QUANTIZATION & AFFINITY

We have two goals this week. The first is to introduce quantizations of symplectic varieties, the second to introduce a good category of representations that may have a chance to be "affine" as in the Beilinson–Bernstein theorem. In the next lectures we will give examples.

§ Quantization

<u>Def^n</u> : A <u>quantization</u> of a Poisson variety $\mathfrak{X} \xrightarrow{\pi} S$ is a sheaf $\mathcal{O}_h$ of flat $\pi^{-1}S[[h]]$-algebras on $\mathfrak{X}$, complete in the h-adic topology, with an isom $\frac{\mathcal{O}_h}{(h)} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}$ $\quad f \longmapsto \bar{f}$ such that

- for $f, g \in \mathcal{O}_h$ we have $\overline{[f,g]} = h\{\bar{f}, \bar{g}\}$.

where $\{\,;\,\}$ is the Poisson bracket on $\mathfrak{X}$ over S.

<u>Example</u> : $S = \operatorname{Spec}\mathbb{C}$, $\mathfrak{X} = T^*V$ with V a vector space with usual symplectic form. Then $\mathcal{O}_{\mathfrak{X}} = \mathbb{C}[x_1 \dots x_n, \xi_1 \dots \xi_n]$. A quantization is

$$W_{T^*V} = \mathbb{C}[x_1, \dots, x_n, \xi_1 \dots \xi_n][[h]] \Big/ \left([x_i, x_j] = 0 = [\xi_i, \xi_j],\ [x_i, \xi_j] = \delta_{ij} h\right)$$

is a quantization of $\mathfrak{X}$.

<u>Example</u> : $\mathfrak{X} = T^*M$ for a smooth variety M. Let $W_{T^*M} = \mathcal{O}_{T^*M}[[h]]$ as a $\mathbb{C}[[h]]$-module and define multiplication by

$$f * g = \sum_{\alpha \in \mathbb{N}^n} \frac{h^{|\alpha|}}{\alpha!} \partial_\xi^\alpha f \, \partial_x^\alpha g$$

in some local coordinate system. This is a quantization of $\mathfrak{X}$ with its usual symplectic form.

There are 2 crucial differences between D_M & W_{T^*M}:

a) the appearance of $\mathbb{C}[[h]]$ in W_{T^*M}

b) D_M is a sheaf on M, W_{T^*M} on T^*M.

In fact, these two counterbalance one another.

It is basically a theorem of Kontsevich that quantizations exist. However, we want something more special and more tractable

<u>Thm</u> (Bezrukavnikov–Kaledin) Let $\pi : X \to Y$ be a symplectic resolution of Y, an affine symplectic singularity. Let $\mathfrak{X}$ be the universal Poisson deformation of X, defined over $\widehat{H^2(X,\mathbb{C})}$. Then there exists a canonical quantization $W_{\mathfrak{X}}$ of $\mathfrak{X}$ such that: for any quantization W_X of X, $\exists$ a unique section

$$s : \mathrm{Spec}\ \mathbb{C}[[h]] \xrightarrow{\ \ } \mathrm{Spec}\ \mathbb{C}[[h]] \hat{\otimes} \mathbb{C}[\widehat{H^2(X,\mathbb{C})}]$$

of the projection p such that $p^* W_{\mathfrak{X}} \cong W_X$.

In other words, quantizations of X are classified by

$$[\omega_X] + h\, H^2(X, \mathbb{C})[[h]]$$

The attachment Quantization $\longrightarrow [\omega_X] + h\, H^2(X,\mathbb{C})[[h]]$ is called the <u>PERIOD</u> map.

<u>Proof</u>: This is an algebraic version of Fedosov quantization. Formally locally the symplectic variety has functions of the form

$$A = \mathbb{C}[[x_1 - x_n, \xi_1 - \xi_n]] \text{ with symplectic form } \Omega'$$

This is unique in the sense that if Ω is another symplectic form (e.g. usual one) on A then $\langle A, \Omega \rangle \cong \langle A, \Omega' \rangle$. So we can attach to X, a symplectic variety, $M_s \longrightarrow X$ where a point of M_s is a pair (x, φ) with $\varphi : \hat{\mathcal{O}}_{X,x} \longrightarrow A$ a symplectic IM. Thus M_s is a $\mathrm{Symp}(A)$-torsor over X.

The quantization of A is also unique:

$$D = \mathbb{C}[[x_1 - x_n, \xi_1 - \xi_n, h]] \big/ \langle [x_i, x_j] = 0 = [\xi_i, \xi_j],\ [x_i, \xi_j] = \delta_{ij} h \rangle$$

So given W_X, a quantization of X, we get $M_q \longrightarrow X$ whose points are (x,ψ) where $\psi : \widehat{W}_{X,x} \longrightarrow D$ is a $\mathbb{C}[[h]]$ algebra isomorphism. This is an $\mathrm{Aut}(D)$-torsor over X.

Now set $\mathrm{Aut}_l(D)$ = automorphisms of D that are the identity on $D/h^l D$, and let $\mathrm{Aut}^l(D) = \mathrm{Aut}(D)/\mathrm{Aut}_{l+1}(D)$ for any $l \geq 0$. We have

$$\mathrm{Aut}^0(D) = \mathrm{Symp}(A)$$

Thus the problem becomes basically to lift the $\mathrm{Aut}^0(D)$-torsor $\mathcal{M}_s$ on X to the $\mathrm{Aut}(D)$-torsor $\mathcal{M}_q$ on X. This is an obstruction theory problem whose solution is classified by $H^2(X,\mathbb{C})$ at each step $l \mapsto l+1$. $\square$

Recall that when we studied $G_{T^*\mathcal{B}}$ we could twist differential operators by $H^2(T^*\mathcal{B},\mathbb{C})$. Here we can twist by $H^2(T^*\mathcal{B},\mathbb{C})[[h]]$!!

Recall if Y is conical symplectic singularity then X inherits a $\mathbb{C}^*$-action with the property that $t \cdot \omega_X = t^n \omega_X$ (where n was the degree of the Poisson bracket on Y)

<u>Def^n</u> : A $\mathbb{C}^*$-equivariant quantization of X is a quantization of X together with a set of isomorphisms

$$a_t : T_t^{-1} W_X \xrightarrow{\sim} W_X \qquad \forall t \in \mathbb{C}^*$$

s.t. $a_{t_1} \circ a_{t_2} = a_{t_1 t_2}$ and such that $a_t(h) = t^n h$.

The following is then standard:

<u>Corollary</u>: The $\mathbb{C}^*$-equivariant quantizations of X are parametrized by

$$H^2(X,\mathbb{C}) = [\omega_X] + h H^2(X,\mathbb{C}) \longleftrightarrow [\omega_X] + h H^2(X,\mathbb{C})[[h]]$$

So we recover the Lie theoretic parametrization. Whenever we are in the $\mathbb{C}^*$-equivariant situation we will write $\mathcal{W}_X^\lambda$ for the $\mathbb{C}^*$-equivariant quantization corresponding to $\lambda \in H^2(X,\mathbb{C})$.

§ Module Categories

There is no hope that we can find classical representation theory categories directly as $\mathcal{W}_X$-modules. This is simply because of $\mathbb{C}[[h]]$ appearing. We want to get rid of that, and to do so we use $\mathbb{C}^*$ again.

As part of this process, suppose we want to quantise a point $\rightsquigarrow \mathbb{C}[[h]]$. We'd then like to take something like "graded" $\mathbb{C}[[h]]$-modules and then take the degree 0 weight space : but this produces objects like $h\,\mathbb{C}[[h]]$ which would then go to zero : so we remove that problem by passing to $\mathbb{C}((h)) = \mathbb{C}[h^{-1}; h]]$.

<u>Def^n</u> : Given $\mathcal{W}_X$, a quantization of X, set $\widetilde{\mathcal{W}}_X = \mathcal{W}_X[h^{-1}]$. Let

$$\widehat{\mathcal{W}}_X(m) = h^{-m}\mathcal{W}_X \subseteq \widetilde{\mathcal{W}}_X \qquad (\text{so } \widehat{\mathcal{W}}_X(m) \subset \widehat{\mathcal{W}}_X(m+1))$$

Note now that in the case $X = T^*V$ we get $\mathcal{D}_V \longrightarrow \pi_* \widetilde{\mathcal{W}}_{T^*V}$ via

$$x_i \longmapsto x_i, \quad \partial_i \longmapsto h^{-1}\xi_i \qquad \text{where } \pi: T^*V \longrightarrow V \text{ is projection.}$$

In particular we get a functor $I: \mathcal{D}_V\text{-mod} \longrightarrow \widetilde{\mathcal{W}}_{T^*V}\text{-mod}$ by

$$N \longmapsto \widetilde{\mathcal{W}}_{T^*V} \underset{\pi^{-1}\mathcal{D}_V}{\otimes} \pi^{-1}N =: \widetilde{N}$$

If in addition N had a filtration, then $\widetilde{N}$ has a lattice $\widetilde{N}(0)$ i.e. $\widetilde{N}(0) \subset \widetilde{N}$ a $\mathcal{W}_{T^*V} = \widehat{\mathcal{W}}_{T^*V}(0)$-module s.t.

$$\widetilde{\mathcal{W}}_{T^*V} \underset{\widehat{\mathcal{W}}^{\circ}_{T^*V}(0)}{\otimes} \widetilde{N}(0) \xrightarrow[\text{mult}^n]{\sim} \widetilde{N}$$

viz. $\widetilde{N}(0) = \sum_k \widehat{\mathcal{W}}_{T^*V}(-k) \otimes F^{k+l} N$ (for any l we get a lattice)

<u>Def^n</u> : $\mathcal{W}_X\text{-mod}_g$ is the full subcategory of $\widetilde{\mathcal{W}}_X$-modules $\mathcal{N}$ admit a coherent $\mathcal{W}_X$-lattice $\mathcal{N}(0)$ (i.e. locally finitely generated)

Observe if $\mathcal{N} \in \mathcal{W}_X\text{-mod}_g$, then $\mathcal{N}(0)/h\mathcal{N}(0)$ is $\mathcal{O}_X$ coherent (and conversely) so this is a replacement for the category of D-modules with good filtrations (we never specify the lattice, just as we never specify the filtration).

<u>Def^n</u>: Suppose that X has a $\mathbb{C}^*$-action with $t\cdot\omega_X = t^n\omega_X$ for some positive integer n. Then $(\mathcal{W}_X, \mathbb{C}^*)$-$\mathrm{mod}_g$ is the category of $\mathbb{C}^\times$-equivariant $\widetilde{\mathcal{W}}_X$-modules that admit a $\mathbb{C}^\times$-equivariant coherent $\mathcal{W}_X$-lattice.

This is a category we should love:

<u>SANITY CHECK</u>: Let $X = T^*M$ with $\mathbb{C}^*$-action given by inverse dilation in cotangent (as earlier) so that $t\cdot\omega_X = t\,\omega_X$. Then for any $\lambda\in H^2(T^*M,\mathbb{C}) = H^2(M,\mathbb{C})$

$$(\mathcal{W}^\lambda_{T^*M}, \mathbb{C}^*)\text{-}\mathrm{mod}_g \overset{\sim}{\longleftrightarrow} D_M^{\lambda+\Omega^{1/2}}\text{-}\mathrm{mod}$$

<u>Proof</u>: The first thing to explain is the shift by "half forms": this arises because $D_X^{op} \cong \Omega\otimes D_X\otimes\Omega^{-1}$. When we replace $\mathcal{W}_{T^*M}$ with $\mathcal{W}_{T^*M}^{op}$ it's another quantization but the parameter $[\omega_X]+\lambda$ changes to $[\omega_X]-\lambda$. Then we deduce in ptc that if $\lambda=0$, we should have an isomorphism $\mathcal{W}_{T^*M}\cong\mathcal{W}_{T^*M}^{op}$ and hence $(D_X^{\mathcal{L}})^{op} = (\mathcal{L}\otimes D_X\otimes\mathcal{L}^{-1})^{op} = \mathcal{L}^{-1}\otimes\Omega\otimes D_X\otimes\Omega^{-1}\otimes\mathcal{L}$ is the only candidate i.e $\mathcal{L}\cong\mathcal{L}^{-1}\otimes\Omega$ i.e $\mathcal{L}^2\cong\Omega$.

The functors are $I: D_M^{\mathcal{L}}\text{-mod}\longrightarrow(\widetilde{\mathcal{W}}_{T^*M}^{\mathcal{L}+\Omega^{1/2}}, \mathbb{C}^\times)\text{-}\mathrm{mod}_g$ and

$$(\pi_*-)^{\mathbb{C}^\times}: (\widetilde{\mathcal{W}}_{T^*M}^{\mathcal{L}+\Omega^{1/2}}, \mathbb{C}^\times)\text{-mod}\longrightarrow D_M^{\mathcal{L}}\text{-mod}.$$

The crucial point to check is that $(\pi_*\widetilde{\mathcal{W}}_{T^*M}^{\mathcal{L}+\Omega^{1/2}})^{\mathbb{C}^\times} = D_M^{\mathcal{L}}$ and this follows from the mapping that we've seen

$$x_i\longmapsto x_i,\ \partial_i\longmapsto h^{-1}\xi_i. \quad \square$$