Exercises on complete reducibility

1. We have two matrices left to calculate in the basis $(e_1+e_2+e_3, e_1, e_2)$

$$(13)\, e_1 = e_3 = (e_1+e_2+e_3) - e_1 - e_2$$
$$(13)\, e_2 = e_2$$

$$\therefore (13) \longmapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$(132)\, e_1 = e_3 = (e_1+e_2+e_3) - e_1 - e_2$$
$$(132)\, e_2 = e_1$$

$$\therefore 132 \longmapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

For the final calculation, in the basis $(e_1+e_2+e_3, e_1-e_2, e_2-e_3)$ we need two more matrices:

$$(13)\cdot(e_1-e_2) = (e_3-e_2) = -(e_2-e_3)$$
$$(13)\cdot(e_2-e_3) = (e_2-e_1) = -(e_1-e_2)$$

$$\therefore (13) \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$(132)\cdot(e_1-e_2) = (e_3-e_1) = -(e_1-e_2) - (e_2-e_3)$$
$$(132)\cdot(e_2-e_3) = (e_1-e_2)$$

$$\therefore (132) \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

2. $U = \mathbb{C}(e_1+e_2+e_3)$; we extend to a basis $(e_1+e_2+e_3, e_1, e_2)$ (this choice is completely free at the moment; different choices will produce different p's, and

$$\left.\begin{array}{l} p(e_1) = \cancel{p((e_1+e_2+e_3)-e_2-e_1)}\ 0 \\ p(e_2) = 0 \\ p(e_3) = p((e_1+e_2+e_3)-e_1-e_2) = e_1+e_2+e_3 \end{array}\right\} \begin{array}{l} p(u+w) = u \\ u \in U \\ w \in sp(e_1, e_2) \end{array}$$

Now $q := \frac{1}{6} \sum_{\sigma \in S_3} \rho(\sigma) \cdot p \cdot \rho(\sigma^{-1})$

$$\therefore\ q(e_1) = \frac{1}{6}\left(p(e_1) + \rho((12))\, p(\rho((12))^{-1} \cdot e_1) + \cdots \right)$$

$\rho((12))e_1 = e_2$, $\rho(13)e_1 = e_3$, $\rho(23)e_1 = e_1$
$\rho(123)e_1 = e_2$ $\rho(132)e_1 = e_3$

$\therefore$ $p(\rho(12)e_1) = 0$ $p(\rho(13)e_1) = e_1+e_2+e_3$, $p(\rho(23)e_1) = 0$

$p(\rho(123)e_1) = 0$ $p(\rho(132)e_1) = e_1+e_2+e_3$

$$\therefore\ q(e_1) = \frac{1}{6}\left(0 + 0 + (e_1+e_2+e_3) + 0 + 0 + (e_1+e_2+e_3)\right)$$
$$= \frac{1}{3}(e_1+e_2+e_3)$$

Now a very similar argument gives $q(e_2) = \frac{1}{3}(e_1+e_2+e_3)$, $q(e_3) = \frac{1}{3}(e_1+e_2+e_3)$.

Since q is linear we get

$$q\left(\sum_{i=1}^{3} \lambda_i e_i\right) = \sum_{i=1}^{3} \lambda_i\, q(e_i) = \frac{1}{3}\left(\sum_{i=1}^{3} \lambda_i\right)(e_1+e_2+e_3)$$

Now $\sum_{i=1}^{3} \lambda_i e_i \in \ker q \Leftrightarrow \sum_{i=1}^{3} \lambda_i = 0$. A basis for this space is $(e_1 - e_2, e_2 - e_3)$ which is W.

(3) $W = sp(e_1 - e_2, e_2 - e_3)$. Extend this to a basis with e_1 so that

$$p : V \to V, \qquad p(w+u) = w \quad \text{where } w \in W,\ u \in \operatorname{span}(e_1)$$

We do the same calculation as in (2):

$$\begin{aligned}
p(e_1) &= 0 \\
p(e_2) &= p(-(e_1-e_2)+e_1) = -(e_1-e_2) = e_2 - e_1 \\
p(e_3) &= p(-(e_1-e_2)-(e_2-e_3)+e_1) = -(e_1-e_2)-(e_2-e_3) \\
&= e_3 - e_1
\end{aligned}$$

Now $q(e_1) = \frac{1}{6}\left(0 + \rho(12)p(\rho(12)^{-1}e_1) + \rho(23)p(\rho(23)^{-1}e_1) + \dots\right)$

$$\begin{aligned}
&= \tfrac{1}{6}\big(0 + \rho(12)(e_2-e_1) + \rho(23)0 + \rho(13)(e_3-e_1) \\
&\qquad + \rho(123)(e_3-e_1) + \rho(132)(e_2-e_1)\big) \\
&= \tfrac{1}{6}\left((e_1-e_2) + (e_1-e_3) + (e_1-e_2) + (e_1-e_3)\right) \\
&= \tfrac{1}{3}\left((e_1-e_2) + (e_1-e_3)\right)
\end{aligned}$$

Similarly,

$$\begin{aligned}
q(e_2) &= \tfrac{1}{6}\big((e_2-e_1) + \rho(12)0 + \rho(23)(e_3-e_1) + \rho(13)(e_2-e_1) \\
&\qquad + \rho(123)0 + \rho(132)(e_3-e_1)\big) \\
&= \tfrac{1}{6}\left((e_2-e_1) + (e_2-e_1) + (e_2-e_3) + (e_2-e_3)\right) \\
&= \tfrac{1}{3}\left((e_2-e_1) + (e_2-e_3)\right.
\end{aligned}$$

And finally
$$q(e_3) = \frac{1}{3}\left((e_3-e_1)+(e_3-e_2)\right)$$

Then
$$q\left(\sum_{i=1}^{3}\lambda_i e_i\right) = \frac{1}{3}\Big[\lambda_1(e_1-e_2)+\lambda_1(e_1-e_3)+\lambda_2(e_2-e_1)+\lambda_2(e_2-e_3)$$
$$+\lambda_3(e_3-e_1)+\lambda_3(e_3-e_2)\Big]$$
$$= \frac{1}{3}\Big((2\lambda_1-\lambda_2-\lambda_3)e_1+(-\lambda_1+2\lambda_2-\lambda_3)e_2$$
$$+(-\lambda_1-\lambda_2+2\lambda_3)e_3\Big)$$

Thus, $\ker q = \left\{\sum_{i=1}^{3}\lambda_i e_i : \begin{array}{l} 2\lambda_1-\lambda_2-\lambda_3=0 \\ -\lambda_1+2\lambda_2-\lambda_3=0 \\ -\lambda_1-\lambda_2+2\lambda_3=0 \end{array}\right\}$

This has a 1-D sol^n space $\lambda_1=\lambda_2=\lambda_3$.

$\therefore \ker q = \mathrm{sp}\{e_1+e_2+e_3\}$.

(4) Suppose ρ is not irreducible. This means that we can find some basis of $\mathbb{C}^2$ s.t ρ has the form
$$\rho(g) = \left(\begin{array}{c|c} \rho_1(g) & 0 \\ \hline 0 & \rho_2(g) \end{array}\right) \qquad \forall g\in G$$
where $\rho_1(g), \rho_2(g) \in GL_1(\mathbb{C})$

i.e $\rho(g)$ is a diagonal matrix $\forall g \in G$. But then
$$\rho(g)\rho(h) = \rho(h)\rho(g) \qquad \forall h,g\in G$$
since diagonal matrices commute with one another. This contradiction proves that ρ must have been irreducible.

Unitary Representations

(5) $\langle x, x \rangle = \sum_{g \in G} (\rho(g)(x), \rho(g)(x)) = \sum_{g \in G} \|\rho(g)(x)\|^2 > 0$

unless $\rho(g)(x) = 0$ i.e unless $x = 0$.

Therefore the form is positive definite.

$$\langle x, y \rangle = \sum_{g \in G} (\rho(g)(x), \rho(g)(y)) = \sum_{g \in G} \overline{(\rho(g)(y), \rho(g)(x))}$$
$$= \overline{\langle y, x \rangle}$$

$$\langle \alpha x + \beta y, z \rangle = \sum_{g \in G} (\rho(g)(\alpha x + \beta y), \rho(g)(z))$$
$$= \sum_{g \in G} (\alpha \rho(g)(x) + \beta \rho(g)(y), \rho(g)(z))$$
$$= \alpha \sum_{g \in G} (\rho(g)(x), \rho(g)(z)) + \beta \sum_{g \in G} (\rho(g)(y), \rho(g)(z))$$
$$= \alpha \langle x, z \rangle + \beta \langle y, z \rangle$$

So the form is hermitian.

Now let $h \in G$. We have

$$\langle \rho(h)(x), \rho(h)(y) \rangle = \sum_{g \in G} (\rho(g)(\rho(h)(x)), \rho(g)(\rho(h)(y)))$$
$$= \sum_{g \in G} (\rho(gh)(x), \rho(gh)(y))$$

because $G = \{g : g \in G\} = \{gh : g \in G\}$

$$= \sum_{g \in G} (\rho(g)(x), \rho(g)(y))$$
$$= \langle x, y \rangle$$

so it is G-invariant.

(6) We started with $(V, (,))$ a given vector space with hermitian form. Now we have found another form $\langle\, , \rangle$ which is G-invariant. However by applying an element from $GL(V)$ we can move from $\langle\, , \rangle$ to $(,)$

i.e. $\exists X \in GL(V)$ s.t. $\langle x, y \rangle = (Xx, Xy)$

(just think of bases: pick an orthonormal basis $\{e_i\}$ wrt $(,)$ and $\{f_i\}$ wrt $\langle\, , \rangle$; then X represents the mapping sending $f_i \mapsto e_i$)

Thus if we let $\sigma : G \longrightarrow GL(V)$ be defined by

$$\sigma(g) = X^{-1} \rho(g) X$$

then we find

$$\begin{aligned} (\sigma(g)(x), \sigma(g)(y)) &= (X^{-1}\rho(g)Xx, X^{-1}\rho(g)Xy) \\ &= \langle \rho(g)Xx, \rho(g)Xy \rangle \\ &= \langle Xx, Xy \rangle = (x, y) \end{aligned}$$

and so σ is a unitary rep^n w.r.t. $(,)$ and it is equivalent to ρ.

(7) Let $v \in U^\perp$. Then $\forall u \in V$ we have

~~$\langle \rho(g)u, v \rangle \overset{G\text{-invariance}}{=} \langle u, \rho(g^{-1})v \rangle = 0$ since $u \in$~~

$\langle u, \rho(g)v \rangle \underset{G\text{-invariance}}{=} \langle \rho(g)^{-1}u, v \rangle = 0$ since $\rho(g)^{-1}u \in U$ and $v \in U^\perp$

$\therefore \rho(g)v \in U^\perp$ and so $U^\perp$ is G-invariant.

Now $V = U \oplus U^\perp$ because

(i) $U \cap U^\perp = \{0\}$: let $u \in U \cap U^\perp$; then $\langle u, u \rangle = 0$, and so $u = 0$.

(ii) $\langle\, , \rangle$ restricts to a non-degenerate pairing

$V/_{U^\perp} \times U \to \mathbb{C}$ $\quad \langle v + U^\perp, u \rangle := \langle v, u \rangle$ (non-degeneracy follows since if $\langle v + U^\perp, u \rangle = 0$ $\forall u \in U$

then $\langle v,u\rangle = 0$ $\forall u \in U$ and so $v \in U^\perp$ and $v + U^\perp = 0 + U^\perp$.
Thus $\dim V/_{U^\perp} = \dim U$ and so $\dim V = \dim U + \dim U^\perp$
and hence

$$V = U \oplus U^\perp \quad (\text{since already } U \oplus U^\perp \subseteq V)$$

(alternatively, just pick ~~an basis~~ restrict $\langle\, ,\, \rangle$ to U, then pick an orthonormal basis of U $\{e_1, \ldots e_t\}$ and then extend this to an orthonormal basis of V $\{e_1, \ldots e_t, e_{t+1}, \ldots e_n\}$
The elements $\{e_{t+1}, \ldots e_n\} \in U^\perp$ and so we get $\dim U + \dim U^\perp \geq \dim V$ and together with $U \oplus U^\perp \subseteq V$ this gives $U \oplus U^\perp = V$.)

(8) Let V be a rep^n of G. ~~WLOG~~ Without loss of generality we can assume that V is unitary. Then let U be a subrep^n. Apply (7) shows that U has a G-invariant complement!