Exercises for Lecture 8

(1) $C_3 \times C_3$ is abelian so each element in a conjugacy class and each irred. is one dimensional.
Let's write $C_3 \times C_3$ as generated by x ($x^3 = e$) and y ($y^3 = e$) (with $xy = yx$). Let $\rho = \exp(2\pi\sqrt{-1}/3)$.
For an irreducible of $C_3 \times C_3$ we have two choices to make: where x goes and where y goes. Now $x^3 = e$ means x must go to a third root of 1, and the same for y; so we get

$$\begin{array}{l} x \longmapsto \rho^i \\ y \longmapsto \rho^j \end{array} \qquad 0 \leq i, j \leq 2. \quad (\text{labelled } (i,j))$$

This gives us the 9 different irreducibles we need.

$C_3 \times C_3$	e	x	x^2	y	xy	x^2y	y^2	xy^2	x^2y^2
	1	1	1	1	1	1	1	1	1
$\chi_{(0,0)}$	1	1	1	1	1	1	1	1	1
$\chi_{(1,0)}$	1	ρ	ρ^2	1	ρ	ρ^2	1	ρ	ρ^2
$\chi_{(2,0)}$	1	ρ^2	ρ	1	ρ^2	ρ	1	ρ^2	ρ
$\chi_{(0,1)}$	1	1	1	ρ	ρ	ρ	ρ^2	ρ^2	ρ^2
$\chi_{(1,1)}$	1	ρ	ρ^2	ρ	ρ^2	1	ρ^2	1	ρ
$\chi_{(2,1)}$	1	ρ^2	ρ	ρ	1	ρ^2	ρ^2	ρ	1
$\chi_{(0,2)}$	1	1	1	ρ^2	ρ^2	ρ^2	ρ	ρ	ρ
$\chi_{(1,2)}$	1	ρ	ρ^2	ρ^2	1	ρ	ρ	ρ^2	1
$\chi_{(2,2)}$	1	ρ^2	ρ	ρ^2	ρ	1	ρ	1	ρ^2

The calculation for $C_2 \times C_2 \times C_2$ is basically the same: we have generators x, y, z (they commute and $x^2 = y^2 = z^2 = e$); each irred. rep^n is then labelled by (i, j, k) $0 \leq i, j, k \leq 1$ with

$$x \longmapsto (-1)^i, \quad y \longmapsto (-1)^j, \quad z \longmapsto (-1)^k.$$

This gives the following character table

$C_2\times C_2\times C_2$	e	x	y	xy	z	xz	yz	xyz
	1	1	1	1	1	1	1	1
$(0,0,0)$	1	1	1	1	1	1	1	1
$(1,0,0)$	1	-1	1	-1	1	-1	1	-1
$(0,1,0)$	1	1	-1	-1	1	1	-1	-1
$(1,1,0)$	1	-1	-1	1	1	-1	-1	1
$(0,0,1)$	1	1	1	1	-1	-1	-1	-1
$(1,0,1)$	1	-1	1	-1	-1	1	-1	1
$(0,1,1)$	1	1	-1	-1	-1	-1	1	1
$(1,1,1)$	1	-1	-1	1	-1	1	1	-1

(2)(a) We follow the technique of the proof of . Diagonalise g to get

$$g \longmapsto \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

where $\lambda_i \in \mathbb{C}$. Now $g^2 = e$ means that each $\lambda_i^2 = 1$ i.e. $\lambda_i = \pm 1$. It follows that

$$\chi(g) = \sum_{i=1}^{n} \lambda_i \in \mathbb{Z}$$

Moreover, $1 \equiv \lambda_i \pmod 2$ for each λ_i and so it follows that

$$\chi(g) = \sum_{i=1}^{n} \lambda_i \equiv \sum_{i=1}^{n} \mathbb{1} = \chi(e) \pmod 2$$

(b) Again, diagonalise g, $g \longmapsto \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ where this time $g^4 = e \Rightarrow \lambda_i^4 = 1$ $\forall i$. i.e. $\lambda_i \in \{\pm 1, \pm\sqrt{-1}\}$.

Now g, g^{-1} conjugate $\Rightarrow \chi(g) = \chi(g^{-1})$ and so $\chi(g) \in \mathbb{R}$.

Thus in $\chi(g) = \sum_{i=1}^{n} \lambda_i$ we must have that if $\pm\sqrt{-1}$ appears as an λ_i then there is ~~~~ another copy $\mp\sqrt{-1}$ which will cancel this out, because otherwise $\chi(g) \in \mathbb{C} \setminus \mathbb{R}$. Hence

$$\chi(g) = \sum_{i=1}^{n} \lambda_i = \sum_{\substack{i=1 \\ \lambda_i = \pm 1}}^{n} \lambda_i \in \mathbb{Z},$$ as req^d.

(c) Again, diagonalise g, $g \longmapsto \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ with $\lambda_i^3 = 1$

i.e. $\lambda_i \in \{1, \omega, \omega^2\}$ where $\omega = \exp(2\pi\sqrt{-1}/3)$. Since g and g^{-1} are conjugate we get $\chi(g) \in \mathbb{R}$. Split up the trace

$$\chi(g) = \sum_{i=1}^{n} \lambda_i = |\{i : \lambda_i = 1\}| 1 + |\{i : \lambda_i = \omega\}| \omega + |\{i : \lambda_i = \omega^2\}| \omega^2$$

Now this is a real number $\iff \chi(g) = \overline{\chi(g)}$
$\iff |\{i : \lambda_i = \omega\}| = |\{i : \lambda_i = \omega^2\}|$

~~Thus~~ But $1 + \omega + \omega^2 = 0$ (ω is a root of $X^2 + X + 1$) and so we find

$$\chi(g) = |\{i : \lambda_i = 1\}| 1 + \tfrac{1}{2}(|\{i : \lambda_i = \omega\}| + |\{i : \lambda_i = \omega^2\}|) \cdot -1$$
$$\in \mathbb{Z}$$

Note too that if $|\{i : \lambda_i = \omega\}| = m$ we have that $|\{i : \lambda_i = \omega^2\}| = m$ and so $|\{i : \lambda_i\}| = n - 2m$ and then

$$\chi(g) = (n - 2m) + m \cdot (-1) = n - 3m \equiv n = \chi(e) \pmod 3$$

(d) Diagonalise g, $g \longmapsto \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$.

Then $|\chi(g)| = |\sum_{i=1}^{n} \lambda_i| \leq \sum_{i=1}^{n} |\lambda_i|$ by the triangle inequality, with equality iff $\lambda_i = \lambda_j$ $\forall i$ and j

But $|\lambda_i| = 1$ since $\lambda_i^{\mathrm{ord}(g)} = 1$. Thus

$$|\chi(g)| \leq n = |\chi(e)| \text{ with equality iff } \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \text{ is}$$

a scalar matrix.

(3) (a) We must check that $\det \rho$ is a homomorphism:

$$(\det \rho)(gg') = \det(\rho(gg')) = \det(\rho(g)\rho(g'))$$
$$= \det \rho(g) \cdot \det \rho(g')$$
$$= (\det \rho)(g)\,(\det \rho)(g') \quad \forall g, g' \in G.$$

(b) Since G is simple & $\ker(\det \rho) \triangleleft G$ we must have $\ker(\det\rho) = \{e\}$ or $\ker(\det \rho) = G$. In the first case, $\det\rho$ is injective and so G is isomorphic to a subgroup of $\mathbb{C}^*$. But $\mathbb{C}^*$ is abelian and so this would force G to be abelian, contradicting our initial hypothesis. It follows that $\ker(\det \rho) = G$ and so $\det \rho(g) = 1$ $\forall g \in G$.

(4) $\Leftarrow$: If $\rho_V(g) = \mathrm{Id}$ then clearly $\chi_V(g) = \chi_V(e)$ and so $g \in \ker \chi_V$.

$\Rightarrow$: Suppose $g \in \ker \chi_V$. Then $\chi_V(g) = \chi_V(e)$ and so $|\chi_V(g)| = |\chi_V(e)|$ and so by 2(d) $\rho_V(g)$ is a scalar matrix $\begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}$. But now if this is $\rho_V(g)$ we get $\chi_V(g) = \mathrm{Tr}\left(\begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}\right) = n\lambda$ and so we get $\chi_V(g) = n\lambda$ and $\chi_V(e) = n$. Thus $\lambda = 1$ and we deduce that $\rho_V(g) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = \mathrm{Id}$. $\square$

Now we've just proved that $\ker \chi_V = \{g \in G : \rho_V(g) = \mathrm{Id}\}$. But this is, by definition, $\ker \rho_V$. As this is the kernel of a group homomorphism it follows that $\ker \rho_V$ is a normal subgroup of G, and so $\ker \chi_V \triangleleft G$, as required.