Ch 3 Symplectic Singularities.

Recall from the previous section we saw the appearance of:

- $\mathcal{N} = \{x \in \mathfrak{g} : x \text{ nilpotent}\}$ i.e. $ad(x) \in \mathfrak{gl}(\mathfrak{g})$ is nilpotent
- $T^*\mathcal{B}$ cotangent bundle of $\mathcal{B}$, and resolution of singularities $T^*\mathcal{B} \xrightarrow{\mu} \mathcal{N}$
- equivalence $\mathcal{U}(\mathfrak{g})_0\text{-mod} \xrightarrow{\sim} D_{\mathcal{B}}\text{-mod}$ which "quantizes" the resolution above, in the sense that $\mathcal{U}(\mathfrak{g})_0$, respectively $D_{\mathcal{B}}$, is a quantization of $\mathcal{O}_{\mathcal{N}}$, resp. $\pi_*\mathcal{O}_{T^*}$.

From a representation theoretic point of view it is not natural to restrict only to representations with trivial central characters: for instance the only finite dimensional (irred) representation with trivial central character is the trivial rep^n $\mathbb{C} = L(0)$, but there are infinitely many fin. dim. irreducible representations $L(\lambda) = \Gamma(X, \mathcal{L}_\lambda)$ with $\lambda \in -P^+ \subset \mathfrak{h}^*$ ($\mathfrak{h}$ Cartan of $\mathfrak{g}$)

$$\mathcal{U}(\mathfrak{g})_\lambda \xrightarrow{\sim} D_X^\lambda\text{-mod}$$

where $\lambda \in \mathfrak{h}^*$ such that $\langle \lambda, \alpha^\vee \rangle \notin \{-1, 0, 1, \dots\}$ $\forall \alpha \in R^+$. ($\exists$ a version of this theorem for arbitrary λ, but it becomes weaker and weaker: the real problem is the α for which $\langle \lambda, \alpha^\vee \rangle = -1$ since on the LHS of the equivalence we have $\mathcal{U}(\mathfrak{g})_\lambda = \mathcal{U}(\mathfrak{g})_{w.\lambda}$ for any $w \in W$ and so we can ensure always that $\langle \lambda, \alpha^\vee \rangle \notin \{0, 1, \dots\}$.)

So we see quantisations of $\mathcal{N}$ labelled by $\mathfrak{h}^*/W$, quantisations of $T^*\mathcal{B}$ labelled by $\mathfrak{h}^*$, and an equivalence between them whenever we avoid a certain explicit set of discrete parameters.

- the space $\mathfrak{h}^*/W$ and $\mathfrak{h}^*$ are deformation spaces of $\mathcal{N}$ and $T^*\mathcal{B}$

e.g. $H^2(T^*\mathcal{B}, \mathbb{C}) = \mathfrak{h}^*$;

- $\mathfrak{h} \xrightarrow{\tau} \mathfrak{h}^*/W$ is generically étale
- $\operatorname{Pic}(T^*\mathcal{B}) \underset{\text{Grauert–Riemenschneider}}{\cong} H^2(T^*\mathcal{B}, \mathbb{Z}) \subseteq H^2(T^*\mathcal{B}, \mathbb{C})$
- There is an equivalence iff $\lambda +$ (eff. div.) is never in the ramification locus of τ.

IS THIS POSSIBLY GENERAL??

DEF^N 35 : A normal affine variety Y has symplectic singularities if

i) $\exists\ \varphi \in \Omega^2_Y$, a 2-form, which restricts on Y_{reg} to a closed ad non-degenerate 2-form (i.e. Y_{reg} is symplectic).

ii) For some (equivalently any) resolution of singularities $f: X \longrightarrow Y$ we have that $f^{-1}(\varphi)$ extends from $f^{-1}(Y_{reg})$ to a closed 2-form on all of X.

Y has conical symplectic singularities if in addition

iii) $\exists\ \mathbb{C}^*$-action on Y such that $\lambda \cdot \varphi = \lambda^i \varphi$ for some $i \in \mathbb{N}_+$ and all $\lambda \in \mathbb{C}^*$ and such that $\mathbb{C}[Y] = \bigoplus_{j \in \mathbb{Z}} \mathbb{C}[Y]_j$ with $\mathbb{C}[Y]_j = 0$ for $j < 0$ ad $\mathbb{C}[Y]_0 = \mathbb{C}$.

This surprisingly weak definition does have some mileage, and in particular it does imply that the variety Y has rational singularities i.e. $\mathcal{O}_Y \xrightarrow{\sim} Rf_* \mathcal{O}_X$.

Lemma 36 Suppose Y has symplectic singularities. Then Y is a Poisson variety i.e $\mathbb{C}[Y]$ has a Poisson bracket.

Proof: Since Y is normal $\mathbb{C}[Y] = \mathbb{C}[Y_{sm}]$: then the symplectic form φ on Y_{sm} produces a Poisson bracket on Y_{sm} via $\{f, g\} = \varphi(\xi_f, \xi_g)$, where $\varphi(\xi_f, -) = -df$ $\square$

Examples 37 :

(A) $\mathcal{N}$ has conical symplectic singularities. There is a Poisson bracket on $\mathfrak{g}^*$ given by extending

$$\{X, Y\} = [X, Y] \longleftrightarrow \text{2-covector on } \mathfrak{g}^* \ni x \longmapsto \begin{pmatrix} \mathfrak{g} \wedge \mathfrak{g} = T^*_x \mathfrak{g}^* \wedge T^*_x \mathfrak{g}^* \to \mathbb{C} \\ (A, B) \longmapsto x([A, B]) \end{pmatrix}$$

for $X, Y \in \mathfrak{g} \subset \mathbb{C}[\mathfrak{g}^*]$. Since $\mathfrak{g}$ is simple, we have $\mathfrak{g}^* \cong \mathfrak{g}$ via the Killing form, and this isomorphism is G-equivariant. So we may interchange $\mathfrak{g}$ ad $\mathfrak{g}^*$ at will. We're already seen that $\mathbb{C}[\mathcal{N}] = \dfrac{\mathbb{C}[\mathfrak{g}]}{\langle \mathbb{C}[\mathfrak{g}]^G_+ \rangle}$. It's then obvious that $\mathbb{C}[\mathcal{N}]$ inherits a Poisson structure from $\mathbb{C}[\mathfrak{g}]$: $\{\mathbb{C}[\mathfrak{g}]^G, Y\} = 0$ since $\mathbb{C}[\mathfrak{g}]^G \subseteq \mathbb{C}[\mathfrak{g}]^{\operatorname{ad} Y}$. It's well-known that the (adjoint / coadjoint) G-orbits on $\mathfrak{g}$ are symplectic : take $x \in \mathfrak{g}^*$

$$T^*(G.x) \wedge T^*(G.x) = \mathfrak{g}/\mathfrak{g}_x \wedge \mathfrak{g}/\mathfrak{g}_x \longrightarrow \mathbb{C} \quad \text{with } (A + \mathfrak{g}_x, B + \mathfrak{g}_x) \longmapsto x([A, B])$$

where $\mathfrak{g}_x = \{Y : x([Y, -]) = 0\}$. This is non-degenerate by construction.

Now, in $\mathcal{N}$ we have a unique G-orbit of smooth points, the regular nilpotent elements $(= G.(\sum_{\alpha \in \Phi^+} e_\alpha))$. It is a theorem of Kostant that $\mathcal{N}$ is normal. Finally

$\mu : T^*\mathcal{B} \longrightarrow \mathcal{N}$ is a resolution of singularities and $T^*\mathcal{B}$ is symplectic.

The conical structure comes from $\lambda \cdot X = \lambda X$ for $X \in \mathfrak{g}$. Then $\mathbb{C}[N]_j = \begin{cases} \mathbb{C} & j=0 \\ 0 & j<0 \end{cases}$ and

$$\{\lambda.X, \lambda.Y\} = \{\lambda X, \lambda Y\} = \lambda^2\{X,Y\}$$
$$= \lambda(\lambda\cdot\{X,Y\})$$

so it has weight one.

(B) $\mathbb{C}^2$ with canonical symplectic form $\omega(x,y)=1$. Then $SL_2(\mathbb{C})$ preserves this bracket. Any finite subgroup $G \leq SL_2(\mathbb{C})$ then produces a normal variety $Y = \mathbb{C}^2/G = \operatorname{Spec} \mathbb{C}[X,Y]^G$ and there is a Poisson bracket induced from $\{X,Y\}=1$ which, since it is G-equivariant passes to $\mathbb{C}[X,Y]^G$. Then $Y_{reg} = Y \setminus \{0\}$ and the bracket is non-degenerate there and it is inherited from a non-degenerate bracket under a G-equiv covering map $\mathbb{C}^2\setminus\{0\} \longrightarrow \mathbb{C}^2\setminus\{0\}/G$. There is a conical $\mathbb{C}^*$-structure via $\lambda(x,y) = (\lambda^{-1}x, \lambda^{-1}y)$ which has weight 2 for the Poisson bracket. Finally, let $f: X \longrightarrow Y$ be the minimal resolution of Y: then $f^*\Omega^2_Y \cong \Omega^2_X$; but $\Omega^2_Y = \mathcal{O}_Y$ because of the form on Y_{reg}, so that $\Omega^2_X \cong \mathcal{O}_X$ and hence X is symplectic.

(C) The above example generalises to $Y = (T^*V)/G$ for any finite $G \leq GL(V)$. It is not true, however, that Y has a resolution of singularities that is complex.

(D) But here is a special case: $V = \mathbb{C}^n$ & $G = \mathfrak{S}_n$. $Y = (T^*V)/\mathfrak{S}_n =: \operatorname{Sym}^n(\mathbb{C}^2)$. This does have a resolution that is symplectic: $\operatorname{Hilb}^n(\mathbb{C}^2) = \{ I \triangleleft \mathbb{C}[x,y] : \dim_{\mathbb{C}} \frac{\mathbb{C}[x,y]}{I} = n\}$. This maps to $\operatorname{Sym}^n \mathbb{C}^2$ by sending I to its support, counted with multiplicity.

(E) Hypertoric varieties, Slodowy slices, Nakajima quiver varieties.

<u>Def^n 38</u>: Let Y have symplectic singularities. Then a symplectic resolution $f: X \longrightarrow Y$ is a resolution of singularities such that the form on $f^{-1}(Y_{reg})$ extends to a closed <u>non-degenerate</u> 2-form on X.

By a result of Fu, this is equivalent to being a crepant resolution of Y.