LECTURE 6:

We want to try to see something about the deformations of conical symplectic varieties and their resolutions.

E.g. 39 Kleinian singularity of type A_{n-1}:

$$G = \left\langle \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \right\rangle \text{ with } \zeta = \exp(2\pi i/n). \quad V = \mathbb{C}^2; \ \mathbb{C}[V] = \mathbb{C}[x,y]$$

$$\mathbb{C}[V/G] = \mathbb{C}[x,y]^G = \mathbb{C}[x^n, y^n, xy] = \mathbb{C}[A,B,C]/(AB - C^n)$$

$AB - C^n$: eq^n for KS.

Let $\widetilde{V/G} = \bigcup_{0 \le i \le n-1} \operatorname{Spec}(\mathbb{C}[x^{n-i}y^{-i}, x^{i+1-n}y^{i+1}]) \longrightarrow \operatorname{Spec}(\mathbb{C}[x^n, xy, y^n])$

This is a resolution of singularities and $\widetilde{V/G}$ is symplectic:

$\{x^{n-i}y^{-i}, x^{i+1-n}y^{i+1}\} = n$ i.e. non-degenerate.

Because V/G is an isolated singularity & complete intersection in $\mathbb{C}^3$ it's easy to write down deformations of it:

$$X = \operatorname{Spec}\left(\mathbb{C}[A,B,C,Z_0,\dots,Z_{n-2}]/(AB - C^n - \textstyle\sum_{i=0}^{n-2} Z_i C^i)\right) \longrightarrow \operatorname{Spec}(\mathbb{C}[Z_0, \dots, Z_{n-2}]) = \mathbb{A}^{n-1}$$

This is flat over $\mathbb{A}^{n-1}$ i.e. has a basis which is same as $\mathbb{C}$-basis for $\mathbb{C}[A,B,C]/(AB - C^n)$, namely $\{A^iC^j\}_{i\ge 0, j\ge 0} \cup \{B^iC^j\}_{i>0, j\ge 0}$

Has Poisson bracket: $\{Z_i, -\} = 0$, $\{A, B\} = n^2C^{n-1} + n\sum_{i=1}^{n-2} i Z_i C^{i-1}$, $\{A, C\} = nA$, $\{B, C\} = -nB$

i.e. this produces examples of Poisson deformations:

DEF^N 40: Let X be a Poisson scheme i.e. $\exists \ \{\cdot,\cdot\}: \Lambda^2 \mathcal{O}_X \longrightarrow \mathcal{O}_X$ satisfying Poisson axioms. Then a <u>Poisson deformation</u> of X over (T, t), where $T = \operatorname{Spec}(S)$ cl. s $t \in T$, is a scheme $\mathcal{X}$ flat over T and a Poisson structure $\{\,,\}_T: \Lambda^2_{\mathcal{O}_T} \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}$ over T together with an isomorphism of Poisson schemes $\mathcal{X} \times_T \{t\} \xrightarrow[j]{\sim} X$.

2 Poisson deformations over T, $\mathcal{X}$ & $\mathcal{X}'$, are equivalent if there exists are Poisson isomorphism over T, $\mathcal{X} \xrightarrow{\sim} \mathcal{X}'$, s.t.

$$\begin{tikzcd} \mathcal{X} \times_T \{t\} \arrow[rr, "\sim"] \arrow[dr, "j'"', "\sim"] & & \mathcal{X}' \times \{t\} \arrow[dl, "j'", "\sim"'] \\ & X & \end{tikzcd}$$

commutes.

Restricting to local Artin algebras over $\mathbb{C}$ produces the Poisson deformation functor

$$PD_{(X, \{\,\})}: (\mathrm{Art})_{\mathbb{C}} \longrightarrow (\mathrm{Sets})$$

Deforming singular affine varieties is <u>bad</u> in general : spaces you wish to be finite are not.⊗ Isolated singularities are examples that break the world! But typically symplectic varieties will not have isolated singularities. Nonetheless :

<u>E.g. 41</u> Adjoint orbits in $\mathfrak{g} = \mathfrak{sl}_n$ Recall $\mathcal{N} = \{\text{nilpotent matrices}\}$ and each orbit has structure of symplectic mfld. Orbits are in correspondence with <u>partitions</u> of n via the Jordan canonical form : $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ gives $\begin{bmatrix} J_{\lambda_1} & & \\ & J_{\lambda_2} & \\ & & \ddots \end{bmatrix}$ where $J_m = \overbrace{\begin{bmatrix} 0 & & 0 \\ 1 & \ddots & \\ 0 & 1 & 0 \end{bmatrix}}^{m} \Big\} m$

It's easy to check that $\dim(\mathcal{O}_\lambda) = n^2 - |\lambda^t|^2 \Longrightarrow \exists$ max^l orbit of dim^n $n^2 - n$, $\lambda = (n)$ and a ! next smallest orbit of dim^n $n^2 - n - 2$, $\lambda = (n-1, 1)$ "subregular". It has a representative $f = \sum_{i=1}^{n-2} i(n-1-i) E_{i+1\, i}$, which together with $h = \sum_{i=1}^{n-1} (n-2i) E_{ii}$ and $e = \sum_{i=1}^{n-2} E_{i\, i+1}$ forms an $\mathfrak{sl}_2$-triple

Now decompose $\mathfrak{g}$ into $\mathrm{ad}(h) = [h, -]$ weight spaces, $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$, so in particular $e \in \mathfrak{g}(2)$, $f \in \mathfrak{g}(-2)$ and all diagonal matrices are in $\mathfrak{g}(0)$. Then $T_f \mathfrak{g} = f + \mathfrak{g}$ and $T_f(G \cdot f) = f + [\mathfrak{g}, f]$. We find a complement to this by $\mathfrak{sl}_2$-representation theory as $f + \mathfrak{g}^e$ where $\mathfrak{g}^e = \{x \in \mathfrak{g} : [e, x] = 0\}$

$S_f := f + \mathfrak{g}^e$ is called the <u>Slodowy slice</u> to $\mathcal{O}(f)$ at f. Explicitly it is the affine space of matrices

$$\begin{bmatrix} \gamma & \lambda_1 & \lambda_2 & \cdots & \lambda_{n-2} & \alpha \\ n-2 & \gamma & \lambda_1 & & & \\ & 2(n-3) & \gamma & \ddots & & \\ & & \ddots & \ddots & \lambda_1 & \\ & & & (n-2) & \gamma & \\ & & & \beta & (1-n) & \gamma \end{bmatrix} \quad \text{for } \alpha, \beta, \gamma, \lambda_1 \dots \lambda_{n-2} \in \mathbb{C}$$

This admits a map to $\mathbb{A}^{n-1}$ by sending this matrix to the coefficients of its characteristic polynomial

$$\pi : S_f \longrightarrow \mathbb{A}^{n-1}$$

Let $\overline{S_f} = \pi^{-1}(0) = S_f \cap \mathcal{N}$. Then, by work of Brieskorn, $\overline{S_f}$ is a Kleinian singularity of the form $AB - C^n$:

i.e. for $n = 3$: $\pi(\alpha, \beta, \gamma, \lambda_1) = (2\gamma^3 - 2\lambda_1\gamma - \alpha\beta, -3\gamma^2 - \lambda_1)$

so that $\pi^{-1}(0)$ sets $\lambda_1 = -3\gamma^2$ and insists then that $8\gamma^3 - \alpha\beta = 0$

In fact : $\begin{tikzcd} S_f \arrow[r, leftrightarrow, "\sim"] \arrow[dr] & X_n \arrow[d] \\ & \mathbb{A}^{n-1} \end{tikzcd}$. For $n=3$ $A \mapsto \alpha$, $B \mapsto -\beta$, $C \mapsto -2\gamma$, $Z_0 \mapsto 2\gamma^3 - 2\lambda_1\gamma - \alpha\beta$ and $Z_1 \mapsto -3\gamma^2 - \lambda_1$

Note that it is possible to define a $\mathbb{C}^*$-action on S_f by $\lambda \cdot x = \lambda^{i+2} \cdot x$ where $x \in \mathfrak{g}(i)$. This make everything equivariant.

So in this example we see:

- $\mathcal{N} \supset (\text{reg. orb} \cup \text{sub reg. orb})$ has codimension at least 4
- inside this set all points are smooth or look locally like $(S,0)\times(\mathbb{C}^{2d},0)$ where S is a Kleinian singularity and $\mathbb{C}^{2d}$ is the usual symplectic space.

Now we make one further observation

Lemma (Namikawa) Let $S = \{f(a,b,c)=0\} \subset \mathbb{C}^3$ by an isolated hypersurface singularity which admits a Poisson structure, let $(\mathbb{C}^{2d},0)$ be a symplectic manifold with the usual symplectic structure. Let $V = (S,0)\times(\mathbb{C}^{2d},0)$ with the product Poisson structure. Let $\mathcal{V}$ be a first order Poisson deformation of V (i.e $\mathcal{V} \in PD_V(\mathbb{C}[\varepsilon]/_{\varepsilon^2})$). Then

$$\mathcal{V} \cong (\mathcal{S},0)\times(\mathbb{C}^{2d},0)$$

as a flat Poisson defomation, where $(\mathcal{S},0)$ is a 1st order Poisson deformation of S.

<u>Proof</u>: Let $Z_1 \ldots Z_{2d}$ be the co-ords of $\mathbb{C}^{2d}$ with $\{Z_1,Z_2\}=\{Z_3,Z_4\}=\ldots=\{Z_{2d-1},Z_{2d}\}=1$ and other brackets involving Z's 0 as appropriate. Note that V is a complete intersection in $\mathbb{C}^{3+2d}$, cut out by f, so by ⊛ if we take $f_1 \ldots f_k$ to be a basis of $\dfrac{\mathbb{C}[A,B,C]}{(f,\frac{\partial f}{\partial A},\frac{\partial f}{\partial B},\frac{\partial f}{\partial C})}$ then the first order deformation may be defined by

$$f(A,B,C)+\varepsilon\big(f_1(A,B,C)g_1(Z)+\cdots+f_k(A,B,C)g_k(Z)\big)$$

But we need to check the Poisson compatibility. Let $\{\ ,\ \}'$ be the bracket on this extension

$$\{A,Z_i\}=\varepsilon\alpha_i,\ \{B,Z_i\}=\varepsilon\beta_i,\ \{C,Z_i\}=\varepsilon\gamma_i$$

for functions $\alpha_i,\beta_i,\gamma_i$ depending on $A,B,C,Z_1 \ldots, Z_{2d}$. Now we work on

$$0=\{f+\varepsilon(f_1g_1+\cdots+f_kg_k),Z_i\}'=\frac{\partial f}{\partial A}\{A,Z_i\}'+\frac{\partial f}{\partial B}\{B,Z_i\}'+\frac{\partial f}{\partial C}\{C,Z_i\}'$$
$$+\varepsilon\Big(\sum_j f_j\{g_j,Z_i\}'+\sum_j g_j\{f_j,Z_i\}'\Big)$$

We can change $\{\ ,\ \}'$ by $\{\ ,\ \}$ in the 2nd line since $\varepsilon^2=0$ and so we deduce for even i

$$\varepsilon\Big(\frac{\partial f}{\partial A}\alpha_i+\frac{\partial f}{\partial B}\beta_i+\frac{\partial f}{\partial C}\gamma_i-\sum_j f_j\frac{\partial g_j}{\partial Z_{i-1}}\Big)=0$$

Forget now ε. In $\mathbb{C}[A,B,C,Z_1,\ldots Z_{2d}]/_{(f,\frac{\partial f}{\partial A},\frac{\partial f}{\partial B},\frac{\partial f}{\partial C})}$ we see that $\sum_j f_j\frac{\partial g_j}{\partial Z_{i-1}}=0$

But the f_j's are a basis, so $\frac{\partial g_j}{\partial Z_{i-1}}=0\ \forall j$. Similarly for odd $i \Rightarrow g_j$ are constant $\square$

So we see the presence of the Poisson structure produces finite dimensional deformation spaces, that were infinite before; they also simplify considerably the geometry appearing in the deformation as we can e.g. under the Kleinian singularity case.

(*) Let $R = S/(f)$ where $S = \mathbb{C}[[x_1 \dots x_n]]$. Then let $(f_\tau)_\tau$ be a $\mathbb{C}$-basis of $S/(f, f_{x_1} \dots f_{x_n})$. Any 1st-order deformation of R has the form $R' = \dfrac{S[\varepsilon]}{(f + \varepsilon(\sum t_\tau f_\tau))}$ for some scalars t_τ.

<u>Proof</u>: Any 1st-order deformation has to be of the form $\dfrac{S[\varepsilon]}{(f + \varepsilon\psi(f))}$ where $\psi : (f) \longrightarrow \dfrac{S}{(f)}$.
i.e $\psi(f) \in S/(f)$. But then applying the automorphism of S

$$x_i \mapsto x_i + \varepsilon p_i \quad \text{for } p_i \in S$$

shows that we can replace $f + \varepsilon\psi(f)$ by $f + \varepsilon\left(\psi(f) + \sum_{i=1}^{n} p_i f_{x_i}\right)$ i.e. the deformations are controlled by $S/(f, f_{x_1} \dots f_{x_n})$. $\square$