# Exercises for 14 & 15.

1. $\mathrm{Ind}_H^G W = \{ f : G \to W : \quad f(gh^{-1}) = \rho_W(h) f(g) \quad \forall h \in H \ \forall g \in G \}$

Functions from $G \to W$ make a vector space via

$$(\lambda_1 f_1 + \lambda_2 f_2)(g) := \lambda_1 f_1(g) + \lambda_2 f_2(g).$$

Now, to show $\mathrm{Ind}_H^G W$ is a subspace we must prove that if $\lambda_1, \lambda_2 \in \mathbb{C}$ and $f_1, f_2 \in \mathrm{Ind}_H^G(W)$ then $\lambda_1 f_1 + \lambda_2 f_2 \in \mathrm{Ind}_H^G W$

i.e. that $(\lambda_1 f_1 + \lambda_2 f_2)(gh^{-1}) = \rho_W(h)\big((\lambda_1 f_1 + \lambda_2 f_2)(g)\big)$

But this is easy:

$$(\lambda_1 f_1 + \lambda_2 f_2)(gh^{-1}) = \lambda_1 f_1(gh^{-1}) + \lambda_2 f_2(gh^{-1})$$

since $f_1, f_2 \in \mathrm{Ind}_H^G W$ $\quad = \lambda_1 (\rho_W(h) f_1(g)) + \lambda_2 (\rho_W(h) f_2(g))$

since $\rho_W(h)$ is linear $\quad = \rho_W(h)(\lambda_1 f_1(g) + \lambda_2 f_2(g))$

$$= \rho_W(h)\big((\lambda_1 f_1 + \lambda_2 f_2)(g)\big)$$

2. So let's look at the mapping (with $H = G$)

$$\varphi \ \mathrm{Ind}_H^G W \longrightarrow W$$
$$f \longmapsto f(e)$$

Now this is linear since

$$\varphi(\lambda_1 f_1 + \lambda_2 f_2) = (\lambda_1 f_1 + \lambda_2 f_2)(e) \overset{\text{by definition}}{=} \lambda_1 f_1(e) + \lambda_2 f_2(e)$$

It's $G$-intertwining since

$$\psi(g\cdot f) = (g\cdot f)(e) \qquad g \in G, f \in \mathrm{Ind}_H^G W$$

$$= f(g^{-1}e) \qquad \text{def}^n \text{ of } g\cdot f$$

$$= f(eg^{-1})$$

$$= \rho_W(g) f(e) \qquad \text{since } f \in \mathrm{Ind}_H^G W \text{ and } H = G$$

$$= \rho_W(g)\psi(f).$$

Finally, it's injective because if

$$\psi(f_1) = \psi(f_2)$$

then $f_1(e) = f_2(e)$ and so

$$\rho_W(g^{-1})(f_1(e)) = \rho_W(g^{-1})(f_2(e)) \qquad \forall g \in G$$

and hence (since $f_1, f_2 \in \mathrm{Ind}_H^G W$ and $g \in H = G$)

$$f_1(eg) = f_2(eg)$$

i.e. $f_1(g) = f_2(g) \qquad \forall g \in G$

In other words $f_1 = f_2$.

So we have an injective $G$-intertwining homomorphism $\psi : \mathrm{Ind}_H^G W \to W$. Since $\dim \text{LHS} = [G:H]\dim W$
$= \dim W = \dim \text{RHS}$
($[G:H] = 1$ since $G = H$) both sides have the same dim$^n$ and so $\psi$ is an isomorphism.

3. We need to show that $\langle \chi_V, \chi_1 \rangle = \langle \chi_V, \chi_2 \rangle = 1$ where $V = \mathrm{Ind}_{A_4}^{A_5} \mathbb{C}$.

$\chi_V = \chi_{triv}\big\uparrow_{A_4}^{A_5}$. Using Frobenius reciprocity we have

$$\langle \chi_V, \chi_i \rangle = \langle \chi_{triv}\big\uparrow_{A_4}^{A_5}, \chi_i \rangle = \langle \chi_{triv}, \chi_i \big\downarrow_{A_4}^{A_5} \rangle$$

Now $A_4$ has four conjugacy classes with representatives and sizes

| | $e$ | $(1\,2\,3)$ | $(1\,3\,2)$ | $(1\,2)(3\,4)$ |
|---|---|---|---|---|
| | 1 | 4 | 4 | 3 |
| $\chi_{triv}$ | 1 | 1 | 1 | 1 |
| $\chi_1\big\downarrow_{A_4}^{A_5}$ | 1 | 1 | 1 | 1 |
| $\chi_2\big\downarrow_{A_4}^{A_5}$ | 4 | 1 | 1 | 0 |

Thus

$$\langle \chi_{triv}, \chi_1\big\downarrow_{A_4}^{A_5} \rangle = \frac{1}{12}(1.1 + 4.1.1 + 4.1.1 + 3.1.1) = 1$$

$$\langle \chi_{triv}, \chi_2\big\downarrow_{A_4}^{A_5} \rangle = \frac{1}{12}(1.4.1 + 4.1.1 + 4.1.1 + 3.1.0) = 1$$

So $I_1 \oplus I_2 \subseteq V$. $(*)$

But $\dim V = \dim \mathrm{Ind}_{A_4}^{A_5} \mathbb{C} = [A_5 : A_4] \dim \mathbb{C} = 5$ and $\dim I_1 = 1$, $\dim I_2 = 4$ so the LHS and RHS of $(*)$ have the same dim$^n$. Thus

$$I_1 \oplus I_2 = V.$$

4. $\mathrm{Ind}_H^G (W_1 \oplus W_2) \xrightarrow{\Psi} \mathrm{Ind}_H^G W_1 \oplus \mathrm{Ind}_H^G W_2$

$$f \longmapsto (f_1, f_2)$$

where

$$f(g) = (f_1(g), f_2(g)) \in W_1 \oplus W_2.$$

We need to check that $\psi$ is a $G$-intertwining linear bijection.

The bijection is obvious since $f$ is determined by $f_1$ and $f_2$ and conversely $f_1$ and $f_2$ are determined by $f$.

We need to prove linearity:

$$\psi(\lambda f + \lambda' f') = ((\lambda f + \lambda' f')_1, (\lambda f + \lambda' f')_2)$$

where

$$\begin{aligned}
((\lambda f + \lambda' f')_1(g), (\lambda f + \lambda' f')_2(g)) &= (\lambda f + \lambda' f')(g) \\
\text{by def}^n \quad &= \lambda f(g) + \lambda' f'(g) \\
\text{by def}^n \quad &= \lambda (f_1(g), f_2(g)) + \lambda' (f_1'(g), f_2'(g)) \\
&= (\lambda f_1(g) + \lambda' f_1'(g), \lambda f_2(g) + \lambda' f_2'(g)) \\
&= ((\lambda f_1 + \lambda' f_1')(g), (\lambda f_2 + \lambda' f_2')(g))
\end{aligned}$$

Thus $(\lambda f + \lambda' f')_1 = \lambda f_1 + \lambda' f_1'$, $(\lambda f + \lambda' f')_2 = \lambda f_2 + \lambda' f_2'$

and so

$$\begin{aligned}
\psi(\lambda f + \lambda' f') &= (\lambda f_1 + \lambda' f_1', \lambda f_2 + \lambda' f_2') \\
&= \lambda (f_1, f_2) + \lambda' (f_1', f_2') \\
&= \lambda \psi(f) + \lambda' \psi(f')
\end{aligned}$$

It follows that $\psi$ is linear.

For G-intertwining

$$\psi(g\cdot f) = ((g\cdot f)_1, (g\cdot f)_2) \qquad \forall g\in G.$$

Here

$$\begin{aligned}((g\cdot f)_1(h), (g\cdot f)(h)) &= (g\cdot f)(h) \qquad \forall h\in G\\ &= f(g^{-1}h)\\ &= (f_1(g^{-1}h), f_2(g^{-1}h))\\ &= ((g\cdot f_1)(h), (g\cdot f_2)(h))\end{aligned}$$

Thus $(g\cdot f)_1 = g\cdot f_1$, $(g\cdot f)_2 = g\cdot f_2$ and so

$$\psi(g\cdot f) = (g\cdot f_1, g\cdot f_2) = g\cdot(f_1, f_2) = g\cdot\psi(f)$$

as required.

5) $\psi: \mathrm{Ind}_K^G(\mathrm{Ind}_H^K W) \longrightarrow \mathrm{Ind}_H^G W$

$$\psi(F)(g) = F(g)(e_K).$$

To check G-intertwining we take an arbitrary element of G, say $g_1$, and we show that

$$\psi(g_1\cdot F) = g_1\cdot\psi(F)$$

On the LHS the action is on $F\in \mathrm{Ind}_K^G(\mathrm{Ind}_H^K W)$; on the RHS the action is on $\mathrm{Ind}_H^G W$.

Let's go!

$$\psi(g_1 \circ F)(g) = (g_1 \cdot F)(g)(e_K)$$

$$= F(g_1^{-1} g)(e_K) \qquad \text{definition of } g_1 \circ F$$

$$(g_1 \circ \psi(F))(g) = \psi(F)(g_1^{-1} g)$$

$$= F(g_1^{-1} g)(e_K)$$

$$\therefore \quad \psi(g_1 \cdot F) = g_1 \cdot \psi(F) \qquad \forall g_1 \in G$$

6. Again we'll use Frobenius reciprocity; we have to calculate

$$\langle \chi_{triv} \uparrow_{A_3}^{A_5}, \chi_i \rangle_{A_5} = \langle \chi_{triv}, \chi_i \downarrow_{A_3}^{A_5} \rangle_{A_3}$$

for $i = 1 \longmapsto$

The ccls for $A_3$ are $e, (123), (132)$, ~~go~~ each with size one so we find

| | $e$ | $(123)$ | $(132)$ |
|---|---|---|---|
| $\chi_1 \downarrow_{A_3}^{A_5}$ | 1 | 1 | 1 |
| $\chi_2 \downarrow_{A_3}^{A_5}$ | 4 | 1 | 1 |
| $\chi_3 \downarrow_{A_3}^{A_5}$ | 5 | -1 | -1 |
| $\chi_4 \downarrow_{A_3}^{A_5}$ | 3 | 0 | 0 |
| $\chi_5 \downarrow_{A_3}^{A_5}$ | 3 | 0 | 0 |

$$\langle \chi_{triv}, \chi_1 \downarrow_{A_3}^{A_5} \rangle = \frac{1}{3}(1.1.1 + 1.1.1 + 1.1.1) = 1$$

$$\langle \chi_{triv}, \chi_2 \downarrow_{A_3}^{A_5} \rangle = \frac{1}{3}(\cancel{4.}4 + 1 + 1) = 2$$

$$\langle \chi_{triv}, \chi_3 \downarrow^{A_5}_{A_3} \rangle = \frac{1}{3}(5 - 1 - 1) = 1$$

$$\langle \chi_{triv}, \chi_4 \downarrow^{A_5}_{A_3} \rangle = \frac{1}{3}(3 + 0 + 0) = 1$$

$$\langle \chi_{triv}, \chi_5 \downarrow^{A_5}_{A_3} \rangle = \frac{1}{3}(3 + 0 + 0) = 1$$

$$\therefore \operatorname{Ind}^{A_5}_{A_3} \mathbb{C} \cong I_1 \oplus I_2^{\oplus 2} \oplus I_3 \oplus I_4 \oplus I_5 .$$