Lemma A *Let G be a finite group, $A \lhd G$ and V an irreducible representation of G. Then $V = Ind_H^G W$ for some subgroup $A \leq H \leq G$ (no big deal) and an irreducible representation W such that $res_A^H W = I \oplus I \oplus \cdots \oplus I$ for some irreducible representation I of A.*

Proof: Take V and consider $res_A^G V$. It has an isotypic decomposition by Theorem 6.3 $V = \bigoplus_{j \in J} V_j$. (So here each V_j is a direct sum of a number of copies of the same irreducible representation I_j of A and I_j and I_k are non-isomorphic if $j \neq k$.) If there is just a single summand V_j (i.e. $|J| = 1$) then we take $H = G$ and the lemma is proved.

So now we assume that there are several summands. Given $S \subseteq V$, an A-subrepresentation, we define

$$^gS := \{g \cdot s : s \in S\} \subseteq V.$$

Then ^gS is an A-subrepresentation of V (possibly different from S) because it is A-invariant. Indeed let $a \in A, s \in S$ and $g \in G$. Then since A is normal in G we have $g^{-1}ag = a' \in A$. Now

$$a \cdot (g \cdot s) = (ag) \cdot s = (ga') \cdot s = g \cdot (a' \cdot s) \in {}^gS$$

since $a' \cdot s \in S$ as S is an A-subrepresentation. This procedure has the following obvious, but important properties

(1) $^{g^{-1}}(^gS) = S$

(2) if $S = S_1 \oplus S_2$ then $^gS = {}^gS_1 \oplus {}^gS_2$.

It follows from (1) and (2) that if S is an irreducible representation of A then so too is ^gS. Indeed if ^gS decomposes as $^gS = X \oplus Y$ and then we'd have $S = {}^{g^{-1}}(^gS) = {}^{g^{-1}}X \oplus {}^{g^{-1}}Y$. But S is irreducible which means that either $^{g^{-1}}X$ or $^{g^{-1}}Y$ is zero, and then this forces (by (1)) X or Y to be zero.

So now we can apply this procedure to the isotypic components $V_j = I_j \oplus \cdots \oplus I_j$. We find that ^gV_j is the direct sum $^gI_j \oplus \cdots \oplus {}^gI_j$ and so is contained in one of the isotypic components, say $V_{g(j)}$. But applying g^{-1} we get

$$V_j = {}^{g^{-1}}(^gV_j) \subseteq {}^{g^{-1}}V_{g(j)} \subseteq V_j$$

and so we have equality throughout. In other words $^gV_j = V_{g(j)}$ and so G permutes the isotypic components V_j and we get an action of G on the indexing set J.

We claim that the action of G on J is transitive. Let $j \in J$ and set $J_1 = \{g(j) : g \in G\}$. We have

$$\bigoplus_{k \in J_1} V_k \subseteq V \qquad \text{and} \qquad g \cdot \left(\bigoplus_{k \in J_1} V_k \right) = \bigoplus_{k \in J_1} {}^gV_k = \bigoplus_{k \in J_1} V_{g(k)} = \bigoplus_{k \in J_1} V_k.$$

Therefore $\oplus_{k\in J_1} V_k$ is a non-zero G-invariant subspace of V and so equals V since V is an irreducible representation of G. It follows that $J_1 = J$, as required.

Now continue with $j \in J$ and let $H = Stab_G(j)$. Note that since V_j is a representation of A we have ${}^aV_j = V_j$ for all $a \in A$ and thus $a(j) = j$ for all $a \in A$. It follows that $A \leq H$. We claim that

$$V = Ind_H^G V_j$$

which will prove the lemma. First note that by the orbit-stabiliser theorem $|J| = |G(j)| = [G : H]$. Thus

$$\dim V = |J| \dim V_j = [G : H] \dim V_j = \dim Ind_H^G V_j.$$

But by Frobenius reciprocity we have

$$\langle \chi_V, \chi_{V_j} \uparrow_H^G \rangle_G = \langle \chi_V \downarrow_H^G, \chi_{V_j} \rangle_H > 0$$

where the inequality follows since $V_j \subseteq V$ is a subrepresentation of H. It follows that $Ind_H^G V_j$ appears at least once as a summand of V, and by the dimension equality it appears at most once. Thus it equals V. □

Lemma B *If G is nilpotent and non-abelian then G contains a normal, abelian, non-central subgroup.*

Proof: The centre $Z = Z(G)$ is a normal subgroup of G. Let $G^{(t)}$ be the smallest subgroup not containing Z in a chain (*) (which defines G being nilpotent). By definition $G^{(t)}/G^{(t-1)}$ is central in $G/G^{(t-1)}$ and then since $G^{(t-1)} \subseteq Z$ we have $G^{(t)}/Z$ is central in G/Z. So if $h \in G^{(t)} \setminus Z$ then we have $(gZ)(hZ) = (hZ)(gZ)$ and so

$$ghg^{-1} \in hZ \qquad \text{for all } g \in G. \tag{1}$$

Now the subgroup generated by h and Z is

- a subgroup
- abelian (Z commutes with *everything*, h commutes with Z and with itself)
- non-central (since $h \notin Z$)
- normal (by (1)).

Thus this subgroup does what the lemma asserts. □