So let us assume we have a symplectic resolution $X \xrightarrow{\pi} Y$ of a a conical symplectic singularity Y. Let $\mathfrak{X} \longrightarrow \mathcal{Y}$ be its universal Poisson deformation and let $\mathcal{W}_X^\lambda$ be a $\mathbb{C}^\times$-equivariant quantization of X, with period $[\omega_X] + h\lambda$ $\quad \lambda \in H^2(X, \mathbb{C})$.

We have the category $(\mathcal{W}_X^\lambda, \mathbb{C}^\times)$-mod$_g$ which we consider to be a reasonable category.

Now $\pi_* \mathcal{W}_X^\lambda$ is a $\mathbb{C}[[h]]$-flat sheaf of algebras on Y. Moreover since $H^i(X, \mathcal{O}_X) = 0$ $\; i > 0$ (rat^l singularities) it follows that

$$\frac{\pi_* \mathcal{W}_X^\lambda}{h \pi_* \mathcal{W}_X^\lambda} \cong \pi_* \frac{\mathcal{W}_X^\lambda}{h \mathcal{W}_X^\lambda} = \pi_* \mathcal{O}_X = \mathcal{O}_Y$$

and so $\pi_* \mathcal{W}_X^\lambda$ is a $\mathbb{C}^\times$-equivariant quantization of Y. We set $W^\lambda(Y) := H^0(Y, \pi_* \widetilde{\mathcal{W}}_X^\lambda)^{\mathbb{C}^\times}$

$$= H^0(X, \widetilde{\mathcal{W}}_X^\lambda)^{\mathbb{C}^\times}$$

This is a filtered ring, where $F^m W^\lambda(Y) = H^0(X, \widetilde{\mathcal{W}}_X^\lambda(m))^{\mathbb{C}^\times}$ where because of our assumptions on the $\mathbb{C}^\times$-action, $F^m = 0$ for $m < 0$.

By construction $gr_F W^\lambda(Y) = \mathcal{O}(Y)$, so $W^\lambda(Y)$ is noetherian.

QUESTION : For which $\lambda \in H^2(X, \mathbb{C})$ is the functor

$$(\mathcal{W}_X^\lambda, \mathbb{C}^\times)\text{-mod}_g \longrightarrow W^\lambda(Y)\text{-mod}$$

$$M \longmapsto H^0(X, M)^{\mathbb{C}^\times}$$

an equivalence of categories?

Obviously, there is an adjoint functor $N \longmapsto \mathcal{W}_X^\lambda \underset{W^\lambda(Y)}{\otimes} N$

e.g. $X = T^*\mathcal{B}$ then the above becomes

$$\mathcal{D}_{\mathcal{B}}^{\lambda - \rho}\text{-mod} \xrightarrow{\Gamma} \mathcal{U}(\mathfrak{g})_{\lambda - \rho}\text{-mod}$$

and we recover the setting of Beilinson-Bernstein.

In general, it is expected that there is a finite set of hyperplanes inside $H^2(X, \mathbb{C}) \cong \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ s.t. if $\lambda + (\text{effective}) \notin$ hyperplanes then we get the equivalence.

The key questions really are:
- why does such an equivalence hold?
- precisely when does it hold?

Probably understanding the 2^{nd} point will clarify the first, and one should certainly expect a connection between the symplectic deformation & cohomology of X.

Examples that are understood in varying levels of detail are:
- Hilbert schemes of points on $\mathbb{C}^2$ (or some resolution of $\mathbb{C}^2/\Gamma$ to replace $\mathbb{C}^2$) $\rightsquigarrow$ Cherednik algebras
- resolutions of Slodowy slices $\rightsquigarrow$ finite W-algebras
- hypertoric varieties $\rightsquigarrow$ hypertoric algebras
- generic case of arbitrary resolution (announced Braden-Proudfoot-Webster, Kremnizer) $\rightsquigarrow$??

Of particular interest in here are Nakajima quiver varieties.

§ Quiver Varieties

a) GIT: Suppose G is a (reductive, connected) group acting algebraically on a variety X. If X were affine, $X = \operatorname{Spec} R$, with $G \circlearrowright R$ then we'd take X/G to be $\operatorname{Spec} R^G$: this is often the wrong space i.e. $\mathbb{C}^\times \circlearrowright \mathbb{C}^N$ via mult^n $\rightsquigarrow \mathbb{C}^N/\!/_{\mathbb{C}^\times} = \{pt\}$. To get round this, regard $G(\mathbb{C}^N)$ as sections of trivial line bundle and lift action of G to the total space, but let the action be lifted non-trivially

i.e. $\mathbb{C}^* \times (\mathbb{C}^N \times \mathbb{C}) \longrightarrow \mathbb{C}^N \times \mathbb{C}$ $\quad \lambda\cdot(z,l) = (\lambda z, \lambda^{-a} l)$ $\quad a \in \mathbb{Z}$

Call this $\mathcal{L}_a$.

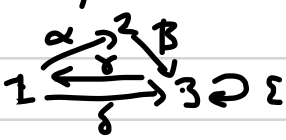
e.g. if $a=1$ then $x_1, \dots x_N$ become invariant. Set S_m = sections of $\mathcal{L}_a^{\otimes m}$

$$S(a) = \bigoplus_{m\geq 0} S_m \supseteq S(a)^{\mathbb{C}^*}$$

$$\operatorname{Proj} S(a)^{\mathbb{C}^*} = \operatorname{Proj} \mathbb{C}[x_1 \_ x_N] = \mathbb{P}^{N-1} =: \mathbb{C}^N/\!/_a \mathbb{C}^*$$

Choice of different line bundles over X, plus lift of action, is a <u>linearisation</u>
Different linearisations $\rightsquigarrow$ different quotients (i.e. orbit spaces)

Now let $Q = (I, A)$ be a quiver $\qquad t, h : A \longrightarrow I$

e.g. 

Then given $\underline{d} \in \mathbb{N}_0^I$, set

$$\mathrm{Rep}(Q, \underline{d}) = \{ (X_\alpha)_{\alpha \in A} : X_\alpha : \mathbb{C}^{d_{t(\alpha)}} \longrightarrow \mathbb{C}^{d_{h(\alpha)}} \} = \mathbb{A}^N$$

with $N = \sum_\alpha d_{t(\alpha)} d_{h(\alpha)}$. This is the space of rep^{ns} of Q of dim^n vector $\underline{d}$. There is a notion of isomorphism of quiver representation: it corresponds to the action of $G(\underline{d}) = \prod_{i \in I} GL_{d_i}(\mathbb{C})$ by base-change

$$g \cdot (X_\alpha) = (g_{h(\alpha)} X_\alpha g_{t(\alpha)}^{-1})$$

A <u>framing</u> of a rep^n of dim^n $w \in \mathbb{N}_0^I$ is

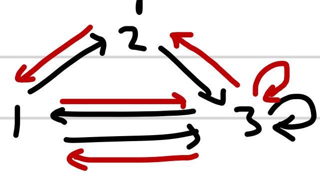
$$a_i : \mathbb{C}^{w_i} \longrightarrow \mathbb{C}^{d_i} \quad \text{for } i \in I$$

The set is denoted $L(\underline{w}, \underline{d})$. Note that $G(\underline{d})$ also acts.

$$\mathrm{FRep}(Q, \underline{d}, \underline{w}) := \mathrm{Rep}(Q, \underline{d}) \times L(\underline{w}, \underline{d})$$

Then $M_{\underline{d}, \underline{w}} := T^* \mathrm{FRep}(Q, \underline{d}, \underline{w}) \cong \mathrm{Rep}(\overline{Q}, \underline{d}) \times L(\underline{w}, \underline{d}) \times L(\underline{d}, \underline{w})$

where $\overline{Q}$ = double of Q = for each $\alpha \in A$, add $\alpha^* \in \overline{A}$ s.t. $\alpha^* : h(\alpha) \to t(\alpha)$



As before, we have a moment map attached to action of $GL(\underline{d})$ on FRep:

$$\mu : M_{\underline{d}, \underline{w}} \longrightarrow \mathfrak{gl}(\underline{d})$$

$$(X, Y, a, b) \longmapsto [X, Y] + ab \qquad \text{(ADHM eq}^{\underline{n}}\text{)}$$

which arises from G-invariant symplectic form on $M_{\underline{d}, \underline{w}}$ given by

$$\omega((X, Y, a, b), (X', Y', a', b')) = \mathrm{tr}(XY' - YX' + ab' - ba')$$

<u>Def^n</u> (Nakajima): $\mathfrak{M}_0(\underline{d}, \underline{w}) = \mathfrak{M}_0 = \mu^{-1}(0) /\!\!/ G(\underline{d}) = \mathrm{Spec}\, [\mu^{-1}(0)]^{G(\underline{d})}$

To get a GIT quotient we need to choose a G-equivariant line bundle on $M(\underline{d},\underline{w})$: labelled by $\mathbb{X}(G(\underline{d})) = \mathbb{Z}^I$ where $(\theta_i)\in\mathbb{Z}^I$ produces $(g_i)\longmapsto \prod \det(g_i)^{\theta_i}$. Nakajima chooses $\theta = (-\mathbb{1})$

Def^n (Nakajima) $\mathcal{M} = \mathcal{M}(\underline{d},\underline{w}) = \mu^{-1}(0)/\!\!/\!\!/\!\!/_\theta\, G(\underline{d}) = \mathrm{Proj}\left(\bigoplus_{n\geq 0} \mathbb{C}[\mu^{-1}(0)]^{G,\det^{-n}}\right)$

Prop^n: (i) $\mu^{-1}(0)/\!\!/\!\!/\!\!/_\theta\, G(\underline{d}) \cong \mu^{-1}(0)^{ss}/G(\underline{d})$ where $(X,Y,a,b)\in\mu^{-1}(0)^{ss}$

iff $\nexists$ non-trivial $S_i\subseteq\mathbb{C}^{d_i}$ for each $i\in I$ s.t. $b(S_i)=0$ $\forall i$ & $X,Y(\underline{S})\subseteq\underline{S}$

(ii) $G(\underline{d})$ acts freely on $\mu^{-1}(0)^{ss}$.

(iii) $\mathcal{M}_0$ is a singular Poisson variety with contracting $\mathbb{C}^*$-action, $\mathcal{M}$ is symplectic variety & $\exists\ \pi:\mathcal{M}\longrightarrow\mathcal{M}_0$ projective morphism that is $\mathbb{C}^*$-equivariant & Poisson

Proof : (i) is deduced by the Hilbert-Mumford criterion

(ii) If (g_i) stabilises $(X,Y,a,b)\in\mu^{-1}(0)^{ss}$, then $\mathrm{Im}(g_i - id) =: S_i$ violates the cond^n of (1) unless $g_i = id$.

(iii) One uses cond^n (i) again to prove that $\mu: M(\underline{d},\underline{w})\to\mathfrak{g}(\underline{d})$ is smooth at points of $\mu^{-1}(0)^{ss}$, and then it follows by hamiltonian reduction that $\mathcal{M}$ is symplectic, $\mathcal{M}_0$ Poisson. The $\mathbb{C}^*$-action arises from $\lambda(X,Y,a,b) = (\lambda X,\lambda Y,\lambda a,\lambda b)$ $\square$

In many cases π is a symplectic resolution of a conical symplectic singularity.

E.g. $Q = 1\longrightarrow 2\longrightarrow 3\longrightarrow\cdots\longrightarrow n-1$; $\underline{d} = (n-1, n-2,\ldots,1)$, $\underline{w} = (n, 0\ldots 0)$

i.e.
$$\begin{tikzcd}
\mathbb{C}^{n-1} \arrow[r, shift left, "X_1"] \arrow[d, shift left, "b"] & \mathbb{C}^{n-2} \arrow[l, shift left, "Y_1"] \arrow[r, shift left, "X_2"] & \cdots \arrow[l, shift left, "Y_2"] & \mathbb{C}^2 \arrow[r, shift left, "X_{n-2}"] & \mathbb{C} \arrow[l, shift left, "Y_{n-2}"] \\
\mathbb{C}^n \arrow[u, shift left, "a"] & & & &
\end{tikzcd}$$

If $(X,Y,a,b)\in\mu^{-1}(0)^{ss}$, then Y & b are injective :

For Y_k : set $S_i = \begin{cases}\ker Y_k & i=k+1\\ 0 & i\neq k+1\end{cases}$.

assuming Y_j inj for $j>k$

Then $Y(S)\subseteq S$ obviously, $b(S)=0$ & $Y_{k+1}X_{k+1}(S_{k+1}) = X_k Y_k(S_{k+1}) = 0$

$\Rightarrow X_{k+1}(S_{k+1}) = 0$

$\therefore \Rightarrow \ker(Y_k) = 0 \;\cdots$ □

Now $\{(X, Y, i, j) : [X,Y] + ab = 0,\ Y\text{'s} \,\&\, b \text{ inj.}\}/_{GL(\underline{d})}$

$$\cong$$

$$T^*\mathcal{B} = \{(\phi, F^\cdot) : \phi(F^i) \subset F^{i-1}\ \forall i\} \subseteq \mathfrak{gl}_n \times \mathcal{B}.$$

The IM is given by $F^i = \mathrm{Im}(bY_1Y_2 \ldots Y_{n-1-i})$ for $i = 1, \ldots, n-1$

$$\phi = ba$$

(Note $babY_1 \ldots Y_{n-1-i} = bY_1X_1Y_1 \ldots Y_{n-1-i} = bY_1 \ldots Y_{n-1-i}Y_{n-i}X_{n-i}$ so map is well-defined)

Injectivity: $GL(\underline{d})$ action allows $b = b'$, $Y_i = Y_i'$ $\forall i$. Then b inj $\Rightarrow a = a'$;
$Y_1X_1 + ab = Y_1'X_1' + a'b' \Rightarrow Y_1X_1 = Y_1'X_1' \Rightarrow X_1 = X_1' \ldots$

Surjectivity: both spaces map properly to $\mathcal{M}_0 = \mathcal{N}$

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\Psi"] \arrow[dr] & T^*\mathcal{B} \arrow[d] \\ & \mathcal{N} \end{tikzcd}$$

but $\dim \mathcal{M} = \dim T^*\mathcal{B}$ & ψ proper $\Rightarrow \psi$ surj.

□

E.g. $Q =$ , $d = n$, $w = 1$.

$$\mathcal{M} \cong \mathrm{Hilb}^n(\mathbb{C}^2) \xrightarrow{\pi} \mathcal{M}_0 = \mathrm{Sym}^n(\mathbb{C}^2) = (\mathbb{C}^2)^n/\mathfrak{S}_n$$

$$\mathrm{Hilb}^n(\mathbb{C}^2) = \{I \triangleleft \mathbb{C}[x,y] : \dim_{\mathbb{C}} \tfrac{\mathbb{C}[x,y]}{I} = n\} \longrightarrow (\mathbb{C}^2)^n/\mathfrak{S}_n$$

$$I \longmapsto \mathrm{supp}\, I$$

Here $X \circlearrowleft \mathbb{C}^n \underset{a}{\overset{b}{\rightleftarrows}} \mathbb{C}$, $Y \circlearrowleft$ + $[X,Y] + ab = 0$ & $I \mapsto X =$ mult by x on $\frac{\mathbb{C}[x,y]}{I}$

$Y =$ mult^n by y

$a(1) = 1 + I$

$b = 0$

§ Quantization of $\mathrm{Hilb}^n(\mathbb{C}^2)$.

$H^2(X, \mathbb{C})$ is 1-dimensional, generated by $\Lambda^n T$ where T is the tautological bundle on $\mathrm{Hilb}^n \mathbb{C}^2$ whose fibre above I is $\mathbb{C}[x,y]/I$, or equivalently $T = \mathbb{C}^n \times_{GL_n} \mu^{-1}(0)^{ss}$.

One constructs $W^{\lambda}_{\mathrm{Hilb}^n\mathbb{C}^2}$ $(\lambda \in \mathbb{C} = H^2(X,\mathbb{C}))$ by quantum hamiltonian reduction:

- take the canonical quantization of $M(n,1) = \{(X,Y,a,b)\}$, $\mathcal{W}_{M(n,1)} =: W$

- the action of $GL(n)$ on $M(n,1)$ produces a quantum moment map:

$$\eta(E_{ij}) = h^{-1}\sum_{t=1}^{n}(X_{it}\partial_{X_{jt}} - X_{jt}\partial_{X_{ti}}) + h^{-1}a_i\partial_{a_j} + \tfrac{1}{2}\delta_{ij}$$

where X_{ij} are co-ords for X, a_i co-ords for a.

- Let $\mathcal{U} = M(n,1)'' = \{(X,Y,a,b) : \nexists$ non-trivial $S \subseteq \mathbb{C}^n$ stable under X, Y & killed by $b\}$

so that $\mu^{-1}(0) \cap \mathcal{U} = \mu^{-1}(0)^{ss} \xrightarrow{\psi} \mathrm{Hilb}^n\mathbb{C}^2$

- $W^{\lambda}_{\mathrm{Hilb}^n\mathbb{C}^2} = \psi_* \mathrm{End}_{W|_{\mathcal{U}}}(\mathcal{L}_\lambda)^{GL_n}$ where

$$\mathcal{L}_\lambda = W|_{\mathcal{U}} \Big/ W|_{\mathcal{U}}(h\eta(x) - h\lambda\,\mathrm{tr}(x) : x \in \mathfrak{gl}_n)$$

<u>Theorem</u> (Etingof-Ginzburg, G-Stafford, Gan-Ginzburg, Kashiwara-Rouquier)

i) $(W^{\lambda}_{\mathrm{Hilb}^n\mathbb{C}^2}, \mathbb{C}^\times)\text{-mod}_g \longrightarrow W^\lambda(\mathrm{Sym}^n\mathbb{C}^2)\text{-mod}$

is an equivalence if $(\lambda + \mathbb{N}) \cap \{-\frac{1}{2} + \{\frac{m}{d} : m, d \in \mathbb{Z}, 2 \le d \le n, (m,d) = 1, -d < m < 0\}\} = \emptyset$

ii) $W^\lambda(\mathrm{Sym}^n\mathbb{C}^2) \cong U_c(S_n)$, a spherical rational Cherednik algebra at parameter $c = \lambda - \frac{1}{2}$ $\square$