Exercises for 8 & 9.

1. Observations: $A \in \mathbb{N}$ because it is the degree.
$B, D \in \mathbb{Z}$ because of Ex lect 8 Qn 2(a)

Now $\langle \chi_1, \chi_1 \rangle = \frac{1}{24}(A^2 + 6B^2 + 8 + 3 \times 4 + 6 \times 0) = 1$

i.e $A^2 + 6B^2 = 4$. Since $A, B \in \mathbb{Z}$ we get
$A = \pm 2$, $B = 0$ as only sol^n.
As $A \in \mathbb{N}$ we get $A = 2$.

Now $\langle \chi_2, \chi_2 \rangle = \frac{1}{24}(9 + 6 + 8C.\bar{C} + 3D^2 + 6) = 1$

$\langle \chi_1, \chi_2 \rangle = \frac{1}{24}(3A + 6B - 8C + 3 \times 2D + 0) = 0$

ie $8|C|^2 + 3D^2 = 3$ and $6D - 8C = -6$
The second equation $\Rightarrow C \in \mathbb{Q}$ and so $|C|^2 = C^2$.
We deduce, since $D \in \mathbb{Z}$, then $D = 0, 1$ or -1. By Ex lect 8 Qn 2(a) we get $D = \pm 1$ and thus $C^2 = 0$ i.e. $C = 0$.
Now $6D = -6 \Rightarrow D = -1$.

$\therefore A = 2, B = 0, C = 0, D = -1$.

2. e, (12), $(12)(34)$, (123), $(123)(45)$, (1234), (12345)
since conjugacy classes are labelled by cycle type.

We just work out the centralisers of each of the given elements and then use the formula which follows from the orbit-stabiliser theorem

$|Cl(x)| = |G| / |C_G(x)|$ where $Cl(x)$ means the conjugacy class containg x.

e.g $Cl(e) = \{e\}$.

$C_G(12) = \langle (12) \rangle \times S_{\{3,4,5\}}$ $\therefore |Cl((12))| = {}^{120}/_{2 \times 6} = 10$

└ permutations of 3,4,5

Similar calculations give
$|\mathrm{Ccl}((12)(34))| = 15$, $|\mathrm{Ccl}((123))| = 20$, $|\mathrm{Ccl}((123)(45))| = 20$
$|\mathrm{Ccl}((1234))| = 30$, $|\mathrm{Ccl}((12345))| = 24$.

(c) V on $\mathbb{C}^5$ permuting the co-ordinates. If we pick a basis ~~then~~ $(e_1, \dots, e_5)$ then

$$e \longmapsto \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \quad (12) \longmapsto \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \quad (12)(34) \longmapsto \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & 1 \end{pmatrix}$$

$$(123) \longmapsto \begin{pmatrix} 0 & 0 & 1 & & \\ 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \quad (123)(45) \longmapsto \begin{pmatrix} 0 & 0 & 1 & & \\ 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix} \quad (1234) \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 & \\ 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ & & & & 1 \end{pmatrix}$$

$$(12345) \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(put zeros where there are empty spaces)

$\therefore \chi_V(e) = 5$, $\chi_V((12)) = 3$, $\chi_V((12)(34)) = 1$, $\chi_V((123)) = 2$
$\chi_V((123)(45)) = 0$ $\chi_V((1234)) = 1$, $\chi_V((12345)) = 0$.

(d) $$\langle \chi_V, \chi_V \rangle = \frac{1}{120}\left(1 \times 5^2 + 10 \times 3^2 + 15 \times 1^2 + 20 \times 2^2 + 20 \times 0^2 + 30 \times 1^2 + 24 \times 0^2\right)$$

$$= \frac{1}{120}(25 + 90 + 15 + 80 + 30) = 2.$$

Since this is not 1, χ_V is not irreducible.

(e) We calculate $$\langle \chi_T, \chi_V \rangle = \frac{1}{120}\left(1 \times 1 \times 5 + 10 \times 1 \times 3 + 15 \times 1 \times 1 + 20 \times 1 \times 2 + 20 \times 1 \times 0 + 30 \times 1 \times 1 + 24 \times 1 \times 0\right)$$

$$= \frac{1}{120}(5 + 30 + 15 + 40 + 30) = 1$$

$\therefore T$ appears with multiplicity one in V.

(f) We have $V = T \oplus W$ and so $\chi_V = \chi_T + \chi_W$.

Now
$$\underset{\underset{2}{\shortparallel}}{\langle \chi_V, \chi_V \rangle} = \underset{\underset{1}{\shortparallel}}{\langle \chi_T, \chi_T \rangle} + 2 \underset{\underset{?}{\shortparallel}}{\langle \chi_T, \chi_W \rangle} + \underset{\underset{?}{\shortparallel}}{\langle \chi_W, \chi_W \rangle}$$

Now $\langle \chi_T, \chi_W \rangle$ is a non-negative integer (since it counts the copies of T in W) and $\langle \chi_W, \chi_W \rangle$ is a positive integer. The only possibility is $\langle \chi_W, \chi_W \rangle = 1$ and $\langle \chi_T, \chi_W \rangle = 0$. Thus χ_W is irreducible.

3). $$\langle \chi, \chi \rangle = \sum_{1 \le i,j \le k} d_i d_j \langle \chi_i, \chi_j \rangle$$
$$= \sum_{i=1}^{k} d_i^2 \langle \chi_i, \chi_i \rangle \qquad \text{(since } \langle \chi_i, \chi_j \rangle = 0 \text{ for } i \neq j)$$
$$= \sum_{i=1}^{k} d_i^2 \qquad \text{(since } \langle \chi_i, \chi_i \rangle = 1)$$

Thus $\langle \chi, \chi \rangle = 1 \Rightarrow d_i = 1$ for some i, $d_j = 0$ $\forall j \neq i$.

$= 2 \Rightarrow d_i = 1, d_{i'} = 1$ for some i, i', $d_j = 0$ $\forall j \neq i, i'$

$= 3 \Rightarrow d_i = 1 = d_{i'} = d_{i''}$ some i, i', i'' and $d_j = 0$ $\forall j \neq i, i', i''$.

$= 4 \Rightarrow$ <u>either</u> $d_i = 2$ for some i and $d_j = 0$ $\forall j \neq i$

<u>or</u> $d_i = 1 = d_{i'} = d_{i''} = d_{i'''}$ and $d_j = 0$ for all other j.

4) The ~~els~~ conjugacy classes for D_4 have representatives:
e, a, a^2, b, ab

From the given matrices we see $\rho^{(1)}(a^2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\rho^{(1)}(ab) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
and $\rho^{(2)}(a^2) = \begin{pmatrix} 83 & 130 \\ -53 & -83 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\rho^{(2)}(ab) = \begin{pmatrix} -71 & -112 \\ 45 & 71 \end{pmatrix}$

So let χ_1 be char. of $\rho^{(1)}$, χ_2 char. of $\rho^{(2)}$. Then

$$\chi_1(e) = 2 = \chi_2(e); \quad \chi_1(a) = 0 = \chi_2(a); \quad \chi_1(a^2) = -2 = \chi_2(a^2);$$
$$\chi_1(b) = 0 = \chi_2(b); \quad \chi_1(ab) = 0 = \chi_2(ab).$$

So the characters are equal and so the representations must be isomorphic.

5.a) A_4 has order 12 $(= 24/2)$ and representatives for its conjugacy classes

$$e, \quad (123), \quad (132), \quad (12)(34).$$

These correspond to e, s, s^2 and t. The sizes are $1, 4, 4, 3$ respectively.

b) We look for group homomorphisms

$$A_4 = \langle s, t : s^3 = t^2 = e, \ sts = ts^{-1}t \rangle \longrightarrow \mathbb{C}^*$$
$$s \longmapsto \alpha, \quad t \longmapsto \beta.$$

Now we need $\alpha^3 = 1$, $\beta^2 = 1$ and $\alpha\beta\alpha = \beta\alpha^{-1}\beta$.
This gives first $\beta = \pm 1$ and $\alpha^2\beta = \beta^2\alpha^{-1} = \alpha^{-1} = \alpha^2$
$$\therefore \beta = 1.$$

$\alpha = 1, \omega, \omega^2$ all possible where $\omega = \exp(2\pi\sqrt{-1}/3)$.

So there are 3 different 1-D representations.

	e	s	s^2	t
χ_1	1	1	1	1
χ_2	1	ω	ω^2	1
χ_3	1	ω^2	ω	1

c) # irred. rep^{ns} of A_4 = # conjugacy classes A_4 $\therefore \exists$ one more irreducible rep^n, and we will call its character χ_4.

Moreover $|A_4| = \chi_1(e)^2 + \chi_2(e)^2 + \chi_3(e)^2 + \chi_4(e)^2$

i.e. $12 = 1 + 1 + 1 + 9$.

$\therefore \chi_4(e) = 3$ (since $\chi_4(e) > 0$).

d). We have

	e	s	s^2	t
χ_4	3	A	B	C

Now $\langle \chi_4, \chi_4 \rangle = \frac{1}{12}\left(1.3^2 + 4|A|^2 + 4|B|^2 + 3|C|^2\right) = 1$ (1)

$\langle \chi_4, \chi_3 \rangle = \frac{1}{12}\left(1.3.1 + 4.A.\overline{\omega} + 4.B.\overline{\omega}^2 + 3.C.1\right) = 0$ (2)

$\langle \chi_4, \chi_2 \rangle = \frac{1}{12}\left(1.3.1 + 4A.\overline{\omega}^2 + 4B\overline{\omega} + 3.C.1\right) = 0$ (3)

$\langle \chi_4, \chi_1 \rangle = \frac{1}{12}\left(1.3.1 + 4.A.1 + 4.B.1 + 3.C.1\right) = 0$ (4)

~~i.e. $4A + 4B + 3C = -3$.~~

Now use $1 + \omega + \omega^2 = 0$ to find, by adding (2)+(3)+(4),

$\frac{1}{12}(3+3+3 + 3C + 3C + 3C) = 0$ i.e. $C = -1$.

Now (4) becomes $4(A+B) = 0 \implies A = -B$

(3) becomes $4A\overline{\omega}^2 + 4B\overline{\omega} = 0 \implies -B\overline{\omega}^2 + B\overline{\omega} = 0$

i.e. $B = 0 = A$.

	e	s	s^2	t
$\therefore \chi_4$	3	0	0	-1.

(6) $D_4 = \langle a, b : a^4 = b^2 = e,\ bab^{-1} = a^{-1} \rangle$.

Conj ~~classes~~ representatives (size) : $e\,(1)$, $a\,(2)$, $a^2(1)$, $b(2)$, $ab(2)$

We calculated on an earlier sheet that there were 4 1-D rep^{ns} given by $a \mapsto \pm 1$, $b \mapsto \pm 1$ independently. This gives

	e	a	a^2	b	ab
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	1	1	1	-1	-1
χ_4	1	-1	1	-1	1

and leaves us with one irred. rep^n, χ_5, to find.

Because of $\sum_{i=1}^{5} \chi_i(e)^2 = 8$ we get $\chi_5(e) = 2$ and so we're looking for a two dimensional irreducible. (We know already that it should be the natural rep^n of D_4, but let's just proceed by orthogonality).

χ_5	2	A	B	C	D.

$$8\langle \chi_5, \chi_1 \rangle = 2 + 2A + B + 2C + 2D = 0 \qquad (1)$$

$$8\langle \chi_5, \chi_2 \rangle = 2 - 2A + B + 2C - 2D = 0 \qquad (2)$$

$$8\langle \chi_5, \chi_3 \rangle = 2 + 2A + B - 2C - 2D = 0 \qquad (3)$$

$$8\langle \chi_5, \chi_4 \rangle = 2 - 2A + B - 2C + 2D = 0 \qquad (4)$$

and $$8\langle \chi_5, \chi_5 \rangle = 4 + 2|A|^2 + |B|^2 + 2|C|^2 + 2|D|^2 = 8 \qquad (5)$$

Now (1) + (2) + (3) + (4) gives $8 + 4B = 0$ i.e $B = -2$

thus from (5) we get $4 + 2|A|^2 + 4 + 2|C|^2 + 2|D|^2 = 8$.

Hence $A = C = D = 0$ ad we've finished.

χ_5	2	0	-2	0	0

7) Q was discussed earlier (Ex. Lect 5+6) ad we got 5 conj. classes represented by

e,	a,	a^2,	b,	ab
1	2	1	2	2

We also fond that there were 4 1-D rep^ns

	e	a	a^2	b	ab
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	1	1	1	-1	-1
χ_4	1	-1	1	-1	1

Now the calculation is <u>exactly</u> the same as the calculation for D_4 in the previous question. In particular D_4 and Q have the same character table! But they are not isomorphic (D_4 has 5 elements of order 2, 2 of order 4; Q has at least 3 elements of order 4 — a, b, a^3)

8) a) We need to check that $\rho_V(g)$ is linear:

$$\rho_V(g)\left(\left(\sum_{x\in X}\lambda_x v_x\right) + \left(\sum_{x\in X}\mu_x v_x\right)\right) = \rho_V(g)\left(\sum_{x\in X}(\lambda_x+\mu_x)v_x\right)$$

$$= \sum_{x\in X}(\lambda_x+\mu_x)\, v_{g\cdot x}$$

$$= \sum_{x\in X}\lambda_x v_{g\cdot x} + \sum_{x\in X}\mu_x v_{g\cdot x}$$

$$= \rho_V(g)\left(\sum_{x\in X}\lambda_x v_x\right) + \rho_V(g)\left(\sum_{x\in X}\mu_x v_x\right)$$

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We also have to show that $\rho_V(gh) = \rho_V(g) \circ \rho_V(h)$:

$$\rho_V(gh)\Big(\sum_{x\in X} \lambda_x v_x\Big) = \sum_{x\in X} \lambda_x v_{(gh)\cdot x}$$

$$= \sum_{x\in X} \lambda_x v_{g\cdot(h\cdot x)} \qquad (G \text{ \underline{acts} on } X)$$

$$= \rho_V(g)\Big(\sum_{x\in X} \lambda_x v_{h\cdot x}\Big)$$

$$= \big(\rho_V(g)\circ\rho_V(h)\big)\Big(\sum_{x\in X} \lambda_x v_x\Big) \qquad \checkmark$$

So it's a representation!

(b) This is the definition of the regular representation we gave in lectures for the basis we called (v_g) there (lecture 5, slide 5)

(c) We work with the basis v_x. Now the character χ_V ($V = \mathbb{C}[X]$) is obtained by taking traces of the matrices $\rho_V(g)$. Traces are sums of diagonal elements in these matrices ad so we really need to know what the diagonal entries of $\rho_V(g)$ are.

Well, the columns of the matrix $\rho_V(g)$ come from expressing $\rho_V(g)(v_x)$ in terms of the basis $\{v_x\}$

i.e. if $\rho_V(g) = A$ then we label columns by elements of X and then

$$\rho_V(g)(v_x) = \sum_{y\in X} A_{xy} v_y$$

Thus the diagonal entries are just A_{xx} i.e. how often

v_x appears in $\rho_V(g)(v_x)$. But by definition

$\rho_V(g)(v_x) = v_{gx}$.

$$\therefore \quad A_{xx} = \begin{cases} 0 & g.x \neq x \\ 1 & g.x = x \end{cases}$$

We deduce that

$\chi_V(g) = \mathrm{Tr}(\rho_V(g))$