Exercises for Lectures 5 and 6.

(1) We know that $\mathbb{C}[G]$ is supposed to see everything, that $\dim \mathbb{C}[G] = |G| = n$ and that G has n distinct irreducible rep^{ns}. So we should expect that $\mathbb{C}[G]$ is the direct sum of ~~each~~ all irred. rep^{ns}. Let's prove this:

set $\zeta = \exp(2\pi\sqrt{-1}/n)$ and then define the following elements of $\mathbb{C}[G]$ for $0 \le i \le n-1$

$$E_i := \sum_{j=0}^{n-1} \zeta^{ij} e_{x^j} \qquad (\text{where } G = \langle x : x^n = e \rangle)$$

Then

$$\rho_{reg}(x)(E_i) = \sum_{j=0}^{n-1} \zeta^{ij} \rho(x) e_{x^j}$$
$$= \sum_{j=0}^{n-1} \zeta^{ij} e_{x^{j+1}} = \sum_{j=0}^{n-1} \zeta^{[(j+1)-1]i} e_{x^{j+1}}$$
$$= \sum_{j=0}^{n-1} \zeta^{(j+1)i} e_{x^{j+1}} \zeta^{-i} = \zeta^{-i} E_i .$$

So E_i spans a 1-D subrep^n of $\mathbb{C}[G]$ corresponding to the rep^n $x \mapsto \zeta^{-i}$. Thus each E_i spans a different subrep^n (i.e. diff e-value) and so we have $\bigoplus_{i=0}^{n-1} \mathbb{C}E_i \subseteq \mathbb{C}[G]$. But both sides have dim^n n and so we have equality

$$\mathbb{C}[G] = \bigoplus_{i=0}^{n-1} \mathbb{C}E_i .$$

Moreover this is the isotypic decomp^n and so it is unique.

(2) Using the relation $b^{-1}ab = a^{-1}$ (or rather $ab = ba^3$) we can write any word in a's + b's in the form $a^i b^j$ for some $i, j \in \mathbb{Z}$ (e.g. $ab^2a^3ba^3 = aea^3ba^3 = a^4ba^3 = ab$, etc) Moreover we can assume that $0 \le i \le 3$ and $0 \le j \le 1$ since $a^4 = e$ and $b^2 = a^2$. This proves that there are at most eight elements: $e, a, a^2, a^3, b, ab, a^2b, a^3b$. We'll show in

part that these elements are distinct. In the meantime we'll just assume it.

(b) $\{e\}$. a^2 is central since it commutes with a ad it commutes with b (since $a^2 = b^2$) $\therefore$ $\{a^2\}$ is a conjugacy class.

a, a^3 are conjugate because of the relation $b^{-1}ab = a^3$,
ab, a^3b are conjugate because of the relation $b^{-1}abb = a^3b$
b, a^2b are conjugate since $a a^2 b a^{-1} = a a^2 a b = b$.

We'll show in (c) that these are the 5 conjugacy classes.

(c) Over $\mathbb{C}$: $a \mapsto \alpha \in \mathbb{C}^*$, $b \mapsto \beta \in \mathbb{C}^*$ s.t $\alpha^4 = 1$, $\alpha^2 = \beta^2$, $\beta^{-1}\alpha\beta = \alpha^{-1}$ i.e $\alpha = \alpha^{-1}$

$\therefore$ $\alpha \mapsto \pm 1$, $\beta \mapsto \pm 1$ are the possibilities

N.B. We see that the conjugacy classes are distinct since choosing $\alpha = -1$, $\beta = 1$ we see that $a, a^3 \mapsto -1$ but $b, a^2b \mapsto 1$ (whereas conjugate elements must be sent to the same element in 1-D rep^{ns} (why?)) ad choosing $\alpha = 1$, $\beta = -1$ $a, a^3 \mapsto 1$, $ab, a^3b \mapsto -1$ ad finally choosing $\alpha = -1, \beta = 1$ $a^3, ab \mapsto -1$ $b, a^2b \mapsto 1$.

(d) Over $\mathbb{F}_2$ we must find $\alpha, \beta \in \mathbb{F}_2^\times$ with $\alpha^4 = 1$ $\alpha^2 = \beta^2$. $\alpha = \alpha^{-1}$
But there's only 1 element in $\mathbb{F}_2^\times$! So there's only one possibility $\alpha, \beta = 1 \in \mathbb{F}_2$

Over $\mathbb{F}_3$ we have 4 choices again $\alpha \mapsto \pm 1$ (or, in other words 1, 2) ad $\beta \mapsto \pm 1$.

(d) We need first to check the relations:

$$a \mapsto \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} \qquad b \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Well, $a^4 = Id$, $a^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = b^2$, and $b^{-1}ab = \begin{pmatrix} -\sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix} = a^{-1}$.

So it's a rep^n. In fact

$$a^i b^j \longmapsto \begin{cases} \begin{pmatrix} (\sqrt{-1})^i & 0 \\ 0 & (-\sqrt{-1})^i \end{pmatrix} & j = 0 \\ \begin{pmatrix} 0 & (\sqrt{-1})^i \\ -(-\sqrt{-1})^i & 0 \end{pmatrix} & j = 1 \end{cases}$$

Thus, since each element $a^i b^j$ is mapped to a different matrix we must have that each $a^i b^j$ $0 \le i \le 3$, $0 \le j \le 1$ is distinct. This finishes off part (a).

Now to irreducibility: any proper subrep^n ~~of~~ would have to be one-dimensional and so would have to be spanned by a simultaneous e-vector of both $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. But the e-vectors of $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and neither of these are e-vectors for $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Thus the rep^n is irreducible.

(3) Im ϕ is certainly a subspace and so we have to show that it is G-invariant. So let $g \in G$ and $x \in \operatorname{Im}\phi$. Then $x = \phi(v)$ for some $v \in V$.

$$\rho_W(g)(x) = \rho_W(g)(\phi(v)) \overset{\text{G-homom. property}}{=} \phi(\rho_V(g)(v)) \in \operatorname{im}\phi$$

Thus $\operatorname{im}\phi$ is G-invariant.

(4) Let $\zeta_j = \exp(2\pi\sqrt{-1}/p_j^{i_j})$ for $j=1,\dots,t$.

Then let $C_{p_j^{i_j}} = \langle x_j : x_j^{p_j^{i_j}} = e \rangle$ so that a typical element of G looks like

$$(x_1^{a_1}, x_2^{a_2}, \dots, x_t^{a_t}) \quad \text{with } 0 \le a_j \le p_j^{i_j}-1 \text{ for } 1\le j\le t.$$

Now we get an irreducible repⁿ of G labelled by a t-tuple of elements ~~between~~

$$(r_1, r_2, \dots, r_t)$$

with $0 \le r_j \le p_j^{i_j}-1$ by sending

$$(x_1^{a_1}, x_2^{a_2}, \dots, x_t^{a_t}) \longmapsto (\zeta_1^{a_1})^{r_1}(\zeta_2^{a_2})^{r_2}\cdots(\zeta_t^{a_t})^{r_t}.$$

This produces all the possible irreducible repⁿˢ of G since each x_j must go to some $\zeta_j^{r_j}$ for some integer r_j (and once we know where the generators $x_1 \dots x_t$ go we know where everything goes because of the ~~gp~~ homomorphism property).

Moreover all of these irred. repⁿˢ are distinct since in each case the generators are sent to different things.

(5). We already know that $\mathrm{Hom}(V,W)$ is a vector space under the given operations, so the content of the question is to show that $\mathrm{Hom}_G(V,W)$ is a subspace under these operations.

Well, if $\phi, \psi \in \mathrm{Hom}_G(V,W)$ and $\lambda, \mu \in F$ then we must show that $\lambda\phi + \mu\psi \in \mathrm{Hom}_G(V,W)$. So let $v \in V$, $g \in G$

$$(\lambda\phi+\mu\psi)(\rho_V(g)(v)) = \lambda(\phi(\rho_V(g)(v))) + \mu(\psi(\rho_V(g)(v)))$$

by definition

$$= \lambda\big(\rho_W(g)(\phi(v))\big) + \mu\big(\rho_W(g)(\psi(v))\big)$$

(since ϕ, ψ are G-intertwining)

$$= \rho_W(g)\big(\lambda(\phi(v))\big) + \rho_W(g)\big(\mu(\psi(v))\big)$$

(since $\rho_W(g)$ is F-linear)

$$= \rho_W(g)\big((\lambda\phi + \mu\psi)(v)\big) \quad \text{(by definition)}$$

$\therefore$ $\lambda\phi + \mu\psi$ is G-intertwining, i.e belongs to $\mathrm{Hom}_G(V, W)$.

(6) Qn (5) proves that $\mathrm{End}_G(V) = \mathrm{Hom}_G(V,V)$ is a subspace of the F-algebra $\mathrm{End}(V)$. We just need to check that $\mathrm{End}_G(V)$ is a subalgebra i.e closed under multiplication ((5) does closed under addition, subtraction and scalar multiplication).

So let $\psi, \phi \in \mathrm{End}_G(V)$, $g \in G$ and $v \in V$.

Then $(\psi\phi)(\rho_V(g)(v)) = \psi(\phi(\rho_V(g)(v)))$ (by definition)

$$= \psi\big(\rho_V(g)(\phi(v))\big) \quad \text{since } \phi \in \mathrm{End}_G(V)$$

$$= \rho_V(g)\big(\psi(\phi(v))\big) \quad \text{" } \psi \text{ "}$$

$$= \rho_V(g)\big((\psi\phi)(v)\big)$$

Thus $\psi\phi \in \mathrm{End}_G(V)$ as req^d.

Now there is an <u>oversight</u> in the rest of the question. $\mathrm{End}_G(V)$ is not generally ~~a division~~ algebra unless V is irreducible.

If V is irreducible Theorem tells us that each $\phi : V \to V$ which is G-intertwining (i.e. each $\phi \in \mathrm{End}_G(V)$) is either zero or invertible. Said another way, each non-zero element of $\mathrm{End}_G(V)$ is invertible.

(7)(a) $\mathrm{End}_G(V) = \mathbb{C}$ by Schur's lemma!

(b) Let $X = \rho_V(x)$ (so it's the matrix $\begin{pmatrix} \cos 2\pi/3 & -\sin 2\pi/3 \\ \sin 2\pi/3 & \cos 2\pi/3 \end{pmatrix}$)

Then $\mathrm{End}(V) = \mathrm{Mat}_2(\mathbb{R})$ and so $E \in \mathrm{End}_G(V)$

iff $E X^i v = X^i E v \quad \forall v \in \mathbb{R}^2, \; 0 \le i \le 2$

i.e iff $E X^i = X^i E \quad 0 \le i \le 2$

iff $EX = XE$

Now $E = \mathrm{Id}, X, X^2$ clearly works!

Moreover there is a relation between these there matraces

$$X^2 + X + \mathrm{Id} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(you should check this if you don't see why: really X corresponds to $e^{2\pi i/3} \in \mathbb{C}$)

So we have found 2 linearly independent matrices in $\mathrm{End}_G(V)$. Now

$$\begin{pmatrix} \alpha & 0 \\ \beta & 0 \end{pmatrix} X \neq X \begin{pmatrix} \alpha & 0 \\ \beta & 0 \end{pmatrix} \qquad \text{unless } \alpha = \beta = 0$$

and so $\mathrm{End}_G \cap \left\{ \begin{pmatrix} \alpha & 0 \\ \beta & 0 \end{pmatrix} : \alpha, \beta \right\} = \{0\}$. Thus we have

$\mathrm{End}_G(V) \oplus \left\{ \begin{pmatrix} \alpha & 0 \\ \beta & 0 \end{pmatrix} : \alpha, \beta \right\} \subseteq \mathrm{End}(V)$. But $\dim \mathrm{End}(V) = 4$ and $\dim \left\{ \begin{pmatrix} \alpha & 0 \\ \beta & 0 \end{pmatrix} : \alpha, \beta \right\} = 2$ and $\dim \mathrm{End}_G(V) \geq 2$ (by above). Thus $\dim \mathrm{End}_G(V) = 2$ and $\mathrm{End}_G(V)$ is generated by Id and X. Now we can identify $\mathrm{End}_G(V)$ with $\mathbb{C}$ via

$$\mathrm{Id} \xmapsto{\theta} 1$$
$$X \xmapsto{\theta} \omega = \exp(2\pi\sqrt{-1}/3).$$

Both sides are 2-dim^l over $\mathbb{R}$ and ~~since~~ the above map is a ring isomorphism

$$\theta\big((a\mathrm{Id} + bX)(c\mathrm{Id} + dX)\big) = \theta\big(ac\mathrm{Id} + (bc+ad)X + bdX^2\big)$$
$$= \theta\big(ac\mathrm{Id} + (bc+ad)X + bd(-\mathrm{Id} - X)\big)$$
since $X^2 + X + \mathrm{Id} = 0$

$$= (ac - bd) + (bc + ad - bd)\omega$$
$$= ac1 + (bc+ad)\omega + bd\omega^2$$
$(\text{since } \omega^2 + \omega + 1 = 0)$

$$= (a1 + b\omega)(c1 + d\omega)$$
$$= \theta(a\mathrm{Id} + bX)\,\theta(c\mathrm{Id} + dX).$$

Thus $\mathrm{End}_G(V) \cong \mathbb{C}$.

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(c) We're supposed to think of the representation

$$a \longmapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$b \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

over $\mathbb{C}$ as a representation over $\mathbb{R}$. Since it had 2 complex dimensions it will have four real dimensions. Before hand our basis was

$$\begin{pmatrix} \alpha + i\beta \\ \gamma + i\delta \end{pmatrix}$$

ad now our basis will be

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \qquad \alpha, \beta, \gamma, \delta \in \mathbb{R}.$$

~~The matrix~~ We have

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \alpha + i\beta \\ \gamma + i\delta \end{pmatrix} = \begin{pmatrix} -\beta + i\alpha \\ \delta - i\gamma \end{pmatrix}$$

Therefore the real matrix we need must send

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \text{ to } \begin{pmatrix} -\beta \\ \alpha \\ \delta \\ -\gamma \end{pmatrix}$$

i.e. $a \longmapsto \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} =: A$

A similar calculation shows

$$b \longmapsto \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} =: B$$

Now you need to find which type of 4×4 matrix commutes with both A and B. This is a brute-force calculation and produces

$$\begin{pmatrix} x & y & z & w \\ -y & x & w & -z \\ -z & -w & x & y \\ -w & z & -y & x \end{pmatrix}$$

This is a four dimensional vector space over $\mathbb{R}$ with basis

$$Id = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$K = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

You can check that $I^2 = J^2 = K^2 = -Id$ and that $IJ = K$, $JI = -K$.

This is called the <u>quaternions</u> and Q in the name of the group becomes apparent.