Lecture 7: Deformations of symplectic singularities.

THM (Namikawa '05) Let X be a symplectic singularity. Then

$$PD_X \cong \mathrm{Hom}_{Art}(R, -)$$

where $R = \mathbb{C}[\![HP^2(X,\mathbb{C})]\!]$, a formal power series ring in the finite dimensional space of 2nd Poisson cohomology of X.

Remark:

i) If X is actually smooth (i.e. symplectic) it's easy enough to understand this theorem. For all (local artinian) deformations of X are trivial as varieties i.e. $\mathcal{X} \cong X \times S$, so all that can happen is that the symplectic form varies. But up to 1st-order it has the form $\omega_X + h\omega'$ for some closed 2-form ω' on X (non-degeneracy is obvious), while we find $\omega_X + h\omega'$ and $\omega_X + h\omega''$ are equiv. iff they have ω' and ω'' differing by an infinitesimal symplectic automorphism i.e. by a Hamiltonian vector field ξ and via the identification of T_X with T_X^* this is just gotten by $\omega'' = \omega' + d\xi^\#$. i.e. $H^2(X,\mathbb{C})$ classifies the first order deform$^{\text{ns}}$ and this iterates to higher order.

ii) Kaledin-Verbitsky have generalized (i) to X symplectic variety with $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$.

I want to recall some classic deformation theoretic language of Schlessinger:

Def^n 44: We call PD_X universal if $PD_X \Leftarrow \mathrm{Hom}_{Art}(R,-)$ for some complete local k-algebra. In other words the canonical homomorphisms $R \to R/\mathfrak{m}^n$ produce elements $X_n \in PD_X(R/\mathfrak{m}^n)$ s.t. for any other suite of compatible deformations $(Y_n) \in PD_X(S/\mathfrak{m}^n)$ $\exists!$ alg. map $R \longrightarrow S$ s.t. $Y_n \simeq X_n \times_{R/\mathfrak{m}^n} S/\mathfrak{m}^n$.

We call PD_X unobstructed if $PD_X(S) \twoheadrightarrow PD_X(S')$ for any $S \twoheadrightarrow S'$.

Schlessinger gave criteria for a functor (such as PD_X) to be universal:

(H1) $PD_X(\mathbb{C}) =$ singleton

(H2) Suppose $0 \to (\varepsilon) \to A'' \to A \to 0$. Then $PD_X(A' \times_A A'') \twoheadrightarrow PD_X(A') \times_{PD_X(A)} PD_X(A'')$

(H3) H2 is a bijection whenever $A'' = \mathbb{C}[\varepsilon]$ & $A = \mathbb{C}$

(H4) $PD_X(\mathbb{C}[\varepsilon])$ is finite dimensional

(H5) Suppose $0 \to (\varepsilon) \to A'' \to A \to 0$ and $\eta \in PD_X(A)$ with $p^{-1}(\eta) \neq \emptyset$. Then $PD_X(\mathbb{C}[\varepsilon])$ acts simply transitively on $p^{-1}(\eta)$.

In our situation (H1) is trivial; (H2) and (H3) are dealt with exactly as for deformations of varieties (see Hartshorne Thm 18.1). It is (H4) and (H5) that are difficult. Given (H4) standard arguments (Hartshorne Theorem 18.2) show that (H5) is equivalent to: if $\mathfrak{X}/_T \in PD_X(T)$, $\overline{T} \hookrightarrow T$ closed subscheme, then any automorphism of $\mathfrak{X} \times_T \overline{T}$ that restricts to the identity on X, can be extended to $\mathfrak{X}$.

§ Why $PD(\mathbb{C}[\varepsilon])$ is finite dimensional.

By definition, $PD(\mathbb{C}[\varepsilon])$ in the set of infinitesimal (or 1st order) deformations. It is a result of Kaledin (hinted at above) that

- $X_{reg} \cup Sing(X)_{reg} \supseteq U$, has codim $(U \subset X) \geq 4$ (finiteness of number of symplectic leaves)
- For $p \in Sing(U)$ there is an analytic neighbourhood of p which is isomorphic to $(S, 0) \times (\mathbb{C}^{\dim X - 2}, 0)$ where S is a Kleinian singularity.

__Lemma 45__ Take U as above (the crucial point is codim $(U \subset X) \geq 3$. Then

$$PD_X(\mathbb{C}[\varepsilon]) \cong PD_U(\mathbb{C}[\varepsilon])$$

__Proof__: X has rational singularities and so is Cohen-Macaulay, and so satisfies the Serre's conditions S_i $\forall i$. It follows (Kollar-Mori) that

$$Ext^1_X(\Omega_X, \mathcal{O}_X) \xrightarrow{\sim} Ext^1_U(\Omega_U, \mathcal{O}_U)$$

But these spaces classify infinitesimal deformations of X and U respectively (e.g. cotangent complex / Kodaira spencer map). Hence an inf. Poisson deformation $\mathcal{U}$ of U will give rise to an infinitesimal deformation $\mathfrak{X}$ of X. But then $\mathcal{O}_{\mathfrak{X}} = j_* \mathcal{O}_{\mathcal{U}}$ because $\mathcal{U} \subset \mathfrak{X}$ has codim ≥ 2 and so the Poisson bracket on $\mathcal{U}$ extends to $\mathfrak{X}$ uniquely. $\square$

Therefore we may reduce to the case where everything looks locally like $(\mathbb{C}^{2n}, 0)$ or $(S, 0) \times (\mathbb{C}^{2n-2}, 0)$.

Now consider the map of sheaves $PT^1_{X^{an}} \xrightarrow{For} T^1_{X^{an}}$ where $PT^1_{X^{an}}$ and $T^1_{X^{an}}$ denote the sheaves of first order deformations and the map is forgetting the Poisson structure. Let $\mathcal{H} = \mathrm{im}(For)$.

__Lemma__ $\mathcal{H}$ is a locally constant sheaf on $\Sigma = Sing(X)$.

__Proof__: It is supported on Σ since there are no non-trivial deformations of smooth spaces. Now cover Σ with open sets from X such that $U_\alpha \overset{\varphi_\alpha}{\cong} S \times \mathbb{C}^{n-2}$.

Now $T^1_{\mathcal{U}^{an}_\alpha} \cong (p_1 \circ \varphi_\alpha)^* T^1_S$. But in the last lecture we saw that $\operatorname{Im}(PT^1_{\mathcal{U}^{an}_\alpha} \longrightarrow T^1_{\mathcal{U}^{an}_\alpha}) = (p_1 \circ \varphi_\alpha)^{-1} T^1_S$ and this space is a finite dimensional vector space since S has isolated singularities. $\square$

We have an exact sequence

$$0 \to PD_{lt,X}(\mathbb{C}[\varepsilon]) \longrightarrow PD_X(\mathbb{C}[\varepsilon]) \longrightarrow H^0(X, \mathcal{H})$$

where "lt" stands for locally trivial.

__Lemma__ $PD_{lt,X}(\mathbb{C}[\varepsilon]) = H^2(X^{an}, \mathbb{C})$

__Proof__: We've already seen for X_{reg} that $PD_{lt,X_{reg}}(\mathbb{C}[\varepsilon]) \cong H^2(X^{an}_{reg}, \mathbb{C})$ via the sketch of the Kaledin – Verbitsky argument. Rather, we should consider it as the first cohomology of $(\Lambda^{\geq 1}\Omega^1_{X_{reg}}, d)$, the truncated de-Rham complex:

$$\Omega^1_{X_{reg}} \xrightarrow{d} \Omega^2_{X_{reg}} \xrightarrow{d} \Omega^3_{X_{reg}} \xrightarrow{d} \cdots$$

So then $PD_{lt,X}(\mathbb{C}[\varepsilon]) = \mathbb{H}^2(X, j_* \Lambda^{\geq 1}\Omega^1_{X_{reg}})$ where $j : X_{reg} \hookrightarrow X$ is the inclusion

However, this yields

$$j_*\Omega^{\geq 1}_{X_{reg}} \longrightarrow j_*\Omega^{\geq 1}_{X_{reg}} \longrightarrow j_*\underset{\substack{\mathcal{O}_X\\ \wr\wr}}{\mathcal{O}_{X_{reg}}} \xrightarrow{+1}$$

which gives

$$H^1(\mathcal{O}_X) \longrightarrow \mathbb{H}^2(j_*\Omega^{\geq 1}_{X_{reg}}) \longrightarrow \mathbb{H}^2(j_*\Omega_{X_{reg}}) \longrightarrow H^2(\mathcal{O}_X)$$

By hypothesis on X, $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$ (we need care here are X is no longer just a symplectic singularity)

$$\Rightarrow \mathbb{H}^2(j_*\Omega^{\geq 1}_{X_{reg}}) \cong \mathbb{H}^2(j_*\Omega_{X_{reg}}) \overset{\text{Grothendieck's de Rham theorem}}{\cong} H^2(X^{an}, \mathbb{C}). \quad \square$$

$\Longrightarrow$ (H4) holds! That $PD_X(\mathbb{C}[\varepsilon])$ is finite dimensional.

§ __The surjectivity of automorphisms.__

If X is smooth, the truncated de Rham complex $(\Omega^{\geq 1}_{X/T}, d)$ can be identified with the so-called Lichnerowicz – Poisson complex $0 \to \mathcal{T}_{X/T} \xrightarrow{\delta_1} \Lambda^2 \mathcal{T}_{X/T} \xrightarrow{\delta_2} \cdots$ where

$$\delta_i f(da_1 \wedge \cdots \wedge da_{i+1}) = \sum_j (-1)^{j+1} \{a_j, f(da_1 \wedge \cdots \wedge \widehat{da_j} \wedge \cdots \wedge da_{i+1})\} - \sum_{j<k} (-1)^{j+k+1} f(d\{a_j, a_k\} \wedge da_1 \wedge \cdots \wedge \widehat{da_j} \wedge \cdots \wedge \widehat{da_k} \wedge \cdots \wedge da_{i+1})$$

This identification is induced by the isomorphism $\Omega^i_{X/T} \xrightarrow{\sim} \Lambda^i \mathcal{T}_{X/T}$ which is induced in turn by $\Omega^1_{X/T} \longrightarrow \mathcal{T}_{X/T} : df \longmapsto \xi_f$.

In particular for $f \in \mathcal{T}_{X/T}$

$$\delta f(da_1 \wedge da_2) = \{a_1, f(da_2)\} + \{f(da_1), a_2\} - f(d\{a_1, a_2\})$$

so that $\operatorname{Ker} \delta_1 =: P\mathcal{T}_{X/T}$ can be thought of as derivatives on $\mathcal{O}_{X/T}$ that satisfy

$\varphi(\{a_1, a_2\}) = \{a_1, \varphi(a_2)\} + \{\varphi(a_1), a_2\}$. $\circledast$

Let $PG_{X/T}$ be the sheaf (of sets) consisting of Poisson automorphisms of X/T as already introduced.

<u>Lemma</u> Let $X \in PD_X(T)$ where X is smooth. Then set $P\mathcal{T}^{\circ}_{X/T}$ to be the subsheaf of $P\mathcal{T}_{X/T}$ consisting of sections vanishing on X. Then $P\mathcal{T}^{\circ}_{X/T} \xrightarrow[\alpha]{\sim} PG_{X/T}$.

<u>Proof</u>: Given $\varphi \in P\mathcal{T}^{\circ}_{X/T}$, set $\alpha(\varphi) = \exp(\varphi) \in \operatorname{Aut}(X/T)$. This is a well-defined automorphism because it equals id on X, and because $\{\exp(\varphi)(f), \exp(\varphi)(g)\}$ $= \exp\varphi(\{f,g\})$ thanks to $\circledast$.

To prove α is surjective we assume X is affine and iteratively lift one step at a time: given $\phi \in PG_{X/T}$ we have $\phi|_{X_1} = id + \varphi_1$ where $X_1 = X \times_T T/m^2$ and $\varphi_1 \in m \cdot P\mathcal{T}_X$. We claim that φ_1 lifts to $\widetilde{\varphi}_1 \in P\mathcal{T}_{X/T}$ and then we'd have $\phi|_{X_2} = \alpha(\widetilde{\varphi}_1)|_{X_2} + \varphi_2$ with $\varphi_2 \in m^2 P\mathcal{T}_{X/T}$ ....

<u>Claim</u>: $P\mathcal{T}_{X/T} \twoheadrightarrow P\mathcal{T}_{\overline{X}/\overline{T}}$ where $\overline{T} \hookrightarrow T$ and $\overline{X} = X \times_T \overline{T}$:

Use the truncated de Rham complex & the triangle

$$\Omega^{\geq 1}_{X/T} \to \Omega_{X/T} \to \mathcal{O}_X \xrightarrow{+}$$

We need to calculate $H^1(\Omega^{\geq 1}_{X/T})$. By the associated long exact sequence we have

$$H^0(X^{an}, T) \to H^0(\mathcal{O}_X) \to H^1(\Omega^{\geq 1}_{X/T}) \to H^1(X^{an}, S) \to \underset{\text{affineness}}{0}$$

But $H^i(X^{an}, T) \cong H^i(X^{an}) \otimes \mathcal{O}_T$, $H^i(X^{an}, \overline{T}) \cong H^i(X^{an}) \otimes \mathcal{O}_{\overline{T}} \cong H^i(X^{an}, T) \otimes_{\mathcal{O}_T} \mathcal{O}_{\overline{T}}$ and $H^0(\mathcal{O}_{\overline{X}}) = H^0(\mathcal{O}_X) \otimes_{\mathcal{O}_T} \mathcal{O}_{\overline{T}}$. It follows by a diagram chase (4 lemma!) that we get surjectivity. $\square$

PROOF OF SURJECTIVITY OF AUTOMORPHISMS (provided $H^1(X^{an}, \mathbb{C})$) for general X.

Let $\mathcal{U}$ be the smooth locus of X over T. Since $j_* \mathcal{O}_{\mathcal{U}} = \mathcal{O}_X$ for codim^n reasons we deduce that $H^0(X, PG_{X/T}) \cong H^0(\mathcal{U}, PG_{\mathcal{U}/T})$ (i.e. Poisson automorphisms extend)

Then it is enough to work in the smooth case and so we need, by the previous argument, to show that $H^0(\mathcal{U}, P\mathcal{T}_{\mathcal{U}/T}) \twoheadrightarrow H^0(\mathcal{U}, P\mathcal{T}_{\overline{\mathcal{U}}/\overline{T}})$. Now we argue as before, but without being able to use affinity. From the sequence

$$\begin{array}{ccccccc} H^0(\mathcal{U}^{an}) \otimes G_T & \longrightarrow & H^0(\mathcal{O}_{\mathcal{U}}) & \longrightarrow & \mathbb{H}^1(\mathcal{U}, \Omega^{\geq 1}_{\mathcal{U}/T}) & \longrightarrow & H^1(\mathcal{U}^{an}) \otimes G_T \\ & & \wr\vert & & \wr\vert & & \\ & & H^0(\mathcal{O}_{\mathcal{X}}) & & H^0(\mathcal{U}, PJ_{\mathcal{U}/T}) & & \end{array}$$

we see that it will be enough to prove that $H^1(\mathcal{U}^{an}) = H^1(\mathcal{U}^{an}, \mathbb{C}) = 0$.

Let $f : \hat{X} \longrightarrow X$ be a resolution of singularities with exceptional locus E a divisor with simple normal crossings. Since $f^{-1}(\mathcal{U}) \cong \mathcal{U}$ we have a long exact sequence

$$H^1(\hat{X}^{an}, \mathbb{C}) \longrightarrow H^1(\mathcal{U}^{an}, \mathbb{C}) \longrightarrow H^2_E(\hat{X}^{an}, \mathbb{C}) \longrightarrow H^2(\hat{X}^{an}, \mathbb{C}).$$

But X has rational singularities so that $H^1(\hat{X}^{an}, \mathbb{C}) \cong H^0(X^{an}, R^1 f_* \mathbb{C}_{\hat{X}}) \cong 0$. Then relatively standard homological algebra shows $H^2_E(\hat{X}^{an}, \mathbb{C}) \longrightarrow H^2(\hat{X}^{an}, \mathbb{C})$ is an inclusion: the image of $H^2_E(\hat{X}^{an}, \mathbb{C}) = \bigoplus \mathbb{C}[E_i]$ is the span of the $[E_i]$. But the $[E_i]$'s are linearly independent. $\square$