

Exercises for 8 & 9.

1. Observations: $A \in \mathbb{N}$ because it is the degree.
 $B, D \in \mathbb{Z}$ because of Exert 8 Qn 2(a)

$$\text{Now } \langle X_1, X_1 \rangle = \frac{1}{24} (A^2 + 6B^2 + 8 + 3 \times 4 + 6 \times 0) = 1$$

i.e. $A^2 + 6B^2 = 4$. Since $A, B \in \mathbb{Z}$ we get
 $A = \pm 2, B = 0$ as only solⁿ.

$\Rightarrow A \in \mathbb{N}$ we get $A = 2$.

$$\text{Now } \langle X_2, X_2 \rangle = \frac{1}{24} (9 + 6 + 8C \cdot \bar{C} + 3D^2 + 6) = 1$$

$$\langle X_1, X_2 \rangle = \frac{1}{24} (3A + 6B - 8C + 3 \times 2D + 0) = 0$$

$$\text{i.e. } 8|C|^2 + 3D^2 = 3 \text{ and } 6D - 8C = -6$$

The second equation $\Rightarrow C \in \mathbb{Q}$ and so $|C|^2 = C^2$.

We deduce, since $D \in \mathbb{Z}$, then $D = 0, 1$ or -1 . By Exert 8 Qn 2(a) we get $D = \pm 1$ and thus $C^2 = 0$ i.e. $C = 0$.

$$\text{Now } 6D = -6 \Rightarrow D = -1.$$

$$\therefore A = 2, B = 0, C = 0, D = -1.$$

(12)(34)

2. $e, (12), (123), (123)(45), (1234), (12345)$
 since conjugacy classes are labelled by ~~the~~ cycle type.

We just work out the centralisers of each of the given elements and then use the formula which follows from the orbit-stabiliser theorem

$$|C_G(x)| = |G| / |C_G(x)|$$

where $C_G(x)$ means the conjugacy class containing x .

e.g. $C_G(e) = \{e\}$.

$$C_G(12) = \langle (12) \rangle \times S_{\{3,4,5\}} \quad \therefore |C_G((12))| = \frac{120}{2 \times 6} = 10$$

\hookrightarrow permutations of 3, 4, 5

Similar calculations give

$$|\text{Ccl}((12)(34))| = 15, \quad |\text{Ccl}((123))| = 20, \quad |\text{Ccl}((123)(45))| = 20$$

$$|\text{Ccl}((1234))| = 30, \quad |\text{Ccl}((12345))| = 24.$$

(c) V on \mathbb{C}^5 permuting the co-ordinates. If we pick a basis ~~then~~ (e_1, \dots, e_5) then

$$e \mapsto \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \quad (12) \mapsto \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \quad (12)(34) \mapsto \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & 1 \end{pmatrix}$$

$$(123) \mapsto \begin{pmatrix} 0 & 0 & 1 & & \\ 0 & 1 & 0 & & \\ 1 & 0 & 0 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \quad (123)(45) \mapsto \begin{pmatrix} 0 & 0 & 1 & & \\ 0 & 1 & 0 & & \\ 1 & 0 & 0 & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix} \quad (1234) \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 & \\ 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$(12345) \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(put zeros where there are empty spaces)

$$\therefore \chi_V(e) = 5, \quad \chi_V((12)) = 3, \quad \chi_V((12)(34)) = 1, \quad \chi_V((123)) = 2$$

$$\chi_V((123)(45)) = 0, \quad \chi_V((1234)) = 1, \quad \chi_V((12345)) = 0.$$

$$(d) \langle \chi_V, \chi_V \rangle = \frac{1}{120} (1 \times 5^2 + 10 \times 3^2 + 15 \times 1^2 + 20 \times 2^2 + 20 \times 0^2 + 30 \times 1^2 + 24 \times 0^2)$$

$$= \frac{1}{120} (25 + 90 + 15 + 80 + 30) = 2.$$

Since this is not 1, χ_V is not irreducible.

$$(e) \text{ We calculate } \langle \chi_T, \chi_V \rangle = \frac{1}{120} (1 \times 1 \times 5 + 10 \times 1 \times 3 + 15 \times 1 \times 1 + 20 \times 1 \times 2 + 20 \times 1 \times 0 + 30 \times 1 \times 1 + 24 \times 1 \times 0)$$

$$= \frac{1}{120} (5 + 30 + 15 + 40 + 30) = 1$$

$\therefore T$ appears with multiplicity one in V .

(f) We have $V = T \oplus W$ and so $\chi_V = \chi_T + \chi_W$.

$$\text{Now } \langle \chi_V, \chi_V \rangle = \langle \chi_T, \chi_T \rangle + 2 \langle \chi_T, \chi_W \rangle + \langle \chi_W, \chi_W \rangle$$

$\begin{array}{cccc} \parallel & \parallel & \parallel & \parallel \\ 2 & 1 & ? & ? \end{array}$

Now $\langle \chi_T, \chi_W \rangle$ is a non-negative integer (since it counts the copies of T in W) and $\langle \chi_W, \chi_W \rangle$ is a positive integer. The only possibility is $\langle \chi_W, \chi_W \rangle = 1$ and $\langle \chi_T, \chi_W \rangle = 0$. Thus χ_W is irreducible.

$$\begin{aligned} 3) \quad \langle X, X \rangle &= \sum_{1 \leq i, j \leq k} d_i d_j \langle \chi_i, \chi_j \rangle \\ &= \sum_{i=1}^k d_i^2 \langle \chi_i, \chi_i \rangle && \text{(since } \langle \chi_i, \chi_j \rangle = 0 \text{ for } i \neq j) \\ &= \sum_{i=1}^k d_i^2 && \text{(since } \langle \chi_i, \chi_i \rangle = 1) \end{aligned}$$

$$\begin{aligned} \text{Thus } \langle X, X \rangle = 1 &\Rightarrow d_i = 1 \text{ for some } i, d_j = 0 \forall j \neq i. \\ = 2 &\Rightarrow d_i = 1, d_{i'} = 1 \text{ for some } i, i', d_j = 0 \forall j \neq i, i'. \\ = 3 &\Rightarrow d_i = 1 = d_{i'} = d_{i''} \text{ some } i, i', i'' \text{ and } d_j = 0 \forall j \neq i, i', i''. \\ = 4 &\Rightarrow \text{either } d_i = 2 \text{ for some } i \text{ and } d_j = 0 \forall j \neq i \\ &\quad \text{or } d_i = 1 = d_{i'} = d_{i''} = d_{i'''} \text{ and } d_j = 0 \text{ for all other } j. \end{aligned}$$

4) The conjugacy classes for D_4 have representatives: e, a, a^2, b, ab

From the given matrices we see $\rho^{(1)}(a^2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\rho^{(1)}(ab) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
 and $\rho^{(2)}(a^2) = \begin{pmatrix} 83 & 130 \\ -53 & -83 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\rho^{(2)}(ab) = \begin{pmatrix} -71 & -112 \\ 45 & 71 \end{pmatrix}$

So let χ_1 be char. of $\rho^{(1)}$, χ_2 char. of $\rho^{(2)}$. Then

$$\begin{aligned} \chi_1(e) = 2 = \chi_2(e); \quad \chi_1(a) = 0 = \chi_2(a); \quad \chi_1(a^2) = -2 = \chi_2(a^2); \\ \chi_1(b) = 0 = \chi_2(b); \quad \chi_1(ab) = 0 = \chi_2(ab). \end{aligned}$$

So the characters are equal and so the representations must be isomorphic.

5.a) A_4 has order 12 ($= 24/2$) and representatives for its conjugacy classes
 $e, (123), (132), (12)(34)$.

These correspond to e, s, s^2 and t . The sizes are 1, 4, 4, 3 respectively.

b) We look for group homomorphisms

$$\begin{aligned} A_4 = \langle s, t : s^3 = t^2 = e, sts = ts^{-1}t \rangle \longrightarrow \mathbb{C}^* \\ s \longmapsto \alpha, \quad t \longmapsto \beta. \end{aligned}$$

Now we need $\alpha^3 = 1, \beta^2 = 1$ and $\alpha\beta\alpha = \beta\alpha^{-1}\beta$.
 This gives first $\beta = \pm 1$ and $\alpha^2\beta = \beta^2\alpha^{-1} = \alpha^{-1} = \alpha^2$
 $\therefore \beta = 1$.

$\alpha = 1, \omega, \omega^2$ all possible where $\omega = \exp(2\pi i/3)$.

So there are 3 different 1-D representations.

	e	s	s^2	t
χ_1	1	1	1	1
χ_2	1	ω	ω^2	1
χ_3	1	ω^2	ω	1

c) # irred. rep^s of $A_4 = \#$ conjugacy classes $A_4 \therefore$ 3 more

irreducible repⁿ, and we will call its character χ_4 .

Moreover $|A_4| = \chi_1(e)^2 + \chi_2(e)^2 + \chi_3(e)^2 + \chi_4(e)^2$

i.e. $12 = 1 + 1 + 1 + 9$.

$\therefore \chi_4(e) = 3$ (since $\chi_4(e) > 0$).

d). We have

	e	s	s ²	t
χ_4	3	A	B	C

Now $\langle \chi_4, \chi_4 \rangle = \frac{1}{12} (1 \cdot 3^2 + 4|A|^2 + 4|B|^2 + 3|C|^2) = 1$ (1)

$\langle \chi_4, \chi_3 \rangle = \frac{1}{12} (1 \cdot 3 \cdot 1 + 4 \cdot A \cdot \bar{\omega} + 4 \cdot B \cdot \bar{\omega}^2 + 3 \cdot C \cdot 1) = 0$ (2)

$\langle \chi_4, \chi_2 \rangle = \frac{1}{12} (1 \cdot 3 \cdot 1 + 4A \cdot \bar{\omega}^2 + 4B \bar{\omega} + 3 \cdot C \cdot 1) = 0$ (3)

$\langle \chi_4, \chi_1 \rangle = \frac{1}{12} (1 \cdot 3 \cdot 1 + 4 \cdot A \cdot 1 + 4 \cdot B \cdot 1 + 3 \cdot C \cdot 1) = 0$ (4)

~~use $4A + 4B + 3C = 0$.~~

Now use $1 + \omega + \omega^2 = 0$ to find, by adding (2)+(3)+(4),

$\frac{1}{12} (3+3+3 + 3C+3C+3C) = 0$ i.e. $C = -1$.

Now (4) becomes $4(A+B) = 0 \implies A = -B$

(3) becomes $4A\bar{\omega}^2 + 4B\bar{\omega} = 0 \implies -B\bar{\omega}^2 + B\bar{\omega} = 0$

i.e. $B = 0 = A$.

$\therefore \chi_4$

	e	s	s ²	t
	3	0	0	-1