1 DIFFERENTIAL OPERATORS ON $\mathbb{A}^n$

1.1. Def^{ns}

To begin with, we will work over $\mathbb{C}$. Let $V = \mathbb{C}^n = \mathbb{A}^n$

DEF^N 1: The ring of differential operators on V, or the nth Weyl algebra, is the $\mathbb{C}$-subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[x_1, \dots, x_n])$ generated by

- multiplication by $p \in \mathbb{C}[x_1 \dots x_n]$
- partial differentiations $\partial_i = \frac{\partial}{\partial x_i}$ $\quad i = 1, \dots, n$

We write D_V or A_n.

THM 2 For $\alpha \in \mathbb{N}^n$ let $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$.

i) A_n is a free $\mathbb{C}[x_1 \dots x_n]$-module with basis $\{\partial^\alpha : \alpha \in \mathbb{N}^n\}$

ii) A_n is defined as a $\mathbb{C}$-algebra by generators $x_1 \dots x_n, \partial_1 \dots \partial_n$ subject to relations

$$[\partial_i, \partial_j] = 0 = [x_i, x_j], [\partial_i, x_j] = \delta_{ij} \qquad \forall i, j.$$

Proof: Observe that there is a surjection

$$\mathbb{C}\langle x_1 \dots x_n, \partial_1, \dots \partial_n \rangle \xrightarrow{\psi} A_n$$

s.t. $\psi[\partial_i, \partial_j] = 0 = \psi[x_i, x_j]$ & $\psi([\partial_i, x_j] - \delta_{ij}) = 0$. (prod. rule)

So A_n is certainly spanned by $\{\partial^\alpha\}$. Now suppose $\psi(\sum_\alpha p_\alpha(x)\partial^\alpha) = 0$. Pick β min^l in lexicographic ordering subject to $p_\beta \neq 0$. Then

$$\psi(\sum p_\alpha(x)\partial^\alpha) . x^\beta = \beta_1! \dots \beta_n! \, p_\beta(x) \neq 0. \quad ↯ \quad \square$$

From one point of view A_n is a very noncommutative algebra:

- Suppose $I \triangleleft A_n$ and assume $I \neq 0$. Then $\sum p_\alpha(x)\partial^\alpha \in I$ can be reduced to containing ∂^α only by applying $[\partial_i, -]$; then applying $[x_i, -]$ reduces this further to $1 \in I$ i.e. $I = A_n$.

So one does not immediately relate this to algebraic geometry as in comm. alg.

1.2 Filtrations

DEF^N 3: $F_l A_n := \{ \sum_{\substack{\alpha \text{ s.t.} \\ \sum \alpha_i \le l}} p_\alpha(x) \partial^\alpha \}$. Then $\dots \subset F_l A_n \subset F_{l+1} A_n \subset \dots$ subject to the following:

1) $F_l A_n = \{0\}$ $i < 0$ $\bigcup F_l A_n = A_n$
2) $F_1 A_n = \mathbb{C}[x_1 \dots x_n]$
3) $(F_l A_n)(F_m A_n) \subseteq F_{l+m} A_n$
4) $F_l A_n$ is a f.g. $\mathbb{C}[x_1 \dots x_n]$-module
5) $[F_l A_n, F_m A_n] \subseteq F_{l+m-1} A_n$.

This is called the operator filtration.

Ex 3.5 Show $\exists$ intrinsic inductive def^n of filt^n $F_l A_n$ via $F_l A_n = 0$ for $l < 0$ and $F_l A_n = \{ \theta \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}[x_1 \dots x_n]) : [\theta, p] \in F_{l-1} A_n \ \forall p \in \mathbb{C}[x_1 \dots x_n] \}$ $l \ge 0$.

THM4 Set $gr A_n = \bigoplus_l gr_l A_n = \bigoplus_l F_l A_n / F_{l-1} A_n$. Then this is a $\mathbb{C}$-algebra and is isomorphic to

$$\mathbb{C}[x_1 \dots x_n][\xi_1 \dots \xi_n]$$

where $\xi_i = \partial_i + F_0 A_n \in gr_1 A_n$

Proof: Cond^n (5) $\Rightarrow$ $gr A_n$ is commutative. As A_n is generated by ∂_i and $\mathbb{C}[\underline{x}]$ we find then

$$\mathbb{C}[\underline{x}][\xi_1 \dots \xi_n] \twoheadrightarrow gr A_n$$

and then the PBW thm $\Rightarrow$ this is an isomorphism. $\square$

Corollary 5: A_n is noetherian (left & right), has no zero divisors and has finite homological dimension.

Proof: e.g. $I \triangleleft_l A_n$. Then define a filt^n $F_l I := I \cap F_l A_n$ so that $gr I \triangleleft gr A_n$

Then if $I_1 \subseteq I_2 \subseteq \dots$ is a chain of left ideals of A_n we have $gr I_1 \subseteq gr I_2 \subseteq \dots$ and by noeth. of $gr A_n$ we see $gr I_j = gr I_{j+1}$ $\forall j >> 0$. Then $F_0(I_j) = gr_0 I_j = gr_0 I_{j+1} = F_0(I_{j+1})$, and by ind^n $gr_t I_j = gr_t I_{j+1}$ then yields $F_t I_j = F_t I_{j+1}$ which then implies $I_j = I_{j+1}$ $\square$

Remark 6 the grading on $gr A_n$ is equivalent to a $\mathbb{C}^*$-action via:

$$z.p = z^{-j} p \quad \forall p \in gr_j A_n \quad \forall z \in \mathbb{C}^*.$$

Then $(gr A_n)^{\mathbb{C}^*} = gr_0 A_n = \mathbb{C}[x_1, \dots, x_n]$

A good intuition (and technical tool) is provided by the Rees ring construction:

Suppose $\{F_l A\}_{l \ge 0}$ is a filt^n on some algebra.

$$\hat{A} := \bigoplus_{l \ge 0} F_l A t^l \subseteq A[t]$$

Then this is a graded $\mathbb{C}[t]$-algebra with $\deg(t)=1$ containing $t = 1.t \in F_1 A t$

Then $\hat{A}/t\hat{A} \cong \mathrm{gr} A$, while $\hat{A}[t^{-1}] = A[t, t^{-1}]$ so $\frac{\hat{A}}{(t-\lambda)\hat{A}} \cong A \quad \lambda \in \mathbb{C}^*$

i.e we have a (flat) family of algebras over $\mathbb{A}^1$ ($= \mathrm{Spec}\, \mathbb{C}[t]$) whose fibre at 0 is $\mathrm{gr} A$ and whose generic fibre is A; it has a $\mathbb{C}^*$-action that contracts to the special fibre.

Passing from generic to special : DEGENERATION

special to generic : DEFORMATION

1.3 Symplectic structures

$$\mathbb{C}[x_1 \ldots x_n][\xi_1 \ldots \xi_n] = \mathrm{Sym}_{\mathbb{C}[\underline{x}]}(\mathrm{gr}_1 A_n) \cong \mathbb{C}[T^*V] = \mathbb{C}[V \times V^*]$$

cotangent bundle of V

This is because ξ_i are degenerations of $\frac{\partial}{\partial x_i}$ which pair naturally against covectors (i.e 1-forms).

FACTS • A vector bundle has a $\mathbb{C}^*$-action which dilates its fibres : here $z(v,\lambda) = (v, z\lambda)$

This induces a $\mathbb{C}^*$-action on functions (symmetric algebra on v.b.) which is precisely the grading we introduced on $\mathrm{gr} A_n$.

• T^*V is a symplectic variety i.e. $\exists$ non. degenerate closed 2-form ω on T^*V:

take $\lambda \in T^*V$ (belonging to $T^*_x V$ where $x \in V$). There is an induced map

$$\pi_*: T_\lambda(T^*V) \longrightarrow (\pi^* TV)_\lambda = T_{\pi(\lambda)} V \quad (\mathrm{proj}^n)$$

Define $L : T(T^*V) \longrightarrow \mathbb{C}[V]$ via $L(\xi) = \langle (\pi_* \xi), \lambda \rangle$ for $\xi \in T_\lambda(T^*V)$

Take $\omega = dL$.

In this example it all boils down to $\langle (x,\lambda), (x',\lambda') \rangle = \lambda(x') - \lambda'(x)$, a symplectic form on the vector space $V \times V^*$.

Algebraically this corresponds to a Poisson algebra structure on $\mathbb{C}[T^*V]$:

$$\{\ ,\ \} : \mathbb{C}[T^*V] \otimes \mathbb{C}[T^*V] \longrightarrow \mathbb{C}[T^*V]$$

satisfying : i) Leibniz rule $\{f, gh\} = \{f,g\}h + \{f,h\}g$

ii) $\{\ ,\ \}$ is a Lie bracket.

It's given by $\{f,g\} := \omega(\xi_f, \xi_g)$ where ξ_f is defined by $\omega(\xi_f, -) = -df$.

Explicitly $\{p_\alpha(x)\xi^\alpha, p_\beta(x)\xi^\beta\} = \sum_{i=1}^n \frac{\partial}{\partial \xi_i}(p_\alpha(x)\xi^\alpha)\frac{\partial}{\partial x_i}(p_\beta(x)\xi^\beta) - \frac{\partial}{\partial \xi_i}(p_\beta(x)\xi^\beta)\frac{\partial}{\partial x_i}(p_\alpha(x)\xi^\alpha)$

which is gotten from repeated use of Leibniz and antisymmetry and linearity plus $\{\xi_i, x_j\} = 1$, $\{\xi_i, \xi_j\} = \{x_i, x_j\} = 0$

One can also construct a Poisson bracket from A_n:

take $z_1, z_2 \in gr_i A_n$ & $gr_j A_n$ respectively. Lift these to $\hat{z}_1, \hat{z}_2 \in F_i A_n, F_j$

Then $[\hat{z}_1, \hat{z}_2] \in F_{i+j-1} A_n$. Set $\{z_1, z_2\} := [\hat{z}_1, \hat{z}_2] + F_{i+j-2} A_n \in gr_{i+j-1} A_n$

Exercise 7: a) Check this is a well-defined Poisson bracket on $gr A_n$

b) Show that it equals the one arising from the symplectic structure on T^*V

So, the noncommutative algebra does see something of the symplectic structure.

1.4 Representations

DEF^N 8: Let A_n-mod denote the category of finitely generated left A_n-modules. A filtration on $M \in A_n$-mod is $\dots \subseteq F_\ell M \subseteq F_{\ell+1} M \subseteq \dots$ $(\ell \in \mathbb{Z})$ s.t.

(1) $M = \bigcup_\ell F_\ell M$; $F_\ell M = 0$ $\ell << 0$

(2) $F_\ell A_n \cdot F_j M \subseteq F_{\ell+j} M$

Then $gr M$ becomes a $gr A_n$-module. The filt^n is good if $gr M$ is f.g. over $gr A_n$.

Lemma/Def^n 9: Take a good filt^n on M. The characteristic variety / singular support of M is

$$V(\mathrm{Ann}\, gr M) = \{\lambda \in T^*V : f(\lambda) = 0 \ \forall f \in \mathrm{Ann}\, gr M\} \subseteq T^*V$$

This is independent of the choice of good filtration. Write $Ch(M)$.

e.g. $n=1$ $M = \dfrac{A_n}{A_n(x\partial - \lambda)}$ where $\lambda \in \mathbb{C}$

e.g. $M = \mathbb{C}[x]$, $F_\ell M = \begin{cases} 0 & \ell < 0 \\ M & \ell \geq 0 \end{cases}$;
$gr M = \dfrac{\mathbb{C}[x, \xi]}{(\xi)}$ so $Ch\, M = T_V^* V = V \times \{0\}$.

Then we get an "induced" filt^n $F_i M = F_i A_n + A_n(x\partial - \lambda)$ and $gr M = \dfrac{\mathbb{C}[x, \xi]}{x\xi}$ so $Ch\, M = \{(v, \lambda) : v = 0 \text{ or } \lambda = 0\}$.

Exercise 10: Prove M is irreducible (i.e no nonzero proper subrep^{ns}) if $\lambda \notin \mathbb{Z}$.

DEF^N/LEMMA 11 $Y \subseteq Ch\, M$ an irreducible component (so $Y = V(\rho)$ for ρ a min^l prime of $\mathrm{Ann}(gr M)$) Then $\dim\left(\mathrm{Quot}\left(gr A_n/\rho\right) \otimes_{gr A_n} gr M\right)$ is called the multiplicity of M along Y. It is well-defined. $\underline{ch}\, M = \sum_{Y \text{ irr cpt}} (\text{mult}_Y M)\, Y$, characteristic cycle.

THM 12 (Gabber) Let $M \in A_n$-mod have a good filtration, and set $I = \mathrm{Ann}(gr M)$. Then $\{\sqrt{I}, \sqrt{I}\} \subseteq \sqrt{I}$.

This is called the "Involutivity of the characteristic variety".

This is a subtle result. You should be able to prove easily that $\{I,I\}\subseteq I$. But generally for ideals satisfying this relation it's not true that $\{\sqrt{I},\sqrt{I}\}\subseteq\sqrt{I}$ e.g. $I=\langle x^2, x\xi, \xi^2\rangle$.

<u>DEFN 13</u> Suppose ω is a symplectic form on a v.s. W (e.g. T^*V or $T_p(T^*V)$). Then for $U\subset W$ a subspace set

$$U^\perp=\{w\in W : \omega(U,w)=0\}$$

so that in ptc. $W/_{U^\perp}\xrightarrow{\sim} U^*$

U is i) isotropic if $U\subseteq U^\perp$ ($\therefore \dim U\leq n$)

ii) coisotropic if $U\supseteq U^\perp$ ($\dim U\geq n$)

iii) lagrangian if $U=U^\perp$ ($\dim U=n$)

This generalizes to subvarieties $Y\subseteq T^*V$ by saying Y is isotropic/coisotropic/lagrangian if $T_pY\subseteq T_p(T^*V)$ is for all p in the smooth locus of Y.

<u>REMARK 14</u>: $I_Y:=\{f\in\mathbb{C}[T^*V] : f(\lambda)=0\ \forall\lambda\in Y\}$. Then Y is coisotropic iff I_Y is involutive

<u>Pf</u>: (We'll only prove "if") Let $p\in Y$ be a smooth point; $w\in(T_pY)^\perp$ so that $\omega_p(w,T_pY)=0$ i.e. $\omega_p(w,-)=\sum_i\alpha_i\, d_pF_i$ where $\{F_i\}$ is a generating set for I_Y (because the sol^{ns} of $d_pF_i=0\ \forall i$ define T_pY). Then set $-g=\sum\alpha_iF_i\in I_Y$. We have

$$\omega_p(\xi_g,-)=\sum\alpha_i d_pF_i=\omega_p(w,-)\text{ so that }\xi_g(p)=w.$$

Then $\forall f\in I_Y$ we have $d_pf(w)=\omega_p(\xi_g,\xi_f)=\{g,f\}(p)=0$ so that $w\in T_pY$. □

<u>Corollary 15</u> (Bernstein's inequality) $\dim(\mathrm{Ch}\,M)\geq\dim V$. □

Sketch of a proof of Gabber's theorem: We reduce to the following (the infinitesimal nbhd of the special fibre in the Rees ring):

$$A=\bigoplus_{i\geq 0}F_iA_n/_{F_{i-2}A_n}\quad,\quad M=\bigoplus\frac{F_iM}{F_{i-2}M}\ \text{(abuse of notation)}$$

So A is an algebra over the dual numbers $D=\mathbb{C}[t]/_{t^2}$ where $t=1+F_{-1}A_n\in A$ deg 1; both A and M are free over D (for M this is because $\overline{M}:=\mathrm{gr}\,M=\frac{M}{tM}\xrightarrow{\times t}tM$ is an isomorphism)

We'll prove the result in this generality i.e. $\sqrt{\mathrm{Ann}\,\mathrm{gr}\frac{M}{tM}}$ is involutive.

$\sqrt{\mathrm{Ann}\,\frac{M}{\varepsilon M}} = \bigcap\limits_{\wp \text{ min'l prime of } \mathrm{Ann}\frac{M}{\varepsilon M}} \wp$ so ETS $\{\wp,\wp\}\subseteq\wp$ for each min'l prime.

Let $x_1,\dots,x_\ell \in A$ s.t. $\bar{x}_i+\wp$ form maximal algebraically independent se- Set $R=\mathbb{C}[\bar x_1,\dots,\bar x_n]\subseteq \bar A$ so that $R \hookrightarrow \bar A/\wp =: B$, is a map of finite rank $\exists f\in R$ s.t. B_f is free over R_f of finite rank. $\bar M_f$ is an $\bar A_f$-module with min'l prime $\wp_f$, so possibly after localising further we find

$$\bar M_f \supseteq \wp \bar M_f \supseteq \wp^2 \bar M_f \supseteq \dots \supseteq \wp^s \bar M_f = 0$$

with each subquotient free over B_f. So we have $m_1,\dots,m_s\in M$ with $\bar m_i$ an R_f-basis of $\bar M_f$ & $\wp\, \bar m_i \subseteq \sum_{j<i} R\bar m_j$ where $j<i$ means $\bar m_j\in\wp^s\bar M_f$ and $m_i\notin\wp^s \bar M_f$. A brute force lemma shows

Lemma: $\forall a,b\in A$ with $\bar a,\bar b\in\wp$, $\exists n_1,n_2,n_3\geq 0$ & $e_{ij}\in A$ with $\bar e_{ij}\in R$ s.t.

$$f^{n_1}[f^{n_2}a, f^{n_3}b]\,m_i = \hbar\sum_j e_{ij}m_j \quad \& \sum \bar e_{ii}=0.$$

Given this: $f^{n_1}[f^{n_2}a,f^{n_3}b]=\hbar c$ with $c\in f^N\{\bar a,\bar b\}+\wp$ $\quad N=n_1+n_2+n_3$. Now $c \curvearrowright \bar M_f$ via the (e_{ij}) (by the lemma) and this is traceless. As $\wp$ acts nilpotently we get that trace $\{\bar a,\bar b\}$ on M_f is zero. So $\forall x\in\bar A_f$ we have

$$\mathrm{tr}_{\bar M_f}(x\{\bar a,\bar b\})=0$$

But $\mathrm{tr}_{\bar M_f}(x\{\bar a,\bar b\}) = \sum_i \mathrm{tr}_{\wp^i\bar M_f/\wp^{i+1}\bar M_f}(x\{\bar a,\bar b\}) = \sum \mathrm{tr}_{B_f^{\ell_i}}(x\{\bar a,\bar b\})$
$= \sum \ell_i\, \mathrm{tr}_{B_f}(x\{\bar a,\bar b\})$

$$\Rightarrow \mathrm{tr}_{B_f}(x\{\bar a,\bar b\})=0 \quad \forall x\in\bar A_f$$

But this trace form should be non-degenerate as B_f is separable over R_f

$$\Rightarrow \{\bar a,\bar b\}=0 \text{ in } B_f \quad \text{i.e. } \{\bar a,\bar b\}\in\wp. \qquad \square$$

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Ginzburg: Lecture notes on D-modules. (Google it!)