Def^n 16 D_V-mod$_h$ is the category of all D_V-representations whose characteristic cycle has dimension $\dim V$ (or 0). Such representations are called holonomic.

Theorem 17 D_V-mod$_h$ is an abelian category, closed under extensions. It is artinian.

Proof: What the first claim means is that if $0 \to M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \to 0$ is a short exact sequence in D_V-mod, then $M_1, M_3 \in D_V\text{-mod}_h \iff M_2 \in D_V\text{-mod}_h$

So to see this take a good filtration on M_2 and set $F_i M_1 = \alpha^{-1}(F_i M_2)$ and $F_i M_3 = \beta(F_i M_2)$, the induced filtrations. Then it is elementary to see

$$0 \longrightarrow \operatorname{gr} M_1 \xrightarrow{\operatorname{gr}\alpha} \operatorname{gr} M_2 \xrightarrow{\operatorname{gr}\beta} \operatorname{gr} M_3 \longrightarrow 0$$

(e.g. $\operatorname{gr}\alpha(x + F_{i-1}M_1) = \alpha(x) + F_{i-1}M_2$ and if $\alpha(x) \in F_{i-1}M_2$ then $x \in F_{i-1}M_1$ by construction....). Now commutative algebra shows that

$$\underset{\substack{\parallel \\ Ch(M_2)}}{\operatorname{Supp}(\operatorname{gr} M_2)} = \underset{\substack{\parallel \\ Ch(M_1) \cup Ch(M_3)}}{\operatorname{Supp}(\operatorname{gr} M_1) \cup \operatorname{Supp}(\operatorname{gr} M_3)}$$

and so the first claim is proved.

It is also true that $\forall \mathfrak{p}$ min^l in $\operatorname{Supp} M_2$ (i.e. irred cpts of $Ch(M_2)$) we have

$$\operatorname{mult}_{\mathfrak{p}} M_1 + \operatorname{mult}_{\mathfrak{p}} M_3 = \operatorname{mult}_{\mathfrak{p}} M_2$$

So it follows that

$$\underline{Ch}\, M_2 = \underline{Ch}\, M_1 + \underline{Ch}\, M_3.$$

Thus if we have a descending chain $N = N_1 \supseteq N_2 \supseteq N_3 \supseteq \dots$ we necessarily have $\underline{Ch}\, N_i \geq \underline{Ch}\, N_{i+1}$ and so the chain must stabilise. Thus the category is artinian. $\square$

Thus the first key problem is

CLASSIFY ALL SIMPLE HOLONOMIC D_V-MODULES.

Recall simple means there are no submodules other than 0 and itself. The theorem shows that each holonomic module has a finite composition series with simple sections

$$N = N_1 \supset N_2 \supset N_3 \supset \dots \supset N_r \supset N_{r+1} = 0$$

s.t N_i/N_{i+1} simple. The Jordan-Hölder theorem states that although this filtration will not be unique, the number of times any simple appears is well-defined.

Let's look first at possible characteristic varieties of holonomic modules. Observe that since $\mathrm{gr} M$ is graded we must have that $\mathrm{Ch}\, M$ is $\mathbb{C}^*$-stable and so each irred. component of $\mathrm{Ch} M$ is $\mathbb{C}^*$-stable.

DEF^N 18 Let $Y \subseteq V$ be smooth irreducible and set

$$T_Y^* V = \{ \alpha \in (T^*V)|_Y \ : \ \alpha(\underset{\substack{\cap \\ TV|_Y}}{TY}) = 0 \}$$

This is the conormal bundle of Y. (Observe that over $y \in Y$, the fibre has dimension $\dim V - \dim Y$. Thus $\dim(T_Y^* V) = \dim Y + (\dim V - \dim Y) = \dim V$.)

Thm (Kashiwara) 19 i) $T_Y^* V$ is $\mathbb{C}^*$-stable and lagrangian in $T^* V$.

ii) Suppose $\Lambda \subset T^*V$ is a closed irreducible $\mathbb{C}^*$-stable lagrangian. Then

$$\Lambda = \overline{T^*_{\pi(\Lambda)_{reg}} V} \qquad \text{where } \pi : T^*V \longrightarrow V.$$

Proof : i) $T_Y^* V$ is obviously $\mathbb{C}^*$-stable. Let $\xi \in T_\lambda(T_Y^* V)$ with $\lambda \in T_Y^* V$. Recall L in the def^n of the symplectic structure on $T^* V$

$$L(\xi) = \langle \pi_* \xi, \lambda \rangle = 0 \quad \text{since } \pi_* \xi \in TY$$

$$\therefore L|_{T_Y^* V} = 0 \Rightarrow \omega = dL|_{T_Y^* V} = 0 \Rightarrow T_Y^* V \text{ is isotropic} \Rightarrow \text{lagrangian}.$$

ii) Set $Y = \pi(\Lambda)_{reg}$. The $\mathbb{C}^*$ action on T^*V differentiates to a vector field on T^*V which is given by $\sum_{i=1}^n \xi_i \frac{\partial}{\partial \xi_i} =: Eu$. Then since Λ is $\mathbb{C}^*$-stable we have that Eu is tangent to $T\Lambda_{reg}$ i.e $Eu \in T\Lambda_{reg}$ and the lagrangian cond^n then gives $0 = \omega(Eu, T\Lambda_{reg})$. But a simple calculation (with co-ordinates) shows that $\omega(Eu, -) = L(-)$. Thus $L|_\Lambda = 0$.

Now $\exists$ dense open set $\Lambda_{sm} \subset \Lambda_{reg}$ s.t. $\pi : \Lambda_{sm} \longrightarrow \pi(\Lambda_{sm}) \subset Y$ is smooth i.e. $\forall \alpha \in \Lambda_{sm}$ $\pi_* : T_\alpha \Lambda \twoheadrightarrow T_{\pi(\alpha)} Y$. Then

$\forall \beta \in T_\alpha \Lambda$ $\quad 0 = L(\beta) = \langle \pi_*(\beta), \alpha \rangle \Rightarrow \alpha(T_{\pi(\alpha)} Y) = 0$ i.e. $\alpha \in T_Y^* V \Rightarrow \Lambda_{sm} \subseteq T_Y^* V$

$\therefore \overline{\Lambda_{sm}} \subseteq \overline{T_Y^* V}$ and equality of dim^n + irred. gives $=$. $\square$

This statement now generalises to :

Cor 20 Let M be holonomic. Then $\exists$ a stratification of $V = \bigcup_\alpha V_\alpha$ by locally closed sets s.t. $\mathrm{Ch} M \subseteq \bigcup_\alpha \overline{T^*_{V_\alpha} V}$ (just work component by component) e.g. if $X \hookrightarrow V$ is a closed embedding then $\exists$ a D_V-module $\mathcal{B}_{X|V}$ whose characteristic variety is $T^*_X V$

i.e. $\{p\} \hookrightarrow V$ $\mathcal{B}_{\{p\}|V} = D_V / D_V m_p$ where $m_p \triangleleft \mathbb{C}[x_1 \_ x_n]$ is the corresponding max^l ideal.

$\rightsquigarrow$ suggests there may be an inductive description, but we'll need to deal with varieties more general than affine spaces V.

Start again!

Let X be a smooth variety.

D_X = sheaf of differential operators on X. It is defined as the subsheaf of $\mathrm{End}_{\mathbb{C}}(\mathcal{O}_X)$ generated locally by $\mathcal{O}_X$ and $\mathcal{T}_X$, the tangent sheaf. In other words, on an affine open set $\mathcal{U}$ we will have $D_X(\mathcal{U})$ generated by

$$A = \mathcal{O}_X(\mathcal{U}), \ \mathcal{T}_X(\mathcal{U}) = \mathrm{Der}(A,A) = \{\theta : A \to A \ \mathbb{C}\text{-linear s.t. } \theta(ab) = a\theta(b) + b\theta(a)\}$$

$$\underset{f}{\Downarrow} \qquad \underset{\theta}{\Downarrow}.$$

subject to $[\theta, f] = \theta(f)$ (noting that $[\theta_1, \theta_2] \in \mathrm{Der}(A,A)$)

(In fact, around any point p one can find an open nbhd $\mathcal{U}$ and local co-ordinates meaning that $\mathcal{T}_X(\mathcal{U}) = \bigoplus_{i=1}^{n} \mathcal{O}_X(\mathcal{U})\partial_i$ with $[\partial_i, \partial_j] = 0$ and elements $x_1 \_ x_n \in \mathcal{O}_X(\mathcal{U})$ s.t. $\partial_i(x_j) = \delta_{ij}$ and $(x_1, \ldots, x_n)$ generate m_p in $\mathcal{O}_{X,p}$. (To prove this take the gen. set $(x_1 \_ x_n)$ which the theory of regular local rings give, then observe that $dx_1 \_ dx_n$ is an $\mathcal{O}_{X,p}$-basis of $(\mathcal{T}_{X,p})^*$ and so can take an open set $\mathcal{U}$ where this freeness continues to hold.)

e.g. $\mathbb{A}^1 \setminus \{0\} = X$ $\quad A = \mathcal{O}(X) = \mathbb{C}[x^{\pm 1}]$, $\mathrm{Der}\,A = \mathbb{C}[x^{\pm 1}]\partial_x$. $D_X = A_1[x^{-1}]$

e.g. $\mathbb{P}^1 = \mathbb{A}^1_0 \cup \mathbb{A}^1_\infty$ $\quad \mathcal{O}(\mathbb{A}^1_0) = \mathbb{C}[x]$, $\therefore D_{\mathbb{P}^1}(\mathbb{A}^1_0) = A_1$ (variable x); $\mathcal{O}(\mathbb{A}^1_\infty) = \mathbb{C}[x^{-1}]$ so $D_{\mathbb{P}^1}(\mathbb{A}^1_\infty) = A_1$ (variable x^{-1})

QN: $D_{\mathbb{P}^1}(\mathbb{P}^1)$ = diff ops that exist both on $\mathbb{A}^1_0$ and $\mathbb{A}^1_\infty$: $\partial_{x^{-1}} = -x^2\partial_x$ (apply to x^i)
so find $x^2\partial_x, x\partial_x, \partial_x$ common to both.

- there is a filtration by degree of operator: $\mathrm{gr}\,D_X = \pi_* \mathcal{O}_{T^*X}$ where $\pi : T^*X \to X$
- D_X-mod: quasicoherent $\mathcal{O}_X$-modules such that D_X acts and is locally finitely generated over D_X.

We get good filtrations and $Ch(M) \subset T^*X$ $\mathbb{C}^*$-stable, involutive and therefore Bernstein's inequality. (meaning $\dots F_i M \subseteq F_{i+1} M \subseteq \dots$ each $\mathcal{O}_X$ coherent in particular)

<u>Prop 21</u> Suppose that $\mathcal{M} \in D_X$-mod is coherent over $\mathcal{O}_X$. Then $\mathcal{M}$ is locally free over $\mathcal{O}_X$.

<u>Pf</u>: Being locally free is a local cond^n: we want to prove that $\forall p \in X$ $\mathcal{M}_p$ is a free in a free $\mathcal{O}_{X,p}$-module. Without loss of generality we may take local coordinates around p: $\{x_1 \_ x_n\}$ $[\partial_1 \_ \partial_n]$ where $D_X = \sum_\alpha^\oplus \mathcal{O}_X \partial^\alpha$ and $m_p \triangleleft \mathcal{O}_{X,p}$ is generated by $(x_1 \_ x_n)$.

M_p is fin. gen. so pick a basis $\overline{m}_1, \dots \overline{m}_t \in \frac{\mathcal{M}_p}{m_p \mathcal{M}_p}$ by Nakayama's lemma $m_1 \_ m_t$ generate $\mathcal{M}_p$ as an $\mathcal{O}_{X,p}$-module.

We will prove they are linearly independent. So suppose $\exists f_i \in \mathcal{O}_{X,p}$ s.t. $\sum_{i=1}^t f_i m_i = 0$ in M_p.

Define $\mathrm{ord}(f_i) = \max\{l : f_i \in m^l\}$. Now $0 = \partial_j (\sum_i f_i m_i) = \sum_i (\partial_j(f_i) m_i + f_i \partial_j m_i)$

Let $\partial_j m_i = \sum_{k=1}^t h_{ij}^k m_k$, so that

$$0 = \sum_{s=1}^t \Big[\underbrace{\partial_j(f_s)}_{\substack{\mathrm{ord} < l \\ \text{for some } j \\ \text{and } s \\ \text{unless } l=0}} + \underbrace{\sum_{i=1}^t f_i h_{ij}^s}_{\mathrm{ord} \geq l} \Big] m_s$$

Let $l = \min\{\mathrm{ord}(f_i) : i = 1, \_ t\}$

$\therefore$ can replace f_i's with lower order terms. Iterate. $\Rightarrow l = 0$. Now factor out m_p to get $\sum \overline{f}_i \overline{m}_i = 0$ with at least one $\overline{f}_i \neq 0$ $\lightning$ unless all $f_i = 0$ to begin with. $\square$

<u>DEF^N 22</u> Any D_X-mod that is coherent over $\mathcal{O}_X$ is an <u>integrable connection</u>:

i.e. $\mathcal{T}_X \longrightarrow \mathrm{End}(\mathcal{M})$: $\theta \longmapsto \nabla_\theta$

(i) $\nabla_{f\theta}(m) = f \nabla_\theta(m)$

ii) $\nabla_\theta(fm) = \theta(f) m + f \nabla_\theta(m)$

} connection

iii) $[\nabla_\theta, \nabla_{\theta'}] = \nabla_{[\theta, \theta']}$ integrable

So we get a geometric restriction. We call the category of such objects $\mathrm{Conn}(X)$

<u>Prop 2.3</u> Let $M \in D_X$-mod. The following are equivalent

i) $\mathrm{Ch}\, M = T_X^* X$

ii) M is coherent over $\mathcal{O}_X$

iii) M is a vector bundle on X of finite rank, with an $\overset{\text{integrable}}{\vee}$ connection.

<u>Proof</u>: ii) $\Leftrightarrow$ iii) is what we've just seen.

ii) $\Rightarrow$ i) Take $F_i M = \begin{cases} M & i \geq 0 \\ 0 & i < 0 \end{cases}$. Then $\mathrm{gr} M = F_0 M / F_{-1} M = M$ so it's a good filtration as this is f.g. $/\mathcal{O}_X$. Then $\mathcal{J}_X \cdot F_0 M \subseteq \underset{F_0 M}{\overset{\shortparallel}{F_1 M}} \Rightarrow \mathcal{J}_X \subset \mathrm{Ann}\, \mathrm{gr} M \Rightarrow$
$\mathrm{Ch}\, M = T_X^* X$.

i) $\Rightarrow$ ii) We work locally, so that $D_X = \sum_\alpha^{\oplus} \mathcal{O}_X \partial^\alpha$ and $\pi_* \mathcal{O}_{T^*X} = \mathcal{O}_X[\xi_1 \ldots \xi_n]$

The cond^n $\mathrm{Ch}\, M = T_X^* X$ is equivalent to

$$\sqrt{\mathrm{Ann}_{\mathcal{O}_X[\xi]}(\mathrm{gr} M)} = (\xi_1 \ldots \xi_n) \text{ for any good filt}^n \text{ of } M.$$

Then, because of noetherianity, $(\xi_1 \ldots \xi_n)^m \subseteq \mathrm{Ann}_{\mathcal{O}_X[\xi]}(\mathrm{gr} M)$ for $m >> 0$

It follows that $(F_m D_X)(F_i M) \subseteq F_{i+m-1} M \quad \forall i$

But recall that $\mathrm{gr} M$ is fin. gen. so that we can find all generators in $F_i M$ for i big enough. Then, for any j we have $(F_j D_X)(F_i M) = F_{i+j} M$. From these two observations it follows that $F_{i+m} M = F_{i+m-1} M$ for all i big enough. In other words the good filtration terminates. Hence $M = F_{i+m} M$ is a coherent $\mathcal{O}_X$-module. $\square$

So we have a description of sorts of all holonomic D-modules with characteristic variety $T_X^* X$ as $\mathrm{Conn}(X)$

<u>Thm 2.4</u> Let Y be a locally closed smooth $\overset{\text{irreducible}}{\vee}$ subvariety of X s.t. $Y \longrightarrow X$ is affine. Let N be an irreducible element of $\mathrm{Conn}(Y)$. Then

i) $\exists$ a "minimal extension" $L(Y, N) \in D_X$-mod which is irreducible holonomic.

ii) Every irreducible holonomic D_X-module is of the form $L(Y, N)$ for some Y and N.

iii) $L(Y, N) \cong L(Y', N')$ iff $\overline{Y} = \overline{Y'}$ and $\exists$ open dense $U \subseteq Y \cap Y'$ s.t $N|_U \cong N'|_U$.