§ D-modules and differential equations.

DEF^N 25: Let M be a $\mathcal{D}_X$-module. We set

$$Sol_X(M) = \mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{O}_X)$$

This is a sheaf of $\mathbb{C}$-vector spaces on X.

Why is this called Sol? Let $P = a_n(x)\partial^n + \cdots + a_1(x)\partial + a_0(x) \in D_{\mathbb{A}^1}$. Then we produce a left $D_{\mathbb{A}^1}$-module $M_P := \dfrac{D_{\mathbb{A}^1}}{D_{\mathbb{A}^1}P}$. Then

$$Sol_{\mathbb{A}^1}(M) = Hom_{\mathbb{A}^1}(M, \mathcal{O}_{\mathbb{A}^1}) \cong \{f : Pf = 0\} \text{ via } \theta \mapsto \theta(1).$$

This highlights a couple of issues:

a) If we only allow polynomial functions, there will be trouble

e.g. $P = \partial - 1$: $\{f \in \mathbb{C}[x] : f' = f\} = \begin{cases} 0 & \text{if we allow polynomial sol}^{ns}; \\ \mathbb{C}e^x & \text{" holomorphic "} \end{cases}$

b) We need to work locally because of multi-valuedness (and singularities)

e.g. $P = x\partial - \alpha$: $xf' = \alpha f$ i.e $Sol(M_P) = \mathbb{C}x^\alpha$.

<u>Theorem 26</u> Let X be a smooth irreducible variety, X^{an} its analytic version considered with metric topology.

$$\begin{tikzcd}
\left\{\begin{matrix}\text{integrable} \\ \text{connections} \\ \text{on } X^{an}\end{matrix}\right\} \arrow[d, leftrightarrow, "\wr"] & (V, \nabla) \arrow[d, mapsto] & (\mathcal{O}_X \otimes_{\mathbb{C}_X} \mathcal{L}, d \otimes 1) & \\
\left\{\begin{matrix}\text{local} \\ \text{systems} \\ \text{on } X^{an}\end{matrix}\right\} \arrow[d, leftrightarrow, "\wr"] & V^\nabla = \{v \in V : \nabla v = 0\} & \mathcal{L} \arrow[u, mapsto] \arrow[d, mapsto] & \tilde{X} \times_{\pi_1(X^{an})} V \longrightarrow X \\
\left\{\begin{matrix}\text{Representations} \\ \text{of } \pi_1(X^{an})\end{matrix}\right\} & & \begin{matrix}\text{monodromy of} \\ \text{local} \\ \text{system}\end{matrix} & \rho : \pi_1(X^{an}) \longrightarrow GL(V) \arrow[u, mapsto]
\end{tikzcd}$$

universal cover of X

Recall that a local system is a sheaf of vector spaces that, on some open cover $\{U_i\}$ we have $\mathcal{L}|_{U_i}$ is constant. The monodromy of a local system is then gotten by taking some loop and tracking how a given element transforms along the isomorphisms $(\mathcal{L}|_{U_i})(U_i \cap U_j) \xrightarrow{\sim} \mathcal{L}|_{U_j}(U_i \cap U_j)$ as the path moves from U_i to U_j.

(the sections of the system $\tilde{X} \underset{\pi_1(X^{an})}{\times} V$ on U are $f : \pi^{-1}(U) \longrightarrow V$ with $f(\gamma x) = \gamma f(x)$.)

Example 27 : $P_\alpha = x\partial - \alpha$. Note first that this doesn't produce an integrable connection since $Ch\, M_{P_\alpha} = \{xy = 0\} = T^*_{\mathbb{A}^1}\mathbb{A}^1 \cup T^*_0 \mathbb{A}^1$. To remove the "vertical line" $T^*_0\mathbb{A}^1$ we pass to $\mathbb{A}^1 \setminus \{0\}$ i.e. invert x. Then

$$\mathcal{M}_{P_\alpha} := \frac{D_{\mathbb{A}^1\setminus\{0\}}}{D_{\mathbb{A}^1\setminus\{0\}} P} \cong \mathbb{C}[x^{\pm 1}] \quad \text{and the local system is } \mathbb{C}x^\alpha,\ \alpha \in \mathbb{C}$$

To calculate the monodromy we then take a loop around 0 : $t \mapsto e^{2\pi i t}$ $t \in [0,1]$ and see how we transform : $\lambda 1 \longmapsto \lambda e^{2\pi i \alpha t}\big|_{t=1}$ i.e. $\pi_1(\mathbb{A}^1\setminus\{0\}) \longrightarrow \mathbb{C}^*$, $T \longmapsto e^{2\pi i \alpha}$

Note this predicts then that $\mathcal{M}_{P_\alpha} \cong \mathcal{M}_{P_{\alpha+n}}$ for any $n \in \mathbb{Z}$. This is achieved by

$$1 + D_{\mathbb{A}^1\setminus\{0\}} P_\alpha \longmapsto x^{-n} + D_{\mathbb{A}^1\setminus\{0\}} P_{\alpha - n}. \qquad \text{(Check !)}$$

In fact, we see that multiplying by any invertible function on $\mathbb{A}^1\setminus\{0\}$ must produce isomorphisms between $\mathcal{M}_P$'s. Again, which function space we choose tells us which type of functions we should use :

e.g $1 + D_{\mathbb{A}^1\setminus\{0\}^{an}} P_\alpha \longmapsto e^{\frac{1}{x}} + D_{\mathbb{A}^1\setminus\{0\}^{an}} \tilde{P}_\alpha$ produces $\mathcal{M}^{an}_{P_\alpha} \xrightarrow{\sim} \mathcal{M}^{an}_{\tilde{P}_\alpha}$ where $\tilde{P}_\alpha = x^2\partial - \alpha x - 1$.

But note $\mathcal{M}_{P_\alpha} \not\cong \mathcal{M}_{\tilde{P}_\alpha}$ since e.g. $Hom_D(\mathcal{M}_{P_0}, \mathcal{O}_{\mathbb{A}^1\setminus\{0\}}) = \mathbb{C}$ while $Hom_D(\mathcal{M}_{\tilde{P}_0}, \mathcal{O}_{\mathbb{A}^1\setminus\{0\}}) = 0$.
(the sol^n should be $e^{1/x}$)

So what we see

$\{D_X\text{-mod, coh over } \mathcal{O}_X\} \xrightarrow{\sim} Conn(X) \longrightarrow Conn(X^{an}) \xrightarrow{\sim} \pi_1(X^{an})\text{-reps}$

is not an equivalence!! Some solutions don't have much to do with monodromy. To get a tight relation between algebraic and holomorphic information we need to pass to projective varieties and use GAGA : $X \hookrightarrow \overline{X}$ where $\overline{X}$ is smooth projective and $\overline{X} \setminus D = X$ for some divisor D.

$$\underset{\text{"non-trivial"}}{Conn(X^{an}) \xrightarrow{\sim}} Conn^{reg}(\overline{X}^{an}; D) \xrightarrow[\text{"GAGA"}]{\sim} Conn^{reg}(\overline{X}; D) \xrightarrow[\text{"trivial"}]{\sim} Conn^{reg}(X)$$

Def^n 28 (Deligne) : a) Let X be a smooth curve (M, ∇) an integrable connection on X. It has regular singularities iff given a completion $i : X \hookrightarrow \overline{X}$ with $\overline{X}$ smooth projective, we have the $D_{\overline{X}}$-module j_*M is a union of coherent $\mathcal{O}_{\overline{X}}$-modules that are stable under $x_p \nabla_{\partial_p}$ for each $p \in \overline{X} \setminus X$.

ii) Let X be a smooth variety, and (M, ∇) an integrable connection on X. Then it is <u>regular</u> if for all $i : C \hookrightarrow X$ with C a smooth curve, $(i^+M, i^+\nabla)$ has regular singularities.

<u>THM 29</u> (Deligne) Let X be a smooth irreducible variety. $\mathrm{Conn}^{reg}(X) \xrightarrow{\sim} \pi_1(X^{an})$-mod.

<u>Example 30</u>:

i) For our sanity: $X = \mathbb{A}^1 \overset{j}{\hookrightarrow} \mathbb{P}^1$ and for $q \in \mathbb{C}[x]$ set $M_q = \frac{D_X}{D_X(\partial - q)} \cong \mathbb{C}[x]$

$(j_* M_q)(U_\infty) = \mathbb{C}[x^{\pm 1}]$ and we have $D_{\mathbb{P}^1}(U_\infty) = \langle \partial_{x^{-1}}, x^{-1} \rangle$ acting:

$$\partial_{x^{-1}} \cdot x^{-i} = i x^{-i+1} + x^{-i} \partial_{x^{-1}} \cdot 1 = i x^{-i+1} - x^{-i+2} \partial_x \cdot 1$$
$$= i x^{-i+1} - x^{-i+2} q$$

Hence $x^{-1}\partial_{x^{-1}} : x^{-i} \longmapsto i x^{-i} - x^{-i+1} q$ and so for regularity we'd need:

$$\mathbb{C}[x^{\pm 1}] = \bigcup_{i \le 0} x^{-i}\mathbb{C}[x^{-1}]$$ with each of these being stable under $x^{-1}\partial_{x^{-1}}$ i.e. $q \in x^{-1}\mathbb{C}[x^{-1}]$. But $q \in \mathbb{C}[x]$ $\therefore$ $q = 0$.

ii) $X = \mathbb{A}^1 \setminus \{0\} \hookrightarrow \mathbb{P}^1$ with M_q now having $q \in \mathbb{C}[x^{\pm 1}]$

regularity at ∞: $q \in x^{-1}\mathbb{C}[x^{-1}]$

at 0: $(x\partial)x^i = i x^i + x^{i+1} q \Rightarrow q \in x^{-1}\mathbb{C}[x]$

$\therefore$ $q = \alpha x^{-1}$.

<u>Def^n 30</u> Recall from Theorem 24 the irreducible holonomic D_X-modules $L(Y, N)$ where $Y \hookrightarrow X$ is affine and $N \in \mathrm{Conn}(Y)$ irreducible.

A holomorphic D-module is <u>regular</u> if each composition factor has the form $L(Y, N)$ for $Y \hookrightarrow X$ affine and $N \in \mathrm{Conn}^{reg}(Y)$.

e.g. if X is a curve then we get $L(x, \mathbb{C})$ $\forall x \in X$, plus regular irred conn^{ns} on open sets.

<u>THEOREM 31</u> (Kashiwara + others) Let X be a smooth irreducible algebraic variety. Then

$$\mathrm{Sol}_X : D^b_{rh}(D_X)^{\circ} \xrightarrow{\sim} D^b_c(X^{an})$$

with $D_X\text{-mod}_{rh} \xrightarrow{\sim} \mathrm{Perv}(X^{an})[-\dim X]$

& $L(Y,N) \longmapsto IC(Y,N^*)[\dim Y - \dim X]$

Here $D_X\text{-mod}_{rh}$ is the category of regular holonomic D_X-modules. It is abelian, closed under extension (by construction). $D^b_{rh}(D_X)$ is the bounded derived category of D_X-modules whose cohomology belongs to $D_X\text{-mod}_{rh}$. $D^b_c(X)$ is the bounded derived category of sheaves of vector spaces on X^{an}, whose cohomology is constructible with respect to some stratification of X by locally closed subvarieties. $\mathrm{Perv}(X^{an})$ is the category of "Perverse Sheaves" and $IC(Y,N)$ is the Intersection cohomology complexes.

CHAPTER 2: Rep^n Theory

We want to explain the representation theory and geometry of this theorem.

THEOREM 32 (Beilinson-Bernstein) Let $\mathfrak{g}$ be a simple Lie algebra, $X = G/B$ the corresponding flag variety. Then

$$D_X\text{-mod} \xleftrightarrow{\sim} \mathfrak{g}\text{-rep}^{ns} \text{ with trivial centre character}$$

$$\begin{array}{ccc} & & \cup \\ D_X\text{-mod}^B_{rh} & \longleftrightarrow & \mathcal{O}_0(\mathfrak{g}) \\ \updownarrow 2 & & \\ \mathrm{Perv}_B(X) & & \end{array}$$