line bundle for $Y \xrightarrow{\pi} X$, this being unobstructed since $H^2(Y, \mathcal{O}_Y) = 0$) and so $\exists\ \hat{\mathcal{Y}} \longrightarrow \hat{\mathcal{X}}$, completions of schemes Y and X over R_Y and R_X which produce Y_n & X_n.

4) The $\mathbb{C}^\times$-action on X induces one on: Y (as we've seen), on R_X & R_Y and on Y_n and X_n, and on $\hat{\mathcal{Y}}$ and $\hat{\mathcal{X}}$. The basic observation that if (A, m) is a complete local ring with $\mathbb{C}^\times$-action such that all weights on m are non-negative, then

$$A\text{-mod}^{f.g.}_{gr} \xleftrightarrow{\sim} A^{fin}\text{-mod}^{f.g.}_{gr}$$

$$M \longmapsto M^{fin}$$

$$A \otimes_{A^{fin}} N \longleftarrow N$$

where A^{fin} = span{ all eigenvectors for $\mathbb{C}^\times$-action}, and similarly for M^{fin}.
This allows one to produce $\mathbb{A}^d$ from R_X, $\mathbb{A}^d$ from R_Y, $\mathcal{X}$ from $\hat{\mathcal{X}}$, $\mathcal{Y}$ from $\hat{\mathcal{Y}}$, and the required diagram in the theorem.

5) To see the map $\mathbb{A}^d \xrightarrow{f} \mathbb{A}^d$ is Galois one considers again

$$\begin{array}{ccc} PD_Y & \longrightarrow & PD_{\tilde{\hat{u}}} \\ \downarrow & & \downarrow \\ PD_X & \longrightarrow & PD_u \end{array}$$

and reduces to the case $PD_{\hat{u}} \longrightarrow PD_u$. Then there will be several components for Σ. Let B be one such. Then the Poisson deformations are related to the global sections of the local system $(R^2\pi_*\mathbb{C})_p$ for $p \in B$. This is $H^2(\pi^{-1}(p), \mathbb{C})$ which is a system of $\mathbb{P}^1$'s glued together via Lie theoretic data of type A, D, E depending on the Kleinian singularity ∴ the monodromy produces an automorphism of this data i.e τ a graph automorphism of the root system. The covering group is then $\prod_B W_B$ where $W_B = \{w \in W : \tau w \tau^{-1} = w\}$ i.e. takes picture for deform^n of KS

Weyl gp type A, D, E

and then introduces global data of trivial monodromy.

6) The local triviality in C^∞-setting is achieved by checking in the terminal singularities case that $PD = PD_{et}$: both unobstructed, IM on tangent space and $PD_{et} \rightarrow PD$.

7) It's more complicated to get the last claim: one introduces "twistor deformations" of Y i.e. $H^2(Y, \mathbb{Z}) \simeq Pic(Y)$ and so line bundles produce quantizations. Ampleness of a line bundle ⇝ affineness of corresp. deformation □

⊛ The Period Map: The local triviality of Y_t's $\Rightarrow$ $H^2(Y_t, \mathbb{C}) \xrightarrow[\varphi_t]{\sim} H^2(Y, \mathbb{C})$
But each Y_t carries ω_t $\Rightarrow$ get a morphism $\mathbb{A}^d \longrightarrow H^2(Y, \mathbb{C})$ $t \mapsto \varphi_t(\omega_t)$ which is called the <u>Period Map</u>. It is an isomorphism around the origin of $\mathbb{A}^d$ and $[\omega] \in H^2(Y, \mathbb{C})$: used originally by Kaledin-Verbitsky.