The Beilinson–Bernstein theorem is actually 2 results in 1

- it's an identification of the global sections of $X = G/B$ as a quotient of an enveloping algebra of a simple Lie algebra
- it's a statement of <u>D-affinity</u> of flag manifolds.

As usual, it's possible to be explicit for the case of SL_2.

§ <u>Example of SL_2:</u>

$$G = SL_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\} \supseteq B = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \supseteq T = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right\}$$

$V = \mathbb{C}^2$, natural representation: $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

<u>Claim</u>: $G/B \cong \mathbb{P}^1 = \{ 0 \subset V_1 \subset V : \dim V_1 = 1 \}$. For this is certainly a homogeneous G-space (Möbius transformations) and if we choose a base-point $V_1 = \mathbb{C}e_1$ then you see $\mathrm{Stab}_G(0 \subset \mathbb{C}e_1 \subset V) = B$. □

We already studied $D_{\mathbb{P}^1}(\mathbb{P}^1)$: it's generated by $x^2\partial, x\partial, \partial$.

Recall $sl_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + d = 0 \right\}$

We construct a morphism $sl_2 \longrightarrow F^1 D_{\mathbb{P}^1}(\mathbb{P}^1)$ by differentiating the action of SL_2 on $\mathcal{O}_{\mathbb{P}^1}$: $f \in \mathcal{O}_{\mathbb{P}^1}$, $z \in sl_2 \rightsquigarrow \exp(tz) \in SL_2$ $\quad \mu_z(f) = \frac{d}{dt}(\exp(tz) \cdot f)\Big|_{t=0}$

- $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad \mu_e(x^i) = \frac{d}{dt}\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} y \\ z \end{pmatrix}^i \right)\Big|_{t=0} = \frac{d}{dt}\left(\frac{y - tz}{z} \right)^i\Big|_{t=0} = -i\left(\frac{y-tz}{z} \right)^{i-1}\Big|_{t=0} = -ix^{i-1}$

$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\quad \mu_f(x^i) = ix^{i+1}$, $\quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mu_h(x^i) = -2ix^i$

So recall $\mathcal{U}(sl_2) = \dfrac{T(sl_2)}{\langle YZ - ZY - [Y,Z] \rangle} = \dfrac{\mathbb{C}\langle e, h, f \rangle}{\langle [h,e] = 2e, [h,f] = -2f, [e,f] = h \rangle}$

Then we get $\mathcal{U}(sl_2) \xrightarrow{\ \mu\ } D_{\mathbb{P}^1}(\mathbb{P}^1)$

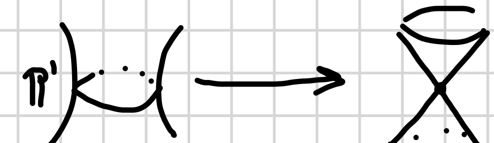
$$e \longmapsto -\partial \qquad f \longmapsto x^2\partial \qquad h \longmapsto -2x\partial$$

(e.g. $[-\partial, x^2\partial] = -[\partial, x^2]\partial = -2x\partial$)

So we get a surjective morphism and it even preserves the natural filtrations on both sides $F_p\mathcal{U}(sl_2)$ = products of p or fewer elements from sl_2.

$$\begin{array}{ccccc} \mathrm{gr}\, \mathcal{U}(sl_2) & \longrightarrow & \mathrm{gr}\, D_{\mathbb{P}^1}(\mathbb{P}^1) & \longrightarrow & \mathrm{gr}\, D_{\mathbb{P}^1} \\ \mathbb{C}[sl_2^*] = \mathrm{Sym}(sl_2) & \twoheadrightarrow & \mathcal{O}_{T^*\mathbb{P}^1}(T^*\mathbb{P}^1) & \longrightarrow & \pi_* \mathcal{O}_{T^*\mathbb{P}^1} \end{array}$$

$T^*\mathbb{P}^1$



$\mathbb{C}[-\xi, 2x\xi, x^2\xi] \cong \dfrac{\mathbb{C}[X,Y,Z]}{\langle 4XZ+Y^2\rangle}$ 2-D algebra; singularity at the origin.

It's quick to see that SL_2 acts on $D_{\mathbb{P}^1}(\mathbb{P}^1)$ and there is a unique invariant vector, the identity, up to scalar. $\therefore\ \psi(\mathcal{U}(sl_2)^{SL_2}) = \mathbb{C}.1$

But $\mathcal{U}(sl_2)^{SL_2} \cong \mathbb{C}[\Omega]$ where $\Omega = 4ef + h(h-2)$ is the Casimir element.

It's obvious that $\psi(\Omega) = 0$ by letting the differential operators act on 1.

So we find $\dfrac{\mathcal{U}(sl_2)}{\langle\Omega\rangle} \twoheadrightarrow D_{\mathbb{P}^1}(\mathbb{P}^1)$ and this is an isomorphism because it is on associated graded algebras.

Next, we want to see D-affinity. This is the statement that

$$D_{\mathbb{P}^1}\text{-mod} \xrightarrow{\Gamma} D_{\mathbb{P}^1}(\mathbb{P}^1)\text{-mod}$$ (i.e. taking global sections is an equivalence)

Its inverse will be the "localization functor" $M \in D_{\mathbb{P}^1}(\mathbb{P}^1)\text{-mod} \longmapsto D_{\mathbb{P}^1} \underset{D_{\mathbb{P}^1}(\mathbb{P}^1)}{\otimes} M.$

If Γ is going to be an equivalence then it will satisfy two properties:

a) Γ is exact i.e $0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$ is a short exact sequence in $D_{\mathbb{P}^1}$-mod, then $0 \to \Gamma(\mathcal{M}_1) \to \Gamma(\mathcal{M}_2) \to \Gamma(\mathcal{M}_3) \to 0$ is exact. [Normally Γ is left exact; being exact characterises affine schemes if we take arbitrary s.e.s. of quasi-coherent $\mathcal{O}_X$-modules.]

b) If $\mathcal{M} \neq 0$ then $\Gamma(\mathcal{M}) \neq 0$.

It's not hard to show that these are actually equivalent to D-affinity.

Now recall $p: \underset{\tilde{X}}{\mathbb{C}^2\setminus\{0\}} \longrightarrow \underset{X}{\mathbb{P}^1}$, factoring by $\mathbb{C}^*$, and recall that $\mathbb{C}^2\setminus\{0\}$ is not affine. Let $\mathcal{M} \in D_{\mathbb{P}^1}$. Then $p^*\mathcal{M} \in D_{\mathbb{C}^2\setminus\{0\}}$-mod There is an action of $\mathbb{C}^*$ on $\Gamma_{\tilde{X}}(p^*\mathcal{M})$ which we can detect by the euler element: $eu = x_1\partial_1 + x_2\partial_2 \in D_{\tilde{X}}(\tilde{X})$, that we saw earlier.

$$\Gamma(p^*\mathcal{M}) = \bigoplus_{i\in\mathbb{Z}} \gamma(\mathcal{M})^i$$

where $u \in (p^*\mathcal{M})^i$ when $eu.u = iu$. Since $p: \mathbb{C}^2\setminus\{0\} \to \mathbb{P}^1$ is a principal $\mathbb{C}^*$-bundle we have $\gamma(\mathcal{M})^0 = \Gamma_X(\mathcal{M})$. Finally, observe that $x_i\,\gamma(\mathcal{M})^i \subseteq \gamma(\mathcal{M})^{i+1}$ and $\partial_i\,\gamma(\mathcal{M})^i \subseteq \gamma(\mathcal{M})^{i-1}$.

Take $0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$; this lifts to $0 \to p^*\mathcal{M}_1 \to p^*\mathcal{M}_2 \to p^*\mathcal{M}_3 \to 0$

on $\mathbb{C}^2\setminus\{0\}$. Let $j: \mathbb{C}^2\setminus\{0\} \hookrightarrow \mathbb{C}^2$ and then take

$$0 \to j_* p^* M_1 \to j_* p^* M_2 \to j_* p^* M_3 \to Z \to \cdots$$

where $Z = R^1 j_* p^* M_1$ is a $D_{\mathbb{C}^2}$-module supported on $\mathbb{C}^2 \setminus (\mathbb{C}^2\setminus\{0\}) = \{0\}$. Then this has characteristic cycle $T_0\mathbb{C}^2$ and has the form $\mathbb{C}[\partial_1,\partial_2] \otimes_{\mathbb{C}} N$ for some vector space N (cf Fourier transform and reg. connections on $\mathbb{A}^2$) with

$$x_i \cdot f(\partial_1,\partial_2) \otimes n = -(\text{deriv. w.r.t. } x_i)\, f(\partial_1,\partial_2) \otimes n.$$

$$\Longrightarrow (x_1\partial_1 + x_2\partial_2)\, \partial^\alpha \otimes n = (-\alpha_1 - \alpha_2 - 2)\, \partial^\alpha \otimes n$$

Now since $\mathbb{C}^2$ is affine we have

$$0 \to \Gamma_{\mathbb{C}^2}(j_* p^* M_1) \to \Gamma_{\mathbb{C}^2}(j_* p^* M_2) \to \Gamma_{\mathbb{C}^2}(j_* p^* M_3) \to \Gamma_{\mathbb{C}^2}(Z) \to \cdots$$

is exact, and then take 0 e-value for eu we see that $\Gamma_{\mathbb{C}^2}(Z)^0 = 0$ as all its weights are negative $\Rightarrow$ $0 \to \Gamma_X(M_1) \to \Gamma_X(M_2) \to \Gamma_X(M_3) \to 0$ i.e Γ_X is exact!

Now suppose $\Gamma_X(M) = 0$ i.e $\Gamma_{\tilde{X}}(p^*M)^0 = 0$. If $p^*M \neq 0$ then $\Gamma_{\tilde{X}}(p^*M) = \Gamma_{\mathbb{C}^2}(j_* p^*M) \neq 0$. But then if $\Gamma_{\tilde{X}}(p^*M)^i \neq 0$ with $i > 0$, then pick such i minimal: $\therefore$ eu $\Gamma_{\tilde{X}}(p^*M) = 0$ (as ∂_1, ∂_2 drop degree) ↯ If $i < 0$ then similarly picking a max'l value we'd get $x_1, x_2\, \Gamma_{\tilde{X}}(p^*M)^i = 0$ which means that elements of $\Gamma_{\tilde{X}}(p^*M)^i$ are supported at the origin and so on $\mathbb{C}^2\setminus\{0\}$ they should be zero ↯. □

§ The two main ingredients of BB.

1. As before we can define $\mu: \mathcal{U}(\mathfrak{g}) \to D_X(X)$ by differentiating the action of G on X. And again it's clear that $\mathcal{U}(\mathfrak{g})^G_+$ acts as zero. This gives the morphism $\mathcal{U}(\mathfrak{g})_0 \to D_X(X)$. Then take associated graded:

$$\frac{\mathbb{C}[\mathfrak{g}^*]}{\mathbb{C}[\mathfrak{g}^*]^G_+} \longrightarrow \mathcal{O}_{T^*X}(T^*X) \longrightarrow \mathcal{O}_{T^*X}$$

Then it is known that i) $\tilde{\mu}: T^*X \to \mathfrak{g}^*$ is the "moment map" of the action of G on T^*X, ii) it produces a resolution of singularities $T^*X \to \mathcal{N}$ = nilpotent elements of $\mathfrak{g}$ ($\cong \mathfrak{g}^*$ via Killing form). This implies that the associated graded is an isomorphism ($\mathbb{C}[\mathcal{N}] \cong \mathbb{C}[\mathfrak{g}^*]/\mathbb{C}[\mathfrak{g}^*]^G_+ \cong \mathcal{O}(T^*X)$ is normal) and so the original homomorphism is too.

2. The D-affinity is a consequence of the Borel-Weil theorem:

given a $\lambda: B \longrightarrow \mathbb{C}$ we get $\mathcal{L}(\lambda)$ on X via associated bundle:

$$G \times \mathbb{C}_\lambda \text{ has a } B\text{-action}: (g,v)b = (gb, \lambda(b^{-1})v)$$

$$\mathcal{L}(\lambda) = G \times \mathbb{C}_{w_0 \cdot \lambda}/B \longrightarrow X \qquad [g,v] \longmapsto gB$$

Then $\mathcal{L}(\lambda)$ has a left G-action $g_1[g,v] = [g_1 g, v]$ $\forall g_1, g, v$. Then all of its invariants are a) finite dimensional, b) G-rep^{ns}

Thm (Borel-Weil) Let $P = \mathrm{Hom}(T, \mathbb{C}^*)$, the weight lattice $(T = B/[B,B])$

Then $W = N_G(T)/T$, the Weyl group, acts on this and there is a fundamental domain for the action P^+ (the dominant chamber) such that

i) $G_X \otimes \Gamma(X, \mathcal{L}(\nu)) \twoheadrightarrow \mathcal{L}(\nu)$ i.e. $\mathcal{L}(\nu)$ is generated by its global sections, for $\nu \in P^+$

ii) $\mathcal{L}(\lambda)$ is ample for $\nu \in P^+ + \rho = \{\nu \in P^+ \text{ s.t } w(\nu) \neq \nu \ \forall w \in W\}$

iii) $H^0(X, \mathcal{L}(\nu)) = \begin{cases} L(\nu) & \nu \in P^+ \\ 0 & \text{o/w} \end{cases}$ with $L(\nu)$ a f.d. irred. $\mathcal{U}(\mathfrak{g})$-module

and this exhausts all f.d. rep^{ns} of $\mathcal{U}(\mathfrak{g})$. □

It is a very clever but entirely elementary proof to get D-affinity from here.

§ What's it good for?

We'd like to apply the RH correspondence. But how do we control regular singularities?

Lemma: Let K be an algebraic group acting on Y with finitely many orbits. Then let $\mathfrak{k} = \mathrm{Lie}(K)$. If $M \in D_Y$-mod has a compatible K-action then $M \in D_Y\text{-mod}_{reg.hol}$.

Proof: The compatibility states that the $\mathfrak{k}$-action arising from the K-action agrees with the one from $\psi: \mathcal{U}(\mathfrak{k}) \longrightarrow D_Y(Y)$. In particular it must be locally finite, since it extends to a K-action. Then begin with the observation that if $Y = K/K'$ then $\mathrm{Ch} M \subseteq T^*Y = K \times_{K'} (\mathfrak{k}/\mathfrak{k}')^*$ whilst local finiteness of $\mathcal{U}(\mathfrak{k})$-action translates to the characteristic variety is supported on $(\mathfrak{k}/\mathfrak{k}')^* = 0$ i.e. M is a K-equivariant local system on Y. Now pull M back to $K \overset{\pi}{\twoheadrightarrow} Y$ $\pi^* M$, a K-equiv. local system on K. In this case the local system must be trivial since the group action allows one to follow everything. Hence $\pi^* M$ has regular singularities. But then so too does M because $\pi: K \longrightarrow Y$ is a smooth morphism, so curves lift and we see given $i: C \to Y$ $\exists$ $i': C \to K$ and so $i^* M = i'^* \pi^* M$ has regular singularities.

$$\begin{array}{ccc} C & \xrightarrow{i'} & K \\ & \searrow^{i} & \downarrow \pi \\ & & Y \end{array}$$

Now one argues by induction on the number of orbits by taking M, a closed orbit O and its complement $Y \setminus O = U$. Then we split M up into parts supported on O and on U. $\square$

There is a famous category $\mathcal{O} \in U(\mathfrak{g})_0$-mod: ones for which the action of $\mathfrak{n} = [\text{Lie}\, B, \text{Lie}\, B]$ is locally nilpotent and $\mathfrak{h} = \text{Lie}\,B/\mathfrak{n}$ acts semisimply. In particular we see that $\mathcal{O} = (U(\mathfrak{g})_0, B)$-mod and so BB produces an equivalence

$$\mathcal{O} \xrightarrow{\sim} D_X\text{-mod}^B \xrightarrow[\sim]{RH} \text{Perv}_X^B$$

The RHS deals with the behaviour of local systems on G/B stable under the B-action, and the like: $G/B = \coprod_{w \in W} X_w$ with $X_w = BwB/B$: Schubert stratification; and this is the start of Kazhdan-Lusztig theory, etc.

§ Disappointment:

It's a conjecture that the only D-affine projective varieties are generalized flag manifolds G/P for P a parabolic. So there are lots of examples of interesting spaces where we couldn't go with this technology ... or perhaps?