§ The unobstructedness of PD_X.

Again, we need to study the two dimensional case, which is rather famous.

We have already seen that the Slodowy slice $\mathcal{S} \subset \mathfrak{g}$ to the subregular nilpotent orbit produces a Poisson deformation of S, a Kleinian singularity.

Now, in this case we have a short exact sequence

$$0 \longrightarrow PD_{lt,S}(\mathbb{C}[\varepsilon]) \longrightarrow PD_S(\mathbb{C}[\varepsilon]) \longrightarrow T^1_S \longrightarrow 0$$

(it is surjective since we saw all deformations inherited a Poisson structure). But we also have $PD_{lt,S}(\mathbb{C}[\varepsilon]) = H^2(S^{an}, \mathbb{C}) = 0$ by the $\mathbb{C}^\times$-action.

Hence $PD_S(\mathbb{C}[\varepsilon]) \cong T^1_S$. Now $\mathcal{S}$ is the semi-universal deformation, and it lives over $\mathfrak{h}/W = \mathfrak{g}/\!/G$. Thus we deduce that it is the universal Poisson deformation of S.

Now consider the following spaces:

$$\begin{array}{c} \tilde{\mathfrak{g}} := G \times_B \mathfrak{b} = \{ (\tilde{\mathfrak{b}}, x) \in G/B \times \mathfrak{g} \ : \ x \in \tilde{\mathfrak{b}} \} \subset G/B \times \mathfrak{g} \\ \downarrow \mu \\ \mathfrak{g} \end{array}$$

This is called the Grothendieck-Springer simultaneous resolution. It admits a morphism

$$\tilde{\mathfrak{g}} \longrightarrow \mathfrak{h} \qquad (\hat{\mathfrak{b}}, x) \mapsto x + [\hat{\mathfrak{b}}, \hat{\mathfrak{b}}] \in \hat{\mathfrak{b}}/[\hat{\mathfrak{b}}, \hat{\mathfrak{b}}] \cong \mathfrak{h}$$

In this way we have
$$\begin{array}{ccc} \tilde{\mathfrak{g}} & \longrightarrow & \mathfrak{h} \\ \downarrow & & \downarrow \\ \mathfrak{g} & \longrightarrow & \mathfrak{h}/W \end{array}$$
i.e. $\tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}$ and this is a simultaneous resolution of $\mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h} \longrightarrow \mathfrak{h}$

There is a symplectic form on $\tilde{\mathfrak{g}}$ by pullback of the form on $\mathfrak{g}$.

Now $\tilde{\mathcal{S}} := \mu^{-1}(\mathcal{S} \times_{\mathfrak{h}/W} \mathfrak{h})$; then $\mu|_{\tilde{\mathcal{S}}} : \tilde{\mathcal{S}} \longrightarrow \mathcal{S} \times_{\mathfrak{h}/W} \mathfrak{h}$ is a simultaneous res^n and the symplectic form persists.

In ptc we get $\tilde{\mathcal{S}} \longrightarrow \mathfrak{h}$, a Poisson (= symplectic) deformation of the resolution $\tilde{S}$ of S. Moreover Slodowy proved that $\tilde{\mathcal{S}}$ was a semiuniversal deformation of the algebraic variety S. But then this is classified by $H^1(\tilde{S}, T_{\tilde{S}}) \cong H^1(\tilde{S}, \Omega^1_{\tilde{S}}) \cong H^2(\tilde{S}, \mathbb{C}) = PD_{\tilde{S}}(\mathbb{C}[\varepsilon])$. So again $\tilde{\mathcal{S}}$ is the universal Poisson deformation.

These two results then extend to $S \times \mathbb{C}^{2n-2}$ and $\widetilde{S} \times \mathbb{C}^{2n-2}$: in the first case by Lemma ?, the second because $H^2(\widehat{S} \times \mathbb{C}^{2n-2}) = H^2(\widehat{S})$.

<u>Theorem</u> (Namikawa) Let X be a symplectic singularity. Then PD_X is unobstructed

<u>Proof</u>: Let $\mathcal{U}$ be the locus of smooth or rational double points, as before. It can be resolved minimally by $\widehat{\mathcal{U}} \xrightarrow{\pi} \mathcal{U}$ (where in a nbhd of a rational double point we just see the resolution of singularities from before).
Let $Z = X \setminus \mathcal{U}$, $\operatorname{codim} Z \geq 4 \Rightarrow H^j_Z(\mathcal{O}_X) = 0 \quad j \leq 3$ by Cohen-Macaulayness

$$\therefore \to H^i(X, \mathcal{O}_X) \to H^i(\mathcal{U}, \mathcal{O}_\mathcal{U}) \to H^{i+1}_Z(X, \mathcal{O}_X) \to \cdots$$

& rat^l sings of $X \Rightarrow H^i(\mathcal{U}, \mathcal{O}_\mathcal{U}) = 0$ for $i = 1, 2$. Then $H^{i+j}(\widetilde{\mathcal{U}}, \mathcal{O}_{\widetilde{\mathcal{U}}}) \Leftarrow H^i(\mathcal{U}, R^j f_* \mathcal{O}_{\widetilde{\mathcal{U}}})$ collapses as $\mathcal{U}$ has rat. sings $\Rightarrow H^1(\widehat{\mathcal{U}}, \mathcal{O}_{\widetilde{\mathcal{U}}}) = H^2(\widehat{\mathcal{U}}, \mathcal{O}_{\widetilde{\mathcal{U}}}) = 0$.

$*$: $\exists\ PD_{\widetilde{\mathcal{U}}} \to PD_\mathcal{U}$. This arises from a result of Wahl which says a 1st order deform^n of $\widehat{\mathcal{U}}$ (not. nec Poisson) induces one on $\mathcal{U}$; then one takes the Poisson structure on $\widehat{\mathcal{U}}$, restricts to smooth locus, passes down to $\mathcal{U}$ and extends, just as usual!
By earlier style analysis, both $PD_{\widetilde{\mathcal{U}}}$ & $PD_\mathcal{U}$ are universal, represented by $R_{\widetilde{\mathcal{U}}}$ & $R_\mathcal{U}$ respectively. $\therefore R_\mathcal{U} \to R_{\widetilde{\mathcal{U}}}$.

$*$ $H^{i+j}(\widehat{\mathcal{U}}, \mathbb{C}) \Leftarrow H^i(\mathcal{U}, R^j \pi_* \mathbb{C})$. But $R^1 \pi_* \mathbb{C} = 0$ (forced by rat^l sings) and so one gets an exact sequence

$$\begin{array}{ccccccc}
0 \to & H^2(\mathcal{U}, \overset{\pi_* \mathbb{C}}{\overset{\parallel}{\mathbb{C}}}) & \longrightarrow & H^2(\widehat{\mathcal{U}}, \mathbb{C}) & \longrightarrow & H^0(\mathcal{U}, R^2 \pi_* \mathbb{C}) & \longrightarrow 0 \\
& & & \parallel & & & \\
& \Big\downarrow & & PD_{\widehat{\mathcal{U}}}(\mathbb{C}[\varepsilon]) & & & \\
& & & \Big\downarrow & & & \\
0 \to & PD_{lt, \mathcal{U}}(\mathbb{C}[\varepsilon]) & \longrightarrow & PD_\mathcal{U}(\mathbb{C}[\varepsilon]) & \longrightarrow & \mathcal{H}^0(\Sigma, \mathcal{H}) &
\end{array}$$

Note that $R^2 \pi_* \mathbb{C}$ is supported on Σ: it turns out the two RHS are closely related i.e. the same dimension:

$$\dim PD_\mathcal{U}(\mathbb{C}[\varepsilon]) \leq \dim PD_{\widehat{\mathcal{U}}}(\mathbb{C}[\varepsilon])$$

$*$ Now we want to deduce that $\operatorname{Spec}(R_{\widetilde{\mathcal{U}}}) \to \operatorname{Spec}(R_\mathcal{U})$ is "quasi-finite" i.e. closed fibre is finite. So take $\alpha: R_{\widetilde{\mathcal{U}}} \to \mathbb{C}[[t]]$ corresponding to a mapping from a curve s.t. $R_\mathcal{U} \to R_{\widetilde{\mathcal{U}}} \xrightarrow{\alpha} \mathbb{C}[[t]]$ is trivial i.e. factors through $R_\mathcal{U}/m_\mathcal{U} = \mathbb{C}$. Restrict to U_p & $\tilde{U}_p$ for any $p \in \Sigma$ (germs of resolution of double pts) we can induce deformations

i.e $R_{u_p} \to R_{\tilde{u}_p} \xrightarrow{\alpha} \mathbb{C}[[t]]$ factoring through R_{u_p}/m_{u_p}. But $\mathrm{Spec}(R_{\tilde{u}_p}) \to \mathrm{Spec}(R_{u_p})$ is Galois by the analysis we looked at at the beginning of the lecture $\Rightarrow$ the deformation of $\hat{u}_p$ is also trivial, as a Poisson deformation

$\Rightarrow$ whole deformation is trivial.

* tan.sp. $PD_{\hat{u}}$ controlled by $H^2(\tilde{u}^{an}, \mathbb{C})$ (as we've seen). But then Gauss-Manin implies

$$H^2(\hat{u}_n^{an}, T_n) = H^2(\hat{u}^{an}, \mathbb{C}) \underset{\mathbb{C}}{\otimes} T_n \quad \text{for a deform}^n\ \tilde{u}_n \text{ over } T_n$$

and so it is straightforward to check unobstructedness

(Write $R_{\tilde{u}} = \mathbb{C}[[x_1 \dots x_r]]/J$. Let $0 \to (\varepsilon) \to A' \to A \to 0$ be a sequence in Art.

Then generally

$$\mathrm{Hom}(R_{\hat{u}}, A') \to \mathrm{Hom}(R_{\hat{u}}, A) \xrightarrow{ob} (J/m_R J)^* \otimes I$$

is exact. Now we have $A' = \dfrac{\mathbb{C}[t]}{t^{n+1}}$, $A = \dfrac{\mathbb{C}[t]}{t^n}$ giving

$$\begin{array}{ccccccccc} 0 & \to & (t^{n+1}) & \to & A' & \to & A & \to & 0 \\ & & \downarrow\wr & & \downarrow {\scriptstyle t \mapsto t+\varepsilon} & & \downarrow & & \\ 0 & \to & (t^n \varepsilon) & \to & A[\varepsilon] & \to & A''[\varepsilon] \times_{A''} A & \to & 0 \end{array}$$

which on applying $\mathrm{Hom}(R_{\hat{u}}, -)$ gives

where $A'' = \dfrac{\mathbb{C}[t]}{t^{n-1}}$

$$\begin{array}{ccccc} PD_{\tilde{u}}(A') & \to & PD_{\tilde{u}}(A) & \to & t^{n+1} \otimes (J/m_R J)^* \\ \downarrow & & \downarrow & & \downarrow\wr \\ PD_{\tilde{u}}(A[\varepsilon]) & \to & PD_{\hat{u}}(A''[\varepsilon] \times_{A''} A) & \to & t^n \varepsilon \otimes (J/m_R J)^* \end{array}$$

surjective by above observation $\quad \therefore PD_{\tilde{u}}(A') \to PD_{\tilde{u}}(A)$ surjective.)

It follows that $R_{\tilde{u}}$ is regular. (i.e. $\mathbb{C}[PD_{\tilde{u}}(\mathbb{C}[\varepsilon])]^\wedge_0$.

* $\dim R_{\tilde{u}} \leq \dim R_u + \dim \pi^{-1}(0) = \dim R_u \leq \dim PD_u(\mathbb{C}[\varepsilon]) \leq \dim PD_{\tilde{u}}(\mathbb{C}[\varepsilon]) = \dim R_{\tilde{u}}$

$\Rightarrow R_u$ is regular i.e PD_u unobstructed.

* Now, by Lemma?, $PD_u(U_n/T_n, T_n[\varepsilon]) \simeq PD_X(X_n/T_n, T_n[\varepsilon])$ so unobstructedness passes to PD_X. $\square$

We will now pass from formal deformations to actual varities by using a $\mathbb{C}^*$-action. So from now on assume that X is a conical symplectic singularity

Lemma (Namikawa). Let X be a conical symplectic singularity and $f: Y \longrightarrow X$ a partial resolution s.t. Y has terminal singularities. Then the $\mathbb{C}^*$-action on X lifts to Y.

(<u>Def</u>^{n}: A variety has terminal singularities if for any resolution $\tilde{f}: \hat{Y} \to Y$ we have $K_{\hat{Y}} = \tilde{f}^* K_Y + \sum_{a_i > 0} a_i E_i$ where the E_i are the components of the exceptional divisor.)

<u>Proof</u>: $X \supset U$, the good set as usual with $\operatorname{codim} Z \geqslant 4$ for $Z = X \setminus U$. Moreover $f|_{f^{-1}(U)} : \hat{U} := f^{-1}(U) \longrightarrow U$ is the min^l resolution. Therefore, by uniqueness of this resolution, we have that the $\mathbb{C}^\times$-action extends to $\tilde{U}$. However $\operatorname{codim}(Y \setminus \hat{U}) \geq 2$ since Namikawa proved that crepant partial resolutions of symplectic singularities are semi-small. (i.e $\operatorname{codim}(X \setminus U) \leqslant \frac{1}{2} \operatorname{codim}(Y \setminus f^{-1}(U))$). Thus we get a rational map $\sigma: \mathbb{C}^\times \times Y \dashrightarrow Y$ which is defined at least on $\hat{U}$, and so in codim 1.
Now let $\mathcal{L}$ be an f-ample line bdle (i.e. produces an embedding in $\mathbb{P}_X(H^0(Y, \mathcal{L}^m))$)
Set $\tilde{\mathcal{L}}_t := \sigma^*(\mathcal{L}|_{\hat{U}})|_{\{t\} \times \hat{U}}$ a line bundle on $\hat{U}$. $\operatorname{Pic}(\hat{U}) \simeq H^2(\hat{U}, \mathbb{Z})$ and so is discrete and so $\tilde{\mathcal{L}}_t \simeq \tilde{\mathcal{L}}_1 = \mathcal{L}|_{\hat{U}}$. As $\mathcal{L}|_{\hat{U}}$ extends to Y, so two does each $\tilde{\mathcal{L}}_t$ and so we get a line bundle on $\mathbb{C}^\times \times Y$ extending $(\sigma|_U)^* \mathcal{L}$. Call it $\sigma^* \mathcal{L}$. Since $\mathcal{L} = (\sigma^* \mathcal{L})|_{\{1\} \times X}$ is f-ample we see that $\mathcal{L}$ is ample on each fibre of f. But then $\mathcal{L}_t = (\sigma^* \mathcal{L})|_{\{t\} \times X}$ are also f-ample since $\operatorname{Pic}(Y)/f^* \operatorname{Pic}(X)$ is discrete. Thus we get

$$Y \dashrightarrow Y \hookrightarrow \mathbb{P}_X(H(\mathcal{L}^{\otimes m})) = Y \hookrightarrow \mathbb{P}_X(H(\mathcal{L}_t^{\otimes m}))$$

and so $\sigma_t : Y \to Y$. Putting these all together produces an action of $\mathbb{C}^\times$: $\sigma_t \sigma_s = \sigma_{t+s}$ since they are equal on $\hat{U}$. $\square$

Now we require a big modern theorem

<u>Theorem</u> (Birkar–Cascini–Hacon–McKernan) Let X have symplectic singularities. Then $\exists\, f: Y \to X$ a partial crepant resolution which is projective and s.t. Y is $\mathbb{Q}$-factorial and has terminal singularities.