Exercises for lecture 16

1. Let $G = G^{(0)} \supset G^{(1)} \supset G^{(2)} \supset \dots \supset G^{(n)} = \{e\}$ be a chain with the properties

- $G^{(i)} \triangleleft G$, (1)
- $G^{(i)}/G^{(i+1)}$ central in $G/G^{(i+1)}$. (2)

a) Intersect the chain with H to get

$$H^{(i)} := H \cap G^{(i)}$$

This gives

$$H = H^{(0)} \geq H^{(1)} \geq \dots \geq H^{(n)} = \{e\}$$

and each $H^{(i)}$ is normal in H ($x \in H^{(i)}$, $h \in H$ then $hxh^{-1} \in H$ since $h, x \in H$ and $hxh^{-1} \in G^{(i)}$ since $x \in G^{(i)}$ and $G^{(i)}$ is normal $\therefore$ $hxh^{-1} \in H \cap G^{(i)} = H^{(i)}$).

Moreover

$$H^{(i)}/H^{(i+1)} \text{ is central in } H/H^{(i+1)}$$

since if $x \in H^{(i)}$, $y \in H$ then

$$yx\, G^{(i+1)} = xy\, G^{(i+1)}$$

by (2) (as $x \in G^{(i)}$, $y \in G$) and so

$$y^{-1}x^{-1}yx \in G^{(i+1)}$$

But $y^{-1}x^{-1}yx \in H$ too and so $y^{-1}x^{-1}yx \in G^{(i+1)} \cap H = H^{(i+1)}$

$$\therefore \quad yx\, H^{(i+1)} = xy\, H^{(i+1)}, \text{ as required.}$$

b) Let $\bar{K}^{(i)} = G^{(i)}K/K \leq G/K$ ($G^{(i)}K \leq G$ since $G^{(i)}$ and K are normal). We have

$$G/K = \bar{K}^{(0)} \geq \bar{K}^{(1)} \geq \dots \geq \bar{K}^{(n)} = \{e_{G/K}\}$$

Now $\bar{K}^{(i)} \triangleleft G/K$ since $G^{(i)}$ and K are normal

(let $gK \in G/K$ and $xK \in \bar{K}^{(i)}$ i.e. $\underset{g_i k}{\overset{\shortparallel}{x}} \in G^{(i)}K$. Then

$$(gK)(xK)(gK)^{-1} = gxg^{-1}K$$

and $gxg^{-1} = g g_i k g^{-1} = g g_i g^{-1} g k g^{-1} \in G^{(i)}K$)

Now let $xK \in \bar{K}^{(i)}$ and $gK \in \bar{K}^{(0)} = G/K$. Then $x = g_i k$ for some $g_i \in G^{(i)}$ and $k \in K$. We find

$$g^{-1}x^{-1}gx = g^{-1}k^{-1}g_i^{-1}g g_i k$$

$$= k' g^{-1} g_i^{-1} g g_i k \quad (\text{since } g^{-1}k^{-1}g \in K \text{ i.e. } g^{-1}k^{-1} = k'g^{-1} \text{ for some } k' \in K)$$

$$= k' x k$$

for some $x \in G^{(i+1)}$ (since $g_i g \cancel{K} G^{(i+1)} = g g_i G^{(i+1)}$)

$$= x' k'' k \quad (\text{since } k' x k'^{-1} = x' \in G^{(i+1)} \text{ as } G^{(i+1)} \triangleleft G)$$

$$\in G^{(i+1)}K$$

$$\therefore \quad xg\bar{K}^{(i+1)} = gx\bar{K}^{(i+1)} \quad \text{as req}^d.$$

(c) No. S_3 is not nilpotent, but $S_3/\langle(123)\rangle \cong C_2$ is nilpotent and $\langle(123)\rangle$ is nilpotent (since they are both abelian).

To see that S_3 is not abelian note that in a nilpotence chain

$$G = G^{(0)} \underset{\triangleleft}{\supsetneq} G^{(1)} \supset \dots \underset{\triangleleft}{\supsetneq} G^{(n-1)} \supset G^{(n)} = \{e\}$$

we must have that $G^{(n-1)}$ is central in G (and not trivial). But the centre of S_3 is trivial and so no such subgroup can exist, and therefore S_3 cannot be nilpotent.

(2) This is <u>big fun</u>!

Let's calculate some conjugations first

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \lambda & \mu \\ 0 & 1 & \rho \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \lambda & \mu+\rho a-\lambda c \\ 0 & 1 & \rho \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^{-1} \qquad (a,b,c,\lambda,\mu,\rho \in \mathbb{F}_3) \quad (*)$$

This shows that the elements

$\begin{pmatrix} 1 & \lambda & * \\ 0 & 1 & \rho \\ 0 & 0 & 1 \end{pmatrix}$ make up 1 conjugacy class with three elements (λ, ρ fixed, not both 0, $*$ anything)

whilst

$\begin{pmatrix} 1 & 0 & \mu \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are in conjugacy classes on their own (ie are central elements)

Thus there are 11 conjugacy classes with representatives and sizes

$$\begin{array}{ccccc} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & \rho \\ 0 & 0 & 1 \end{pmatrix} & : \lambda, \rho \in \mathbb{F}_3 \\ 1 & 1 & 1 & 3 & \lambda \,\&\, \rho \text{ not both zero} \end{array}$$

We have to find 11 irreducible representations.

We look for 1-D rep^{ns} first. By earlier work (Exercises for Lectures 10–13) we need to calculate G' and then G/G'.

Well to get at G' we need to look at elements $xyx^{-1}y^{-1}$ i.e. at

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \lambda & \mu \\ 0 & 1 & \rho \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\lambda & \lambda\rho-\mu \\ 0 & 1 & -\rho \\ 0 & 0 & 1 \end{pmatrix}$$

$$\parallel$$

$$\begin{pmatrix} 1 & 0 & \rho a-\lambda c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus as we let $a, b, c, \lambda, \mu, \rho$ vary in $\mathbb{F}_3$ we get elements of the form

$$\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $* \in \mathbb{F}_3$ is arbitrary. Thus

$$G' = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : b \in \mathbb{F}_3 \right\}$$

Moreover $G/G' \xrightarrow{\psi} \mathbb{F}_3 \times \mathbb{F}_3$ (RHS ~~as gp~~ has $\mathbb{F}_3$ as a group under $+$)

$$\psi\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = (a, c)$$

is a group homomorphism $\left(\psi\left(\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & A & B \\ 0 & 1 & C \\ 0 & 0 & 1 \end{pmatrix}\right) = \right.$

$$\left. \psi\begin{pmatrix} 1 & a+A & B+aC+b \\ 0 & 1 & c+C \\ 0 & 0 & 1 \end{pmatrix} = (a+A, c+C) = \psi\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \cdot \psi\begin{pmatrix} 1 & A & B \\ 0 & 1 & C \\ 0 & 0 & 1 \end{pmatrix} \right),$$

which is surjective (obviously) and has kernel $\begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Thus $G/G' \cong \mathbb{F}_3 \times \mathbb{F}_3$. This is an abelian group isomorphic to $C_3 \times C_3$ (generators $(1,0)$ and $(0,1)$) and so we deduce that G/G' has 9 1-D rep^{ns} and hence so too does G. They are described by

	$\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix} 1&0&1\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix} 1&0&2\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix} 1&\lambda&0\\0&1&\mu\\0&0&1\end{pmatrix}$
χ_1	1	1	1	1
χ_2	1	1	1	ω^{λ}
χ_3	1	1	1	$\omega^{2\lambda}$
χ_4	1	1	1	ω^{μ}
χ_5	1	1	1	$\omega^{\lambda+\mu}$
χ_6	1	1	1	$\omega^{2\lambda+\mu}$
χ_7	1	1	1	$\omega^{2\mu}$
χ_8	1	1	1	$\omega^{\lambda+2\mu}$
χ_9	1	1	1	$\omega^{2\lambda+2\mu}$

(the first three elements are in G' so act trivially, the last bunch of elements are coset reps for G/G')
Here $\omega = \exp(2\pi\sqrt{-1}/3)$

Now we have 2 more irreducibles to find. They must satisfy

$$|G| = 27 = \sum_{i=1}^{11} \chi_i(e)^2 = 9 + \chi_{10}(e)^2 + \chi_{11}(e)^2$$

Thus $\chi_{10}(e) = \chi_{11}(e) = 3$. By our results on nilpotent groups we know that both these must be induced for 1-D rep^{ns} of subgroups of G. Moreover the subgroups considered must have index 3 since

$$\dim \mathrm{Ind}_H^G W = [G:H] \dim W = [G:H]$$

is supposed to be 3 in both cases.

One obvious subgroup of G of index 3 is

$$H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a, b \in \mathbb{F}_3 \right\}$$

Moreover

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & A & B \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+A & b+B \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and so the mapping

$$\theta : H \longrightarrow \mathbb{F}_3 \times \mathbb{F}_3$$
$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longmapsto (a, b)$$

is a group isomorphism. In particular H is abelian and we understand all of its irreducible rep^{ns}. We'll pick two special ones V and W where they are both 1-D but

$$\rho_V \left(\begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = \omega^b \quad \text{and} \quad \rho_W \left(\begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = \omega^{2b}.$$

We claim $\chi_{10} = \chi_V \uparrow_H^G$ and $\chi_{11} = \chi_W \uparrow_H^G$

i.e. we claim that both $\chi_V \uparrow_H^G$ and $\chi_W \uparrow_H^G$ are irred. characters of degree 3 and that they are distinct.
To do this we need to calculate:

$$\chi_V \uparrow_H^G (g) = \sum_{r \in R} \dot{\chi}_V (r^{-1} g r)$$ where R is a set of coset representatives for H in G.

We pick the most obvious set for R:

$$r_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad r_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad r_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Now our first calculation in this question, called $\circledast$, shows that

$$r_i^{-1}\begin{pmatrix}1&\lambda&0\\0&1&\rho\\0&0&1\end{pmatrix}r_i = \begin{pmatrix}1&\lambda&i\lambda\\0&1&\rho\\0&0&1\end{pmatrix}$$

and
$$r_i^{-1}\begin{pmatrix}1&0&\mu\\0&1&0\\0&0&1\end{pmatrix}r_i = \begin{pmatrix}1&0&\mu\\0&1&0\\0&0&1\end{pmatrix} \qquad (\ast\ast)$$

Since $\begin{pmatrix}1&\lambda&i\lambda\\0&1&\rho\\0&0&1\end{pmatrix} \in H \iff \rho=0$ we get the following calculation for the characters

$$\chi_v\uparrow_H^G\left(\begin{pmatrix}1&\lambda&0\\0&1&\rho\\0&0&1\end{pmatrix}\right) = \begin{cases}\omega^0+\omega^\lambda+\omega^{2\lambda} & \rho=0\\ 0 & \rho\neq 0\end{cases}$$

λ, ρ not both zero

$$\left[\begin{aligned}&\text{since } \chi_v\left(r_0^{-1}\begin{pmatrix}1&\lambda&0\\0&1&0\\0&0&1\end{pmatrix}r_0\right)+\chi_v\left(r_1^{-1}\begin{pmatrix}1&\lambda&0\\0&1&0\\0&0&1\end{pmatrix}r_1\right)+\chi_v\left(r_2^{-1}\begin{pmatrix}1&\lambda&0\\0&1&0\\0&0&1\end{pmatrix}r_2\right)\\ &= \chi_v\begin{pmatrix}1&\lambda&0\\0&1&0\\0&0&1\end{pmatrix}+\chi_v\begin{pmatrix}1&\lambda&\lambda\\0&1&0\\0&0&1\end{pmatrix}+\chi_v\begin{pmatrix}1&\lambda&2\lambda\\0&1&0\\0&0&1\end{pmatrix}\end{aligned}\right]$$

But if $\rho=0$ then we assume $\lambda\neq 0$ and so
$$\omega^0+\omega^\lambda+\omega^{2\lambda} = 1+\omega+\omega^2=0$$

Thus
$$\chi_v\uparrow_H^G\left(\begin{pmatrix}1&\lambda&0\\0&1&\rho\\0&0&1\end{pmatrix}\right)=0$$

Similarly $\chi_w\uparrow_H^G\left(\begin{pmatrix}1&\lambda&0\\0&1&\rho\\0&0&1\end{pmatrix}\right)=0$

Finally, calculating using the formula for $\chi_v\uparrow_H^G$ and $(\star\star)$

$$\chi_v\uparrow_H^G\left(\begin{pmatrix}1&0&\mu\\0&1&0\\0&0&1\end{pmatrix}\right)=3\omega^{\mu}$$

and

$$\chi_w\uparrow_H^G\left(\begin{pmatrix}1&0&\mu\\0&1&0\\0&0&1\end{pmatrix}\right)=3\omega^{2\mu}.$$

So our characters behave as follows:

	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&2\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&\lambda&0\\0&1&\rho\\0&0&1\end{pmatrix}$: λ,ρ not both zero
$\chi_v\uparrow_H^G$	3	3ω	$3\omega^2$	0
$\chi_w\uparrow_H^G$	3	$3\omega^2$	3ω	0

Finally

$$\langle \chi_v\uparrow_H^G, \chi_v\uparrow_H^G\rangle = \frac{1}{27}\left(1.3.3+1.3\omega.\overline{3\omega}+1.3\omega^2\,\overline{3\omega^2}+3.0.0 + 3.0.0+\dots+3.0.0\right)$$

$$=\frac{1}{27}(9+9+9)=1$$

and similarly

$$\langle \chi_w\uparrow_H^G, \chi_w\uparrow_H^G\rangle = 1$$

So $\chi_v\uparrow_H^G$ and $\chi_w\uparrow_H^G$ are both irreducible and are distinct. Hence they give χ_{10} and χ_{11} and the character table is complete.

(3) A typical element of this subgroup looks like

$$h^{i_1} z_1 \, h^{i_2} z_2 \ldots$$

where $i_1, i_2, \ldots, \in \mathbb{Z}$ and $z_1, z_2 \in Z$.
After commuting z_i's to the right we get a typical element looking like

$$h^i z$$

$i \in \mathbb{Z}$, $z \in Z$.
Now let $g \in G$. Then

$$g h^i z g^{-1} = \underbrace{g h g^{-1} g h g^{-1} \ldots g h g^{-1}}_{i \text{ times}} g z g^{-1}$$

$$\in hZ hZ \ldots hZ z$$

So $g h^i z g^{-1} = h^i z'$ for some $z' \in Z$ and we deduce that the subgroup is normal.

4) We know all about the characters/representations of D_4. ~~There~~
• there are 4 one-dimensional ones
I_1, I_2, I_3, I_4 (corresponding to $\chi_1, \ldots, \chi_4$)
We have $\mathrm{Ind}_G^G I_j \cong I_j$ $(j = 1, \ldots 4)$ and so these are induced from one-dimensional rep^{ns}.
• there is one 2-dim^l rep^n I_5 whose character is

$\cancel{x}$	e	a	a^2	b	ab
χ_5	2	0	-2	0	0

Now we claim $\chi_5 = \chi_i \uparrow_{\langle a \rangle}^{D_4}$ where χ_i is the following character of $\langle a \rangle$

	e	a	a^2	a^3
χ_i	1	$\sqrt{-1}$	-1	$-\sqrt{-1}$

Indeed $\langle \chi_i \uparrow_{\langle a \rangle}^{D_4}, \chi_5 \rangle_{D_4} = \langle \chi_i, \chi_5 \downarrow_{\langle a \rangle}^{D_4} \rangle_{\langle a \rangle}$

$$\begin{array}{lcccc} & e & a & a^2 & a \\ \text{and} \quad \chi_5 \downarrow^{D_4}_{\langle a \rangle} & 2 & 0 & -2 & 0 \end{array}$$

so that

$$\langle \chi_{\sqrt{-1}}, \chi_5 \downarrow^{D_4}_{\langle a \rangle} \rangle = \frac{1}{4}(1.2 + 0.\sqrt{-1} + -2.-1 + 0.\sqrt{-1})$$
$$= 1$$

$\therefore \Phi_5$ is a summand of $\mathrm{Ind}^{D_4}_{\langle a \rangle} \underbrace{\mathbb{C}_{\sqrt{-1}}}_{\text{rep}^n \text{ corresponding to } \chi_{\sqrt{-1}}}$.

But I_5 and $\mathrm{Ind}^{D_4}_{\langle a \rangle} \mathbb{C}_{\sqrt{-1}}$ both have dim^n 2 and so must be equal.

(5) Exactly the same argument works since the character table is the same and since $\langle a \rangle \leq Q$ is a cyclic subgroup of order 4.