We observe some extra structure in the Hilbert scheme case: $\exists\, \mathbb{C}^* \curvearrowright (X,Y,a,b)$ $t\cdot(X,Y,a,b) = (tX, t^{-1}Y, a, b)$. Let $\mathbb{T}$ denote this torus, which is acting in a hamiltonian fashion on $\mathrm{Hilb}^n \mathbb{C}^2$.

Generally suppose that we have a hamiltonian action of a 1-D torus $\mathbb{T}$ on a symplectic resolution $\pi: X \longrightarrow Y$ of a conical symplectic singularity that commutes with the $\mathbb{C}^*$-action on Y defining the conical structure.

Def^n: (Braden-Licata-Proudfoot-Webster) Let $\lambda \in H^2(X,\mathbb{C})$, with W_X^λ and $W^\lambda(Y)$ defined as before. There is a $\mathbb{T}$-action on $W^\lambda(Y)$.

i) Decompose $W^\lambda(Y) = \bigoplus_{i\in\mathbb{Z}} W^\lambda(Y)_i$ under the action of $\mathbb{T}$. Let $G^\lambda(Y)$ be the category of finitely generated $W^\lambda(Y)$-modules such that $W^\lambda(\lambda)_{\geq 0}$ acts locally finitely

ii) Let $X_+ = \{x \in X : \lim_{t\to 0} t\cdot x \text{ exists}\} \subseteq X$. Set $G^\lambda(X)$ to be full subcategory of $(W_X, \mathbb{C}^*)$-mod$_g$ whose objects
  (i) are supported on X_+, i.e. for any lattice $M(0)$ we get $M(0)/hM(0)$ supp on X_+
  (ii) If $\hat{\xi}$ is a lift of the vector field of $\mathbb{T} \curvearrowright X$ to W_X, then $\exists$ lattice $M(0)$ stable under $\hat{\xi}$ (cf. Def^n 28)

Prop^n: Assume equivalences $(W_X^\lambda, \mathbb{C}^*)\text{-mod}_g \underset{\mathrm{Loc}}{\overset{\Gamma(\,)^{\mathbb{C}^*}}{\rightleftarrows}} W^\lambda(Y)\text{-mod}$. Then these restrict to $G^\lambda(X) \rightleftarrows G^\lambda(Y)$ if $\exists$ a finite # of $\mathbb{T}$-fixed points in X.

Proof: If $M \in G^\lambda(X)$ supported on X_+ then let $M(0)$ be a $\hat{\xi}$-stable lattice $\rightsquigarrow$ $M(0)/hM(0)$ is stable under $\mathrm{Lie}(\mathbb{T})$-action & supported on X_+:
but all irreducible components of X_+ have rings of functions that are non-positively graded (by construction) & so $\Gamma(M(0)/hM(0))$ is graded by $\mathrm{Lie}(\mathbb{T})$ with bounded above eigenvalues.

Now $\Gamma(M)^{\mathbb{C}^*}$ has a filt^n $\Gamma(M(i))^{\mathbb{C}^*} = F^i(\Gamma(M)^{\mathbb{C}^*})$ & the

$$\mathcal{M}(i) = h^{-i}\mathcal{M}(0)$$

associated graded $\mathrm{gr}\ \mathcal{M}^{\mathbb{C}^*} = \bigoplus_{i\in\mathbb{Z}} \frac{\mathcal{M}(i)[0]}{\mathcal{M}(i-1)[0]} \xrightarrow{h^i} \bigoplus_{i\in\mathbb{Z}} \frac{\mathcal{M}(0)[i]}{\mathcal{M}(1)[i]} = \frac{\mathcal{M}}{h\mathcal{M}}$

so we see that $\mathrm{gr}\,\Gamma(\mathcal{M})^{\mathbb{C}^*} = \Gamma(\mathcal{M}/h\mathcal{M})$ and so the $\mathrm{Lie}(\mathbb{T})$-action on $\Gamma(\mathcal{M})^{\mathbb{C}^*}$ is bounded above. Now it is clear that $W^\lambda(Y)_{>0}$ acts locally nilpotently, and $W^\lambda(Y)_0$ acts locally finitely as the weight spaces are finite dimensional. (because fixed points are isolated)

Conversely, given $N \in \mathcal{O}^\lambda(Y)$, choose a weight decomp^n (for the $\mathbb{T}$-action) compatible with the filtration (possible since $\mathbb{T}$ & $\mathbb{C}^*$ commute) and then $\mathrm{Loc}(N)(0)$ is preserved by $\hat{\xi}$. We need to show that its support is contained in X_+.

First $X_+ = V(f : f \in \mathcal{O}(X)$ eigenvector with positive $\mathbb{C}^*$-weight, non-neg $\mathbb{T}$-weight$)$
[such an f vanishes on any $\mathbb{C}^*$-fixed pt; $\pi^{-1}(0)$ $\mathbb{C}^*$-stable & projective, so contains fixed pt $\Rightarrow f$ is 0 on $\pi^{-1}(0)$. Now pick $x \in X_+$ and observe $f(x) = (t\cdot f)(t\cdot x) = \lim_{\substack{t\to 0\\ (t\in\mathbb{T})}} (t\cdot f)(t\cdot x) = 0$ since $\lim_{t\to 0} t.f$ exists and $\lim_{t\to 0} t.x$ lies in $\pi^{-1}(0)$ [finiteness again: $\lambda\cdot x$ is a fixed point of $\mathbb{T}$ $\forall\lambda \Rightarrow \lambda\cdot x = x$ $\forall \lambda\in\mathbb{C}^*$);
conversely all fns on X have non-negative $\mathbb{C}^*$-weight by the conical hypothesis and only constants have degree 0. Suppose $x\in X$ vanishes on the above zero set. Then $\lim_{t\to 0}(t^{-1}\cdot f)(x) = f(\lim_{t\to 0} t.x)$ exists $\forall f \in \mathcal{O}(X)$ since either $\lim_{t\to 0} t^{-1}.f$ exists because f has non-positive $\mathbb{T}$-degree, or $t^{-1}.f$ belongs to ideal defining the zero set. Thus $\lim_{t\to 0} t.x$ exists, so $x\in X_+$.]

Now take f as above, let $i_f : X_f \longrightarrow X$ be the inclusion of the principal open set. Then $i_f^{-1}\mathrm{Loc}(N)(0)$ is a sheaf of flat $\mathbb{C}[[h]]$-modules with fibre $\mathcal{O}_{X_f} \otimes_{\mathcal{O}(X)} \mathrm{gr}N$ at $h=0$. Now $\mathrm{gr}N$ has finite dim^l wt spaces that are bded above (for the $\mathrm{Lie}(\mathbb{T})$-action) and so if f has positive $\mathbb{T}$-degree it acts locally nilpotently on $\mathrm{gr}N$, & if f has $\mathbb{T}$-degree 0, then the positivity of the $\mathbb{C}^*$-action shows it acts locally nilpotent. Hence $\mathcal{O}_{X_f} \otimes_{\mathcal{O}(X)} \mathrm{gr}N = 0$ and so $i_f^{-1}\mathrm{Loc}(N)(0) = 0$

i.e N is supported on X_+. $\square$

e.g. $X = T^*\mathcal{B}$, $Y = \mathcal{N}$. We have T - a max^l torus of G - acting on all spaces in a hamiltonian fashion. Let $\mathbb{T} \hookrightarrow T$ be a general one parameter subgroup: then there are $|W|$ fixed points : $T \subset B$ with this giving a choice of positive roots and $\alpha(t) > 0$ $t \in \mathbb{T}$ generating the subgp. Then the the fixed points are $(wB, 0) \in T^*\mathcal{B}$ and the attracting set to one of these is $X_+(w) = T^*_{BwB/B}\, G/B$, the conormal to a Schubert cell.

Moreover, $\mathcal{O}^\lambda(\mathcal{N}) =$ cat $\mathcal{O}$ for $\mathfrak{g}$ with $U(\mathfrak{n})$ acting nilpotently, $U(\mathfrak{h})$ loc. fin & $Z(\mathfrak{g})$ via λ.

So we get back to the geometric description of category $\mathcal{O}$ for Lie algebras that we had before.

It is a fundamental (but non-trivial) property of category $\mathcal{O}$ for a simple Lie algebra that it is Koszul: if we consider a regular block $\mathcal{O}_0$ then the Koszul dual of $\mathcal{O}_0(\mathfrak{g})$ is $\mathcal{O}_0(\mathfrak{g}^L)$ where $\mathfrak{g}^L$ is the Langlands dual of $\mathfrak{g}$.

<u>Conjecture</u> (BLPW): For λ "good enough" $\mathcal{O}^\lambda(Y)$ is Koszul. Its Koszul dual $(\mathcal{O}^\lambda(Y))^!$ is equivalent to $\mathcal{O}^{\lambda^!}(Y^!)$ for some <u>symplectic dual</u> variety $Y^!$.

Examples of symplectic duals are expected to include :

1: $T^*(G/B)$ & $T^*(G^L/B^L)$

2. $\mathrm{Hilb}^n\,\mathbb{C}^2$ is self-dual

3. $T^*(GL(n)/P)$ & resolution of Slodowy slice in $\mathfrak{gl}(n)$ to nilpotent orbit described by P.

"4. Higgs branch of the moduli space of vacua for an $N=4, d=3$ supersymmetric QFT dual to the Coulomb branch of the moduli space of same theory."

BVT note : very few tools i.e. no RH correspondence.

Last example : Hypertoric Varieties, following Bellamy-Kuwabara & Braden-Licata-Proudfoot-Webster.

CAVEAT: many definitions to do with the integral geometry here (i.e $\Lambda, \eta, \lambda, \xi$ to appear) are not stated precisely or with accurate technical hypotheses.

Start with $W_{\mathbb{Z}} = \mathrm{span}_{\mathbb{Z}}\{\varepsilon_1 \dots \varepsilon_n\}$ with $\Lambda \subset W_{\mathbb{Z}}$ a rank k direct summand.

Set $T^n = \mathrm{Spec}\, \mathbb{C}[W_{\mathbb{Z}}]$, an n-dim^l torus. Then $\mathrm{Lie}(T^n)^* =: \mathfrak{t}^*$ is identified with $\mathbb{C}W_{\mathbb{Z}}$, and the character lattice $X(T) = W_{\mathbb{Z}}$.

Observe $T^n \hookrightarrow \mathbb{C}^n$ (we'll use that the dual basis $\{h_1 \dots h_n\}$ of $\{\varepsilon_1 \dots \varepsilon_n\}$ inherits the action $h_i \cdot x_j = \delta_{ij} x_j$ for $\mathbb{C}^n = \mathrm{Spec}\, \mathbb{C}[x_1 \dots x_n]$.)

Let $\mathfrak{k} := (\mathbb{C}W_{\mathbb{Z}}/\mathbb{C}\Lambda_0)^* \subseteq (\mathbb{C}W_{\mathbb{Z}})^* \cong \mathrm{Lie}(T^n) = \mathfrak{t}$, and let K be the corresponding subtorus of T^n that is connected. By construction $X(K) = W_{\mathbb{Z}}/\Lambda_0$ & $X(T/K) = \Lambda_0$

Recall that given $\eta \in X(K)$ we can define a toric variety

$$\mathbb{C}^n /\!\!/_\eta K \quad \circlearrowleft T/K \qquad \text{(a GIT quotient)}$$

Def^n: Keep the above notation. The <u>hypertoric variety</u> attached to the data (Λ_0, η) is the GIT quotient

$$\mathfrak{M} = \mathfrak{M}(\Lambda_0, \eta) := \mu^{-1}(0) /\!\!/_\eta K$$

where $\mu : T^*\mathbb{C}^n \longrightarrow \mathfrak{k}^*$ is the moment map.

Under decent general (and explicit) conditions on η this is a symplectic resolution of $\mathfrak{M}^0 = \mu^{-1}(0)/\!\!/K$.

Def^n: The <u>hypertoric enveloping algebra</u> is $\mathcal{U} = D(\mathbb{C}^n)^K$.

Note that $D(\mathbb{C}^n)^{T^n} = \mathbb{C}[x_1\partial_1, \dots, x_n\partial_n] \subseteq \mathcal{U}$, and we have $\mathrm{gr}\, D(\mathbb{C}^n)^{T^n} = \mathbb{C}[h_1 \dots h_n] = \mathrm{Sym}(\mathfrak{t})$. From the construction of K as the conn. torus attached to $\mathfrak{k}$, we deduce $Z(\mathcal{U}) = \langle h \in F_1 D(\mathbb{C}^n)^{T^n} : h + F_0 D(\mathbb{C}^n)^{T^n} \in \mathfrak{k} \rangle$

Let $\widetilde{W} = \mathrm{Spec}\, D(\mathbb{C}^n)^{T^n}$, so that this is a $\mathbb{C}W_{\mathbb{Z}}$-torsor. Then points of $\mathrm{Spec}\, Z(\mathcal{U})$ are labelled by $\widetilde{W}/\mathbb{C}\Lambda_0$ orbits.

Given data: (Λ_0, λ) with $\lambda \in \widetilde{W}/\mathbb{C}\Lambda_0$, the <u>reduced hypertoric enveloping algebra</u> is

$$\mathcal{U}_\lambda = \frac{\mathcal{U}}{m_\lambda \mathcal{U}}, \quad \text{where } m_\lambda \in \mathrm{Spec}\, Z(\mathcal{U}).$$

By construction $\mathrm{gr}\, \frac{\mathcal{U}}{m_\lambda \mathcal{U}} = \mathrm{Spec}\,(\mu^{-1}(0)/\!\!/K)$. The main difference between the data (Λ_0, λ) & (Λ_0, η) is that $\eta \in X(K) = W_{\mathbb{Z}}/\Lambda_0$ is integral, $\lambda \in \widetilde{W}/\mathbb{C}\Lambda_0$ is not

$\eta \in X(K)$

Def^n: Let $p : \mu^{-1}(0)^{ss} \longrightarrow \mu^{-1}(0)^{ss}/K = \mu^{-1}(0)/\!\!/_{\eta} K$. Let $\lambda \in \widehat{W}/\mathbb{C}\Lambda_0$ and consider as before

$$D(\mathbb{C}^n) \longrightarrow \widetilde{W}_{T^*\mathbb{C}^n} = W_{T^*\mathbb{C}^n}[h^{-1}].$$

Then set $\mathcal{L}_\lambda = \widetilde{W}|_{(T^*\mathbb{C}^n)^{ss}} \big/ m_\lambda \widetilde{W}|_{(T^*\mathbb{C}^n)^{ss}} \quad \circlearrowleft K$

The sheaf of hypertoric algebras associated to $(\Lambda_0, \eta, \lambda)$ is

$$(p_* \mathcal{E}nd_{\widetilde{W}}(\mathcal{L}_\lambda))^K =: \widetilde{W}(\Lambda_0, \eta, \lambda)$$

By construction $\widetilde{W}(\Lambda_0, \eta, \lambda)$ is a quantization of the hypertoric variety attached to (Λ_0, η)!

Thm (B-K) $\Gamma(\ )^{\mathbb{C}^*} : (\widetilde{W}(\Lambda_0, \eta, \lambda), \mathbb{C}^*)-\mathrm{mod}_g \xrightarrow{\sim} U_\lambda-\mathrm{mod}$ is an equivalence, provided a somewhat explicit combinatorial criterion on $(\Lambda_0, \eta, \lambda)$ holds: this in particular provides a relation between η & λ.

To define a category $\mathcal{O}$ we choose a 1-parameter subgroup of T/K which still acts on everything in sight. i.e. we need $\xi \in \Lambda_0^*$ — cocharacters of T/K. Given this, we can follow the recipe from earlier to construct

$$\mathcal{O}(\Lambda_0, \lambda, \xi),$$

and similarly a geometric version of $\mathcal{O}$, supported on $(\mu^{-1}(0)/\!\!/_{\eta} K)_+$, each piece of which is a toric Lagrangian variety.

Thm (BLPW) Under suitable (mild, combinatorial) hypotheses:

(i) $\mathcal{O}(\Lambda_0, \lambda, \xi)$ is a highest weight category (its simple objects are labelled by elements of $H^{mid}_{\Pi}(\mu^{-1}(0)/\!\!/_{\eta} K, \mathbb{C})$
attached to ξ.

(ii) All multiplicities of irreducibles in standards are either 0 or 1.

(iii) $\mathcal{O}(\Lambda_0, \lambda, \xi)$ is Koszul; its Koszul dual is $\mathcal{O}(\Lambda_0^!, \lambda^!, \xi^!)$ where

$$\Lambda_0^! = \Lambda_0^\perp \subset W_{\mathbb{Z}} \quad \text{w.r.t. standard inner product.}$$

$$\lambda^! = -\xi \in \Lambda_0^* \cong W_{\mathbb{Z}}/\Lambda_0^!, \quad \xi^! = -\lambda \in W_{\mathbb{Z}}/\Lambda_0 \cong (\Lambda_0^!)^*.$$